

HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

$$1. I = \int_{-\alpha}^{(\pi-\alpha)} \sin |t| dt \quad \text{where } 2x - \alpha = t \Rightarrow dx = \frac{dt}{2}$$

$$\begin{aligned} &= \frac{1}{2} \int_{-\alpha}^0 -\sin t dt + \frac{1}{2} \int_0^{\pi-\alpha} \sin t dt \\ &= \frac{1}{2} \cos t \Big|_{-\alpha}^0 - \frac{1}{2} \cos t \Big|_0^{\pi-\alpha} \\ &= \frac{1}{2} [1 - \cos \alpha] - \frac{1}{2} [-\cos \alpha - 1] \\ &= \frac{1}{2} (1 - \cos \alpha) + \frac{1}{2} (1 + \cos \alpha) = 1 \end{aligned}$$

$$2. \text{ Note that in } \left(-\frac{1}{2}, \frac{1}{2}\right), \sin^{-1}(3x - 4x^3) = 3 \sin^{-1}x$$

$$\text{and } \cos^{-1}(4x^3 - 3x) = 2\pi - 3 \cos^{-1}x$$

$$\text{hence } f(x) = 3 \sin^{-1}x - 2\pi + 3 \cos^{-1}x = -\frac{\pi}{2}$$

$$\therefore I = -\frac{\pi}{2} \int_{-1/2}^{1/2} dx = -\frac{\pi}{2}$$

$$5. f'(x) = e^{g(x)} \cdot g'(x); g'(x) = \frac{x}{1+x^4};$$

$$f'(x) = e^{g(x)} \cdot \frac{x}{1+x^4}; e^{g(2)} = e^0 = 1$$

$$\text{hence } f'(2) = e^{g(2)} \cdot g'(2) = e^0 \cdot \frac{2}{17} = \frac{2}{17}$$

$$6. \text{ put } \ln(\ln(\ln x)) = t$$

$$\int_1^e \frac{dt}{t} = \ln t \Big|_1^e = 1$$

$$7. I = \int_1^e (x+1) e^{x \ln x} dx = \int_1^e [x \ln x + \ln x + 1 - 1] e^x dx$$

$$\begin{aligned} &= \int_1^e [x \ln x + \ln x + 1] e^x dx - \int_1^e e^x dx \\ &= [x \ln x e^x]_1^e - [e^x]_1^e = ee^e - e^e + e \end{aligned}$$

$$\begin{aligned} 9. \int_2^4 \left[\frac{1}{\log_2 x} - \frac{1}{\ln 2 (\log_2 x)^2} \right] dx \\ = \int_2^4 \left[\frac{1}{\log_2 x} + \frac{(-x)}{x \ln 2 (\log_2 x)^2} \right] dx = \left[\frac{x}{\log_2 x} \right]_2^4 = 0 \end{aligned}$$

$$10. F(x) = \int \frac{\sin x}{x} dx$$

$$\text{Now } I = \int_1^3 \frac{\sin 2x}{x} dx \quad [\text{put } 2x = t]$$

$$= \int_2^6 \frac{2 \sin t}{2t} dt = [F(x)]_2^6 = F(6) - F(2)$$

$$11. \text{ put } e^{x^2} = t; e^{x^2} \cdot 2x dx = dt;$$

$$\int_1^{\pi/2} \cos t dt = [\sin t]_1^{\pi/2} = 1 - (\sin 1)$$

$$13. \int_{-3}^3 \frac{t^2 \sin 2t}{t^2 + 1} dt = 0 \text{ as the integrand is an odd function.}$$

$$\text{also } \int_0^1 \frac{dt}{t^2 + 2t \cos \alpha + 1} = \frac{1}{\sin \alpha} \tan^{-1} \frac{t + \cos \alpha}{\sin \alpha} \Big|_0^1 = \frac{\alpha}{2 \sin \alpha}$$

Thus the given equation reduces to

$$x^2 \frac{\alpha}{2 \sin \alpha} - 2 = 0 \Rightarrow x = \pm 2 \sqrt{\frac{\sin \alpha}{\alpha}}$$

$$14. \text{ put } x = \tan \theta$$

$$I = \int_0^{\pi/2} \frac{d\theta}{1 + (\tan \theta)^a} = \int_0^{\pi/2} \frac{(\cos \theta)^a}{(\sin \theta)^a + (\cos \theta)^a} d\theta \Rightarrow I = \frac{\pi}{4}$$

$$16. f'(x) = \begin{cases} 1 & \text{for } 0 < x \leq 1 \\ x & \text{for } x > 1 \end{cases}$$

$$\text{put } \ln x = t \Rightarrow x = e^t$$

$$\text{for } x > 1; f'(t) = e^t \text{ for } t > 0$$

$$\text{integrating } f(t) = e^t + C; f(0) = e^0 + C$$

$$\Rightarrow C = -1 \quad (\text{given } f(0) = 0)$$

$$\therefore f(t) = e^t - 1 \text{ for } t > 0 \text{ (corresponding to } x > 1)$$

$$\text{hence } f(x) = e^x - 1 \text{ for } x > 0 \dots (1)$$



again for $0 < x \leq 1$

$$f'(x) = 1 \quad (x = e^t)$$

$$f'(t) = 1 \quad \text{for } t \leq 0$$

$$f(t) = t + C$$

$$f(0) = 0 + C$$

$$\Rightarrow C = 0 \Rightarrow f(t) = t \quad \text{for } t \leq 0$$

$$\Rightarrow f(x) = x \quad \text{for } x \leq 0$$

$$18. \quad T_r = \frac{\pi}{6n} \sec^2 \frac{r\pi}{6n}$$

$$S = \sum T_r = \frac{\pi}{6n} \sum_{r=1}^n \sec^2 \frac{r\pi}{6n} = \frac{\pi}{6} \int_0^1 \sec^2 \frac{\pi x}{6} dx = \tan \frac{\pi x}{6} \Big|_0^1$$

$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$19. \quad \lim_{\lambda \rightarrow 0} \left(\int_0^1 (1+x)^\lambda dx \right)^{1/\lambda} = \lim_{\lambda \rightarrow 0} \left(\frac{(1+x)^{\lambda+1}}{\lambda+1} \Big|_0^1 \right)^{1/\lambda}$$

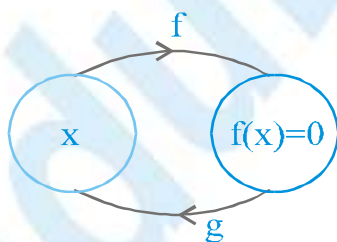
$$= \lim_{\lambda \rightarrow 0} \left(\frac{2^{\lambda+1} - 1}{\lambda+1} \right)^{1/\lambda} \quad (1^\infty \text{ form})$$

$$= e^{\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left(\frac{2^{\lambda+1} - 1}{\lambda+1} - 1 \right)} = e^{\lim_{\lambda \rightarrow 0} \left(\frac{2^{\lambda+1} - 2 - \lambda}{\lambda(\lambda+1)} \right)}$$

$$= e^{\lim_{\lambda \rightarrow 0} \left(\frac{2(2^\lambda - 1)}{\lambda} - 1 \right)} = e^{2 \ln 2 - 1} = e^{\ln \left(\frac{4}{e} \right)} = \frac{4}{e}$$

$$20. \quad f'(x) = \frac{1}{\sqrt{1+x^4}} = \frac{dy}{dx}$$

$$\text{now } g'(x) = \frac{dx}{dy} = \sqrt{1+x^4}$$



$$\text{when } y=0 \text{ i.e. } \int_2^x \frac{dt}{\sqrt{1+t^4}} = 0 \text{ then } x=2 \text{ (think !)}$$

$$\text{hence } g'(0) = \sqrt{1+16} = \sqrt{17}$$

$$22. \quad I = \int_0^{3\pi/4} (\sin x + \cos x) dx + \int_0^{3\pi/4} x (\sin x + \cos x) dx$$

$$= \int_0^{3\pi/4} (\sin x + \cos x) dx + \frac{x}{1} \left(-\cos x - \sin x \right) \Big|_0^{3\pi/4} + \int_0^{3\pi/4} (\sin x + \cos x) dx$$

$$= 2 \int_0^{3\pi/4} (\sin x + \cos x) dx = 2(\sqrt{2} + 1)$$

$$23. \quad \int_0^1 (1 + \cos^8 x)(ax^2 + bx + c) dx$$

$$= \int_0^1 (1 + \cos^8 x)(ax^2 + bx + c) dx + \int_1^2 (1 + \cos^8 x)(ax^2 + bx + c) dx$$

$$\therefore \int_1^2 (1 + \cos^8 x)(ax^2 + bx + c) dx = 0$$

since $1 + \cos^8 x$ is always positive

$$= \int_a^b f(x) dx = 0 \quad (b > a)$$

means $f(x)$ is positive in some portion and negative in some portion from a to b

$\therefore ax^2 + bx + c$ is positive and negative in $(1, 2)$

$\therefore ax^2 + bx + c$ has a root in $(1, 2)$

$$24. \quad y = f(x) \Rightarrow x = f^{-1}(y) \text{ and } dy = f'(x) dx$$

$$\text{now } I = \int_3^7 f^{-1}(x) dx = \int_3^7 f^{-1}(y) dy = \int_2^5 x f'(x) dx ;$$

(when $y=3$ then $x=2$ and $y=7$ then $x=5$)

$$\text{hence } I = \int_2^5 x f'(x) dx. \text{ Integrating by parts gives,}$$

$$I = x f(x) \Big|_2^5 - \int_2^5 f(x) dx$$

$$I = 5 \cdot 7 - 2 \cdot 3 - 17 = 35 - 6 - 17 = 12$$

$$25. \int_a^y \cos t^2 dt = \int_a^{x^2} \frac{\sin t}{t} dt$$

differentiating both sides w.r.t x we get

$$\frac{d}{dx} \int_a^y \cos t^2 dt = \frac{d}{dx} \int_a^{x^2} \frac{\sin t}{t} dt$$

$$\text{RHS} = \frac{\sin[x^2]}{x^2} \frac{dx^2}{dx} = 2x \frac{\sin x^2}{x^2}$$

$$\text{L.H.S.} = \frac{d}{dy} \left(\int_a^y \cos t^2 dt \right) \frac{dy}{dx} = \cos y^2 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 \sin x^2}{x \cos y^2}$$

$$26. g(x+\pi) = \int_0^{x+\pi} \cos^4 t dt$$

$$= \int_0^x \cos^4 t dt + \int_x^{x+\pi} \cos^4 t dt$$

$$= g(x) + \int_0^\pi \cos^4 t dt = g(x) + g(\pi)$$

$$27. S'(x) = \ln x^3 \cdot 3x^2 - \ln x^2 \cdot 2x = 9x^2 \ln x - 4x \ln x = x \ln x (9x - 4).$$

$$\text{Hence } \frac{S'(x)}{x} = \ln x (9x - 4).$$

Now it is obvious that $\frac{S'(x)}{x}$ is continuous and derivable in its domain.

$$29. \int_a^0 3^{-x} (3^{-x} - 2) dx \geq 0$$

$$\text{put } 3^{-x} = t$$

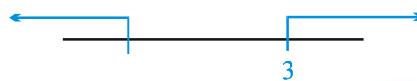
$$\Rightarrow 3^{-x} \ln 3 dx = -dt$$

$$\ln 3 \int_1^{3^{-a}} (t-2) dt \geq 0 \Rightarrow \left[\frac{t^2}{2} - 2t \right]_1^{3^{-a}} \geq 0$$

$$\left(\frac{3^{-2a}}{2} - 2 \cdot 3^{-a} \right) - \left(\frac{1}{2} - 2 \right) \geq 0$$

$$3^{-2a} - 4 \cdot 3^{-a} + 3 > 0$$

$$(3^{-a} - 3)(3^{-a} - 1) > 0$$



$$3^{-a} > 3^1 \Rightarrow a < 1$$

$$\text{or } 3^{-a} < 3^0 \Rightarrow a > 0$$

$$\text{Hence } a \in (-\infty, -1) \cup [0, \infty)$$

30. The given integrand is perfect differential coeff.

$$\text{of } \prod_{r=1}^n (x+r) \Rightarrow I = \left[\prod_{r=1}^n (x+r) \right]_0^1 = (n+1)! - n! = n \cdot n!$$

$$31. \text{ Use } \int_0^a f(x) dx = \int_0^a (a-x) dx \text{ and add two integrals}$$

$$34. a_n = \int_0^{\pi/2} (1 - \sin t)^n \sin 2t dt$$

$$\text{Let } 1 - \sin t = u \Rightarrow -\cos t dt = du$$

$$= 2 \int_0^1 u^n (1-u) du$$

$$= 2 \left(\int_0^1 u^n du - \int_0^1 u^{n+1} du \right)$$

$$= 2 \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$\text{hence } \frac{a_n}{n} = 2 \left(\frac{1}{n(n+1)} - \frac{1}{n(n+2)} \right)$$

$$\lim_{n \rightarrow \infty} \sum_{n=1}^n \frac{a_n}{n}$$

$$= 2 \left(\sum \left(\frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{2} \sum \left(\frac{1}{n} - \frac{1}{n+2} \right) \right)$$

$$= 2 \sum_{n=1}^n \left(\frac{1}{n} - \frac{1}{n+1} \right) - \sum_{n=1}^n \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

$$= 2(1) - \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots \right]$$

$$= 2 - \frac{3}{2} = \frac{1}{2}$$

$$36. I = \int_2^3 \frac{(x+2)^2}{2x^2 - 10x + 53} dx \quad \dots(i)$$

$$I = \int_2^3 \frac{(7-x)^2}{2(5-x)^2 - 10(5-x) + 53} dx$$

$$I = \int_2^3 \frac{(7-x)^2}{2x^2 - 10x + 53} dx \quad \dots(ii)$$

add (i) & (ii)

$$2I = \int_2^3 \frac{(x+2)^2 + (7-x)^2}{2x^2 - 10x + 53} dx$$

$$= \int_2^3 dx = 1 \quad \therefore I = 1/2$$

38. Differentiate both sides w.r.t. x

$$f'(x) = \cos x + f'(x)(2 \sin x - \sin^2 x)$$

$$\therefore (1 + \sin^2 x - 2 \sin x) f'(x) = \cos x$$

$$f'(x) = \frac{\cos x}{1 + \sin^2 x - 2 \sin x} = \frac{\cos x}{(1 - \sin x)^2}$$

$$\text{Integrating } f(x) = \int \frac{\cos x dx}{(1 - \sin x)^2} \quad (\text{Put } 1 - \sin x = t);$$

$$f(x) = - \int \frac{dt}{t^2} = \frac{1}{t} = \frac{1}{1 - \sin x} + C$$

$$\text{also } f(0) = 0, \text{ hence } C = -1$$

$$f(x) = \frac{1}{1 - \sin x} - 1 = \frac{1 - 1 + \sin x}{1 - \sin x}$$

$$= \frac{\sin x}{1 - \sin x}$$

$$39. \text{ Put } \pi x = t \Rightarrow dx = \frac{dt}{\pi}$$

$$I = \frac{1}{\pi} \frac{\pi}{\pi} \int_0^{2008\pi} t |\sin t| dt = \frac{1}{\pi} \int_0^{2008\pi} t |\sin t| dt \quad \dots(1)$$

$$I = \frac{1}{\pi} \int_0^{2008\pi} (2008\pi - t) |\sin t| dt \quad \dots(2)$$

$$(1) + (2) \Rightarrow 2I = \frac{2008\pi}{\pi} \int_0^{2008\pi} |\sin t| dt = (2008)^2.$$

$$\int_0^{\pi} |\sin t| dt$$

$$I = (2008)^2; \text{ hence Here } \sqrt{I} = 2008$$

$$41. I = \int_0^{\pi/2} \sqrt{\tan x} dx \quad \dots(1)$$

$$I = \int_0^{\pi/2} \sqrt{\cot x} dx \quad \dots(2)$$

adding (1) and (2), we get

$$2I = \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}) dx = \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

$$= \sqrt{2} \int_{-1}^1 \frac{dt}{\sqrt{1-t^2}} = 2\sqrt{2} \int_0^1 \frac{dt}{\sqrt{1-t^2}} = \sqrt{2} \pi$$

(where $\sin x - \cos x = t$)

$$\therefore I = \frac{\pi}{\sqrt{2}}$$

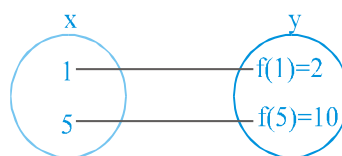
$$42. I = \int_{-1/n\lambda}^{1/n\lambda} \frac{f\left(\frac{x^2}{4}\right) [f(9x) - f(-x)]}{g\left(\frac{x^2}{4}\right) [g(x) + g(-x)]} dx = 0.$$

$$\text{Since } \frac{f\left(\frac{x^2}{4}\right) [f(x) - f(-x)]}{g\left(\frac{x^2}{4}\right) [g(x) + g(-x)]} \text{ is an odd function}$$

$$43. y = f(x) \Rightarrow x = f^{-1}(y) = g(y)$$

$$dy = f'(x) dx$$

$$\therefore I = \int_1^5 f(x) dx + \int_1^5 x f'(x) dx$$



where y is 2 then x = 1
y is 10 then x = 5

$$\therefore I = \int_1^5 (f(x) + x f'(x)) dx$$

$$= x f(x) \Big|_1^5 = 5 f(5) - f(1) = 5 \cdot 10 - 2 = 48$$

$$45. I = \int_{-\pi/2}^{\pi/2} \frac{\cos \theta d\theta}{(2 - \sin \theta) \cos \theta} \quad (\text{putting } x = \sin \theta)$$

$$= \int_0^{\pi/2} \left(\frac{1}{2 - \sin \theta} + \frac{1}{2 + \sin \theta} \right) d\theta$$

$$\left[u \sin g \int_{-a}^a f(x) dx = \int_0^a [f(x) + f(-x)] dx \right]$$

$$= 4 \int_0^{\pi/2} \frac{d\theta}{4 - \sin^2 \theta} = \frac{4}{3} \int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{\frac{4}{3} + \tan^2 \theta} = \frac{4}{3} \int_0^{\infty} \frac{dt}{t^2 + \frac{4}{3}}$$

$$= \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} \cdot \tan^{-1} \frac{\sqrt{3} t}{2} \Big|_0^{\infty} = \frac{2}{\sqrt{3}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

$$46. I = \lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{\pi}{n} \sin \frac{r\pi}{n} = \pi \int_0^1 \sin \pi x dx = \pi \left[-\frac{\cos \pi x}{\pi} \right]_0^1$$

$$= [-\cos \pi + 1] = 2$$

$$47. C_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\tan^{-1}(nx)}{\sin^{-1}(nx)} dx \quad (\text{put } nx = t) \Rightarrow C_n$$

$$= \frac{1}{n} \int_{\frac{n}{n+1}}^1 \frac{\tan^{-1}(t)}{\sin^{-1}(t)} dt$$

$$L = \lim_{n \rightarrow \infty} n^2 \cdot C_n = \lim_{n \rightarrow \infty} n \cdot \int_{\frac{n}{n+1}}^1 \frac{\tan^{-1} t}{\sin^{-1} t} dt \quad (\infty \times 0);$$

$$L = \lim_{n \rightarrow \infty} \frac{\int_{\frac{n}{n+1}}^1 \frac{\tan^{-1} t}{\sin^{-1} t} dt}{\frac{1}{n}} \quad \left(\frac{0}{0} \right)$$

applying Leibnitz rule

$$L = \lim_{n \rightarrow \infty} \frac{0 - \frac{\tan^{-1} \frac{n}{n+1}}{\sin^{-1} \frac{n}{n+1}} \left(\frac{1}{(n+1)^2} \right)}{-\frac{1}{n^2}} = \frac{\pi}{4} \cdot \frac{2}{\pi} = \frac{1}{2}$$

$$49. \text{ Let } f(a) = \int_a^{a^2} \frac{dx}{x + \sqrt{x}}$$

$$f'(a) = \frac{2a}{a + a^2} - \frac{1}{a + \sqrt{a}} = 0$$

$$\Rightarrow 2a^2 + 2a\sqrt{a} = a^2 + a \Rightarrow a^2 + 2a\sqrt{a} - a = 0$$

$$a + 2\sqrt{a} - 1 = 0 \Rightarrow (\sqrt{a} + 1)^2 = 2$$

$$\Rightarrow \sqrt{a} = \sqrt{2} - 1 \Rightarrow \tan \frac{\pi}{8}$$

$$a = (\sqrt{2} - 1)^2 = \tan^2 \left(\frac{\pi}{8} \right)$$

$$50. I = \int_{2-\ln 3}^{3+\ln 3} \frac{\ln(4+x)}{\ln(4+x) + \ln(9-x)} dx$$

$$= \int \frac{\ln(9-x)}{\ln(9-x) + \ln(4-x)} dx$$

$$2I = \int_{2-\ln 3}^{3+\ln 3} 1 \cdot dx = 3 + \ln 3 - (2 - \ln 3) = 1 + 2 \ln 3$$

$$51. \text{ We have } \int_{-1}^1 (px + q)(x^{2n+1} + a_n x + b_n) dx = 0$$

equating the odd component to be zero and integrating we get

$$\frac{2p}{2n+3} + \frac{2a_n p}{3} + 2b_n q = 0 \text{ for all } p, q$$

$$\text{hence } b_n = 0 \text{ and } a_n = -\frac{3}{2n+3}$$

$$52. \int_0^{n\pi+V} \sqrt{\frac{2 \cos^2 x}{2}} dx = \int_0^{n\pi+V} |\cos x| dx$$

$$= \int_0^{n\pi} |\cos x| dx + \int_{n\pi}^{n\pi+V} |\cos x| dx$$

$$= n \int_0^{\pi} |\cos x| dx + \int_0^V |\cos x| dx$$

$$= 2n + \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^V \cos x dx = 2n + 2 - \sin V$$

53. Given $f(x) = \int_{-1}^x e^{t^2} dt$;
 $h(x) = f(1+g(x))$; $g(x) < 0$ for $x > 0$

now $h(x) = \int_{-1}^{1+g(x)} e^{t^2} dt = f(1+g(x))$ (given)

differentiating $h'(x) = e^{(1+g(x))^2} \cdot g'(x)$

$h'(1) = e$ (given)

$e^{(1+g(1))^2} \cdot g'(1) = e$

$\therefore (1+g(1))^2 = 1$

$1+g(1) = \pm 1$

$\Rightarrow g(1) = 0$ (not possible) or $g(1) = -2$

$\Rightarrow$ (C)

54. put $-x/2 = t \Rightarrow dx = -2dt$

$$I = -2 \int_{-\pi/4}^{-\pi/3} \frac{e^t \sqrt{1+\sin 2t}}{1+\cos 2t} dt$$

$$= 2 \int_{-\pi/3}^{-\pi/4} \frac{e^t |\sin t + \cos t|}{2 \cos^2 t} dt$$

$$= - \int_{-\pi/3}^{-\pi/4} \frac{e^t (\sin t + \cos t)}{\cos^2 t} dt$$

$$= - \int_{-\pi/3}^{-\pi/4} e^t (\sec t \tan t + \sec t) dt$$

$$= -[e^t \sec t]_{-\pi/3}^{-\pi/4} = -[e^{-\pi/4} \sqrt{2} - e^{-\pi/3} 2]$$

55. $I = \lim_{h \rightarrow 0} \frac{\int_a^x \ln^2 t \, dt + \int_x^{x+h} \ln^2 t \, dx - \int_a^x \ln^2 t \, dt}{h}$

$$\Rightarrow I = \lim_{h \rightarrow 0} \frac{\int_x^{x+h} \ln^2 t \, dx}{h}$$

Using L hospital we get

$$I = \lim_{h \rightarrow 0} \ln^2(x+h) = \ln^2 x$$

56. Consider $I = \int_{-1/x}^{1/x} \frac{\ln(1+t^2)}{1+e^t} dt$ (1)

$$= \int_{-1/x}^{1/x} \frac{\ln(1+t^2)}{1+e^{-t}} dt \quad (\text{Using King})$$

$$I = \int_{-1/x}^{1/x} \frac{\ln(1+t^2)e^t}{1+e^t} dt \quad \dots(2)$$

(1)+(2)

$$2I = \int_{-1/x}^{1/x} \ln(1+t^2) dt = 2 \int_0^{1/x} \ln(1+t^2) dt$$

$$\Rightarrow I = \int_0^{1/x} \ln(1+t^2) dt$$

hence $I = \lim_{x \rightarrow \infty} x^3 \int_0^{1/x} \ln(1+t^2) dt$

$$= \lim_{x \rightarrow \infty} \frac{\int_0^{1/x} \ln(1+t^2) dt}{x^{-3}} \quad \left(\frac{0}{0} \text{ form} \right)$$

Using L'Hospital's Rule

$$I = \lim_{x \rightarrow \infty} \frac{x^4 \ln\left(1 + \frac{1}{x^2}\right) \cdot \left(-\frac{1}{x^2}\right)}{-3}$$

$$= \frac{1}{3} \lim_{x \rightarrow \infty} x^2 \ln\left(1 + \frac{1}{x^2}\right) = \frac{1}{3} \lim_{x \rightarrow \infty} \ln\left(1 + \frac{1}{x^2}\right)^{x^2} \quad (1^\infty \text{ form})$$

$$= \lim_{x \rightarrow \infty} \frac{1}{3} x^2 \left(1 + \frac{1}{x^2} - 1\right) = \frac{1}{3}$$

57. $f'(x) = \frac{1}{\sqrt{1+g^2(x)}} \cdot g'(x)$;

$$f'\left(\frac{\pi}{2}\right) = \frac{g'(\pi/2)}{\sqrt{1+g^2(\pi/2)}}; \quad g\left(\frac{\pi}{2}\right) = 0$$

$$= g'\left(\frac{\pi}{2}\right)$$

but $g'(x) = [1 + \sin(\cos^2 x)](-\sin x)$

$$g'\left(\frac{\pi}{2}\right) = 1(-1) = -1$$

hence $f'\left(\frac{\pi}{2}\right) = -1$ as $h'(0^+) = -1 \Rightarrow$ (C)

59. put $x = 19 + y \Rightarrow dx = dy$,

$$I = \int_0^{18} (\{19 + y\}^2 + \sin 2y) dy = \int_0^{18} (\{y\}^2 + \underbrace{\sin 2y}_{\text{zero}}) dy$$

$$= 18 \int_0^1 y^2 dy = 6$$

60. $\sqrt{5x - 6 - x^2} + \frac{\pi x}{2} > x \frac{\pi}{2} \Rightarrow 5x - 6 - x^2 > 0$

61. $T_r = \frac{1}{\sqrt{\frac{r}{n}} \cdot n \left(3\sqrt{\frac{r}{n}} + 4 \right)^2}$

$$S = \frac{1}{n} \sum_{r=1}^{4n} \frac{1}{\left(3\sqrt{\frac{r}{n}} + 4 \right)^2 \cdot \sqrt{\frac{r}{n}}} = \int_0^4 \frac{dx}{\sqrt{x} (3\sqrt{x} + 4)^2}$$

put $3\sqrt{x} + 4 = t \Rightarrow \frac{3}{2} \frac{1}{\sqrt{x}} dx = dt$

$$= \frac{2}{3} \int_4^{10} \frac{dt}{t^2} = \frac{2}{3} \left[\frac{1}{t} \right]_4^{10} = \frac{2}{3} \left[\frac{1}{4} - \frac{1}{10} \right] = \frac{2}{3} \cdot \frac{6}{40} = \frac{1}{10}$$

62. $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \frac{2x}{1+x^2} dx \dots (1)$

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \left(\frac{-2x}{1-x^4} \right) dx \quad (\text{using King})$$

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \left(\pi - \cos^{-1} \frac{2x}{1-x^4} \right) dx \dots (2)$$

add (1) and (2)

$$\therefore 2I = \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx$$

$$2I = 2\pi \int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx$$

$$\therefore k = \pi$$

63. Given $U_n = \int_0^1 x^n \cdot (2-x)^n dx$;

$$V_n = \int_0^1 x^n \cdot (1-x)^n dx$$

in U_n put $x = 2t \Rightarrow dx = 2dt$

$$\therefore U_n = 2 \int_0^{1/2} 2^n \cdot t^n \cdot 2^n (1-t)^n dt \dots (1)$$

Now $V_n = 2 \int_0^{1/2} x^n (1-x)^n dx$

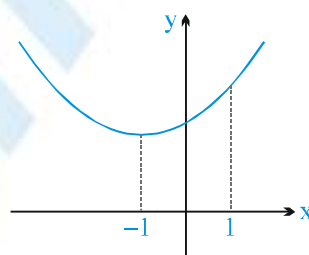
(Using Queen) (2)

From (1) and (2)

$$U_n = 2^{2n} \cdot V_n \Rightarrow (C)$$

65. $A = \int_{-1}^1 (ax^2 + bx + c) dx = 2 \int_0^1 (ax^2 + c) dx$

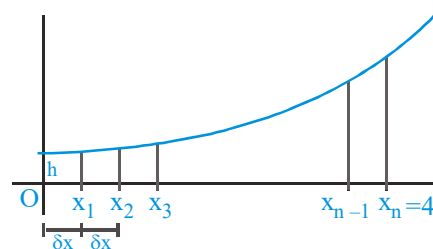
$$= 2 \left[\frac{a}{3} + c \right] = \frac{1}{3} [2a + 6c]$$



$$\therefore A = \frac{1}{3} [f(-1) + 4f(0) + f(1)]$$

67. $L = \lim_{\delta x \rightarrow 0} \delta x (x_1 + x_2 + x_3 + \dots + x_n)$ 77

$$= \lim_{n \rightarrow \infty} \frac{4}{n} \left[\frac{4}{n} + \frac{8}{n} + \frac{12}{n} + \dots + 4 \cdot \frac{n}{n} \right] \left(\delta x = \frac{4}{n} \right)$$



$$= \lim_{n \rightarrow \infty} \frac{16}{n^2} (1 + 2 + 3 + \dots + n) = \lim_{n \rightarrow \infty} \frac{16}{n^2} \cdot \frac{n(n+1)}{2} = 8$$

$$68. S = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2^2}{n^2}\right) \dots \left(1 + \frac{n^2}{n^2}\right) \right]^{1/n}$$

$$\bullet nS =$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[\ln \left(1 + \frac{1}{n^2}\right) + \ln \left(1 + \frac{2^2}{n^2}\right) + \dots + \ln \left(1 + \frac{n^2}{n^2}\right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \ln \left(1 + \frac{r^2}{n^2}\right) = \int_0^1 \ln(1+x^2) dx$$

$$\Rightarrow \bullet nS = \ln[2e^{\frac{\pi}{2}}]$$

$$\Rightarrow S = 2e^{\frac{\pi}{2}} = 2e^{\frac{\pi}{2}} \cdot e^{-2} = \frac{2}{e^2} e^{\frac{\pi}{2}}$$

$$69. I = \int_0^1 x \ln \left(\frac{x+2}{2} \right) dx$$

$$= \int_0^1 x (\ln(x+2) - \ln 2) dx$$

$$\therefore I = \int_0^1 x \ln(x+2) dx - \ln 2 \int_0^1 x dx ;$$

$$\text{hence } I = \ln(x+2) \cdot \frac{x^2}{2} \Big|_0^1 - \int_0^1 \frac{x^2}{x+2} dx - \frac{\ln 2}{2}$$

$$= \frac{1}{2} \ln 3 - \int_0^1 \frac{x^2 - 4 + 4}{x+2} dx - \frac{\ln 2}{2}$$

$$\Rightarrow \frac{1}{2} \ln \frac{3}{2} - \int_0^1 \left((x-2) + \frac{4}{x+2} \right) dx \text{ now proceed}$$

$$70. I(a) = \int_0^{\pi} \left(\frac{x^2}{a^2} + a^2 \sin^2 x + 2x \sin x \right) dx$$

$$\left(\int_0^{\pi} x \sin x dx = \pi \right)$$

$$\therefore I(a) = \frac{\pi^3}{3a^2} + \frac{\pi a^2}{2} + 2\pi = \pi \left[\frac{\pi^2}{3a^2} + \frac{a^2}{2} \right] + 2\pi$$

$$= \pi \left[\left(\frac{\pi}{\sqrt{3a}} - \frac{a}{\sqrt{2}} \right)^2 + \frac{2\pi}{\sqrt{6}} \right] + 2\pi$$

$$I(a) \text{ is minimum when } \frac{\pi}{\sqrt{3a}} = \frac{a}{\sqrt{2}} \Rightarrow a^2 = \pi \sqrt{\frac{2}{3}}$$

$$\Rightarrow a = \sqrt{\pi \sqrt{\frac{2}{3}}}$$

$$\text{Also } I(a)_{\min} = 2\pi + \pi^2 \sqrt{\frac{2}{3}}$$

$$71. \left| \int_{10}^{19} \frac{\sin x}{1+x^8} dx \right| \leq \int_{10}^{19} \frac{|\sin x|}{1+x^8} dx \leq \int_{10}^{19} \frac{dx}{1+x^8} < \int_{10}^{19} \frac{dx}{x^8}$$

$$= \left[\frac{x^{-7}}{-7} \right]_{10}^{19}$$

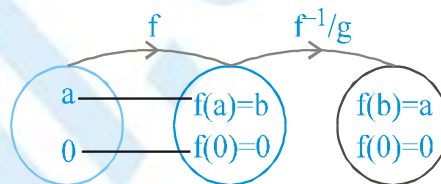
$$= -\frac{1}{7} [19^{-7} - 10^{-7}] = \frac{1}{7} [10^{-7} - 19^{-7}]$$

$$\text{Ans: } < 10^{-7}$$

$$73. y=f(x) \Rightarrow x=g(y) \text{ and } dy=f'(x) dx$$

$$I = \int_0^a f(x) dx + \int_0^b g(y) dy ; y=f(x) \Rightarrow x=f^{-1}(y)=g(y)$$

$$= \int_0^a f(x) dx + \int_0^a x f'(x) dx$$



$$= \int_0^a (f(x) + x f'(x)) dx = [x f(x)]_0^a = a f(a) = ab$$

$$75. L = \lim_{n \rightarrow \infty} \frac{n}{(n!)^{1/n}} = \lim_{n \rightarrow \infty} \frac{n}{(1 \cdot 2 \cdot 3 \cdot \dots \cdot n)^{1/n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \dots \cdot \frac{n}{n} \right)^{1/n}}$$

$$\ln L = -\frac{1}{n} \left[\ln \left(\frac{1}{n} \right) + \ln \left(\frac{2}{n} \right) + \ln \left(\frac{3}{n} \right) + \dots + \ln \left(\frac{n}{n} \right) \right]$$

general term of $\ln L$ is

$$T_r = -\frac{1}{n} \ln \frac{r}{n}$$

$$\therefore S = \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{r=1}^n \ln \frac{r}{n} = -\int_0^1 \ln x dx = -[x \ln x - x]_0^1$$

$$= -[(0-1) - (0)] = 1$$

$$\therefore \text{Hence } \ln L = 1 \Rightarrow L = e$$

EXERCISE - 2

Part # I : Multiple Choice

1. $I = \int_0^1 (10x^4 - 3x^2 - 1) dx = 2x^5 - x^3 - x \Big|_0^1 = 0 \Rightarrow A.$

Since $f(x)$ is even hence must have a root in $(-1, 0)$ also

$\Rightarrow A$ and $B \Rightarrow C$

3. Given $f(f(x)) = -x + 1$

replacing $x \rightarrow f(x)$

$$f(f(f(x))) = -f(x) + 1$$

$$f(1-x) = -f(x) + 1$$

$$f(x) + f(1-x) = 1 \dots (1) \Rightarrow (A)$$

now $J = \int_0^1 f(x) dx = \int_0^1 f(1-x) dx$

$$2J = \int_0^1 (f(x) + f(1-x)) dx; 2J = \int_0^1 dx = 1 \Rightarrow J = \frac{1}{2}$$

4. $v = \int_0^\infty \frac{x^2 dx}{x^4 + 7x^2 + 1}$

Put $x = \frac{1}{t} \Rightarrow dx = \frac{-1}{t^2} dt$

$$v = - \int_\infty^0 \frac{\frac{1}{t^2} \cdot \frac{1}{t^2} dt}{\frac{1}{t^4} + \frac{7}{t^2} + 1} = \int_0^\infty \frac{dx}{x^4 + 7x^2 + 1}$$

$v = u$

Hence $2u = \int_0^\infty \left(\frac{x^2 + 1}{x^4 + 7x^2 + 1} \right) dx$

$$= \int_0^\infty \left(\frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} + 7} \right) dx = \int_0^\infty \frac{d\left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^2 + 3^2} = \int_0^\infty \frac{dt}{t^2 + 9}$$

$$= \frac{2}{3} \left[\tan^{-1} \frac{t}{3} \right]_0^\infty$$

$2u = \pi/3$

5. $f(x+\pi) = \int_0^{x+\pi} (\cos^4 t + \sin^4 t) dt$

$$= \int_0^x (\cos^4 t + \sin^4 t) dt + \int_x^{x+\pi} (\sin^4 t + \cos^4 t) dt$$

$$= f(x) + \int_x^{x+\pi} (\sin^4 t + \cos^4 t) dt$$

put $t = x + y = f(x) + \int_0^\pi (\sin^4 y + \cos^4 y) dy = f(x) + f(\pi)$

6. Given $(f'(x))^2 + (g(x))^2 = 1$

$$f(x) + \int_0^x g(t) dt = \sin x (\cos x - \sin x)$$

differentiating both sides

$$f'(x) + g(x) = \cos 2x - \sin 2x \dots (1)$$

squaring (1)

$$(f'(x))^2 + (g(x))^2 + 2f'(x) \cdot g(x) = 1 - \sin 4x$$

$$1 + 2f'(x) \cdot g(x) = 1 - \sin 4x$$

$$\therefore 2f'(x)g(x) = -\sin 4x$$

now, substituting $g(x) = -\frac{\sin 4x}{2f'(x)}$ in equation (1)

$$f'(x) - \frac{\sin 4x}{2f'(x)} = \cos 2x - \sin 2x$$

put $f'(x) = t$

$$2t^2 - 2(\cos 2x - \sin 2x)t - \sin 4x = 0$$

$$\Rightarrow t = \frac{2(\cos 2x - \sin 2x) \pm \sqrt{4(1 - \sin 4x) + 8\sin 4x}}{4}$$

$$\therefore 4t = 2(\cos 2x - \sin 2x) \pm \sqrt{4(1 - \sin 4x) + 8\sin 4x} \Rightarrow$$

$$2t = (\cos 2x - \sin 2x) \pm \sqrt{1 + \sin 4x}$$

taking (+)ve sign, $2t = \cos 2x - \sin 2x + \cos 2x + \sin 2x$

$$\Rightarrow t = \cos 2x$$

taking (-) ve sign, $t = -\sin 2x$

hence $f'(x) = \cos 2x$ or $f'(x) = -\sin 2x$

Integrating

$$f(x) = \frac{1}{2} \sin 2x + C_1 \quad \text{or} \quad f(x) = \frac{\cos 2x}{2} + C_2$$

$$f(0) = 0 \Rightarrow C_1 = 0 \quad \text{and} \quad C_2 = -1/2$$

$$\therefore f(x) = \frac{1}{2} \sin 2x \quad \text{or} \quad f(x) = \frac{\cos 2x - 1}{2}$$

if $f'(x) = \cos 2x$ then $g(x) = -\sin 2x$

if $f'(x) = -\sin 2x$ then $g(x) = \cos 2x$

i.e. $f(x) = \frac{1}{2} \sin 2x$ and $g(x) = -\sin 2x \Rightarrow (C)$

$$f(x) = \frac{\cos 2x - 1}{2} \quad \text{and} \quad g(x) = \cos 2x \Rightarrow (D)$$



$$8. \quad (A) \quad I = \int_0^{\pi/2} \ln(\cot x) dx \Rightarrow I = \int_0^{\pi/2} \ln(\tan x) dx$$

$$I = - \int_0^{\pi/2} \ln(\cot x) dx \Rightarrow I = -I \Rightarrow I = 0$$

$$(B) \quad I = \int_0^{2\pi} \sin^3 x dx = - \int_0^{2\pi} \sin^3 x dx \Rightarrow I = 0$$

$$(C) \text{ at } x=1/t, I = \int_c^{1/c} \frac{-(1/t^2)dt}{-1/t(\ln t)^{1/3}} = - \int_{1/c}^c \frac{dt}{t(\ln t)^{1/3}}$$

$$I = -I \Rightarrow I = 0$$

$$(D) \quad \sqrt{\frac{1+\cos 2x}{2}} > 0 \Rightarrow \int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}} dx > 0$$

$$9. \quad \text{Numerator} = 2(x^2 + 2x + 2) - (x + 1)$$

$$10. \quad f(x) = 2 \int_0^1 (1-t) \cos(xt) dt$$

$$= 2 \left[(1-t) \frac{\sin xt}{x} \Big|_0^1 + \frac{1}{x} \int_0^1 \sin xt dt \right] = 2 \left[0 - \frac{1}{x^2} \cos xt \Big|_0^1 \right]$$

$$f(x) = 2 \left[\frac{1 - \cos x}{x^2} \right] \quad (x \neq 0)$$

$$\text{if } x=0 \text{ then } f(x) = \int_{-1}^1 (1-|t|) dt = 2 \int_0^1 (1-t) dt = 1$$

$\Rightarrow$ (C) is correct

$$\text{hence } f(x) = \begin{cases} 2 \left(\frac{1 - \cos x}{x^2} \right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$\therefore$ f is continuous at $x=0 \Rightarrow$ (D) is correct]

$$12. \quad \text{We have } f(x) = x^2 + ax^2 + bx^3$$

$$\text{where } a = \int_{-1}^1 t \cdot f(t) dt \text{ and } b = \int_{-1}^1 f(t) dt$$

$$\text{How } a = \int_{-1}^1 t[(a+1)t^2 + bt^3] dt$$

$$a = 2b \int_0^1 t^4 dt = \frac{2b}{5} \quad \dots\dots\dots (1)$$

$$\text{Again } b = \int_{-1}^1 f(t) dt = \int_{-1}^1 ((a+1)t^2 + bt^3) dt$$

$$= 2 \int_0^1 (a+1)t^2 dt$$

$$b = \frac{2(a+1)}{3} \quad \dots\dots\dots (2)$$

From (1) and (2)

$$\frac{5a}{2} = \frac{2(a+1)}{3} \left(\frac{5}{2} - \frac{2}{3} \right) a = \frac{2}{3} \Rightarrow \frac{11}{6} a = \frac{2}{3}$$

$$a = \frac{4}{11} \text{ and } b = \frac{10}{11}$$

$$\text{Hence } \int_{-1}^1 t \cdot f(t) dt = \frac{4}{11} \text{ and } \int_{-1}^1 f(t) dt = \frac{10}{11}$$

$$\therefore f(x) = (a+1)x^2 + bx^3$$

$$\begin{aligned} f(1) &= (a+1) + b \\ f(-1) &= (a+1) - b \end{aligned} \Rightarrow f(1) + f(-1) = 2(a+1) = \frac{30}{11}$$

$$\text{and } f(1) - f(-1) = 2b = \frac{20}{11} \Rightarrow \text{B, D correct.}$$

$$13. \quad f(-x) = -f(x) \quad \dots (1)$$

$$f(x+2) = f(x) \quad \dots (2)$$

$$g(2n) = \int_0^{2n} f(t) dt = n \int_0^2 f(t) dt$$

$$\Rightarrow g(2n) = n g(2) \quad \dots (3)$$

$$\text{Now } g(-x) = \int_0^{-x} f(t) dt$$

$$\text{put } t = -z \Rightarrow dt = -dz$$

$$= \int_0^x f(-z)(-dz) = - \int_0^x f(-z) dz \quad (\text{from (1)})$$

$$= \int_0^x f(t) dt = g(x) \quad \therefore g(-x) = g(x)$$

$$\text{Again } g(x+2) = \int_0^{x+2} f(t) dt + \int_x^{x+2} f(t) dt$$

$$\therefore g(x+2) = \int_0^x f(t) dt + \int_0^2 f(t) dt \quad (\rightarrow f \rightarrow \text{period})$$

$$\Rightarrow g(x+2) = g(x) + g(2) \quad \dots (4)$$

Putting $x=0, 2, \dots$



$$g(2) = g(0 + g(2)) \Rightarrow g(0) = 0$$

$$g(4) = g(2) + g(2) \Rightarrow g(4) = 2g(2)$$

putting $x \rightarrow -x$ we get

$$g(2-x) = g(-x) + g(2) = g(x) + g(2)$$

at $x = 2$

$$g(0) = 2g(2) \Rightarrow g(2) = 0$$

$$\therefore g(0) = g(\pm 2) = g(\pm 4) = \dots = 0$$

from (3) $g(2n) = 0$

& from (4) $g(x+2) = g(x) \Rightarrow$ prd. of $g(x)$ is 2

14. Let $f(x) = \frac{ax^5}{5} + \frac{bx^3}{3} + cx$

It is continuous & differentiable everywhere

Now $f(0) = 0, f(1) = \frac{3a + 5b + 15}{15} = 0$

and $f(-1) = 0$

so $f'(x) = 0$ will have at least one root in $(-1, 0)$ atleast one root in $(0, 1)$, so it will have atleast two roots in $(-1, 1)$

16. $f(x) = \frac{\ln^2 x}{2} = 2 \Rightarrow C, D$

17. $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{(n+r)(n+2r)}$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{n^2 \left(1 + \frac{r}{n}\right) \left(1 + \frac{2r}{n}\right)}$$

$$= \int_0^1 \frac{dx}{(1+x)(1+2x)} = \int_0^1 \left(\frac{2}{1+2x} - \frac{1}{1+x} \right) dx$$

$$= \ln(1+2x) - \ln(1+x) \Big|_0^1 = \ln 3 - \ln 2 = \ln \frac{3}{2}$$

18. $I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x) dx$

Use king

$$I_1 = \int_0^{\pi/2} \cos(\pi \cos^2 x) dx$$

On adding

$$2I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x) + \cos(\pi \cos^2 x) dx$$

$$= \int_0^{\pi/2} 2 \cos\left(\frac{\pi}{2}\right) \cdot \cos\left(\frac{\pi}{2} \cos 2x\right) dx = 0$$

$$\Rightarrow I_1 = 0 \quad \dots(1)$$

$$I_2 = \int_0^{\pi/2} \cos(\pi(1 - \cos 2x)) dx = - \int_0^{\pi/2} \cos(\pi \cos 2x) dx$$

$$= - \frac{1}{2} \int_0^{\pi} \cos(\pi \cos t) dt \quad [\text{Put } 2x = t]$$

$$= - \frac{2}{2} \int_0^{\pi/2} \cos(\pi \cos t) dt = I_3 \Rightarrow I_2 + I_3 = 0$$

$$I_2 = - \int_0^{\pi/2} \cos(\pi \sin t) dt$$

$$\therefore I_2 + I_3 = 0 \quad \dots(2)$$

$$\text{Hence, } I_1 + I_2 + I_3 = 0 \quad \dots(3)$$

29. $I = \int_{-\infty}^0 \frac{ze^{-z}}{\sqrt{1-e^{-2z}}} dz$

put $e^{-z} = \sin \theta$

$$I = - \int_0^{\pi/2} \frac{\ln(\sin \theta)(-\cos \theta) d\theta}{\sqrt{1-\sin^2 \theta}} = \int_0^{\pi/2} \ln \sin \theta d\theta$$

$$= \frac{-\pi}{2} \ln 2$$

21. I.B.P. taking 1 as the 2nd and $\frac{1}{(1+x^2)^n}$ as the 1st function

23. Consider $f(x) = \int_{-x}^x (t \sin at + bt + c) dt$

$$= 2 \int_0^x (t \sin at + c) dt$$

$$= 2 \left[-t \frac{\cos at}{a} \Big|_0^x + \int_0^x \frac{\cos at}{a} dt + ct \Big|_0^x \right] \quad (\text{using I.B.P.})$$

$$= 2 \left[\frac{-x \cos ax}{a} + \frac{1}{a^2} \sin ax + cx \right]$$

$$\therefore \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} 2 \left[-\frac{\cos ax}{a} + \frac{\sin ax}{a \cdot ax} + c \right]$$

$$= 2 \left[-\frac{1}{a} + \frac{1}{a} + c \right] = 2c$$

24. Consider $I = \int_a^\infty \frac{n dx}{n^2 \left(x^2 + \frac{1}{n^2} \right)} = \frac{1}{n} \cdot n \left(\tan^{-1} nx \right)_a^\infty$



$$= \left(\frac{\pi}{2} - \tan^{-1} a \right)$$

$$\therefore L = \lim_{n \rightarrow \infty} \left(\frac{\pi}{2} - \tan^{-1} a_n \right) = \begin{cases} \pi & \text{if } a < 0 \\ \pi/2 & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases}$$

Part # II : Assertion & Reason

2. $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1 + \sin x}$

$$2I = \int_{-\pi/4}^{\pi/4} \frac{2dx}{1 - \sin^2 x} \Rightarrow I$$

$$= \int_{-\pi/4}^{\pi/4} \frac{dx}{\cos^2 x}$$

$$I = 2 \int_0^{\pi/4} \sec^2 x dx \neq 0 \Rightarrow \text{Statement-1 is false}$$

3. $\int_0^t \{x\} dx = \int_0^{[t]} \{x\} dx + \int_{[t]}^t \{x\} dx = [t] \int_0^1 x dx + \int_0^{\{t\}} x dx$

$$= \frac{[t]}{2} + \frac{\{t\}^2}{2}$$

$\therefore$ statement-2 is true.

$$\int_0^{5.5} \{x\} dx = \frac{5}{2} + \frac{(0.5)^2}{2} = \frac{21}{8}$$

$\therefore$ statement-1 is true and is explained by statement-2.

5. Statement-1 :

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$

$$I = - \int_3^{1/3} t \operatorname{cosec}^{99} \left(\frac{1}{t} - t \right) \frac{1}{t^2} dt$$

$$= - \int_{1/3}^3 \frac{1}{t} \operatorname{cosec}^{99} \left(t - \frac{1}{t} \right) dt$$

$$I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$$

6. Let $\int_0^1 f(t) dt = k$, so

$$f(x) = xk + 1, \text{ now}$$

$$\int_0^1 (kt + 1) dt = k$$

$$\Rightarrow \frac{k}{2} + 1 = k, \text{ so } k = 2$$

$$\therefore f(x) = 2x + 1,$$

Also $\int_0^3 f(x) dx = 12$

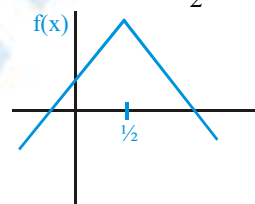
$\Rightarrow$ option (C) is correct.

7. $f(x) = -x^2 + x + 1$

$$f(x) = 1 - 2x$$

$$f(x) > 0 \Rightarrow 1 - 2x > 0 \Rightarrow x < \frac{1}{2}$$

$$f(x) < 0 \Rightarrow 1 - 2x < 0 \Rightarrow x > \frac{1}{2}$$



$\Rightarrow f(x)$ is increasing in $(0, \frac{1}{2})$ and decreasing in $(\frac{1}{2}, 1)$

Now $g(x) = \max \{f(t) ; 0 \leq t \leq x\}$

$$= \begin{cases} x - x^2 + 1 & 0 \leq x \leq \frac{1}{2} \\ \frac{5}{4} & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\int_0^1 g(x) dx = \int_0^{1/2} (x - x^2 + 1) dx + \int_{1/2}^1 \frac{5}{4} dx = \frac{29}{24}$$

8. Statement-1 :

$$I = \int_0^{\pi} x \tan x \cos^3 x dx \quad \dots\dots(i)$$

$$I = \int_0^{\pi} (\pi - x) \tan x \cos^3 x dx \quad \dots\dots(ii)$$

(i) + (ii)

$$2I = \pi \int_0^{\pi} \tan x \cdot \cos^3 x dx$$

$$I = \frac{\pi}{2} \int_0^{\pi} \tan x \cos^3 x dx \quad (\text{true})$$

Statement-2 :

$$I = \int_a^b x f(x) dx \quad \dots\dots(i)$$

$$I = \int_a^b (a+b-x) f(a+b-x) dx \quad \dots\dots(ii)$$

(i) + (ii)

$$2I = (a+b) \int_a^b f(x) dx$$

$$\{ \text{If } f(a+b-x) = f(x) \}$$

$$I = \frac{a+b}{2} \int_a^b f(x) dx$$

Hence Statement-2 false

$$\text{but if } f(a+b-x) \neq f(x), \text{ then } I \neq \frac{a+b}{2} \int_a^b f(x)$$

9. If $f(x)$ is odd $\Rightarrow f'(x)$ is even but converse is not true

e.g. if $f'(x) = x \sin x$

then $f(x) = \sin x - x \cos x + C$

$$f(-x) = -\sin x + x \cos x + C$$

$$f(x) + f(-x) = \text{constant which need not to be zero}$$

for S-1: $f(x) = \int_0^x \sqrt{1+t^2} dt$; $g(x) = \sqrt{1+x^2}$

$$f(-x) = \int_0^{-x} \sqrt{1+t^2} dt; \quad t = -y$$

$$f(-x) = - \int_0^x \sqrt{1+y^2} dy$$

$$\therefore f(x) + f(-x) = 0 \Rightarrow f \text{ is odd and } g \text{ is obviously even}$$

12. $P(x) = ax^2 + bx + c$; $f(x) = \sin^3 x$

$$I = \int_0^{2\pi} P(x) \cdot f''(x) dx$$

Using I.B.P.

$$= \left[P(x) \cdot f'(x) \right]_0^{2\pi} - \int_0^{2\pi} P'(x) \cdot f'(x) dx$$

zero

$$= - \left[P'(x) \cdot f(x) \right]_0^{2\pi} - \int_0^{2\pi} P''(x) \cdot f(x) dx$$

$$= \int_0^{2\pi} P''(x) \cdot f(x) dx = 2 \int_0^{2\pi} \sin^3 x dx = 0$$

13. $\int_{-\pi}^{\pi} (\sin mx \cdot \sin nx) dx = 0$ if $m \neq n$

and $\int_{-\pi}^{\pi} (\sin mx \cdot \sin nx) dx = \pi$ if $m = n$

$$\therefore a = \cos 0 = 1 \text{ and } b = \cos \pi = -1$$

$$\therefore a + b = 0$$

EXERCISE - 3

Part # I : Matrix Match Type

1. (A) Apply L'Hospital's rule twice or use expansion of $e^x \cos x$

(B) $x = u^6 \Rightarrow dx = 6u^5 du$

$$I = \int_0^1 \frac{6u^5 du}{u^3 + u^2} = 6 \int_0^1 \frac{u^3 + 1 - 1}{u + 1} du = 5 - 6 \ln 2$$

$\Rightarrow a + b = 5 - 6 = -1$ Ans.

(C) $e^n \int_0^n e^{-\theta} (\sec^2 \theta - \tan \theta) d\theta = 1$

put $-\theta = t$; $d\theta = -dt$

$$-e^n \int_0^{-n} e^t (\sec^2 t + \tan t) dt = 1$$

[use $\int e^x (f(x) + f'(x)) = e^x f(x)$]

$$-e^n [e^t \tan t]_0^{-n} = 1 \Rightarrow -e^n [-e^{-n} \tan n] = 1$$

$\Rightarrow \tan n = +1$ Ans.

(D) $\lim_{n \rightarrow \infty} \frac{\int_{\frac{1}{n+1}}^{\frac{1}{n}} \tan^{-1}(nx) dx}{\int_{\frac{1}{n+1}}^{\frac{1}{n}} \sin^{-1}(nx) dx}$ (put $nx = t$);

$$\therefore \lim_{n \rightarrow \infty} \frac{\int_{\frac{1}{n+1}}^{\frac{1}{n}} \tan^{-1}(t) dt}{\int_{\frac{1}{n+1}}^{\frac{1}{n}} \sin^{-1}(t) dt} \left(\frac{0}{0} \right)$$

use L'Hospital's rule

$$\lim_{n \rightarrow \infty} \frac{\tan^{-1}\left(\frac{n}{n+1}\right)}{\sin^{-1}\left(\frac{n}{n+1}\right)} = \frac{\frac{\pi}{4}}{\frac{\pi}{2}} = \frac{1}{2}$$

2. (A) Let $a = 2008$

$$I = \int_0^1 (1 + ax^a) e^{x^a} dx ; I = \int_0^1 (e^{x^a} + ax^a e^{x^a}) dx$$

(Note: $ax^a = ax \cdot x^{a-1}$)

$$\therefore I = \int_0^1 (e^{x^a} + e^{x^a} \cdot x \cdot ax^{a-1}) dx = \int_0^1 (f(x) + x f'(x)) dx$$

where $f(x) = e^{x^a}$

hence $I = [xe^{x^a}]_0^1 = e$ Ans. $\Rightarrow$ (S)

- (B) $I = I_1 + I_2$

Consider $I_2 = \int_1^{1/e} \sqrt{-\ln x} dx$

Put $\sqrt{-\ln x} = t \Rightarrow -\ln x = t^2 \Rightarrow x = e^{-t^2}$

$\Rightarrow dx = -2t e^{-t^2} dt$

$$\therefore I_2 = \int_0^1 t \cdot (-2t e^{-t^2}) dt = -t^2 e^{-t^2} \Big|_0^1 - \int_0^1 e^{-t^2} dt$$

$$= -\frac{1}{e} - \int_0^1 e^{-t^2} dt$$

hence $I = \int_0^1 e^{-x} dx + \frac{1}{e} - \int_0^1 e^{-t^2} dt = \frac{1}{e} = e^{-1}$ Ans.

$\Rightarrow$ (P)

note that if $f(x) = e^{-x^2}$ then $f^{-1}(x) = \sqrt{-\ln x}$

(C) $L = \lim_{n \rightarrow \infty} \left(\left(\frac{1}{n} \right)^1 \cdot \left(\frac{2}{n} \right)^2 \cdot \left(\frac{3}{n} \right)^3 \cdots \left(\frac{n}{n} \right)^n \right)^{\frac{1}{n^2}}$

$$\ln L = \lim_{n \rightarrow \infty} \frac{1}{n^2} \left(1 \cdot \ln \left(\frac{1}{n} \right) + 2 \cdot \ln \left(\frac{2}{n} \right) + 3 \cdot \ln \left(\frac{3}{n} \right) + \cdots + n \cdot \ln \left(\frac{n}{n} \right) \right)$$

general term of $\ln L = \frac{r}{n^2} \ln \frac{r}{n}$

$$\text{Sum} = \frac{1}{n} \cdot \sum_{r=1}^n \frac{r}{n} \ln \left(\frac{r}{n} \right)$$

$$\ln L = \int_0^1 x \ln x dx = \left[\frac{x^2 \cdot \ln x}{2} \right]_0^1 - \frac{1}{2} \int_0^1 x^2 \cdot \frac{1}{x} dx = 0$$

$$- \left[\frac{1}{2} \frac{x^2}{2} \right]_0^1 = -\frac{1}{4}$$

$L = e^{-1/4}$ Ans. $\Rightarrow$ (Q)



$$4. f(\theta) = \frac{(x + \sin \theta)^3}{3} \Big|_0^1 = \frac{(1 + \sin \theta)^3 - \sin^3 \theta}{3}$$

$$\text{and } g(\theta) = \frac{(x + \cos \theta)^3}{3} \Big|_0^1 = \frac{(1 + \cos \theta)^3 - \cos^3 \theta}{3}$$

$$f(\theta) = \frac{1 + 3\sin \theta + 3\sin^2 \theta}{3}; g(\theta) = \frac{1 + 3\cos \theta + 3\cos^2 \theta}{3}$$

$$f(\theta) - g(\theta) = (\sin \theta - \cos \theta) + (\sin^2 \theta - \cos^2 \theta)$$

$$f(\theta) - g(\theta) = (\sin \theta - \cos \theta)(1 + \sin \theta + \cos \theta)$$

now verify all matching.

$$5. (A) f(n) = \frac{\log 3}{\log 2} \cdot \frac{\log 4}{\log 3} \cdots \frac{\log n}{\log n-1}$$

$$f(n) = \frac{\log n}{\log 2} = \log_2(n)$$

$$\therefore f(2^k) = \log_2(2^k) = k$$

$$\therefore \sum_{k=2}^{100} f(2^k) = \sum_{k=2}^{100} k = 2 + 3 + 4 + \cdots + 100 = 5050 - 1 = 5049 \text{ Ans.}$$

$$(B) f(x) = \sqrt{1+x} \sqrt{1+(x+1)} \sqrt{1+(x+2)} \sqrt{1+(x+4)}$$

$$= \sqrt{1+x} \sqrt{1+(x+1)} \sqrt{x^2+6x+9}$$

$$= \sqrt{1+x} \sqrt{1+(x+1)} \sqrt{(x+3)^2}$$

$$= \sqrt{1+x} \sqrt{x^2+4x+4} = \sqrt{1+x} \sqrt{(x+2)^2}$$

$$= \sqrt{x^2+2x+1} = (x+1)$$

$$\therefore I = \int_0^{100} (x+1) dx = \frac{(x+1)^2}{2} \Big|_0^{100} = \frac{(101)^2 - 1^2}{2}$$

$$= \frac{100 \times 102}{2} = 5100 \text{ Ans.}$$

$$(C) \text{ A.P. is } a, (a+d), (a+2d), \dots, (a+98d)$$

sum of odd terms = 2550

$$\frac{1}{4}(a+2d) + \frac{1}{4}(a+4d) + \frac{1}{4}(a+6d) + \cdots + \frac{1}{4}(a+98d) = 2550$$

50 terms

$$\frac{50}{2} [2a + 98d] = 2550 \text{ or } 50[a + 49d] = 2550 \text{ or } a + 49d = 51$$

This is the 50th term of the A.P. Hence $S_{99} = 51 \times 99 = 5049$

$$6. (A) \int_0^{\pi/2} \ln(\tan x + \cot x) dx$$

$$= \int_0^{\pi/2} -\ln(\sin x \cdot \cos x) dx$$

$$= - \int_0^{\pi/2} \ln \sin x dx - \int_0^{\pi/2} \ln \cos x dx$$

$$= -2 \left(-\frac{\pi}{2} \ln 2 \right) = \pi \bullet \ln 2$$

$$(B) I = \int_0^{\pi/2} \frac{\sin x - \cos x}{(\sin x + \cos x)^2} dx$$

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{\left(\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right)^2} dx$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx = -I \quad \therefore I = 0$$

$$(C) I = \int_0^{2\pi} x \sin^2 x \cos^2 x dx$$

$$= \int_0^{2\pi} (2\pi - x) \sin^2 x \cos^2 x dx$$

$$I = \pi \int_0^{2\pi} \sin^2 x \cos^2 x dx$$

$$= \frac{\pi}{4} \int_0^{2\pi} 4 \sin^2 x \cos^2 x dx = \frac{\pi}{4} \int_0^{2\pi} \sin^2 2x dx$$

$$= \frac{\pi}{8} \int_0^{2\pi} (1 - \cos 4x) dx = \frac{\pi}{8} \left(x - \frac{\sin 4x}{4} \right) \Big|_0^{2\pi}$$

$$= \frac{\pi}{8} (2\pi) = \frac{\pi^2}{4}$$

$$(D) \int_0^{\pi/2} (2 \ln \sin x - \ln 2 - \ln \sin x - \ln \cos x) dx$$

$$= - \int_0^{\pi/2} \ln 2 dx = - \frac{\pi}{2} \bullet \ln 2$$



$$8. (A) f'(x) = \frac{g'(x)}{\sqrt{1+g^3(x)}}$$

$$\text{and } g'(x) = [1 + \sin(\cos^2 x)](-\sin x)$$

$$\text{hence } f'(x) = \frac{[1 + \sin(\cos^2 x)](-\sin x)}{\sqrt{1+g^3(x)}}$$

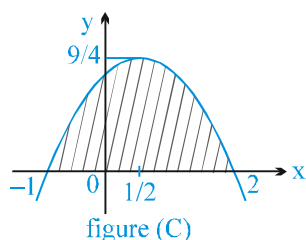
$$f'\left(\frac{\pi}{2}\right) = \frac{1+0}{\sqrt{1+g^3(\pi/2)}} = \frac{-1}{1+0} = -1$$

$$\text{as } g\left(\frac{\pi}{2}\right) = 0$$

$$\therefore f'\left(\frac{\pi}{2}\right) = -1 \text{ Ans.}$$

$$(C) \text{ Maximum when } a = -1; b = 2$$

$$\Rightarrow a + b = 1$$



$$(D) \text{ If } \lim_{x \rightarrow 0} \frac{\sin 2x}{x^3} + a + \frac{b}{x^2} = 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin 2x + ax^3 + bx}{x^3} = 0$$

$$\text{for limit to exist } 2 + b = 0 \Rightarrow \boxed{b = -2}$$

$$\therefore \lim_{x \rightarrow 0} \frac{\sin 2x + ax^3 - 2x}{x^3} = 0$$

$$\text{apply LHS rule, } \lim_{x \rightarrow 0} \frac{2 \cos 2x + 3ax^2 - 2}{3x^2} = 0$$

$$\therefore a = \lim_{x \rightarrow 0} \frac{2(1 - \cos 2x)}{3x^2}$$

$$\Rightarrow \boxed{a = \frac{4 \sin^2 x}{3x^2} = \frac{4}{3}}$$

$$\therefore 3a + b = 3 \cdot \frac{4}{3} - 2 = 2 \text{ Ans.}$$

$$9. (A) I = \int_0^{\pi} x (\sin^2(\sin x) + \cos^2(\cos x)) dx ;$$

$$I = \int_0^{\pi} (\pi - x) (\sin^2(\sin x) + \cos^2(\cos x)) dx$$

$$2I = \pi \int_0^{\pi} (\sin^2(\sin x) + \cos^2(\cos x)) dx ;$$

$$2I = 2\pi \int_0^{\pi/2} (\sin^2(\sin x) + \cos^2(\cos x)) dx \text{ (Using Queen)}$$

$$\text{also } I = \pi \int_0^{\pi/2} (\sin^2(\cos x) + \cos^2(\sin x)) dx ;$$

use king and add,

$$2I = \pi \int_0^{\pi/2} 2 dx \Rightarrow I = \pi \int_0^{\pi/2} dx = \frac{\pi^2}{2} \text{ Ans. } \Rightarrow (Q)$$

$$(B) I = \int_0^{\pi} \frac{\pi - x}{1 + \sin^2 x} dx ;$$

$$\therefore 2I = \pi \int_0^{\pi} \frac{dx}{1 + \sin^2 x}$$

$$2I = 2\pi \int_0^{\pi/2} \frac{dx}{1 + \sin^2 x} \Rightarrow I = \pi \int_0^{\pi/2} \frac{\sec^2 x dx}{1 + 2 \tan^2 x}$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\sec^2 x dx}{\tan^2 x + (1/\sqrt{2})^2} = \frac{\pi\sqrt{2}}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}} \text{ Ans.}$$

$$\Rightarrow (S)$$

$$(C) f(x) = 2 \sin \sqrt{x}$$

$$f'(x) = \frac{2 \cos \sqrt{x}}{2\sqrt{x}} = \frac{\cos \sqrt{x}}{\sqrt{x}}$$

$$x f'(x) = \sqrt{x} \cos \sqrt{x}$$

$$\therefore I = \int_0^{\pi^2/4} (f(x) + x f'(x)) dx$$

$$= [x f(x)]_0^{\pi^2/4} = 2x \sin \sqrt{x} \Big|_0^{\pi^2/4} = \frac{2\pi^2}{4} = \frac{\pi^2}{2} \text{ Ans.}$$

$$\Rightarrow (Q)$$

10. (A) $f(x) = \int x^{\sin x} (1 + x \cos x \cdot \ln x + \sin x) dx$

if $F(x) = x^{\sin x} = e^{\sin x \ln x}$

$\therefore x F'(x) = x^{\sin x} (x \cos x \ln x + \sin x)$

$\therefore f(x) = \int (F(x) + x F'(x)) = x F(x) + C$

$f(x) = x \cdot x^{\sin x} + C$

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cdot \frac{\pi}{2} + C \Rightarrow C = 0$

$\therefore f(x) = x(x)^{\sin x}$; $f(\pi) = \pi(\pi)^0 = \pi$ (irrational)

$\Rightarrow$ (Q)

(B) $g(x) = \int \frac{\cos x (\cos x + 2) + \sin^2 x}{(\cos x + 2)^2} dx$

$= \int \cos x \cdot \frac{1}{(\cos x + 2)} dx + \int \frac{\sin^2 x}{\cos x + 2} dx$

$= \frac{1}{\cos x + 2} \cdot \sin x - \int \frac{\sin^2 x}{(\cos x + 2)^2} dx + \int \frac{\sin^2 x}{(\cos x + 2)^2} dx$

$g(x) = \frac{\sin x}{\cos x + 2} + C$

11. (A) $I = \int_4^{10} \frac{[x^2] dx}{[(14-x)^2] + [x^2]}$ ----- (i)

$I = \int_4^{10} \frac{[(14-x)^2]}{[x^2] + [(14-x)^2]} dx$ ----- (ii)

add (i) & (ii)

$2I = \int_4^{10} dx$

$\Rightarrow 2I = 6 \Rightarrow I = 3$

(B) $\int_{-1}^2 \frac{|x|}{x} dx = \int_{-1}^0 (-1) dx + \int_0^2 (1) dx = 1$

(C) $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r^{99}}{n^{100}} = \int_0^1 x^{99} dx = \left[\frac{x^{100}}{100} \right]_0^1 = \frac{1}{100}$

(D) $5050 \int_{-1}^1 \sqrt{x^{200}} dx = 5050 \times 2 \int_0^1 x^{100} dx$

$= 5050 \times 2 \int_0^1 x^{100} dx$

$= 10100 \times \left[\frac{x^{101}}{101} \right]_0^1 = 100 = \frac{1}{\alpha}$

$\Rightarrow \alpha = \frac{1}{100}$

12. (A) for $a=0$, $I = \int_0^T \sin^2 x dx = \int_0^T \frac{1 - \cos 2x}{2} dx$

$= \frac{x}{2} - \frac{1}{4} \sin 2x \Big|_0^T = \frac{T}{2} - \frac{1}{4} \sin 2T$

$\therefore L = \frac{1}{2} - \lim_{T \rightarrow \infty} \frac{1}{4} \frac{\sin 2T}{T} = \frac{1}{2} \Rightarrow$ (A) $\rightarrow q$

(B) for $a=1$, $\int_0^T 4 \sin^2 x dx$

$\Rightarrow L = 2$, hence (B) $\rightarrow s$

(C) $a=-1$, $\int_0^T 0 dx = 0 \Rightarrow L=0$,

hence (C) $\rightarrow p$

(D) for $a \neq 0, -1, 1$,

$I = \int_0^T (\sin^2 x + \sin^2 ax + 2 \sin x \cdot \sin ax) dx$

$= \int_0^T \left(\frac{1 - \cos 2x}{2} + \frac{1 - \cos 2ax}{2} + \cos(a-1)x - \cos(a+1)x \right) dx$

$= \left[x - \frac{1}{4} \sin 2x - \frac{1}{4a} \sin 2ax + \frac{\sin(a-1)x}{a-1} - \frac{\sin(a+1)x}{a+1} \right]_0^T$

$\therefore L = \lim_{T \rightarrow \infty} \frac{T}{T}$

$= \lim_{T \rightarrow \infty} \frac{1}{T}$

$\left[\frac{1}{4} \sin 2x - \frac{1}{4a} \sin 2ax + \frac{\sin(a-1)x}{a-1} - \frac{\sin(a+1)x}{a+1} \right]_0^T$

$L = 1$ hence (D) $\rightarrow r$



Part # II : Comprehension

Comprehension # 1

$$1. \lim_{x \rightarrow 0} \frac{\int_0^x \frac{t^2}{(a+t^r)^{1/p}} dt}{bx - \sin x}$$

using L'Hospital's Rule

$$\lim_{x \rightarrow 0} \frac{\frac{x^2}{(a+x^r)^{1/p}}}{b - \cos x}$$

 for existence of limit $\lim_{x \rightarrow 0} D^r \rightarrow 0$

$$\therefore b - 1 = 0 \Rightarrow b = 1 \text{ Ans.}$$

$$2. \therefore l = \lim_{x \rightarrow 0} \frac{x^2}{(a+x^r)^{1/p}} \cdot \frac{x^2}{(1-\cos x)} \cdot \frac{1}{x^2}$$

$$= 2 \lim_{x \rightarrow 0} \frac{1}{(a+x^r)^{1/p}} = \frac{2}{a^{1/p}}$$

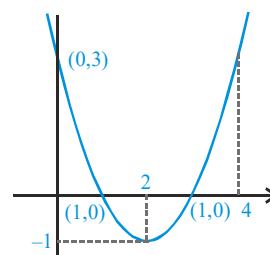
 if $p = 3$ and $l = 1$

$$\therefore 1 = \frac{2}{a^{1/3}} \Rightarrow a = 8 \text{ Ans.}$$

 3. again $p = 2$ and $a = 9$ then

$$l = \frac{2}{9^{1/3}} = \frac{2}{3} \text{ Ans.}$$

Comprehension # 2



$$f(x) = x^2 - 4x + 3$$

$$f(x)|_{\max} = 3 \quad x \in [0, 4]$$

$$f(x)|_{\min} = \begin{cases} x^2 - 4x + 3 & x \in [0, 2) \\ -1 & x \in [2, 4] \end{cases}$$

$$13. (A) L = \lim_{x \rightarrow 0} \frac{\int_0^{\ln(1+x)} (1 - \tan 2y)^{1/y} dy}{\frac{\sin x}{x} \cdot x}$$

$$= \lim_{x \rightarrow 0} \frac{\int_0^{\ln(1+x)} (1 - \tan 2y)^{1/y} dy}{x} \quad \text{Using L'Hospital's Rule}$$

$$L = \lim_{x \rightarrow 0} \frac{[1 - \tan 2(\ln(1+x))]^{\frac{1}{\ln(1+x)}}}{(1+x)} \quad (1^\infty \text{ form})$$

$$L = \lim_{x \rightarrow 0} e^l$$

$$\text{where } l = \lim_{x \rightarrow 0} \frac{-1}{\ln(1+x)} \tan(2(\ln(1+x)))$$

$$= - \lim_{x \rightarrow 0} \frac{2 \cdot \tan(2(\ln(1+x)))}{2 \ln(1+x)} = -2$$

$$\text{hence } L = e^{-2} \Rightarrow (s)$$

$$(B) \ln l = \lim_{x \rightarrow \infty} \frac{\ln(e^{2x} + e^x + x)}{x} = \lim_{x \rightarrow \infty} \frac{2e^{2x} + e^x + 1}{e^{2x} + e^x + 1}$$

$$= \lim_{x \rightarrow \infty} \frac{2 + e^{-x} + e^{-2x}}{1 + e^{-x} + e^{-2x}} = 2$$

$$\Rightarrow l = e^2 \Rightarrow (r)$$

$$(C) f(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n}{n^2 + k^2 x^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \left(\frac{kx}{n}\right)^2}$$

$$= \int_0^1 \frac{dy}{1 + x^2 y^2} = \frac{1}{x^2} \int_0^1 \frac{dy}{\frac{1}{x^2} + y^2}$$

$$= \frac{1}{x^2} \cdot x \cdot \tan^{-1} yx \Big|_0^1 = \frac{\tan^{-1} x}{x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1 \Rightarrow (p)$$

Now, $g(x) = \begin{cases} \frac{x^2 - 4x + 6}{2} & 0 \leq x < 2 \\ \frac{3 - 1}{2} = 1 & 2 \leq x < 4 \\ -x + 5 + x - 4 = 1 & 4 < x < 5 \\ \tan\left(\tan^{-1}\left(\frac{6-x}{1}\right)\right) = 6-x & x \geq 5 \end{cases}$

$$g(x) = \begin{cases} \frac{x^2 - 4x + 6}{2} & 0 \leq x < 2 \\ 1 & 2 \leq x < 5 \\ 6 - x & x \geq 5 \end{cases}$$

1. $\int_2^5 g(x) dx = 5 - 2 = 3$

2. $h(x) = \int_0^{x^2} g(t) dt$

$$h'(x) = g(x^2) \cdot 2x$$

$$g(x^2) = 0 \text{ at } x = \sqrt{6}$$

$$\therefore h'(x) < 0 \text{ in } (\sqrt{6}, 7]$$

and hence $h(x)$ is decreasing

3. $\lim_{x \rightarrow 4} \frac{g(x) - g(2)}{\ln(\cos(4 - x))} \quad \left(\frac{0}{0} \text{ form}\right)$

$$\lim_{x \rightarrow 4} \frac{g'(x)}{\frac{1}{\cos(4 - x)} (\sin(4 - x))}$$

$$= \lim_{x \rightarrow 4} \frac{g'(x)}{\tan(4 - x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\Rightarrow \lim_{x \rightarrow 4} \frac{-g''(x)}{\sec^2(4 - x)} = 0 \quad (Q \ g''(4) = 0)$$

Comprehension # 4

1. $\int_0^1 \frac{\sin x}{x^2} dx - \int_0^1 \frac{\cos x}{x} dx = \sin x \left(-\frac{1}{x}\right) \Big|_0^1 + \int_0^1 \cos x \left(\frac{1}{x}\right) dx$
 $- \int_0^1 \frac{\cos x}{x} dx$
 $= - \left[\frac{\sin x}{x} \right]_0^1 = (1) - \sin(1) \text{ Ans.}$

2. $\lim_{t \rightarrow 0} \frac{\int_0^t f(x) dx}{t^2}$

$$= \lim_{t \rightarrow 0} \frac{\int_0^t \frac{\sin x - x \cos x}{x^2} dx}{t^2}$$

using L'Hospital's rule

$$I = \lim_{t \rightarrow 0} \frac{\sin t - t \cos t}{t^2 \cdot 2t} = \lim_{t \rightarrow 0} \frac{\cos t (\tan t - t)}{2t^3}$$

$$= \frac{1}{2} \lim_{t \rightarrow 0} \frac{\sec^2 t - 1}{3t^2} = \frac{1}{6}$$

Comprehension # 5

1. $D(t) = \left| \frac{1}{\sqrt{(ae^t)^2 + (be^{-t})^2}} \right|$

$$= \frac{1}{\sqrt{a^2 e^{2t} + b^2 e^{-2t}}}$$

$$\frac{1}{(D(t))^2} = (a^2 e^{2t} + b^2 e^{-2t})$$

$$\begin{array}{c} (0, 0) \\ | \\ D(t) \\ \hline (ae^t)x + (be^{-t})y = 1 \text{ line} \end{array}$$

$$\therefore I = \int_0^1 (a^2 e^{2t} + b^2 e^{-2t}) dt = \left[\frac{a^2 e^{2t}}{2} - \frac{b^2 e^{-2t}}{2} \right]_0^1$$

$$= \left(\frac{a^2 e^2 - b^2 e^{-2}}{2} \right) - \left(\frac{a^2 - b^2}{2} \right)$$

$$= \frac{a^2(e^2 - 1) - b^2(e^{-2} - 1)}{2} = \frac{a^2(e^2 - 1) + \frac{b^2}{e^2}(e^2 - 1)}{2}$$

$$= \frac{e^2 - 1}{2} \left(a^2 + \frac{b^2}{e^2} \right)$$

2. now put $b = \frac{1}{a}$

$$I = \frac{e^2 - 1}{2} \left(a^2 + \frac{1}{a^2 e^2} \right)$$

$$= \frac{e^2 - 1}{2} \left(\left(a - \frac{1}{ae} \right)^2 + \frac{2}{e} \right)$$

$$I \text{ is minimum if } a = \frac{1}{ae} \Rightarrow a^2 = \frac{1}{e} \Rightarrow a = \frac{1}{\sqrt{e}}$$

$$\Rightarrow b = \sqrt{e}$$

3. and $I_{\min} = \frac{e^2 - 1}{2} \cdot \frac{2}{e} = e - \frac{1}{e}$

Comprehension # 7

1. $g(x) = \int_0^x f(t) dt$

$$g'(x) = f(x)$$

From the graph it is clear that

$$f(x) > 0 \text{ in } x \in [0, 3] \text{ and}$$

$$f(x) < 0 \text{ in } x \in (3, 7)$$

$\therefore g(x)$ is increasing in $[0, 3]$ and

$g(x)$ is decreasing in $[3, 7]$

$\therefore$ maximum value of $g(x)$ occurs at $x = 3$

$$\therefore g(3) = \int_0^3 f(t) dt$$

$$= \int_0^1 1 \cdot dt + \int_1^2 (2t - 1) dt + \int_2^3 (3t + 9) dt$$

$$= 1 + (t^2 - t)_1^2 + \left(9t - 3 \frac{t^2}{2} \right)_2^3$$

$$= 1 + (4 - 2 - 0) + \left(27 - \frac{27}{2} - 18 + 6 \right) = \frac{9}{2}$$

2. $g(x)$ start decreasing from $x = 3$

$$g(4) = \int_0^4 f(t) dt = \int_0^3 f(t) dt + \int_3^4 f(t) dt$$

$$= \frac{9}{2} + \int_3^4 (-3t + 9) dt = \frac{9}{2} + \left(9t - \frac{3t^2}{2} \right)_3^4$$

$$= \frac{9}{2} + \left(36 - 24 - 27 + \frac{27}{2} \right) = 3$$

Now, $g(x) = \int_0^x f(t) dt$

$$= \int_0^4 f(t) dt + \int_4^x f(t) dt \quad 0 \leq x \leq 6$$

$$= 3 + \int_4^x (-3) dt = 3 - 3(x - 4) = 15 - 3x$$

$$g(x) = 0 \Rightarrow 15 - 3x = 0 \Rightarrow x = 5$$

which lies in $[0, 6]$

3. $g(x)$ becomes zero at $x = 5$

$\therefore g(x)$ will be negative in $(5, 7)$

Comprehension # 8

1. Given $f(x) \cdot f'(-x) = f(-x) \cdot f'(x)$

$$\frac{f'(x)}{f(x)} = \frac{f'(-x)}{f(-x)}$$

Integrating

$$\ln f(x) = -\ln f(-x) + C$$

$$\ln (f(x) \cdot f(-x)) = C \quad f(x) \cdot f(-x) = C$$

but $f(0) = 3$

$$\therefore f^2(0) = C \Rightarrow C = 9$$

$$\therefore f(x) \cdot f(-x) = 9 \Rightarrow \text{(B)}$$

2. Let $I = \int_{-51}^{51} \frac{dx}{3 + f(x)}$

$$= \int_{-51}^{51} \frac{dx}{3 + f(-x)}$$

$$2I = \int_{-51}^{51} \frac{6 + f(x) + f(-x)}{(3 + f(x))(3 + f(-x))} dx$$

$$= \int_{-51}^{51} \frac{6 + f(x) + f(-x)}{9 + 3(f(x) + f(-x)) + f(x) \cdot f(-x)} dx$$

$$2I = \int_{-51}^{51} \frac{6 + f(x) + f(-x)}{18 + 3(f(x) + f(-x))} dx = \frac{1}{3} \int_{-51}^{51} dx = \frac{2 \cdot 51}{3}$$

$$\Rightarrow I = \frac{51}{3} = 17 \text{ Ans.}$$



3. Let $x = \alpha$ be the root of $f(x) = 0$

$$\therefore f(\alpha) = 0$$

$$f(x) \cdot f(-x) = 9$$

$$\text{put } x = \alpha \Rightarrow 0 = 9 \text{ impossible}$$

hence $f(x)$ has no root

$$\text{but } f(0) = 3$$

hence $f(x) > 0 \forall x \in \mathbb{R}$ as f is continuous

possible function $f(x) = 3e^{-x}$ **Ans.**

$$\therefore B = \frac{B}{2}(e^2 - 1) + 1 \Rightarrow 2B = B(e^2 - 1) + 2$$

$$\Rightarrow 3B = Be^2 + 2 \Rightarrow B = \frac{2}{3 - e^2}$$

$\therefore$ from (2)

$$g(x) = \left(\frac{2}{3 - e^2} \right) e^x + x; g(0) = \frac{2}{3 - e^2} \text{ **Ans.**}$$

Comprehension # 11

$$1. f(x) = e^x \int_0^1 e^t \cdot f(t) dt \quad f(x) = Ae^x \quad \dots(1)$$

$$\Rightarrow f(t) = Ae^t$$

$$\text{where, } A = \int_0^1 e^t \cdot f(t) dt$$

$$\Rightarrow A = \int_0^1 e^t \cdot Ae^t dt; A = A \int_0^1 e^{2t} dt$$

$$\text{now } A \left[\int_0^1 e^{2t} dt - 1 \right] = 0 \Rightarrow A = 0 \text{ as } \int_0^1 e^{2t} dt \neq 0$$

$$\text{hence } f(x) = 0 \Rightarrow f(1) = 0 \text{ **Ans.**}$$

$$2. \text{ again } g(x) = e^x \int_0^1 e^t g(t) dt + x$$

$$g(x) = Be^x + x \quad \dots(2)$$

$$\Rightarrow g(t) = Be^t + t$$

$$\text{where } B = \int_0^1 e^t g(t) dt; B = \int_0^1 e^t (Be^t + t) dt;$$

$$B = B \int_0^1 e^{2t} dt + \int_0^1 e^t \cdot t dt$$

$$\text{but } \int_0^1 e^{2t} dt = \frac{1}{2}(e^2 - 1) \text{ and } \int_0^1 te^t dt = 1$$

$$3. g(2) = \frac{2e^2}{3 - e^2} + 2 = \frac{6}{3 - e^2};$$

$$\frac{g(0)}{g(2)} = \frac{2}{3 - e^2} \cdot \frac{3 - e^2}{6} = \frac{1}{3} \text{ **Ans.**}$$

EXERCISE - 4

Subjective Type

1. (I) $-5(\sqrt[3]{16}-1)$

(II) $0.2(e-1)^5$

(III) $\frac{\pi(9-4\sqrt{3})}{36} + \frac{1}{2}\ln\frac{3}{2}$

(IV) $2 - \frac{\pi}{2}$

(V) $\sqrt{2} - \frac{2}{\sqrt{3}} + \ln\frac{2+\sqrt{3}}{1+\sqrt{2}}$

(VI) $\frac{1}{32}\left(\pi + \frac{7\sqrt{3}}{2} - 8\right)$

(VII) $\tan^{-1}\frac{1}{2}$

(VIII) $\frac{11}{6}$

(IX) 2

(X) $\int_0^2 [x^2] dx = \int_0^1 0 \cdot dx + \int_1^{\sqrt{2}} dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 \cdot dx + 3 \int_{\sqrt{3}}^2 dx$

$= 5 - \sqrt{2} - \sqrt{3}$

(XI) $\int_{-1}^1 [\cos^{-1} x] dx = 3 \int_{-1}^{\cos 3} dx + 2 \int_{\cos 3}^{\cos 2} dx + \int_{\cos 2}^{\cos 1} dx + \int_{\cos 1}^0 0 \cdot dx$

$= \cos 1 + \cos 2 + \cos 3 + 3$

(XII) $I = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2 + 1} = \left[\tan^{-1}(x+1) \right]_{-\infty}^{\infty}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

(XIII) $x = \sec \theta$

$\Rightarrow dx = \sec \theta \tan \theta d\theta$

$\Rightarrow I = \int_{\pi/4}^{\pi/2} \frac{\sec \theta \tan \theta d\theta}{\sec \theta \cdot \tan \theta} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

(XIV) $I = \int_0^4 \frac{x^2+1-1}{1+x} dx = \int_0^4 (x-1) dx + \int_0^4 \frac{1}{1+x} dx$

$= \left[\frac{x^2}{2} - x \right]_0^4 + [\ln(1+x)]_0^4 = 4 + \ln 5$

(XV) Let $\cos \theta = t - \sin \theta$ $d\theta = dt$

$I = - \int_1^0 \sqrt{t}(1-t^2) \cdot dt$

$I = - \left[\frac{t^{3/2}}{3/2} - \frac{t^{7/2}}{7/2} \right]_1^0$

$I = - \left(-\frac{2}{3} + \frac{2}{3} \right) = - \left(\frac{-14+6}{21} \right)$

$I = + \frac{8}{2}$

(XVI) $\frac{\pi-2}{2}$

(XVII) $\frac{1}{2} \ln \left(\frac{e}{2} \right)$

(XVIII) 1

(XIX) $\frac{\pi}{6} - \frac{2}{9}$

(XX) 0

(XXI) 0

2. (I) $-\pi \log 2$

(II) t

(III) $\frac{\pi}{8} \log 2$

(IV) $\frac{\pi^2}{8}$

(V) $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \int_0^1 \frac{x^4[(1+x^2)-2x]^2}{1+x^2} dx$

$= \int_0^1 x^4(1+x^2) dx - 4 \int_0^1 x^5 dx + 4 \int_0^1 \frac{x^6}{1+x^2} dx$

$= \left[\frac{x^5}{5} + \frac{x^7}{7} \right]_0^1 - 4 \left[\frac{x^6}{6} \right]_0^1 + 4 \int_0^1 \frac{-dx}{1+x^2} +$

$4 \int_0^1 \frac{(x^2+1)(x^4+1-x^2)}{1+x^2} dx$

$= \left(\frac{1}{5} + \frac{1}{7} \right) - 4 \left(\frac{1}{6} \right) - 4 \left[\tan^{-1} x \right]_0^1 + 4 \left(\frac{1}{5} + 1 - \frac{1}{3} \right)$

$= \frac{22}{7} - \pi$

(VI) $I = \int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin \left(\frac{\pi}{4} + x \right)} dx$

$I = \sqrt{2} \int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx \quad \dots(1)$

$I = \sqrt{2} \int_0^{\pi/2} \frac{a \cos x + b \sin x}{\cos x + \sin x} dx \quad \dots(2)$

add.(1) & (2)

$2I = \sqrt{2}(a+b) \int_0^{\pi/2} dx \Rightarrow I = \frac{(a+b)\pi}{2\sqrt{2}}$



$$(VIII) \frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$$

$$(IX) I = \int_0^1 \frac{1-x}{1+x} \cdot \frac{dx}{\sqrt{x+x^2+x^3}}$$

$$= \int_0^1 \frac{(1-x^2)}{(x^2+2x+1)\sqrt{x+x^2+x^3}}$$

$$= \int_0^1 \frac{\left(\frac{1}{x^2}-1\right) dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$$

Put $x + \frac{1}{x} + 1 = t^2$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = 2t dt$$

$$I = \int_{\infty}^{\sqrt{3}} \frac{-2t dt}{(t^2+1)t} = 2 \int_{\sqrt{3}}^{\infty} \frac{dt}{t^2+1} = 2[\tan^{-1} t]_{\sqrt{3}}^{\infty} = \frac{\pi}{3}$$

$$(X) \frac{\pi}{8} \ln 2$$

$$(XI) I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

Let $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

$$\Rightarrow I = \int_0^{\pi/4} 2\theta \sec^2 \theta d\theta$$

$$= [\theta \tan \theta]_0^{\pi/4} - 2 \int_0^{\pi/4} \tan \theta d\theta = 2\left(\frac{\pi}{4}\right) - 2[\ln \sec \theta]_0^{\pi/4}$$

$$= \frac{\pi}{2} - 2 \ln \sqrt{2}$$

$$= \frac{\pi}{2} - \ln 2$$

$$(XII) \text{ Let } \tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$$

$$\therefore I = \int_0^{\pi/4} \frac{t \tan t \cdot dt}{\sqrt{1+\tan^2 t}} = \int_0^{\pi/4} t \cdot \sin t dt$$

$$= \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4-\pi}{4\sqrt{2}}$$

$$(XIII) I = \int_a^b \sqrt{(x-a)(b-x)} dx$$

put $x = a \sin^2 \theta + b \cos^2 \theta$

$$I = 2 \int_0^{\pi/2} -(b-a)^2 \sin^2 \theta \cos^2 \theta d\theta$$

$$I = -2(b-a)^2 \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$I = -\frac{(b-a)^2}{2} \int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$I = -\frac{(b-a)^2}{4} \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = -\frac{(b-a)^2 \pi}{8}$$

$$(XIV) I = \int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

put $x = \tan \theta$

$$I = \int_0^{\pi/3} \tan^{-1}(\tan 2\theta) \sec^2 \theta d\theta + \int_{\pi/4}^{\pi/3} (\pi - 2\theta) \sec^2 \theta d\theta$$

$$I = \pi \left(1 - \frac{1}{\sqrt{3}}\right) - \frac{\pi}{4}$$

$$(XV) \frac{\pi}{4}$$

$$(XVI) 2\left(\frac{5}{6} - \ln 2\right)$$

$$(XVII) \frac{9}{8}$$

$$(XVIII) \frac{\pi}{2}$$

$$(XIX) \frac{1}{20} \ln 3$$

$$(XX) 0$$

$$(XXI) 0$$

$$(XXII) I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow I = \frac{\pi}{4}$$

(XXIII)

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos x\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$$

$$2I = \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

(XXIV)
$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx = \int_0^a \frac{\sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$$

$$2I = \int_0^a 1 \cdot dx = a \Rightarrow I = \frac{a}{2}$$

(XXV)
$$I = \int_0^{\pi/2} \frac{a \sin x + 6 \cos x}{\sin x + \cos x} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{(a+b) \sin x + (a+b) \cos x}{\sin x + \cos x} dx$$

$$\Rightarrow I = (a+b) \frac{\pi}{4}$$

3. (I) $\frac{\pi}{2}$

(II)
$$I = \int_0^1 \frac{\sin^{-1} \sqrt{x}}{x^2 - x + 1} dx \quad \dots(i)$$

$$I = \int_0^1 \frac{\sin^{-1} \sqrt{1-x}}{(1-x)^2 - (1-x) + 1} dx = \int_0^1 \frac{\sin^{-1} \sqrt{1-x}}{x^2 - x + 1} dx \quad \dots(ii)$$

Add. (i) and (ii)

$$2I = \int_0^1 \frac{\sin^{-1} \sqrt{x} + \sin^{-1} \sqrt{1-x}}{x^2 - x + 1} dx$$

$$\Rightarrow 2I = \frac{\pi}{2} \int_0^1 \frac{dx}{(x-1/2)^2 + (\sqrt{3}/2)^2}$$

$$\Rightarrow I = \frac{\pi}{4} \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{2x-1}{\sqrt{3}} \right]_0^1 = \frac{\pi^2}{6\sqrt{3}}$$

(III)
$$\frac{1}{3} \left(\tan^{-1} \frac{\sqrt{2}}{3} - \tan^{-1} \frac{1}{3} \right)$$

(IV)
$$I = \int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} dx \quad \dots(i)$$

then,

$$I = \int_0^{\pi} \frac{(\pi-x) \sin 2(\pi-x) \sin\left(\frac{\pi}{2} \cos(\pi-x)\right)}{2(\pi-x) - \pi} dx \quad \dots(ii)$$

$$= \int_0^{\pi} \frac{(\pi-x) \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{\pi - 2x} dx$$

$$= \int_0^{\pi} \frac{(x-\pi) \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} dx$$

add equation (i) & (ii)

$$2I = \int_0^{\pi} \sin 2x \sin\left(\frac{\pi}{2} \cos x\right) dx$$

$$\therefore I = \int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2} \cos x\right) dx$$

Put $\frac{\pi}{2} \cos x = t \Rightarrow \sin x dx = -\frac{2}{\pi} dt$

$$\therefore I = -\frac{2}{\pi} \int_{\pi/2}^{-\pi/2} \frac{2t}{\pi} \sin t dt = \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} t \sin t dt$$

$$\Rightarrow I = \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} t \sin t dt = \frac{4}{\pi^2} [-t \cos t + \sin t]_{-\pi/2}^{\pi/2}$$

$$= \frac{4}{\pi^2} \times 2 = \frac{8}{\pi^2}$$

(V)
$$-\frac{3\sqrt{2}}{5} (e^{2\pi} + 1)$$

(VI)
$$\frac{\pi^2}{16} - \frac{\pi}{4} \bullet n2$$

4. (V)
$$\int_a^b \frac{x^{n-1} \{nx^2 - 2x^2 + n(a+b)x - (a+b)x + nab\}}{(x+a)^2(x+b)^2} dx$$

$$= \int_a^b \frac{x^{n-1} \{n(x+a)(x+b) - x(2x+a+b)\}}{(x+a)^2(x+b)^2} dx$$

$$= \int_a^b \frac{nx^{n-1}}{(x+a)(x+b)} dx - \int_a^b \frac{x^n(x+a+x+b)}{(x+a)^2(x+b)^2} dx$$

$$= \int_a^b \left(\frac{d}{dx} \frac{x^n}{(x+a)(x+b)} \right) dx$$

$$= \left[\frac{x^n}{(x+a)(x+b)} \right]_a^b = \frac{b^{n-1} - a^{n-1}}{2(a+b)}$$

$$\begin{aligned}
 \text{(VII)} \int_0^1 x^m (1-x)^n dx &= \left[-x^m \frac{(1-x)^{n+1}}{n+1} \right]_0^1 + \frac{m}{n+1} \int_0^1 x^{m-1} (1-x)^{n+1} dx \\
 &= 0 + \frac{m}{n+1} \int_0^1 x^{m-1} (1-x)^{n+1} dx \\
 &= \frac{m(m-1)}{(n+1)(n+2)} \int_0^1 x^{m-2} (1-x)^{n+2} dx \\
 &= \frac{m(m-1) \dots 1}{(n+1)(n+2) \dots (n+m+1)} = \frac{m!}{(n+m+1)!}
 \end{aligned}$$

7. 5250

$$9. \frac{t^{n+1} - 1}{(t-1)(n+1)}$$

$$10. u_n = \{x(1-x)\}^n$$

$$\frac{du_n}{dx} = n \{x(1-x)\}^{n-1} \{1-2x\}$$

$$\frac{du_n}{dx} = n \cdot u_{n-1} - 2nx u_{n-1}$$

$$\frac{d^2 u_n}{dx^2} = n(n-1) u_{n-2} \{1-2x\}$$

$$\begin{aligned}
 &-2n \{u_{n-1} + x \cdot (n-1) u_{n-2} \{1-2x\}\} \\
 &= n(n-1) u_{n-2} - 2nx(n-1) u_{n-2} \\
 &-2n \cdot u_{n-1} - x 2n(n-1) (1-2x) u_{n-2} \\
 &= n(n-1) u_{n-2} - 2nx(n-1) u_{n-2} \{1+1-2x\} - 2n u_{n-1} \\
 &= n(n-1) u_{n-2} - 4nx(1-x) u_{n-2} (n-1) - 2n u_{n-1} \\
 &= n(n-1) u_{n-2} - 2n u_{n-1} \{2n-1\}
 \end{aligned}$$

$$v_n = \int_0^1 e^x \cdot u_n dx$$

II I

& apply by parts twice

11.

$$F(x) = \begin{cases} \int_0^x (1-t) dt & ; 0 \leq x \leq 1 \\ \int_0^1 (1-t) dt + \int_1^x 0 \cdot dx & ; 1 < x \leq 2 \\ \int_0^1 (1-t) dt + \int_1^2 0 \cdot dx + \int_2^x (2-t)^2 dt & ; 2 < x \leq 3 \end{cases}$$

$$F(x) = \begin{cases} x - \frac{x^2}{2} & ; 0 \leq x \leq 1 \\ \frac{1}{2} & ; 1 < x \leq 2 \\ \frac{(x-2)^3}{3} + \frac{1}{2} & ; 2 < x \leq 3 \end{cases}$$

$$13. I = \int_0^\pi f(x) dx = \int_0^\pi \frac{\sin x}{x} dx \quad \dots (i)$$

$$I = \int_0^\pi f(\pi-x) dx = \int_0^\pi \frac{\sin(\pi-x)}{\pi-x} dx = \int_0^\pi \frac{\sin x}{\pi-x} dx$$

... (ii)

(i) + (ii)

$$\Rightarrow 2I = \int_0^\pi \left\{ \frac{\sin x}{x} + \frac{\sin x}{\pi-x} \right\} dx \Rightarrow I$$

$$= \frac{\pi}{2} \int_0^\pi \frac{\sin x}{x(\pi-x)} dx \quad \dots (iii)$$

$$\text{Now } \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(x) f\left(\frac{\pi}{2}-x\right) dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} \cdot \frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x} dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} \cdot \frac{\cos x}{\frac{\pi}{2}-x} dx = \frac{\pi}{4} \cdot \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{x\left(\frac{\pi}{2}-x\right)} dx$$

$$= \frac{\pi}{8} \int_0^\pi \frac{\sin t}{\frac{t}{2}\left(\frac{\pi}{2}-\frac{t}{2}\right)} dt, \text{ where } t=2x \quad \dots (iv)$$

(iii) + (iv)

$$\Rightarrow \frac{\pi}{2} \int_0^\pi f(x) f\left(\frac{\pi}{2}-x\right) dx = \int_0^\pi f(x) dx$$

14. $(1-x)^n = C_0 - C_1 x + C_2 x^2 - \dots + (-1)^n C_n x^n$

$$x^{n-1}(1-x)^{n+1} = (C_0 x^{n-1} - C_1 x^n + C_2 x^{n+1} - \dots + (-1)^n C_n x^{2n-1})(1-x)$$

$$= (C_0 x^{n-1} - C_1 x^n + C_2 x^{n+1} - \dots + (-1)^n C_n x^{2n-1})$$

$$- (C_0 x^n - C_1 x^{n+1} + C_2 x^{n+2} - \dots + (-1)^n C_n x^{2n})$$

$$\int_0^1 x^{n-1} (1-x)^{n+1} dx$$

$$= \left[\frac{C_0 x^n}{n} - \frac{C_1 x^{n+1}}{n+1} + \frac{C_2 x^{n+2}}{n+2} - \dots - \frac{(-1)^n C_n x^{2n}}{2n} \right]_0^1$$

$$- \left[\frac{C_0 x^{n+1}}{n+1} - \frac{C_1 x^{n+2}}{n+2} + \frac{C_2 x^{n+3}}{n+3} - \dots + \frac{(-1)^n C_n x^{2n+1}}{2n+1} \right]_0^1$$

$$= \left[\frac{C_0}{n} - \frac{C_1}{n+1} + \frac{C_2}{n+2} - \dots + \frac{(-1)^n C_n}{2n} \right]$$

$$- \left(\frac{C_0}{n+1} - \frac{C_1}{n+2} + \frac{C_2}{n+3} - \dots + (-1)^n \frac{C_n}{2n+1} \right)$$

$$= \frac{C_0}{n(n+1)} - \frac{C_1}{(n+1)(n+2)} + \frac{C_2}{(n+2)(n+3)} - \dots$$

upto $(n+1)$ terms

$$\int_0^1 x^{n-1} (1-x)^{n+1} dx$$

put $x = \sin^2 \theta \Rightarrow dx = 2 \sin \theta \cos \theta d\theta$

$$\int_0^1 x^{n-1} (1-x)^{n+1} dx = \int_0^{\pi/2} \sin^{2n-2} \theta \cos^{2n+2} \theta (2 \sin \theta \cos \theta) d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2n-1} \theta \cos^{2n+3} \theta d\theta$$

$$= \frac{2 \Gamma\left(\frac{2n-1+1}{2}\right) \Gamma\left(\frac{2n+3+1}{2}\right)}{2 \Gamma\left(\frac{2n-1+2n+3+2}{2}\right)}$$

$$= \frac{\Gamma(n) \Gamma(n+1)}{\Gamma(2n+2)} = \frac{|n-1| |n+1|}{|2n+1|}$$

15. $\int_1^2 \frac{(x^2-1)dx}{x^3 \sqrt{2x^4-2x^2+1}} = \int_1^2 \frac{x(x^2-1)dx}{x^4 \sqrt{2x^4-2x^2+1}}$

Let $x^2 = t \Rightarrow x dx = dt/2$

$$= \frac{1}{2} \int_1^4 \frac{(t-1)dt}{t^2 \sqrt{2t^2-2t+1}}$$

$$= \frac{1}{2} \int_1^4 \frac{t-1}{t^3 \sqrt{2-\frac{2}{t}+\frac{1}{t^2}}} dt = \frac{1}{2} \int_1^4 \frac{\frac{1}{t^2} - \frac{1}{t^3}}{\sqrt{2-\frac{2}{t}+\frac{1}{t^2}}} dt$$

Let $2 - \frac{2}{t} + \frac{1}{t^2} = z^2 \Rightarrow \left(\frac{2}{t^2} - \frac{2}{t^3} \right) dt = 2z dz$

$$= \frac{1}{2} \int_1^{5/4} \frac{z dz}{\sqrt{z^2}} = \frac{1}{2} \int_1^{5/4} dz = \frac{1}{8} = \frac{U}{V}$$

$$\Rightarrow (1000) \frac{U}{V} = \frac{1000}{8} = 125$$

16. -1

17. $\frac{2\pi}{\sqrt{3}}$

18. $I = \int_{-4}^{-5} e^{(x+5)^2} dx + 3 \int_{1/3}^{2/3} e^{9(x-\frac{2}{3})^2} dx$

Let $I_1 = \int_{-4}^{-5} e^{(x+5)^2} dx$

$$= (-5+4) \int_0^1 e^{((-5+4)x-4+5)^2} dx$$

{using property $\int_a^b f(x) dx = (b-a) \int_0^1 f((b-a)x+a) dx$ }

$$= - \int_0^1 e^{(x-1)^2} dx$$

$$I_2 = \int_{1/3}^{2/3} e^{9(x-\frac{2}{3})^2} dx$$

$$= \left(\frac{2}{3} - \frac{1}{3} \right) \int_0^1 e^{9\left[\left(\frac{2}{3}-\frac{1}{3}\right)x + \frac{1}{3} - \frac{2}{3}\right]^2} dx$$

$$= \frac{1}{3} \int_0^1 e^{(x-1)^2} dx = \frac{-1}{3} I_1$$

where $I = I_1 + 3I_2$

$$= I_1 + 3(-I_1/3) = 0$$

$\therefore I = 0$



19. $x^2 + 2x + 1 = k + 1 + \int_0^1 |t + k| dt$

$$(x + 1)^2 = (k + 1) + \int_0^1 |t + k| dt$$

If $k \geq -1$ R.H.S. ≥ 0

so there will be two real and distinct roots for

$$k \geq -1$$

If $k < -1$

$$(x + 1)^2 = k + 1 - \int_0^1 (t + k) dt$$

$$(x + 1)^2 = 1/2$$

so there will have two real and distinct roots for $k < -1$

$\Rightarrow$ The equation will have two real and distinct roots for $k \in \mathbb{R}$,

20. (II) $\left(0, \frac{5}{2}\right]$ (III) 1, 3

21. (III) $I = \int_0^1 \frac{dx}{2+x^2} + \int_1^2 \frac{dx}{2+x^2}$

$$\frac{1}{3} \leq \int_0^1 \frac{dx}{2+x^2} \leq \frac{1}{2} \quad \dots\dots(1)$$

$$\frac{1}{6} \leq \int_1^2 \frac{dx}{2+x^2} \leq \frac{1}{3} \quad \dots\dots(2)$$

add (1) & (2)

$$\frac{1}{2} \leq I \leq \frac{5}{6}$$

(IV) $\rightarrow \frac{\sin x}{x}$ is monotonic decreasing

$$\therefore \frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx < \frac{\sqrt{2}}{6}$$

(V) Let $f(x) = \sqrt{3+x^3}$

$$f'(x) = \frac{3x^2}{2\sqrt{3+x^3}} > 0 \quad \forall x \in (1, 3) \Rightarrow f \text{ is strictly}$$

increasing in (1, 3)

$\rightarrow m = \text{least value} = f(1) = \sqrt{3+1} = 2$

$$M = \text{greatest value} = f(3) = \sqrt{30}$$

$$\rightarrow m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

$$\Rightarrow 2(2) \leq \int_1^3 \sqrt{3+x^3} dx \leq \sqrt{30} \times 2$$

$$\Rightarrow 4 \leq \int_1^3 \sqrt{3+x^3} dx \leq 2\sqrt{30}$$

22. 0

25. $\sec(1) - 1$

26. $I = \int_0^\infty f\left(\frac{a}{x} + \frac{x}{a}\right) \frac{\ln x}{x} dx$

Let $x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2} dt$

$$\Rightarrow I = \int_0^\infty f\left(t + \frac{1}{t}\right) \frac{\ln\left(\frac{a}{t}\right)}{a/t} \cdot \left(\frac{a}{t^2}\right) dt$$

Again put

$$t = \frac{z}{a} \Rightarrow dt = \frac{dz}{a}$$

$$\Rightarrow I = \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{\ln\left(\frac{a^2}{z}\right)}{\frac{z}{a}} \cdot \frac{dz}{a}$$

$$= \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{[2\ln a - \ln z]}{z} dz$$

$$= \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{2\ln a}{z} dz - \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{\ln z}{z} dz$$

$$= 2 \bullet \ln a \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{dz}{z} - I$$

$$\Rightarrow 2I = 2 \bullet \ln a \int_0^\infty f\left(\frac{a}{z} + \frac{z}{a}\right) \cdot \frac{dz}{z} \Rightarrow I$$

$$= \bullet \ln a \int_0^\infty f\left(\frac{a}{z} + \frac{z}{a}\right) \cdot \frac{dz}{z} = \bullet \ln a \int_0^\infty f\left(\frac{a}{x} + \frac{x}{a}\right) \cdot \frac{dx}{x}$$

$$27. I_n = \int_0^1 e^x \cdot (x-1)^n dx$$

$$= e^x \cdot (x-1)^n \Big|_0^1 - n \int_0^1 e^x (x-1)^{n-1} dx$$

$$I_n = -(-1)^n - n I_{n-1} = (-1)^{n+1} - n I_{n-1}$$

$$n=1, I = \int_0^1 e^x (x-1) dx$$

$$= (x-1)e^x \Big|_0^1 - \int_0^1 e^x dx = 2 - e$$

$$I_2 = -1 - 2(2 - e) = 2e - 5$$

$$I_3 = 1 - 3(2e - 5) = 16 - 6e$$

$$\text{so } n=3$$

$$28. (A) f(x) = \int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} dt$$

Put $t = \sin^2 \theta$ in 1st integral and $t = \cos^2 \phi$ in the second integral

$$\text{then } f(x) = \int_0^x \theta \sin 2\theta d\theta - \int_{\pi/2}^x \phi \sin 2\phi d\phi$$

$$= \int_0^x \theta \sin 2\theta d\theta + \int_x^{\pi/2} \theta \sin 2\theta d\theta$$

$$= \int_0^{\pi/2} \theta \sin 2\theta d\theta = \frac{\pi}{4}$$

$$29. (i) 4\sqrt{2}$$

$$31. J_m = \int_1^e \ln^m x dx = [x \bullet \ln^m x]_1^e - m \int_1^e \ln^{m-1} x \cdot \frac{1}{x} dx$$

$$= e - m J_{m-1}$$

$$34. b\beta - a\alpha$$

$$35. (I) 2e^{(1/2)(\pi-4)} \quad (II) 11 \quad (V) \text{ Let } P = \lim_{n \rightarrow \infty} \left(\frac{\ln}{n^n} \right)^{1/n}$$

$$P = \lim_{n \rightarrow \infty} \left(\frac{1.2.3.4 \dots n}{n.nnn \dots n} \right)^{1/n}$$

$$P = \lim_{n \rightarrow \infty} \left(\left(\frac{1}{n} \right) \left(\frac{2}{n} \right) \left(\frac{3}{n} \right) \dots \left(\frac{n}{n} \right) \right)^{1/n}$$

$$\Rightarrow \ln P = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\log \left(\frac{1}{n} \right) + \log \left(\frac{2}{n} \right) + \dots + \log \left(\frac{n}{n} \right) \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \frac{r}{n} = \int_0^1 \ln x dx = [x \bullet \ln x - x]_0^1$$

$$= (0-1) - \lim_{x \rightarrow 0} (x \bullet \ln x) + 0$$

$$= -1 - \lim_{x \rightarrow 0} \frac{\ln x}{1/x} = -1 - \lim_{x \rightarrow 0} \frac{1/x}{(-1/x^2)}$$

$$= -1 - \lim_{x \rightarrow 0} x = -1 + 0 = -1$$

$$\Rightarrow \bullet np = -1$$

$$P = e^{-1} = 1/e$$

$$(VI) \frac{2}{3}$$

$$(VII) 0$$

$$(VIII) \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}} = \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{1}{n} \cdot \frac{1}{\sqrt{1 - \left(\frac{r}{n} \right)^2}}$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_0^1 = \frac{\pi}{2}$$

$$(IX) \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{3}{n} \sqrt{\frac{n}{n+3r}} = \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{3}{n} \sqrt{\frac{n/r}{3+n/r}}$$

$$= \int_0^1 3 \sqrt{\frac{x}{x+3}} dx$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

$$\begin{aligned}
 3. \quad & \int_0^{\pi} |\sin x| dx + \int_{\pi}^{10\pi} |\sin x| dx - \int_0^{\pi} \sin x dx \\
 &= \int_0^{10\pi} |\sin x| dx - \int_0^{\pi} \sin x dx \\
 &= 10 \int_0^{\pi} |\sin x| dx - \int_0^{\pi} \sin x dx = 9 \int_0^{\pi} |\sin x| dx \\
 &= 9 \times 2 = 18
 \end{aligned}$$

$$\begin{aligned}
 4. \quad I &= \int_0^{\sqrt{2}} [x^2] dx = \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx \\
 &= \int_0^1 0 dx + \int_1^{\sqrt{2}} dx = [x]_1^{\sqrt{2}} = \sqrt{2} - 1
 \end{aligned}$$

11. $f(y) = e^y$, $g(y) = y$; $y > 0$

and $F(t) = \int_0^t f(t-y)g(y)dy$.

$$\begin{aligned}
 &= \int_0^t e^{t-y} y dy = e^t \int_0^t e^{-y} y dy = e^t [-ye^{-y} - e^{-y}]_0^t \\
 &= -e^t [te^{-t} + e^{-t} - 0 - 1] = e^t - (1+t)
 \end{aligned}$$

17. $f(x) = \frac{e^x}{1+e^x}$ $I_1 = \int_{f(-a)}^{f(a)} xg[x(1-x)]dx$

$$I_2 = \int_{f(-a)}^{f(a)} g[x(1-x)]dx$$

$$f(a) = \frac{e^a}{1+e^a}, f(-a) = \frac{e^{-a}}{1+e^{-a}}$$

$$\therefore f(a) + f(-a) = 1$$

$$2I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\}dx + \int_{f(-a)}^{f(a)} \{f(a) + f(-a) - x\}g(1-x)(x)dx$$

$$2I_1 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}dx = I_2$$

→ $f(a) + f(-a) = 1$

$$2I_1 = I_2$$

$$\frac{I_2}{I_1} = 2$$

18. $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r}{n^2} \sec^2 \frac{r^2}{n^2}$

$$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \cdot \frac{r}{n} \sec^2 \frac{r^2}{n^2} \quad \text{Put } \frac{1}{n} = dx; \frac{r}{n} = x$$

lower limit $x = \frac{r}{n}$

$$\begin{array}{ll}
 r=1 & x=1/n \\
 n \rightarrow \infty & x=0 \\
 r=n & x=1
 \end{array}$$

$$= \int_0^1 x \sec^2 x^2 dx$$

Put $x^2 = t$; $2x dx = dt$; $x dx = \frac{dt}{2}$

$$x=0, t=0$$

$$x=1, t=1$$

$$= \frac{1}{2} \int_0^1 \sec^2 t dt$$

$$= \frac{1}{2} (\tan t)_0^1 = \frac{1}{2} \tan 1$$

19. for $0 < x < 1$, $x^2 > x^3$ and
for $1 < x < 2$, $x^3 > x^2$
for $0 < x < 1$, $2x^2 > 2x^3$ and
for $1 < x < 2$, $2x^2 < 2x^3$

$$\therefore \int_0^1 2x^2 dx > \int_0^1 2x^3 dx \text{ and } \int_1^2 2x^2 dx < \int_1^2 2x^3 dx$$

$$\therefore I_1 > I_2 \text{ and } I_3 < I_4$$



21. Putting $-x$ for x

$$I = \int_{\pi}^{-\pi} \frac{\cos^2 x}{1+a^{-x}} (-dx) = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}} dx$$

$$I + I = \int_{-\pi}^{\pi} \cos^2 x \left(\frac{1}{1+a^x} + \frac{1}{1+a^{-x}} \right) dx$$

$$= \int_{-\pi}^{\pi} \cos^2 x dx \Rightarrow 2I = 2 \int_0^{\pi} \cos^2 x dx$$

$$= \int_0^{\pi} (1 + \cos 2x) dx$$

$$2I = \left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$2I = \pi \Rightarrow I = \frac{\pi}{2}$$

25. $= \int_1^2 1 \cdot f'(x) dx + \int_2^3 2 \cdot f'(x) dx + \dots + \int_a^a [a] f'(x) dx$

$$= [f(2) - f(1)] + 2[f(3) - f(2)] + \dots + [a] [f(a)]$$

$$= [a] f(a) - \{f(1) + f(2) + \dots + f[a]\}$$

26. $F(x) = f(x) + f(1/x)$ put $x = e$

$$F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$$

$$\text{let } t = \frac{1}{z} \Rightarrow \frac{dt}{dz} = \left(\frac{-1}{z^2} \right)$$

$$= \int_1^e \frac{\ln t}{(1+t)} dt + \int_1^e \frac{\ln 1/z}{(1+1/z)} \left(\frac{-1}{z^2} \right) dz$$

$$\text{by property } \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$\int_1^e \frac{\ln t}{(1+t)} dt + \int_1^e \frac{\ln t}{t(1+t)} dt = \int_1^e \frac{\ln t}{t} dt = \frac{1}{2}$$

28. Now

$$\sin x < x \Rightarrow \frac{\sin x}{\sqrt{x}} < \sqrt{x}$$

$$\int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \sqrt{x} dx$$

$$I < \left[\frac{2}{3} x^{3/2} \right]_0^1$$

$$I < \frac{2}{3}$$

$$\rightarrow \cos x < 1$$

$$\frac{\cos x}{\sqrt{x}} < \frac{1}{\sqrt{x}} \Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}} dx < [2\sqrt{x}]_0^1 < 2$$

$$J < 2$$

29. $I = \int_0^{\pi} [\cot x] dx \dots \dots (1)$

$$I = \int_0^{\pi} [\cot(\pi - x)] dx = \int_0^{\pi} [-\cot x] dx \dots \dots (2)$$

add (1) & (2)

$$2I = \int_0^{\pi} [\cot x] + [-\cot x] dx \rightarrow [x] + [-x] = -1$$

$$= \int_0^{\pi} -1 dx = -[x]_0^{\pi} \Rightarrow I = -\frac{\pi}{2}$$

32. $\int_0^{1.5} x[x^2] dx$

$$\int_0^1 0 dx + \int_1^{\sqrt{2}} x dx + \int_{\sqrt{2}}^{1.5} 2x dx$$

$$\left[\frac{x^2}{2} \right]_1^{\sqrt{2}} + \left[x^2 \right]_{\sqrt{2}}^{1.5}$$

$$\left(\frac{2}{2} - \frac{1}{2} \right) + (2.25 - 2)$$

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$33. g(x) = \int_0^x \cos 4t \, dt$$

$$g(x + \pi) = \int_0^{x+\pi} \cos 4t \, dt$$

$$= \int_0^x \cos 4t \, dt + \int_x^{x+\pi} \cos 4t \, dt$$

$$= \int_0^x \cos 4t \, dt + \int_0^\pi \cos 4t \, dt = g(x) + g(\pi)$$

Because $g(\pi) = 0$ so $g(x) - g(\pi)$ is also correct Ans.

$$34. \text{Statement-I : } I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$$

$$I = \int_{\pi/6}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(1)$$

$$\text{use } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x} dx}{\sqrt{\cos x} + \sqrt{\sin x}} \quad \dots(2)$$

$$(1) + (2)$$

$$2I = \int_{\pi/6}^{\pi/3} dx$$

$$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}$$

So Statement-I is false.

and statement-II is true as it is property.

Part # II : IIT-JEE ADVANCED

6. Given that $f(x)$ is an even function, then to prove

$$\int_0^{\pi/2} f(\cos 2x) \cos x \, dx = \sqrt{2} \int_0^{\pi/4} f(\sin 2x) \cos x \, dx$$

$$\text{Let } I = \int_0^{\pi/2} f(\cos 2x) \cos x \, dx \quad \dots(1)$$

$$= \int_0^{\pi/2} f\left[\cos 2\left(\frac{\pi}{2} - x\right)\right] \cos\left(\frac{\pi}{2} - x\right) dx$$

$$\left[\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$= \int_0^{\pi/2} f(-\cos 2x) \sin x \, dx$$

$$I = \int_0^{\pi/2} f(\cos 2x) \sin x \, dx \quad \dots(2)$$

[As $f(x)$ is an even function]

adding two values of I in (1) and (2) we get

$$2I = \int_0^{\pi/2} f(\cos 2x) (\sin x + \cos x) dx$$

$$\Rightarrow I = \frac{\sqrt{2}}{2} \int_0^{\pi/2} f(\cos 2x) \left[\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right] dx$$

$$I = \frac{\sqrt{2}}{2} \int_0^{\pi/2} f(\cos 2x) \cos(x - \pi/4) dx$$

$$\text{Let } x - \pi/4 = t \Rightarrow dx = dt$$

$$\therefore I = \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} f[\cos 2(t + \pi/4)] \cos t \, dt$$

$$= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} f[-\sin 2t] \cos t \, dt$$

$$= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} f(\sin 2t) \cos t \, dt$$

[$\rightarrow f$ is an even function]

$$= \frac{2}{\sqrt{2}} \int_0^{\pi/4} f(\sin 2t) \cos t \, dt$$

[$\rightarrow f$ is an even function]

$$= \sqrt{2} \int_0^{\pi/4} f(\sin 2x) \cos x \, dx = \text{R.H.S.}$$

$$8. \text{ (B) } I = \int_{-2}^0 [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)] dx$$

$$= \left[\frac{x^4}{4} + x^3 + \frac{3x^2}{2} + 3x + (x+1)\sin(x+1) + \cos(x+1) \right]_{-2}^0$$

$$= 4$$

9. Let $I = \int_0^{\pi} e^{\frac{1}{2} \cos x} \left[2 \sin \left(\frac{1}{2} \cos x \right) + 3 \cos \left(\frac{1}{2} \cos x \right) \right] \sin x \, dx$

$$= \int_0^{\pi} e^{\frac{1}{2} \cos x} 2 \sin \left(\frac{1}{2} \cos x \right) \sin x \, dx$$

$$+ \int_0^{\pi} e^{\frac{1}{2} \cos x} 3 \cos \left(\frac{1}{2} \cos x \right) \sin x \, dx$$

$$= I_1 + I_2$$

Now using the property that

$$\int_0^{2a} f(x) \, dx = 0 \quad \text{if } f(2a - x) = -f(x)$$

$$= 2 \int_0^a f(x) \, dx \quad \text{if } f(2a - x) = f(x)$$

We get, $I_1 = 0$

and $I_2 = 2 \int_0^{\pi/2} e^{\cos x} 3 \cos \left(\frac{1}{2} \cos x \right) \sin x \, dx$

$$= 6 \int_0^{\pi/2} e^{\cos x} \cos \left(\frac{1}{2} \cos x \right) \sin x \, dx$$

Put $\cos x = t \Rightarrow -\sin x \, dx = dt$, we get

or $I_2 = 6 \int_0^1 e^t \cos \frac{t}{2} \, dt$

$$I_2 = 6 \left[(e^t \cos \frac{t}{2})_0^1 + \frac{1}{2} \int_0^1 e^t \sin \frac{t}{2} \, dt \right]$$

$$= 6 \left[e \cos(1/2) - 1 + \frac{1}{2} \left\{ (e^t \sin t/2)_0^1 - \frac{1}{2} \int_0^1 e^t \cos t/2 \, dt \right\} \right]$$

$$I_2 = 6 \left[e \cos \left(\frac{1}{2} \right) - 1 + \frac{1}{2} \left\{ e \sin(1/2) - \frac{1}{2} \cdot \frac{1}{6} I_2 \right\} \right]$$

$$I_2 + \frac{1}{4} I_2 = 6 \left[e \cos(1/2) + \frac{1}{2} e \sin(1/2) - 1 \right]$$

$$\Rightarrow I_2 = \frac{24}{5} \left[e \cos(1/2) + \frac{1}{2} e \sin \left(\frac{1}{2} \right) - 1 \right]$$

10. $\int_0^{\pi/2} \sin x \, dx = \frac{\left(\frac{\pi}{2} - 0 \right)}{4} \left(\sin 0 + \sin \frac{\pi}{2} + 2 \sin \frac{\pi}{4} \right)$

$$= \frac{\pi}{8} (1 + \sqrt{2})$$

11. $f''(x) < 0, \forall x \in (a, b)$, for $c \in (a, b)$

$$F(c) = \frac{c-a}{2} (f(a) + f(c)) + \frac{b-c}{2} (f(b) + f(c))$$

$$= \frac{b-a}{2} f(c) + \frac{c-a}{2} f(a) + \frac{b-c}{2} f(b)$$

$$\Rightarrow F'(c) = \frac{b-a}{2} f'(c) + \frac{1}{2} f(a) - \frac{1}{2} f(b)$$

$$= \frac{1}{2} [(b-a)f'(c) + f(a) - f(b)]$$

$$F''(c) = \frac{1}{2} (b-a)f''(c) < 0$$

$$[Q f''(x) < 0, \forall x \in (a, b) \text{ and } b > a]$$

$\therefore F(c)$ is max. at the point $(c, f(c))$ where

$$F'(c) = 0 \Rightarrow f'(c) = \left(\frac{f(b) - f(a)}{b - a} \right)$$

12. $\lim_{x \rightarrow a} \frac{\int_a^x f(x) \, dx - \left(\frac{x-a}{2} \right) (f(x) + f(a))}{(x-a)^3} = 0$

$$\lim_{h \rightarrow 0} \frac{\int_a^{a+h} f(x) \, dx - \frac{h}{2} (f(a+h) + f(a))}{h^3} = 0$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(a+h) - \frac{1}{2} [f(a) + f(a+h)] - \frac{h}{2} (f'(a+h))}{3h^2} = 0$$

[Using L'Hospital rule]

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\frac{1}{2} f'(a+h) - \frac{1}{2} f'(a) - \frac{h}{2} f''(a+h)}{3h^2} = 0$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\frac{1}{2} f''(a+h) - \frac{1}{2} f''(a) - \frac{h}{2} f'''(a+h)}{6h} = 0$$

[Using L' Hospital rule]

$$\Rightarrow \lim_{h \rightarrow 0} \frac{-f'''(a+h)}{12} = 0 \Rightarrow f'''(x) = 0, \forall x \in R$$

$\Rightarrow f(x)$ must be of max. degree 1



13. Let $I = \int_0^1 (1 - x^{50})^{100} dx$ and $I' = \int_0^1 (1 - x^{50})^{101} dx$

Then, $I' = \int_0^1 1 \cdot (1 - x^{50})^{101} dx = (x(1 - x^{50})^{101})_0^1$

$+ 101 \int_0^1 50x^{50} (1 - x^{50})^{100} dx$

$= 5050 \int_0^1 x^{50} (1 - x^{50})^{100} dx$

$-I' = 5050 \int_0^1 -x^{50} (1 - x^{50})^{100} dx$

$\Rightarrow 5050I - I' = 5050 \int_0^1 (1 - x^{50})^{100} dx$

$+ 5050 \int_0^1 -x^{50} (1 - x^{50})^{100} dx$

$\Rightarrow 5050 \int_0^1 (1 - x^{50})^{101} dx = 5050 I'$

$\Rightarrow 5050 I = 5050 I' \Rightarrow 5050 \frac{I}{I'} = 5050$

17. $S_n = \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \frac{k}{n} + \frac{k^2}{n^2}}$

$S_n < \int_0^1 \frac{dx}{x^2 + x + 1}$

(→ the function is decreasing)

$S_n < \int_0^1 \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$S_n < \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{2x+1}{\sqrt{3}} \right]_0^1$

$S_n < \frac{2}{\sqrt{3}} \left[\frac{\pi}{3} - \frac{\pi}{6} \right]$

$S_n < \frac{\pi}{3\sqrt{3}}$

Now $T_n - S_n = 1 - \frac{1}{3n} \Rightarrow T_n - S_n > \frac{2}{3}$

$\Rightarrow T_n > S_n + \frac{2}{3}$

as $S_n < \frac{\pi}{3\sqrt{3}}$ so $T_n > \frac{\pi}{3\sqrt{3}}$

18. $\int_0^x \sqrt{1 - (f'(t))^2} dt = \int_0^x f(t) dt, 0 \leq x \leq 1$

differentiating both the sides & squaring

$\Rightarrow 1 - (f'(x))^2 = f^2(x) \Rightarrow \frac{f'(x)}{\sqrt{1 - f^2(x)}} = 1$

$\Rightarrow \sin^{-1} f(x) = x + c$
 $f(0) = 0$

$\Rightarrow f(x) = \sin x \Rightarrow \sin x \leq x$ for $x \in [0, 1]$

$\Rightarrow f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$.

19. $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx$

$I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1 + \pi^x) \sin x} dx$

$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx \dots (i)$

$2I_{n+2} = \int_{-\pi}^{\pi} \frac{\sin(n+2)x}{\sin x} dx \dots (ii)$

$(ii) - (i)$

$\Rightarrow 2(I_{n+2} - I_n) = \int_{-\pi}^{\pi} \cos(n+1)x dx = 0 \Rightarrow I_{n+2} = I_n$

$\sum_{m=1}^{10} I_{2m} = 10 \sum_{m=1}^{10} I_2 = \frac{10}{2} \int_{-\pi}^{\pi} \frac{\sin 2x}{\sin x} dx = 0$

Put $n = 1$ in equation (i)

$2I_1 = \int_{-\pi}^{\pi} \frac{\sin x dx}{\sin x} = 2\pi$

$I_1 = \pi$

$\sum_{m=1}^{10} I_{2m+1} = 10\pi$

20. $f(x) = \int_0^x f(t) dt \dots (i)$

$f'(x) = f(x) \Rightarrow f(x) = k.e^x$

From (i) $f(0) = 0$

$\Rightarrow f(0) = k.e^0 \Rightarrow k = 0 \Rightarrow f(x) = 0$

21. Applying L-Hospital rule,

$$\lim_{x \rightarrow 0} \frac{\int_0^x \frac{t \ln(1+t)}{t^4+4} dt}{x^3} = \lim_{x \rightarrow 0} \frac{x \ln(1+x)}{3x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(1+x)}{3x(x^2+4)} = \frac{1}{12}$$

22. $I = \int_0^1 \frac{x^4(1-2x+x^2)^2}{1+x^2} dx$

$$I = \int_0^1 \frac{x^4 \left\{ (1+x^2)^2 - 4x(1+x^2) + 4x^2 \right\}}{1+x^2} dx$$

$$= \int_0^1 (1+x^2)x^4 dx - \int_0^1 4x^5 dx + 4 \int_0^1 \frac{(x^6+1)-1}{1+x^2} dx$$

$$= \frac{1}{5} + \frac{1}{7} - 4 \cdot \frac{1}{6} + 4 \int_0^1 \frac{(x^2+1)^3 - 3x^2(1+x^2)}{1+x^2} dx - 4 \int_0^1 \frac{dx}{1+x^2}$$

$$= \frac{12}{35} - \frac{2}{3} + 4 \int_0^1 (x^4 + 2x^2 + 1) dx - 12 \int_0^1 x^2 dx - \pi$$

$$= \frac{12}{35} - \frac{2}{3} + 4 \left(\frac{1}{5} + \frac{2}{3} + 1 \right) - 4 - \pi$$

$$= \frac{12}{35} - \frac{2}{3} + \frac{52}{15} - \pi = \frac{22}{7} - \pi$$

23. $f(x) = \begin{cases} \{x\} & \text{when } -9 \leq x < -8; -7 \leq x < -6, \dots \dots \dots \\ 1 - \{x\} & \text{when } -10 \leq x \leq -9; -8 \leq x < -7, \dots \dots \dots \end{cases}$

Since $f(x)$ & $\cos \pi x$ both are periodic functions having period 2.

$$I = \frac{10 \times \pi^2}{10} \left(\int_0^1 (1 - \{x\}) \cos \pi x dx + \int_1^2 \{x\} \cos \pi x dx \right)$$

$$= \pi^2 \left(\int_0^1 (1-x) \cos \pi x dx + \int_1^2 (x-1) \cos \pi x dx \right)$$

$$= \pi^2 \left(\int_0^1 \cos \pi x dx - \int_1^2 \cos \pi x dx + \int_1^2 x \cos \pi x dx - \int_0^1 x \cos \pi x dx \right)$$

$$\Rightarrow I = 4$$

24. $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1} dt$

$$e^{-x} f'(x) - e^{-x} f(x) = \sqrt{x^4+1}$$

$$\Rightarrow f(x) - f(x) = e^x \sqrt{x^4+1}$$

$$\Rightarrow \frac{dy}{dx} = y + e^x \sqrt{x^4+1} \quad (\text{say}) \quad \dots \dots (i)$$

considering $y = f(x)$. so that $x = f(y)$

$$f^{-1}'(2) = \left(\frac{dx}{dy} \right)_{y=2} \quad \dots \dots (ii)$$

$$\text{for } x = 0 \Rightarrow f(x) = 2 \text{ i.e. } y = 2$$

$$\Rightarrow f^{-1}(2) = 0$$

$$\frac{dy}{dx} = 2 + 1\sqrt{1} = 3$$

$$\text{from (2), } f^{-1}'(2) = \frac{1}{3}$$

25. $I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$; put $x^2 = t$

$$\Rightarrow 2x dx = dt$$

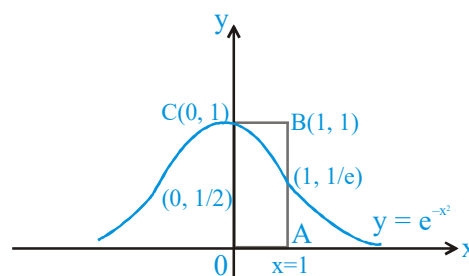
$$\Rightarrow I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)} dt \quad \dots \dots (i)$$

$$\Rightarrow I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - t)}{\sin(\ln 6 - t) + \sin t} dt \quad \dots \dots (ii)$$

Adding equation (i) & (ii)

$$\Rightarrow 2I = \frac{1}{2} \int_{\ln 2}^{\ln 3} dt \Rightarrow I = \frac{1}{4} \ln \left(\frac{3}{2} \right)$$

26. Area (OABC) = 1



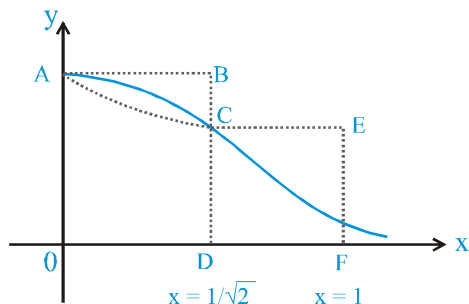
Shaded area is S.

Clearly $S < 1$

$$\text{and } \int_0^1 e^{-x^2} dx > \int_0^1 e^{-x} dx$$

$$\Rightarrow S > 1 - \frac{1}{e} \quad (\therefore \text{(B) is correct})$$

Again $S \geq \text{Area (trapezium ACDO)}$



$$\Rightarrow S \geq \frac{1}{2} \left(1 + \frac{1}{\sqrt{e}} \right) \left(\frac{1}{\sqrt{2}} \right)$$

$$\Rightarrow S \geq \frac{1}{2\sqrt{2}} \left(1 + \frac{1}{\sqrt{e}} \right)$$

$\therefore$ C is wrong

Also $S \leq \text{Sum of areas of rectangles ABDO \& CEFD}$

$$\Rightarrow S \leq \frac{1}{\sqrt{2}} \times 1 + \left(1 - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{e}} \right)$$

$$\Rightarrow S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}} \left(1 - \frac{1}{\sqrt{2}} \right)$$

$\therefore$ (D) is correct

$$27. \int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx + \int_{-\pi/2}^{\pi/2} \ln \left(\frac{\pi+x}{\pi-x} \right) \cos x \, dx$$

$$= \int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx = 2 \int_0^{\pi/2} x^2 \cos x \, dx$$

$$= 2 \left((x^2 \sin x) \Big|_0^{\pi/2} - 2 \int_0^{\pi/2} x \sin x \, dx \right)$$

$$= 2 \left(\frac{\pi^2}{4} - 2 \left(-(x \cos x) \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx \right) \right)$$

$$= 2 \left(\frac{\pi^2}{4} - 2 \int_0^{\pi/2} \cos x \, dx \right)$$

$$= 2 \left(\frac{\pi^2}{4} - 2 \right) = \frac{\pi^2}{2} - 4$$

28.

$$L = \lim_{n \rightarrow \infty} \frac{1^a + 2^a + \dots + n^a}{(n+1)^{a-1} \left[\frac{1^a + 2^a + \dots + n^a}{n} + \frac{1+2+3+\dots+n}{2} \right]}$$

$$= \lim_{n \rightarrow \infty} \frac{\sum_{r=1}^n r^a}{(n+1)^{a-1} \left[\frac{n^2 a + \frac{n(n+1)}{2}}{n} \right]}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} \sum_{r=1}^n \frac{r^a}{n^a} \right) n^{a+1}}{(n+1)^{a-1} \left[\frac{n^2 a + \frac{n(n+1)}{2}}{n} \right]}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} \sum_{r=1}^n \frac{r^a}{n^a} \right)}{\left(\frac{n+1}{n} \right)^{a-1} \left[\frac{n^2 a + \frac{n(n+1)}{2}}{n^2} \right]}$$

$$= \frac{\int_0^1 x^a \, dx}{\left(a + \frac{1}{2} \right)} = \frac{1}{60} \Rightarrow \frac{2}{(a+1)(2a+1)} = \frac{1}{60}$$

$$\Rightarrow 2a^2 + 3a - 119 = 0 \Rightarrow a = 7 \text{ \& } -\frac{17}{2}$$

$$a = -\frac{17}{2} \text{ will be rejected as } \int_0^1 x^{-17/2} \, dx \text{ is not defined.}$$

$$42. \text{ Let } f(x) = \int_0^x \frac{t^2 \, dt}{1+t^4} - 2x + 1$$

$$f(x) = \frac{x^2}{1+x^4} - 2x + 1$$

$$\Rightarrow \frac{-2x^4 + x^2 - 2}{x^4 + 1} < 0 \quad \forall x \in \mathbb{R}$$

$$f(0) > 0, \quad f(1) < 0$$

$\therefore$ One solution in (0, 1)

$$43. I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1 + e^x} dx \quad \dots(i)$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1 + \frac{1}{e^x}} dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1 + \frac{1}{e^x}} dx \quad \dots(ii)$$

(i) and (ii)

$$2I = \int_{-\pi/2}^{\pi/2} x^2 \cos x dx$$

$$I = \int_0^{\pi/2} x^2 \cos x dx \quad (\text{even fn})$$

$$= x^2 \cdot \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x dx$$

$$= \frac{\pi^2}{4} - 2 \left[(-x \cos x) \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) dx \right]$$

$$= \frac{\pi^2}{4} - 2 \left[0 + \sin x \Big|_0^{\pi/2} \right]$$

$$= \frac{\pi^2}{4} - 2[1] = \frac{\pi^2}{4} - 2$$

MOCK TEST

$$1. I = \int_{-1}^1 \left(\tan^{-1} \frac{x}{x^2+1} + \tan^{-1} \frac{x^2+1}{x} \right) dx = \int_1^3 \frac{\pi}{2} dx = (3-1)$$

$$\frac{\pi}{2} = \pi$$

2. (B)

$$p = \lim_{n \rightarrow \infty} \left[\frac{\prod_{r=1}^n (n^3 + r^3)}{n^{3n}} \right]^{1/n}$$

$$\bullet np = \lim_{n \rightarrow \infty} \cdot \frac{1}{n} \sum_{r=1}^n \ln \left(1 + \left(\frac{r}{n} \right)^3 \right) = \int_0^1 \ln(1+x^3) dx$$

$$= \bullet n 2 - 3 + 3\lambda$$

$$3. \text{ Let } 5(4x-5) = t^2 \Rightarrow 20x-25 = t^2$$

$$\Rightarrow x = \frac{t^2 + 25}{20}$$

$$\text{Also } 20 dx = 2t dt \text{ or } dx = \frac{t}{10} dt$$

$$\therefore I = \int_{\sqrt{15}}^{\sqrt{35}} \left(\sqrt{\frac{t^2+25}{10}} - t + \sqrt{\frac{t^2+25}{10}} + t \right) \frac{t}{10} dt$$

$$= \int_{\sqrt{15}}^{\sqrt{35}} \left(\frac{|t-5|}{\sqrt{10}} + \frac{|t+5|}{\sqrt{10}} \right) \frac{t}{10} dt$$

$$= \int_{\sqrt{15}}^5 \left(\frac{-t+5}{\sqrt{10}} + \frac{t+5}{\sqrt{10}} \right) \frac{t}{10} dt$$

$$+ \int_5^{\sqrt{35}} \left(\frac{t-5}{\sqrt{10}} + \frac{t+5}{\sqrt{10}} \right) \frac{t}{10} dt = \frac{1}{\sqrt{10}} \left[\frac{t^2}{2} \right]_{\sqrt{15}}^5$$

$$+ \frac{1}{5\sqrt{10}} \int_5^{\sqrt{35}} t^2 dt$$

$$= \frac{1}{\sqrt{10}} \left[\frac{25}{2} - \frac{15}{2} \right] + \frac{1}{15\sqrt{10}} \left[(\sqrt{35})^3 - 125 \right]$$

$$= \frac{\sqrt{10}}{2} + \frac{7\sqrt{7}-5\sqrt{5}}{3\sqrt{2}}$$