CONTINUITY

EXERCISE # 1

Question based on Continuity

Q.1 Let $f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right|$ when $-2 \le x \le 2$.

Where [.] represents greatest integer function? Then

(A) f(x) is continuous at x = 2

(B) f(x) is continuous at x = 1

(C) f (x) is continuous at x = –1

(D) f(x) is discontinuous at x = 0

Sol. **[D]**

$$f(x) = \left| x[x] + \frac{[x]}{2} \right|$$

[x] is discontinuous at integers

$$f(x) = \begin{cases} 2\left(x + \frac{1}{2}\right); & -2 \le x \le -1 \\ x + \frac{1}{2}; & -1 \le x < -\frac{1}{2} \\ 0; & x = \frac{-1}{2} \\ x + \frac{1}{2}; & \frac{-1}{2} < x < 0 \\ 0; & x = 0 \\ 0; & 0 < x < 1 \\ x + \frac{1}{2}; & 1 < x < 2 \\ 2\left(x + \frac{1}{2}\right); & x = 2 \end{cases}$$

$\therefore$ discontinuous at x = 2, 1, –1, 0

Q.2 The value of x where the function

$f(x) = \frac{\tan x \log(x-2)}{x^2 - 4x + 3}$ is discontinuous are given by:

(A) $(-\infty, 2) \cup \{3\}$

(B) $(-\infty, 2] \cup \{3, n\pi + \frac{\pi}{2}, n \in N\}$

(C) $(-\infty, 2)$

(D) None of these

Sol. **[B]**

$$f(x) = \frac{\tan x \log(x-2)}{(x^2 - 4x + 3)}$$

tanx not defined at $x \in n\pi + \pi/2$; $n \in$ Integer

log(x – 2) not defined at $x \in (-\infty, 2]$

and f(x) not defined at $x \in \{1, 3\}$

Hence, set of discontinuous points would be

$x \in (-\infty, 2] \cup \{3, n\pi + \pi/2 ; n \in I\}$

Hence, option (B) is correct Answer.

Q.3 If $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous on

(A) R (B) ϕ

(C) –1, 1 (D) None of these

Sol. **[B]**

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x \to Q} f(x) = 1$; $x \in Q$ (Rational Number)

$\lim_{x \to Q^C} f(x) = -1$; $x \in Q^C$ (Irrational number)

Since, there is no unique limit. Hence set of continuous points would be empty.

$\therefore$ Option (B) is correct Answer.

Q.4 The set of all points where f(x) = sec 2x + cosec 2x is discontinuous is-

(A) $\{n\pi ; n = 0, \pm 1, \pm 2 ......\}$

(B) $\{\frac{n\pi}{2} ; n = 0, \pm 1, \pm 2 ....\}$

(C) $\{\frac{(2n+1)\pi}{4} ; n = 0, \pm 1, \pm 2 ....\}$

(D) $\{n\pi/4 ; n = 0, \pm 1, \pm 2 ....\}$

Sol. **[D]**

f(x) = sec2x + cosec2x

$$f(x) = \frac{\sin 2x + \cos 2x}{\sin 2x . \cos 2x} \times \frac{2}{2}$$

$$f(x) = \frac{\sin 2x + \cos 2x}{\sin 4x} \times 2$$

f(x) not to be defined at sin4x = 0 = sin nπ; $n \in I$

4x = nπ

$\Rightarrow x = \frac{n\pi}{4}$; $n \in I$

$\therefore$ Option (D) is correct Answer.

Q.5 f(x) = [sin x] + |x| is discontinuous (Here [ ] represents greatest integer function)

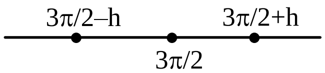
(A) every where (B) at x = 3π/2

(C) at $x = -\pi/2$ (D) at infinite points

Sol. **[D]**

$f(x) = [\sin x] + |x|$

At $x = \frac{3\pi}{2}$



L.H.L. $= \lim_{x\to 3\pi/2^-} f(x) = \lim_{h\to 0} f\left(\frac{3\pi}{2} - h\right)$

$$= \lim_{h\to 0}\left\{\left[\sin\left(\frac{3\pi}{2} - h\right)\right] + \left|\frac{3\pi}{2} - h\right|\right\}$$

$$= \lim_{h\to 0}\left\{\left[-\sin\left(\frac{\pi}{2} - h\right)\right] + h - \frac{3\pi}{2}\right\}$$

$$= \lim_{h\to 0}\{[-(\text{value less than } 1)] + h - 3\pi/2\}$$

$= -1 - 3\pi/2$

R.H.L $= \lim_{x\to \frac{3\pi}{2}^+} f(x) = \lim_{h\to 0} f\left(\frac{3\pi}{2} + h\right)$

$$= \lim_{h\to 0}\left\{\left[\sin\left(\frac{3\pi}{2} + h\right)\right] + \left|\frac{3\pi}{2} + h\right|\right\}$$

$$= \lim_{h\to 0}\left\{\left[-\sin\left(\frac{\pi}{2} + h\right)\right] + \frac{3\pi}{2} + h\right\}$$

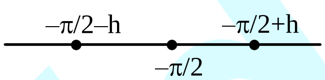
$$= \lim_{h\to 0}\left\{[-\text{Value less than } 1] + \frac{3\pi}{2} + h\right\}$$

$= -1 + 3\pi/2$

Since, L.H.L $\neq$ R.H.L

Hence, f(x) is Discontinuous at $x = 3\pi/2$

At $x = -\pi/2$



L.H.L $= \lim_{x\to \frac{-\pi}{2}^-} f(x) = \lim_{h\to 0} f\left(-\frac{\pi}{2} - h\right)$

$$= \lim_{h\to 0}\left\{\left[\sin\left(-\frac{\pi}{2} - h\right) + \left|-\frac{\pi}{2} - h\right|\right]\right\}$$

$$= \lim_{h\to 0}\left\{\left[-\sin\left(\frac{\pi}{2} + h\right) + \frac{\pi}{2} + h\right]\right\}$$

$$= \lim_{h\to 0}\left\{[-\text{value less than } 1] + \frac{\pi}{2} + h\right\}$$

$= -1 + \pi/2$

R.H.L $= \lim_{h\to \frac{-\pi}{2}^+} f(x) = \lim_{h\to 0} f(-\pi/2 + h)$

$$= \lim_{h\to 0}\left\{[\sin(-\pi/2 + h) + \left|\frac{-\pi}{2} + h\right|]\right\}$$

$$= \lim_{h\to 0}\left\{\left[-\sin\left(\frac{\pi}{2} - h\right)\right] + h - \pi/2\right\}$$

$$= \lim_{h\to 0}\{[-\text{Value less than} 1] + h - \pi/2\}$$

$= -1 - \pi/2$

Sine, L.H.L $\neq$ R.H.L

$\therefore$ f(x) is Discontinuous at $x = -\pi/2$. In general, we can say that greatest Integer function is discontinuous at all points, $x \in n\pi + (-1)^n \frac{\pi}{2}$; $n \in$ Integer

Q.6 f(x) = [cosec x], where [x] represents greatest integer function -

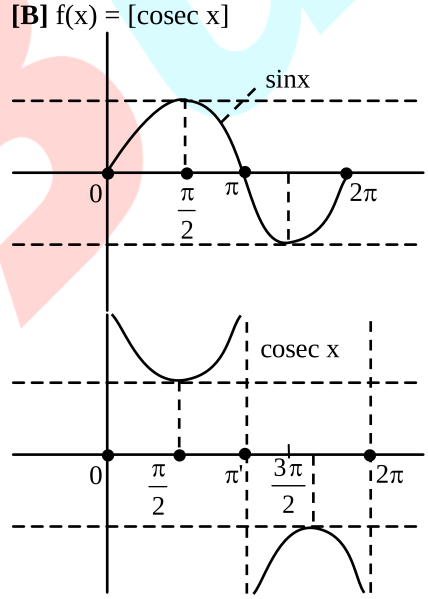
(A) f(x) is discontinuous at $x = \pi/2$

(B) f(x) is continuous at $x = \pi/2$

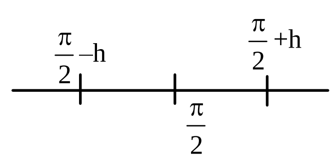
(C) $\lim_{x\to\pi/2} f(x)$ does not exist

(D) f(x) is continuous at $x = 3\pi/2$

Sol. **[B]** f(x) = [cosec x]



At $x = \pi/2$



L.H.L $= \lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2} - h\right)$

$= \lim_{h\to 0} [\text{cosec}(\pi/2 - h)]$

$= \lim_{h\to 0} [1/\sin(\pi/2 - h)]$

= [1/value less than 1]

= 1.

R.H.L = $\lim_{x \to \frac{\pi}{2}^+} (f(x)) = \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right)$

$= \lim_{h \to 0} \left[\text{cosec}\left(\frac{\pi}{2} + h\right)\right]$

$= \lim_{h \to 0} \left[1/\sin\left(\frac{\pi}{2} + h\right)\right]$

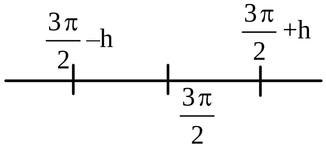
$= \lim_{h \to 0}$ [1 / value less than 1]

= 1

f(π/2) = [cosec π/2] = 1

Hence, f(x) is continuous at x = π/2.

At x = $\frac{3\pi}{2}$



L.H.L = $\lim_{x \to 3\pi/2} f(x) = \lim_{h \to 0} f\left(\frac{3\pi}{2} - h\right)$

$= \lim_{h \to 0} [\text{cosec}(3\pi/2 - h)]$

$= \lim_{h \to 0} \left[\frac{-1}{\sin(\pi/2 - h)}\right]$

= [–1/value less than 1]

= –2

R.H.L = $\lim_{x \to \frac{3\pi}{2}^+} (f(x)) = \lim_{h \to 0} f\left(\frac{3\pi}{2} + h\right)$

$= \lim_{h \to 0} \left[1/\sin\left(\frac{3\pi}{2} + h\right)\right]$

$= \lim_{h \to 0} \left[1/-\sin\left(\frac{\pi}{2} + h\right)\right]$

= [–1/ value less than 1]

= – 2

f(3π/2) = [cosec 3π/2] = [– cosec π/2]

= [–1] = – 1

Since, L.H.L = R.H.L ≠ f (3π/2)

Hence, f(x) is not continuous at x = $\frac{3\pi}{2}$

∴ Option (B) is correct Answer.

Q.7 The set of points of continuity of the function $f(x) = \sqrt{\frac{1}{2} - \cos^2 x}$ is

(A) $\left\{x : \frac{\pi}{4} + 2n\pi \le x \le \frac{3\pi}{4} + 2n\pi,\ n \in I\right\}$

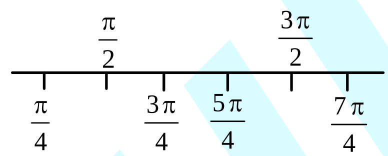
(B) $\left\{x : \frac{5\pi}{4} + 2n\pi \le x \le \frac{7\pi}{4} + 2n\pi,\ n \in I\right\}$

(C) $\left\{x : \frac{\pi}{4} + 2n\pi \le x \le \frac{3\pi}{4} + 2n\pi\right\} \cup \left\{x : \frac{5\pi}{4} + 2n\pi \le x \le \frac{7\pi}{4} + 2n\pi\right\}$

(D) None of these

Sol. **[C]**

$f(x) = \sqrt{\frac{1}{2} - \cos^2 x}$



At x = π/2, f(π/2) = $\sqrt{\frac{1}{2} - 0}$

$= \pm \frac{1}{2}$ (Not a unique value)

Hence, all those points x ∈ 2nπ ± π/2 where limit does not get a unique value, will be discontinuous points.

Hence, Option (C) is correct Answer. Because it includes $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Q.8 The set of points for which $f(x) = |x|\,|x-1| + \frac{1}{[x+1]}$ is discontinuous is -

(A) [–1, 1] (B) [–1, 0) ∪ Z – {0}

(C) {–1, 0, 1} (D) All integral points

Sol. **[B]**

$f(x) = |x|\,|x - 1| + \frac{1}{[x+1]}$

f(x) will not be continuous at those points where (x + 1) will be integers. Also [x + 1] ≠ 0

[x + 1] ≠ 0 ⇒ x ∉ [–1, 0)

x + 1 = 0, –1, ± 2, ± 3 ; ± 4 …..

Because at x = 0, f(x) will be continuous

Hence, points where f(x) is discontinuous

[–1, 0) U integers – {0}.

∴ Option (B) is correct answer.

Question basedon **Discontinuity**

Q.9 A function f(x) is defined as below $f(x) = \dfrac{\cos(\sin x) - \cos x}{x^2}$, $x \neq 0$ and $f(0) = a$

f(x) is continuous at x = 0 if a equals.

(A) 0 (B) 4

(C) 5 (D) 6

Sol. **[A]**

$$f(x) = \frac{\cos(\sin x) - \cos x}{x^2}$$

$$\lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0} \frac{2\sin\left(\frac{\sin x + x}{2}\right)\sin\left(\frac{x - \sin x}{2}\right)}{\frac{\sin x + x}{2} \times \frac{x - \sin x}{2}} \times \frac{x^2 - \sin^2 x}{4x^2}$$

$$\lim_{x\to 0} \frac{x^2 - \sin^2 x}{2x^2} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\lim_{x\to 0} f(0) = f(0) \Rightarrow f(0) = a = 0$$

Q.10 If $f(x) = \begin{cases} \tan^{-1}(\tan x) & x \leq \frac{\pi}{4} \\ \pi[x] + 1 & x > \frac{\pi}{4} \end{cases}$, then jump of discontinuity is

(A) $\frac{\pi}{4} - 1$ (B) $\frac{\pi}{4} + 1$

(C) $1 - \frac{\pi}{4}$ (D) $-1 - \frac{\pi}{4}$

Sol. **[C]**

$$f(x) = \begin{cases} \tan^{-1}(\tan x), & x \leq \frac{\pi}{4} \\ \pi[x] + 1, & x > \frac{\pi}{4} \end{cases}$$

L.H.L = $\frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$

R.H.L = 1

$\therefore$ jump = $1 - \frac{\pi}{4}$.

Q.11 If the function $f(x) = \begin{cases} \dfrac{x^2 - (A+2)x + A}{x - 2}; & x \neq 2 \\ 2 & x = 2 \end{cases}$

is continuous at x = 2 then A =

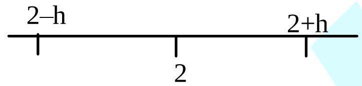
(A) 0 (B) 1

(C) 2 (D) None of these

Sol. **[A]**

$$f(x) = \frac{x^2 - (A+2)x + A}{x - 2} \;;x \neq 2$$

$$= \quad 2 \quad ; x=2$$



L.H.L = $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h)$

$$= \lim_{h\to 0} \frac{(2-h)^2 - (A+2)(2-h) + 2A}{2 - h - 2}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 - (A+2)(2-h) + 2A}{-h}$$

$$= \lim_{h\to 0} \frac{2(2-h)(-1) - (A+2)(-1) + 0}{-1}$$

$$= \lim_{h\to 0} \frac{2(2-h) - (A+2)}{1}$$

= 4 – (A + 2)

= 2 – A

Since, f(x) is continuous (There is no need to calculate both limits)

Hence, L.H.L = f(2)

2 – A = 2

$\therefore$ Option (A) is correct Answer.

Q.12 If $f(x) = \begin{cases} \dfrac{36^x - 9^x - 4^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}}, & x \neq 0 \\ K, & x = 0 \end{cases}$

is continuous at x = 0, then K equals

(A) 16 log2 log 3 (B) $16\sqrt{2}$ log 6

(C) $16\sqrt{2}$ log2 log 3 (D) None of these

Sol. **[C]**

$$f(x) = \begin{cases} \dfrac{36^x - 9^x - 4^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

$$= \frac{9^x.4^x - 9^x - 4^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}} = \frac{9^x(4^x - 1) - 1(4^x - 1)}{\sqrt{2} - \sqrt{1 + \cos x}}$$

$$\Rightarrow \lim_{x\to 0} \frac{(9^x - 1)(4^x - 1)}{x^2(2 - 1 - \cos x)} \times (\sqrt{2} + \sqrt{1 + \cos x}) \times x^2$$

$$= \lim_{x\to 0} \frac{\lambda n9 \times \lambda n4}{\frac{1 - \cos x}{x^2}} (\sqrt{2} + \sqrt{1 + \cos x})$$

$$= 2\lambda n9 \times \lambda n4\, (\sqrt{2} + \sqrt{2})$$

$= 16\sqrt{2}\ \lambda n3\ \lambda n2.$

Q.13 If $f(x) = \dfrac{\sin^4\left(\frac{1}{x}\right) - \sin^2\left(\frac{1}{x}\right) + 1}{\cos^4\left(\frac{1}{x}\right) - \cos^2\left(\frac{1}{x}\right) + 1}$ is to be made continuous at x = 0, then f(0) should be equal to -

(A) 0 (B) 1

(C) 1/3 (D)1/2

Sol. **[B]**

$$f(x) = \frac{\sin^4\left(\frac{1}{x}\right) - \sin^2\left(\frac{1}{x}\right) + 1}{\cos^4\left(\frac{1}{x}\right) - \cos^2\left(\frac{1}{x}\right) + 1}$$

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin^4\frac{1}{x} - \sin^2\frac{1}{x} + 1}{\cos^4\frac{1}{x} - \cos^2\frac{1}{x} + 1}$$

= 1. (by common sense)

because $\lim_{x\to 0}\left(\sin^4\frac{1}{x}, \sin^2\frac{1}{x}, \cos^4\frac{1}{x}, \cos^2\frac{1}{x}\right)$

= Does not exist.

$\therefore$ Option (B) is correct Answer.

Q.14 $f(x) = [\tan^{-1}x]$ where [·] denotes the greatest integer function, is discontinuous at -

(A) $-\frac{\pi}{4}$, 0 and $\frac{\pi}{4}$ (B) $-\frac{\pi}{3}$, 0 and $\frac{\pi}{3}$

(C) –tan 1, tan 1, 0 (D) None of these

Sol. **[C]**

$f(x) = [\tan^{-1} x]$

f(x) will be discontinuous at those points where $\tan^{-1}x$ will become integer i.e.

$\tan^{-1}x = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots.$

x =0, + tan1, + tan2, + tan3

– tan1, –tan2,–tan 3

$\therefore$ Option (C) is correct answer.

Question basedon

Miscellaneous

Q.15 Let f(x) = Sgn (x) and $g(x) = x\,(x^2 - 5x + 6)$. The function f(g (x)) is discontinuous at

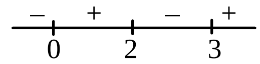
(A) infinitely many points

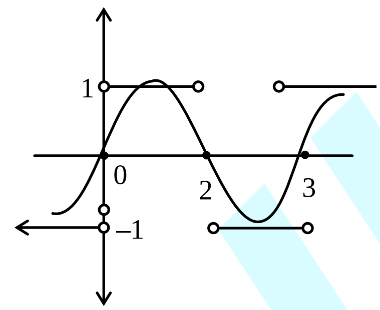
(B) exactly one points

(C) exactly three points

(D) no point

Sol. **[C]**





f(x) = sgn(x), g(x) = x(x – 3) (x – 2)

f(g(x)) = 0 at x = 0, 3, 2

Q.16 The set of all points of discontinuity of the function

$f(x) = \dfrac{\tan x \log x}{1 - \cos 4x}$ contains

(A) $\left\{\frac{n\pi}{2} : n \in Z\right\}$ (B) $\left\{\frac{n\pi}{2} : n \in Q\right\}$

(C) $]-\infty, 0] \cup \left\{\frac{n\pi}{2} : n \in N\right\}$

(D) None of these

Sol. **[D]**

$f(x) = \dfrac{\tan x \log x}{1 - \cos 4x}$

tan x is not defined at $x \in (n\pi \pm \pi/2)$;

n ∈ Integer

logx is not defined at $x \in (-\infty, 0]$

Also $1 - \cos 4x \neq 0 \Rightarrow \cos 4x \neq 1$

$\Rightarrow \cos 4x \neq \cos 2n\pi$; n ∈ Integer

$\Rightarrow x \neq n\pi/2$; n ∈ Integer

Hence, f(x) would be discontinuous at points

$x \in \frac{n\pi}{2}$; n ∈ N

$\therefore$ Option (D) is correct Answer.

Q.17 In [1, 3] the function $[x^2 + 1]$; [x] denoting the greatest integer function, is continuous -

(A) for all x except nine points

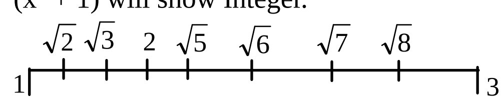
(B) for all x except four points

(C) for all x except seven points

(D) for all x except eight points

Sol. **[D]** $f(x) = [x^2 + 1]$; $x \in [1, 3]$

f(x) will be discontinuous at those points where $(x^2 + 1)$ will show Integer.



There are Nine Integers. But at x = 1. f(x) shows continuous nature as follows :

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[(1+h)^2+1] = 2$$

$$f(1) = [1 + 1] = 2$$

Hence, f(x) is continuous at x = 1

At x = 3, $\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h)$

$$= \lim_{h\to 0}[(3-h)^2+1]$$

$$= 9$$

$$f(3) = [9 + 1] = 10$$

At x = 3, it shows discontinuous nature. Hence, there are eight points (including 3) where f(x) will show discontinuous nature.

∴ Option (D) is correct Answer.

Q.18 y = f(x) is a continuous function such that its graph passes through (a, 0) then

$$\lim_{x\to a} \frac{\log_e(1+3f(x))}{2f(x)} \text{ is}$$

(A) 1 (B) 0

(C) 3/2 (D) 2/3

Sol. **[C]**

y = f(x) at passes through (a, 0) so that f(a) = 0

$$\lim_{x\to a} \frac{\log(1+3f(x))}{2f(x)} \qquad [\frac{0}{0} \text{ form}]$$

Apply L – H Rule, we get

$$\lim_{x\to a} \frac{1}{1+3f(x)} \cdot \frac{3f'(x)}{2f'(x)} = \frac{3}{2} \times \frac{1}{1+3f(a)}$$

$$= \frac{3}{2} \times \frac{1}{1+3\times 0} = \frac{3}{2}$$

∴ Option (C) is correcet Answer.

L.H.L = $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} \frac{\left[\frac{1}{2}+0-h\right]-\left[\frac{1}{2}\right]}{(0-h)}$

$$= \lim_{h\to 0} \frac{0-0}{(-h)} = 0$$

R.H.L = $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} \frac{\left[\frac{1}{2}+0+h\right]-\left[\frac{1}{2}\right]}{(0+h)}$$

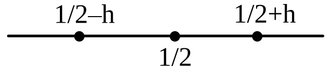
$$= \lim_{h\to 0} \frac{0-0}{(h)} = 0$$

$f(0) = \dfrac{\left[\frac{1}{2}+0\right]-\left[\frac{1}{2}\right]}{0}$ Not defined.

Hence, f(x) is Discontinuous at x = 0

∴ Option (A) is correct Answer.

At x = $\frac{1}{2}$



R.H.L = $\lim_{x\to \frac{1}{2}^+} f(x) = \lim_{h\to 0} f\left(\frac{1}{2}+h\right)$

$$= \lim_{h\to 0} \frac{\left[\frac{1}{2}+\frac{1}{2}+h\right]-\left[\frac{1}{2}\right]}{\left(\frac{1}{2}+h\right)}$$

$$= \lim_{h\to 0} \frac{[1+h]-[1/2]}{\left(\frac{1}{2}+h\right)}$$

$$= \lim_{h\to 0} \frac{1-0}{\left(\frac{1}{2}+h\right)} = 2$$

L.H.L = $\lim_{x\to \frac{1}{2}^-} f(x) = \lim_{h\to 0} f\left(\frac{1}{2}-h\right)$

$$= \lim_{h\to 0} \frac{\left[\frac{1}{2}+\frac{1}{2}-h\right]-\left[\frac{1}{2}\right]}{\left(\frac{1}{2}-h\right)}$$

$$= \lim_{h\to 0} \frac{[1-h]-\left[\frac{1}{2}\right]}{\left(\frac{1}{2}-h\right)}$$

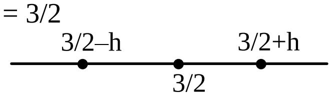
$$= \lim_{h\to 0} \frac{0-0}{\left(\frac{1}{2}-h\right)} = 0$$

Since, L.H.L ≠ R.H.L

∴ f(x) is Discontinuous at x = 1/2

∴ option (B) is correct Answer.

At x = 3/2



L.H.L = $\lim_{x\to \frac{3}{2}^-} f(x) = \lim_{h\to 0} f\left(\frac{3}{2}-h\right)$

$$= \lim_{h\to 0} \frac{\left[\frac{1}{2}+\frac{3}{2}-h\right]-\left[\frac{1}{2}\right]}{(3/2-h)}$$

$$= \lim_{h\to 0} \frac{[2-h]-\left[\frac{1}{2}\right]}{\left(\frac{3}{2}-h\right)}$$

$$= \lim_{h\to 0} \frac{1-0}{\left(\frac{3}{2}-h\right)} = \frac{2}{3}$$

$$\textbf{R.H.L} = \lim_{x\to \frac{3}{2}^+} f(x) = \lim_{h\to 0} f\left(\frac{3}{2}+h\right)$$

$$= \lim_{h\to 0} \frac{\left[\frac{1}{2}+\frac{3}{2}+h\right]-\left[\frac{1}{2}\right]}{\left(\frac{3}{2}+h\right)}$$

$$= \lim_{h\to 0} \frac{2-0}{\left(\frac{3}{2}+h\right)} = 4/3$$

Since, L.H.L ≠ R.H.L.

∴ f(x) is D is continuous at x = 3/2

∴ Option (C) is also correct.

∴ All options are correct.

(D) is correct Answer.

➢ True or False type Questions

Q.19 The function f(x) = p[x + 1] + q [x – 1], (where [.] denotes the greatest integer function) is continuous at x = 1 if p + q = 0

Sol. **[True]**

f(x) = p[x] + p + q[x] – q f(1) = 2p

at x = 1,

RHL : p + p + q – q = 2p

LHL : p – q

if L.H.L. = RHL

2p = p – q

⇒ p + q = 1 ∴ true.

Q.20 The point of discontinuity of the function $f(x) = \lim_{n\to\infty} \frac{(2\sin x)^{2n}}{3^n - (2\cos x)^{2n}}$ is $n\pi \pm \frac{\pi}{6}$, n ∈ N

Sol. **[True]**

$$f(x) = \lim_{n\to\infty} \frac{(2\sin x)^{2n}}{3^n - (2\cos x)^{2n}}$$

f(x) is not defined for $3^n - (2\cos x)^{2n} = o$

$3^n = (2\cos x)^{2n} = ((2\cos x)^2)^n$

⇒ $3 = (2\cos x)^2$

⇒ $\cos x = \pm\sqrt{3}/2$

⇒ x = nπ ± π/6 ; n ∈ Integer

Hence, option is true.

Q.21 If the function

$$f(x) = \begin{cases} \frac{3}{x^2}\sin 2x^2 & \text{if } x<0 \\ 0 & \text{if } x=2 \\ \frac{x^2+3x+K}{1-3x^2} & \text{if } x\ge 0,\ x\ne 2 \end{cases}$$

is continuous at x = 0, then value of K is 6

Sol. **[True]**

$$f(x) = \begin{cases} \frac{3}{x^2}\sin 2x^2 & \text{if } x<0 \\ 0 & \text{if } x=2 \\ \frac{x^2+3x+k}{1-3x^2} & \text{if } x\ge 0,\ x\ne 2 \end{cases}$$

0–h 0 0+h

$$\textbf{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{3}{(0-h)^2}\sin 2(0-h)^2$$

$$= \lim_{h\to 0} \frac{3}{h^2}\times \sin 2h^2$$

∴ $\sin x = x - x^3/3! + x^5/5!$ …..

$\sin 2h^2 = 2h^2 - (2h^2)^3/3! + (2h^2)^5/5!$

$$= \lim_{h\to 0} \frac{3}{h^2}\times(2h^2-(2h^2)^3/3!+(2h^2)^5/5!-\ldots\ldots)$$

$$= \lim_{h\to 0} \frac{6\times h^2}{h^2}\times\left(1-\frac{4h^4}{3!}+\frac{16h^8}{5!}-\ldots..\right)$$

= 6 .

$$\textbf{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{(0+h)^2+3(0+h)+k}{1-3(0+h)^2}$$

$$= \lim_{h\to 0} \frac{h^2+3h+k}{1-3h^2} = k$$

Since, f(x) is continuous at x = 0.

L.H.L.= R.H.L.

⇒ k = 6

∴ Option is true.

Q.22 The function $f(x) = \lim_{p\to\infty} \frac{(1+\sin \pi x)^p - 1}{(1+\sin \pi x)^p + 1}$ is continuous at each point of its domain.

Sol. **[False]**

$$f(x) = \lim_{p\to\infty} \frac{(1+\sin \pi x)^p - 1}{(1+\sin \pi x)^p + 1}$$

f(x) to be defined if $(1 + \sin\pi x)^p + 1 \ne 0$

$(1+\sin\pi x)^p \ne -1$

$1 + \sin\pi x \ne (-1)^{1/p}$

$\sin\pi x \ne (-1)^{1/p} - 1$

$\pi x \ne \sin^{-1}[(-1)^{1/p} - 1]$

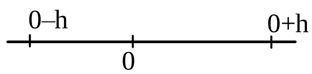
$$x \neq \frac{1}{\pi}\sin^{-1}[(-1)^{1/p}-1]$$

$$x \neq \lim_{p\to\infty}\frac{1}{\pi}\sin^{-1}[(-1)^{1/p}-1]$$

$$x \neq \lim_{p\to\infty}\frac{1}{\pi}\sin^{-1}[(-1)^{o}-1]$$

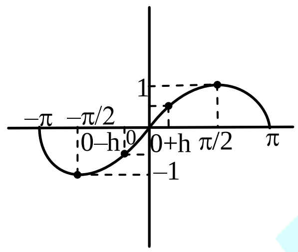
$$x \neq \frac{1}{\pi}\sin^{-1}(1-1)$$

$x \neq 0$.



L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\frac{[1+\sin\pi(0-h)]^p-1}{[1+\sin\pi(0-h)]^p+1}$$



$$= \lim_{h\to 0}\frac{[1+\sin\pi(0-h)]^p-1}{[1+\sin\pi(0-h)]^p+1}$$

$\sin\pi$ (0–h) have some Negative quantity

Hence $1 + \sin\pi(0-h) < 1$.

$$\therefore \lim_{p\to\infty}[1+\sin\pi(0-h)]^p \to 0.$$

$$= \lim_{h\to 0}\frac{0-1}{0+1} = -1.$$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0}\frac{(1+\sin\pi(0+h))^p-1}{[1+\sin\pi(0+h)]^p+1}$$

$\sin\pi(0+h) \to$ have some + ve quantity

Hence, $1+\sin\pi(0+h) > 1$

$$\lim_{p\to\infty}[1+\sin\pi(0+h)]^p \to \infty$$

$$\Rightarrow \lim_{p\to\infty}\frac{1}{[1+\sin\pi(0+h)]^p} \to 0$$

$$= \lim_{p\to\infty}\frac{1-\dfrac{1}{[1+\sin\pi(0+h)]^p}}{1+\dfrac{1}{[1+\sin\pi(0+h)]^p}} = \frac{1-0}{1+0} = 1$$

L.H.L. = –1 and R.H.L. = + 1

Since L.H.L. ≠ R.H.L.

Hence, f(x) would be discontinuous.

$\therefore$ Option is false.

➤ Fill in the blanks type questions

Q.23 If $f(x) = [\alpha + \beta \sin x]$, $x \in (0, \pi)$, $\alpha \in I$, β is a prime number and $[x]$ is the greatest integer function, then number of points at which f(x) is discontinuous is ……………

Sol. $f(x) = [\alpha + \beta \sin x]$; $x \in (0,\pi)$, $\alpha \in$ integer

β is prime number.

f(x) will be discontinuous at those points where $(\alpha + \beta \sin x)$ will become integer.

$\alpha + \beta \sin x$ = Integer.

$\sin x = 0, \pm 1, r/\beta$; $0 < r \leq \beta - 1$

$x = 0, -\pi/2, \pi/2$ $\sin^{-1}(r/\beta)$, $\pi - \sin^{-1}(r/\beta)$

But $x = 0, -\pi/2$ Not included in $x \in (0,\pi)$.

Hence, $x = \frac{\pi}{2}$, $\sin^{-1}r/\beta$, $\pi - \sin^{-1}r/\beta$......

Therefore, total number of discontinuous points will be $[1+2(\beta - 1) = (2\beta - 1)]$

Q.24 The set of points where $f(x) = \sec^{-1}[1 + \sin^2 x]$, where $[\cdot]$ denotes the greatest integer function, is not continuous is..........................

Sol. $f(x) = \sec^{-1}[1+\sin^2 x]$

f(x) will be discontinuous at those points where $1+\sin^2 x$ will be come Integer.

i.e. $1 + \sin^2 x = 0, + 1, + 2$ ……..

↳ 0 and –ve values will be neglected

$\sin^2 x = 0, 1$

$\Rightarrow \sin x = 0, \pm 1$ But $\sin x \neq 0$

$x = n\pi + (-1)^n(\pm\pi/2)$

$$x = \frac{2n\pi \pm \pi}{2} = (2n \pm 1)\pi/2.$$

EXERCISE # 2

Part-A | **Only single correct answer type questions**

Q.1 The function defined by $f(x) = (-1)^{[x^3]}$ (where $[\cdot]$ denotes greatest integer function satisfies)

(A) discontinuous for $x = n^{1/3}$ where n is any integer

(B) $f(3/2) = 1$

(C) $f'(x) = 0$ for $-1 < x < 1$

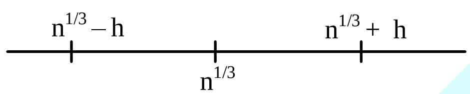
(D) none of these

Sol. **[A]**

$f(x) = (-1)^{[x^3]}$

we have to check for every option

For A :



$$\text{L.H.L} = \lim_{x\to n^{1/3^-}} f(x) = \lim_{h\to 0} f(n^{1/3}-h)$$

$$= \lim_{h\to 0}(-1)^{[(n^{1/3}-h)^3]}$$

$$= \lim_{h\to 0}(-1)^{\left[n\left(1-\frac{h}{n^{1/3}}\right)^3\right]}$$

$$= \lim_{h\to 0}(-1)^{[n\times \text{value less than } 1]}$$

$$= (-1)^{n-1}$$

$$\text{R.H.L} = \lim_{x\to n^{1/3^+}} = \lim_{h\to 0} f(n^{1/3}+h)$$

$$= \lim_{h\to 0}(-1)^{[(n^{1/3+h})^3]}$$

$$= \lim_{h\to 0}(-1)^{\left[n(1+h/n^{1/3})^3\right]}$$

$$= \lim_{h\to 0}(-1)^{[n \text{ value greater than } 1]}$$

$$= (-1)^n$$

Since, L.H.L $\neq$ R.H.L

Hence, f(x) is discontinuous at $x = n^{1/3}$

$\therefore$ Option (A) is correct Answer.

For B : $\lim_{x\to 3/2} f(x) = f(3/2) = (-1)^{[(3/2)^3]}$

$= (-1)^{[27/8]}$

$= (-1)^{[3.5]}$

$= (-1)^3$

$= -1$

$\therefore$ Option (B) is not correct Answer.

For C : $f'(x) = (-1)^{[x^3]}\log(-1) \times \frac{d}{dx}[x^3]$

For $-1 < x < 1$

since log(–1) does not exist.

$\therefore$ Option (C) is not correct Answer

Q.2 The function $f(x) = \begin{cases} \frac{x^2}{a} & ,0 \le x < 1 \\ a & ,1 \le x < \sqrt{2} \\ \frac{(2b^2-4b)}{x^2} & ,\sqrt{2} \le x < \infty \end{cases}$

is continuous for $0 \le x < \infty$ then the most suitable values of a and b are -

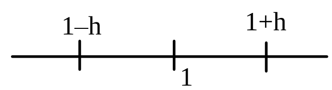
(A) a = 1, b = –1 (B) a = –1, b = $1 + \sqrt{2}$

(C) a = –1, b = 1 (D) None of these

Sol. **[C]**

$$f(x) = \begin{cases} \frac{x^2}{a} & ,0 \le x < 1 \\ a & ,1 \le x < \sqrt{2} \\ \frac{(2b^2-4b)}{x^2} & ,\sqrt{2} \le x < \infty \end{cases}$$

At x = 1



$$\text{L.H.L} = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0}\frac{(1-h)^2}{a} = 1/a$$

$$\text{R.H.L} = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= a$$

value, f(1) = a

Since, f(x) is continuous, then

L.H.L = R.H.L. = f(1)

1/a = a = a

$\Rightarrow$ $a^2 = 1$

$\Rightarrow$ $a = \pm 1$

At $x = \sqrt{2}$

$$\text{L.H.L} = \lim_{x\to \sqrt{2}^-} f(x) = \lim_{h\to 0} f(\sqrt{2}-h)$$

$$= \lim_{h\to 0} a$$

$= a$

R.H.L = $\lim_{x\to\sqrt{2}^+} f(x)$

$= \lim_{h\to 0} f(\sqrt{2}+h) = \lim_{h\to 0} \dfrac{2b^2-4b}{(\sqrt{2}+h)^2}$

$= \dfrac{2b^2-4b}{2}$

Value, $f(\sqrt{2}) = (2b^2-4b)/2$

Since, f(x) is continuous at $x = \sqrt{2}$

then L.H.L = R.H.L = $f(\sqrt{2})$

$a = (2b^2-4b)/2$

$a = (b^2-2b) = \pm 1$

Taking $+1$: $b^2-2b = +1$

$\Rightarrow b^2-2b-1 = 0$

$b = \dfrac{2\pm\sqrt{4+4}}{2\times 1}$

$b = 1 \pm \sqrt{2}$

Taking -1 : $b^2-2b = -1$

$\Rightarrow b^2-2b+1 = 0$

$\Rightarrow (b-1)^2 = 0$

$\Rightarrow b = 1$

Hence, possible combinations will be

$a = 1$; $b = 1+\sqrt{2}$

or

$a = 1$; $b = 1-\sqrt{2}$

or

$a = -1$; $b = 1$

Hence, option (C) is correct answer.

Q.3 If $f(x) = \begin{cases} \dfrac{1-\cos 10x}{x^2}, & x<0 \\ a, & x=0 \\ \dfrac{\sqrt{x}}{\sqrt{625+\sqrt{x}}-25}, & x>0 \end{cases}$, then the value of a so that f(x) may be continuous at x = 0 is -

(A) 25 (B) 50

(C) –25 (D) None of these

Sol. **[B]**

0–h 0 0+h

L.H.L $= \lim_{x\to 0^-} f(x)$

$= \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \dfrac{1-\cos 10(0-h)}{(0-h)^2}$

$= \lim_{h\to 0} \dfrac{1-\cos 10h}{h^2} = \lim_{h\to 0} \dfrac{\sin 10h(10)}{2h}$

$= \lim_{h\to 0} \dfrac{\cos 10h(100)}{2} = 50$

R.H.L = $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} \dfrac{\sqrt{0+h}}{\sqrt{625+\sqrt{0+h}}-25} = \lim_{h\to 0} \dfrac{\sqrt{h}}{\sqrt{625+\sqrt{h}}-25}$

$= \lim_{h\to 0} \dfrac{1/2\sqrt{h}}{\dfrac{1}{2\sqrt{625+\sqrt{h}}}\times\dfrac{1}{2\sqrt{h}}} = \lim_{h\to 0} 2\sqrt{625+\sqrt{h}}$

= 50

Since, f(x) is continuous at x = 0,

L.H.L = R.H.L = f(0)

$\Rightarrow$ 50 = a $\Rightarrow$ a = 50

$\therefore$ Option (B) is correct Answer.

Q.4 If $f(x) = x^P \cos(1/x)$; $x \neq 0$ and f(0) = 0. The condition for P ($P \in I$) is____, which make function f(x) continuous at x = 0

(A) P > 0 (B) P < 0

(C) P = 0 (D) None of these

Sol. **[A]**

$f(x) = x^P.\cos(1/x)$; $x \neq 0$

$= 0$, $x = 0$

$\cos\dfrac{1}{x} = 1 - \dfrac{1}{2!x^2} + \dfrac{1}{4!x^4} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$x^P\cos\dfrac{1}{x} = x^P\left(1-\dfrac{1}{2!x^2}+\dfrac{1}{4x^4}-\ldots\ldots\ldots\right)$

$\lim_{x\to 0} x^P\cos\dfrac{1}{x} = \lim_{x\to 0} x^P\left(1-\dfrac{1}{2!x^2}+\dfrac{1}{4!x^4}-\ldots.\right)$

$= \lim_{x\to 0}\left(x^P - \dfrac{x^P}{2!x^2}+\dfrac{x^P}{4!x^4}-\ldots\ldots\right)$

= 0 (Given)

It menas P > 0

= ∞ It means P < 0

= 1 It means P = 0

$\therefore$ Option (A) is correct Answer.

Q.5 If $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]+1} & \text{for } x>0 \\ \dfrac{\cos\pi/2[x]}{[x]} & \text{for } x<0 \\ k & \text{for } x=0 \end{cases}$

where [x] denotes the greatest integer less than or equal to x, then in order that f(x) be continuous at x = 0, the value of k is -

(A) equal to 0 (B) equal to 1

(C) equal to – 1 (D) indeterminate

Sol. **[A]**

$$f(x) = \begin{cases} \dfrac{\sin[x]}{[x]+1} & \text{for } x > 0 \\ \cos\dfrac{\pi}{2[x]} & \text{for } x < 0 \\ k \quad \text{for } x = 0 \end{cases}$$

L.H.L $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} \cos\dfrac{\pi}{2[0-h]}$

$= \lim_{h\to 0} \cos\dfrac{\pi}{2(-1)}$

$= \lim_{h\to 0} \cos\pi/2 = 0$

R.H.L $= \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} \dfrac{\sin[0+h]}{[0+h]+1}$

$= \lim_{h\to 0} \dfrac{\sin(0)}{0+1} = 0$

Since, f(x) is continuous at x = 0.

L.H.L = R.H.L = f(0) = k

$\Rightarrow$ k = 0

$\therefore$ Option (A) is correct Answer.

Q.6 Let f(x) = $[2x^3 - 5]$ where [ ] denotes the greatest integer function. Then number of points in [1, 2) where the function is discontinuous, is -

(A) 14 (B) 13

(C) 10 (D) None of these

Sol. **[B]**

f(x) = $[2x^3 - 5]$

Greatest Integral function is discontinuous at integer points.

Hence, $2x^3 - 5 = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ ....

$2x^3 - 5 = 0 \Rightarrow x = (2.5)^{1/3}$

$2x^3 - 5 = \pm 1 \Rightarrow x = (3)^{1/3}$ and $(2)^{1/3}$

$2x^3 - 5 = \pm 2 \Rightarrow x = (3.5)^{1/3}$ and $(1.5)^{1/3}$

$2x^3 - 5 = \pm 3 \Rightarrow x = (4)^{1/3}$ and $(1)^{1/3}$

$2x^3 - 5 = \pm 4 \Rightarrow x = (4.5)^{1/3}$ and $(0.5)^{1/3}$

(This value not included)

$2x^3 - 5 = \pm 5 \Rightarrow x = 0$ and $(5)^{1/3}$

Value x = 0 is not included

$2x^3 - 5 = +6 \Rightarrow x = (11/2)^{1/3} = (5.5)^{1/3}$

$2x^3 - 5 = +7 \Rightarrow x = (6)^{1/3}$

$2x^3 - 5 = 8 \Rightarrow x = (6.5)^{1/3}$

$2x^3 - 5 = 9 \Rightarrow x = (7)^{1/3}$

$2x^3 - 5 = 10 \Rightarrow x = (7.5)^{1/3}$

$2x^3 - 5 = 11 \Rightarrow x = (8)^{1/3} = 2$

(This value not included because at 2 open Interval.)

Hence, total values at which function is discontinuous is 14. But 1 is included in this group. Therefore, 13 points at which f(x) is discontinuous in [1, 2)

Q.7 The number of points where f(x) = [sin x + cos x] (where [·] denotes the greatest integer function) x ∈ (0, 2π) is not continuous is-

(A) 3 (B) 4

(C) 5 (D) 6

Sol. **[C]**

f(x) = [sinx + cosx] ; x ∈ (0, 2π)

f(x) will be discontinuous at those points where (sinx + cosx) will be Integer

Hence, sinx + cosx = 0, ± 1

$\sin x = -\cos x \Rightarrow \cos x = -\sin(x) = -\cos\left(\dfrac{\pi}{2} - x\right)$

$\cos x = \cos(\pi + \pi/2 - x)$

$= \cos(3\pi/2 - x)$

$x = 2n\pi \pm \dfrac{3\pi}{2} - x$

$2x = 2n\pi \pm 3\pi/2$

$x = n\pi \pm \dfrac{3\pi}{4}$; n ∈ Integer (1)

sinx + cosx = ± 1

$\sqrt{2}\sin(x + \pi/4) = \pm 1$

$\sin(x + \pi/4) = \pm\dfrac{1}{\sqrt{2}} = \sin(\pm\pi/4)$

$x + \pi/4 = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

from (1) $x = \frac{3\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}$

$x = \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$

But at $x = \pi/4$, $\sin x + \cos x = \sqrt{2} \notin$ Integer.

Hence, points of discontinuity are

$x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$

There are 5 points

$\therefore$ Option (C) is correct Answer.

Q.8 Let $f(x) = \sqrt{\sin^{-1}x} + \sqrt{\cos^{-1}x}$ defined in [0,1] then

(A) f(x– 3) will be continuous in [3,4]

(B) f(x– 3) will be continuous in [0,1]

(C) f(x – 3) will be continuous in [0,1)

(D) f(x– 3) will be continuous in (0, 1)

Sol. **[A]**

$f(x) = \sqrt{\sin^{-1}x} + \sqrt{\cos^{-1}x}$ defined in [0, 1]

Replace x by (x – 3), we get

$f(x-3) = \sqrt{\sin^{-1}(x-3)} + \sqrt{\cos^{-1}(x-3)}$

then $0 \le x - 3 \le 1$

$\Rightarrow 3 \le x \le 4$

$\Rightarrow x \in [3, 4]$

$\therefore$ Option (A) is correct Answer.

Q.9 Let $f(x) = [x^3 - 3]$, where $[\cdot]$ denotes the greatest integer function. Then the number of points in the interval (1, 2) where the function is discontinuous, is-

(A) 4 (B) 2

(C) 6 (D) None of these

Sol. **[C]**

$f(x) = [x^3 - 3]$; $x \in (1, 2)$

f(x) will be discontinuous at those points where $(x^3 - 3)$ will be integer i.e.,

$x^3 - 3 = 0, \pm1, \pm2, \pm3$ ....

$x^3 - 3 = 0 \Rightarrow x = 3^{1/3}$

$x^3 - 3 = \pm 1 \Rightarrow x = 4^{1/3}, 2^{1/3}$

$x^3 - 3 = \pm 2 \Rightarrow x = 5^{1/3}, 1^{1/3}$

this is not included (Because of open Interval)

$x^3 - 3 = \pm 3 \Rightarrow x = 6^{1/3}, 0$

This is not included.

$x^3 - 3 = \pm 4 \Rightarrow x = 7^{1/3}$

$x^3 - 3 = \pm 5 \Rightarrow x = 8^{1/3} = 2$

This is not included because of open Interval.

Hence, $x = 3^{1/3}, 4^{1/3}, 2^{1/3}, 5^{1/3}, 6^{1/3}, 7^{1/3}$ will be discontinuous points.

$\therefore$ Option (C) is correct Answer.

Q.10 Function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \frac{\pi}{4}$. If f(x) is continuous at $x = \frac{\pi}{4}$ then $f\left(\frac{\pi}{4}\right)$ is equal to -

(A) 1 (B) 2

(C) $1/\sqrt{e}$ (D) None of these

Sol. **[C]**

$f(x) = (\sin 2x)^{\tan^2 2x}$

It is the type of $(1)^{\infty}$.

Since, $\lim_{x\to\pi/4} (\sin 2x - 1) = 0$ and $\lim_{x\to\pi/4} \tan 2x \to \infty$

$$\lim_{x\to\pi/4} (\sin 2x)^{\tan^2 2x} = e^{\lim_{x\to\frac{\pi}{4}} (\sin 2x - 1)\times\tan^2 2x}$$

$$= e^{\lim_{x\to\frac{\pi}{4}} \frac{\sin 2x - 1}{\cot^2 2x}}$$

$$= e^{\lim_{x\to\frac{\pi}{4}} \frac{2\cos 2x}{2\cot 2x.(-\mathrm{cosec}^2 2x).2}}$$

$$= e^{\lim_{x\to\frac{\pi}{4}} \frac{1}{2}\times(-\sin^3 2x)}$$

$$= e^{\frac{1}{2}\times(-1)} = e^{-1/2} = 1/\sqrt{e}$$

$\therefore$ option (C) is correct Answer.

Q.11 If [x] denotes the integral part of x and

$$f(x) = [x]\left\{\frac{\sin\frac{\pi}{[x-1]} + \sin \pi[x+1]}{1+[x]}\right\}$$ then-

(A) f(x) is continuous in R

(B) f(x) is continuous in all integral points

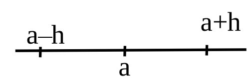
(C) f(x) is discontinuous at all integers

(D) None of these

Sol. **[C]**

$$f(x) = [x]\left\{\frac{\sin\frac{\pi}{[x-1]} + \sin \pi[x+1]}{1+[x]}\right\}$$

Let x = a ; $a \in I$



L. H. L. $= \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$

$$= \lim_{h\to 0}[a-h]\left\{\frac{\sin\frac{\pi}{[a-h-1]}+\sin\pi[a-h+1]}{1+[a-h]}\right\}$$

$$= (a-1)\left\{\frac{\sin\frac{\pi}{(a-2)}+\sin\pi a}{1+(a-1)}\right\}$$

$$= (a-1)\left\{\frac{\sin\frac{\pi}{(a-2)}+\sin\pi a}{a}\right\}$$

R.H.L. = $\lim_{x\to a^+} f((x) = \lim_{h\to 0} f(a+h)$

$$= \lim_{h\to 0}[a+h]\left\{\frac{\sin\frac{\pi}{[a+h-1]}+\sin\pi[a+h+1]}{1+[a+h]}\right\}$$

$$= \lim_{h\to 0} a\left\{\frac{\sin\frac{\pi}{(a-1)}+\sin\pi(a+1)}{1+a}\right\}$$

Since, L.H.L $\neq$ R.H.L

Hence, f(x) is discontinuous at alinteger points

$\therefore$ Option (C) is correct answer.

Q.12 If f(x) is a continuous function $\forall$ x $\in$ R and the range of f(x) is $(2, \sqrt{26})$ and g(x) = $\left[\frac{f(x)}{c}\right]$ is continuous $\forall$ x $\in$ R, then the least positive integral value of c is, ([·] denoted greatest integer function)-

(A) 2 (B) 3

(C) 5 (D) 6

Sol. **[D]**

f(x) is continuous in x $\in$ R

and range of f(x) is $(2, \sqrt{26})$.

g(x) = $\left[\frac{f(x)}{c}\right]$ continuous $\forall$ x$\in$R.

Since, g(x) is continuous. It means $\frac{f(x)}{c}$ must not be integer. If we take c = 6

then g(x) = $\left[\frac{f(x)}{6}\right]$ = 0 for every value of f(x) in $(2, \sqrt{26})$

$\therefore$Option (D) is Correct Answer

Q.13 If f(x) be a continuous function for all real values of x and satisfies $x^2 + (f(x) - 2)x + 2\sqrt{3} - 3 - \sqrt{3}\, f(x) = 0\ \forall\ x \in R$. Then $f(\sqrt{3})$ is equal to-

(A) $2 - \frac{1}{\sqrt{3}}$ (B) $2(1-\sqrt{3})$

(C) $2 + \sqrt{3}$ (D) Zero

Sol. **[B]**

$x^2 + (f(x) - 2)x + 2\sqrt{3} - 3 - \sqrt{3}\, f(x) = 0$
$\forall\ x \in R$

$x^2 + f(x).x - 2x + 2\sqrt{3} - 3 - \sqrt{3}\, f(x) = 0$

$\Rightarrow f(x)(x - \sqrt{3}) = 2x + 3 - 2\sqrt{3} - x^2$

$\Rightarrow f(x) = \frac{2x+3-2\sqrt{3}-x^2}{(x-\sqrt{3})}$

$f(\sqrt{3}) = \lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}} \frac{2x+3-2\sqrt{3}-x^2}{(x-\sqrt{3})}$ $\left(\frac{0}{0}\text{form}\right)$

Apply, L– H Rule weget

$= \lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}} \frac{2-2x}{1}$

$= 2(1-\sqrt{3})$

$\therefore$ Option (B) is correct Answer.

Part-B One or more than one correct answer type questions

Q.14 Given the function f(x) = $\frac{1}{1-x}$, the points of discontinuity of the composite function y = f(f(f(x))) are at x =

(A) 0 (B) 1

(C) 2 (D) –1

Sol. **[A,B]**

$f(x) = \frac{1}{1-x}$; $x \neq 1 \Rightarrow f(f(x)) = 1/1 - f(x)$

$= 1/1 - \frac{1}{1-x} = 1/\frac{1-x-1}{1-x}$

$= \frac{1-x}{-x} = \frac{x-1}{x}$; $x \neq 0$

$f(f(f(x)) = 1/1 - f(f(x)) = 1/1 - \frac{x-1}{x} = 1/\frac{x-x+1}{x}$

= x ; x $\in$ R except x $\neq$ 0, 1

Hence, y = f(f(f(x))) will be discontinuous at x = 0,1

Q.15 Let $f(x) = \left[x\left[\frac{1}{x}\right]\right]$ for x > 0, where [·] denotes the greatest integer function, then f(x) is

(A) $f\left(\frac{1}{2}\right) = 0$

(B) $f\left(\frac{3}{4}\right) = 0$

(C) discontinuous at finite number of points

(D) discontinuous at infinite number of points.

Sol. **[B,D]**

$f(x) = \left[x\left[\frac{1}{x}\right]\right]$ for x > 0

At x = 1/2 $f\left(\frac{1}{2}\right) = \lim_{x\to\frac{1}{2}} f(x) = \lim_{x\to\frac{1}{2}} \left[\frac{1}{2}[2]\right]$

$= \lim_{x\to\frac{1}{2}} \left[\frac{1}{2}\times 2\right] = 1$

L.H.L. $= \lim_{x\to\frac{1}{2}} f(x) = \lim_{h\to 0} f\left(\frac{1}{2}-h\right)$

$= \lim_{h\to 0}\left[\left(\frac{1}{2}-h\right)\times\left[\frac{1}{\frac{1}{2}-h}\right]\right] = \lim_{h\to 0}\left[\left(\frac{1}{2}-h\right)\times 1\right]$

$\left(\text{Because } \frac{1}{\frac{1}{2}-h} > 1\right)$

$= \lim_{h\to 0}$ [value less than 1] = 0

R.H.L. $= \lim_{x\to\frac{1^+}{2}} f(x) = \lim_{h\to 0} f\left(\frac{1}{2}+h\right)$

$= \lim_{h\to 0}\left[\left(\frac{1}{2}+h\right)\left[\frac{1}{\frac{1}{2}+h}\right]\right]$

$= \lim_{h\to 0}\left[\left(\frac{1}{2}+h\right)\times 1\right]$

Because $\left[\frac{1}{\frac{1}{2}+h}\right] = 1 = 0$

Since, L.H.L. = R.H.L. $\neq f\left(\frac{1}{2}\right)$

Hence, f(x) is not continuous at $x = \frac{1}{2}$

At x = 3/4

$f\left(\frac{3}{4}\right) = \lim_{x\to 3/4} f(x) = \lim_{h\to 0} f\left(\frac{3}{4}-h\right)$

$= \lim_{h\to 0}\left[\left(\frac{3}{4}-h\right)\left[\frac{1}{\frac{3}{4}-h}\right]\right]$

$= \lim_{h\to 0}\left[\left(\frac{3}{4}-h\right).1\right]$

Because $\left[\frac{1}{\frac{3}{4}-h}\right] = 1$

$= \lim_{h\to 0}$ [value less than 1] = 0

R.H.L. $= \lim_{x\to\frac{3^+}{4}} f(x) = \lim_{h\to 0} f(3/4 + h)$

$= \lim_{h\to 0}\left[\left(\frac{3}{4}+h\right)\left[\frac{1}{\frac{3}{4}+h}\right]\right]$

$= \lim_{h\to 0}\left[\left(\frac{3}{4}+h\right).1\right]$

Because , $\left[\frac{1}{\frac{3}{4}+h}\right] = 1$

$= \lim_{h\to 0}$ [Value less then 1] = 0

Since, L.H.L. = R.H.L. = f(3/4) = 0

Hence, f(x) is continuous at x = 3/4.

Therefore, we can make conclusions that there will be infinite number of Points where f(x) will be discontinuous.

Hence, Options (B) and (D) are correct answers.

Q.16 Given

$$f(x)=\begin{cases} 3-\left[\cot^{-1}\left(\frac{2x^3-3}{x^2}\right)\right] & \text{for } x>0 \\ \{x^2\}\cos(e^{1/x}) & \text{for } x<0 \end{cases}$$

where { } & [ ] denotes the fractional part and the integral part functions respectively, then which of the following statement does not hold good-

(A) $f(0^-) = 0$

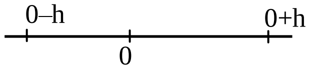
(B) $f(0^+) = 3$

(C) f(0) = 0 ⇒ continuity of f at x = 0

(D) irremovable discontinuity of f at x = 0

Sol. **[B,D]**

$$f(x) = \begin{cases} 3 - \left[\cot^{-1}\left(\dfrac{2x^3 - 3}{x^2}\right)\right] & \text{for } x > 0 \\ \{x^2\}\cos(e^{1/x}) & \text{for } x < 0 \end{cases}$$



L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} ((0-h)^2 - [(0-h)^2]) \times \cos\left(\dfrac{1}{e^{(0-h)}}\right)$

$= \lim_{h \to 0} [h^2 - [h^2]] \times \cos\ (e^{-1/h})$

$= \lim_{h \to 0} [h^2 - [h^2]] \times \cos\ e^{-\infty}$

$= \lim_{h \to 0} [h^2 - h^2] \times \cos 0 = 0 \times 1 = 0$

R.H.L. = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$= \lim_{h \to 0} \left\{3 - \left[\cot^{-1}\left(\dfrac{2(0+h)^3 - 3}{(0+h)^2}\right)\right]\right\}$

$= \lim_{h \to 0} \left\{3 - \left[\cot^{-1}\left(\dfrac{2h^3 - 3}{h^2}\right)\right]\right\}$

$= \lim_{h \to 0} \{3 - [\cot^{-1}(-\infty)]\}$

$= \lim_{h \to 0} \{3 - [3.14]\} = 0$

$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \left\{3 - \left[\cot^{-1}\left(\dfrac{2x^3 - 3}{x^2}\right)\right]\right\}$

$= \lim_{x \to 0} \{3 - [\cot^{-1}(-\infty)]\}$

$= \lim_{x \to 0} \{3 - [\pi]\} = 0$

Hence, L.H.L. = R.H.L. = f(0) = 0

$\therefore$ f(x) is continuous at x = 0

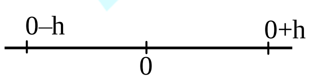
Q.17 Which of the following function(s) not defined at x = 0 has/ have non- removable discontinuity at the origin-

(A) $f(x) = \dfrac{1}{1 + 2^{1/x}}$ (B) $f(x) = \tan^{-1}\dfrac{1}{x}$

(C) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ (D) None of these

Sol. **[A,B,C]**

For A : $f(x) = 1/1+2^{1/x}$



L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \dfrac{1}{1 + 2^{-\frac{1}{h}}} = 1$

R.H.L.= $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$= \lim_{h \to 0} \dfrac{1}{1 + 2^{\frac{1}{h}}}$

$= \lim_{h \to 0} \dfrac{1}{1 + 2^{\infty}} = 0$

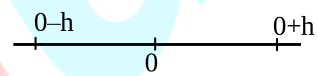
$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{1}{1 + 2^{1/x}} = \lim_{x \to 0} \dfrac{1}{1 + 2^{\infty}}$

= 0

Since, L.H.L. $\neq$ R.H.L.

Hence. It is Discontinuous as well as have non removable discontinuity .

For B : $f(x) = \tan^{-1}\dfrac{1}{x}$



L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \tan^{-1}\left(\dfrac{1}{0-h}\right)$

$= \lim_{h \to 0} \tan^{-1}\left(\dfrac{-1}{h}\right)$

$= \tan^{-1}(\infty) = -\pi/2$

R.H.L.= $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

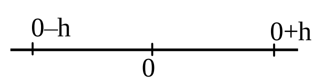
$= \lim_{h \to 0} \tan^{-1}\left(\dfrac{1}{0+h}\right)$

$= \tan^{-1}(\infty) = \pi/2$

Since, R.H.L. $\neq$ L.H.L.

Hence, f(x) is discontinuous and have non-removable discontinuity.

For C : $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$



L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \dfrac{e^{-1/h} - 1}{e^{-1/h} + 1}$

$$= \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{0-1}{0+1} = -1.$$

R.H.L.= $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} \frac{e^{1/h}-1}{e^{1/h}+1}$$

$$= \lim_{h\to 0} \frac{1-e^{-1/h}}{1+e^{-1/h}}$$

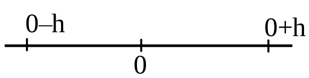
$$= \frac{1-e^{-\infty}}{1+e^{-\infty}} = \frac{1-0}{1+0} = 1.$$

Since, R.H.L. $\neq$ L.H.L.

Hence, f(x) is discontinuous and have non-removable discontinuity

For D : $f(x) = 1/\lambda n|x|$

$$= \begin{cases} 1/\lambda n x; x \geq 0 \\ 1/\lambda n(-x); x < 0 \end{cases}$$



L.H.L. = $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$= \lim\limits_{h\to 0} 1/\lambda n\,(-(0-h))$

$= \lim\limits_{h\to 0} 1/\lambda n\, h = 1/-\infty = 0$

R.H.L. = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$= \lim\limits_{h\to 0} 1/\lambda nh = 1/(-\infty) = 0$

$f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} 1/\lambda n|0| = 1/-\infty = 0$

Hence, f(x) is continuous at x = 0 as

L.H.L. = R.H.L. = f(0) = 0.

$\therefore$ options (A), (B) and (C) are correct answers.

Q.18 Which of the following function(s) not defined at x = 0 has/ have removable discontinuity at the origin-

(A) $f(x) = \dfrac{1}{1+2^{\cos x}}$ (B) $f(x) = \cos\left(\dfrac{|\sin x|}{x}\right)$

(C) $f(x) = x \sin\dfrac{\pi}{x}$ (D) $f(x) = \dfrac{1}{\lambda n\,|x|}$

Sol. **[B,C,D]**

For A: $f(x) = \dfrac{1}{1+2^{\cos x}}$

$f(0) = \lim\limits_{h\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{1}{1+2^{\cos 0}} = \dfrac{1}{3}$

L.H.L. = $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} \frac{1}{1+2^{\cos(0-h)}}$$

$$= \lim_{h\to 0} \frac{1}{1+2^{\cos h}} = \frac{1}{1+2} = \frac{1}{3}$$

R.H.L. = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} \frac{1}{1+2^{\cos(0+h)}}$$

$$= \lim_{h\to 0} \frac{1}{1+2^{\cos h}}$$

$$= \frac{1}{1+2} = \frac{1}{3}$$

Since, L.H.L. = R.H.L. = f(0) = 1/3

Therefore, $f(x)=\dfrac{1}{1+2\cos x}$ is continuous at x =0

For B : $f(x) = \cos\left(\dfrac{|\sin x|}{x}\right)$

It is discontinuous function at x = 0

have removable discontinuity as follows :

L.H.L = $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} \cos\left(\frac{-\sin(0-h)}{0-h}\right)$$

$$= \lim_{h\to 0} \cos\left(\frac{\sin h}{-h}\right)$$

As $\lim\limits_{h\to 0} \dfrac{\sin h}{-h} = -1$

$= \lim\limits_{h\to 0} \cos(-1) = \cos 1.$

R.H.L. = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} \cos\left(\frac{\sin(0+h)}{h}\right)$$

$$= \lim_{h\to 0} \cos\left(\frac{\sin h}{h}\right)$$

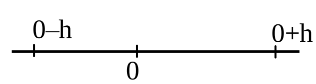
As $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$

= cos 1

Sincec, R.H.L. = L.H.L. = cos1.

Hence, it has removable discontinuity

For C : $f(x) = x \sin\pi/x$. It is not defined at x = 0.



L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} (0-h)\sin\left(\frac{\pi}{0-h}\right)$

$= \lim_{h\to 0} (-h)\times\sin\left(\frac{\pi}{-h}\right)$

$= \lim_{h\to 0} h \times \sin\pi/h$

$= \lim_{h\to 0} \frac{\sin\pi/h}{\pi/h}\times\pi$

$= 1\times\pi \quad = \pi$

R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} (0+h)\sin\left(\frac{\pi}{0+h}\right)$

$= \lim_{h\to 0} h\sin\frac{\pi}{h}$

$= \lim_{h\to 0} \frac{\sin\pi/h}{\pi/h}\times\pi$

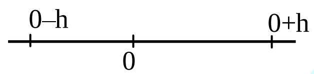
$= 1\times\pi = \pi$

Since, L.H.L. = R.H.L. = π

Hence, it has removable discontinuity.

For D : f(x) = 1/λn|x| = $\begin{cases} 1/\lambda n x; x>0 \\ 1/\lambda n(-x); x<0 \end{cases}$

It is not defined at x = 0.



L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} 1/\lambda n(-(0-h))$

$= \lim_{h\to 0} 1/\lambda n\, h$

$= 1/(-\infty)$

$= 0$

R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} 1/\lambda n(0+h)$

$= \lim_{h\to 0} 1/\lambda nh$

$= 1/(-\infty) = 0$

Since, R.H.L. = L.H.L. = 0

Hence, it has removable discontinuity.

options (B), (C), and (D) are correct answers.

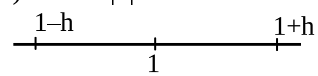
Q.19 Which of the following function(s) has/have removable discontinuity at x = 1

(A) $f(x) = \frac{1}{\lambda n|x|}$ (B) $f(x) = \frac{x^2-1}{x^3-1}$

(C) $f(x) = 2^{-2^{\frac{1}{1-x}}}$ (D) $f(x) = \frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

Sol. **[B,D]**

For A : f(x) = 1/λn |x|



L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} 1/\lambda n(1-h)$ = – ve quantity

(as λn (1 – h) = –ve)

R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$

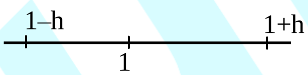
$= \lim_{h\to 0} 1/\lambda n(1+h)$ = + ve quantity

Since, R.H.L. ≠ L.H.L.

Hence, f(x) is discotinuous and have non-removable discontinuity

∴ Option (A) is not correct answer.

For B : $f(x) = \frac{x^2-1}{x^3-1}$. It is not defined at x = 1.



L.H.L. = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} \frac{(1-h)^2-1}{(1-h)^3-1}\left(\frac{0}{0}\text{form}\right)$

Apply L–H Rule, weget.

$= \lim_{h\to 0} \frac{2(1-h)(-1)-0}{3(1-h)^2(-1)-0} = 2/3$

R.H.L.= $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$

$= \lim_{h\to 0} \frac{(1+h)^2-1}{(1+h)^3-1}\left(\frac{0}{0}\text{form}\right)$

Apply L – H Rule, weget

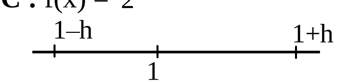
$= \lim_{h\to 0} \frac{2(1+h)-0}{3(1+h)^2-0} = 2/3$

Since, L.H.L. = R.H.L. = 2/3

Hence, it has removable discontinuity.

∴ Option (B) is correct answer.

For C : $f(x) = 2^{-2^{\frac{1}{1-x}}}$



L.H.L. = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} 2^{-2^{\frac{1}{1-(1-h)}}} = \lim_{h\to 0} 2^{-2^{\frac{1}{h}}}$

$= 2^{-2^{\infty}} = 2^{-\infty} = o$

R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$

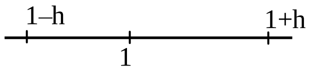
$= \lim_{h\to 0} 2^{-2^{\frac{1}{1-(1+h)}}} = \lim_{h\to 0} 2^{-2^{-1/h}} = 2^{-2^{-\infty}} = 2^{-0} = 1$

Since, R.H.L. ≠ L.H.L.

Hence, It has non removable discontinuity.

For D : $f(x) = \dfrac{\sqrt{x+1}-\sqrt{2x}}{x^2 - x}$.

It is not defined at x = 1.



L.H.L. $= \lim\limits_{x\to1^-} f(x) = \lim\limits_{h\to0} f(1-h)$

$$= \lim_{h\to0}\frac{\sqrt{1-h+1}-\sqrt{2(1-h)}}{(1-h)^2-(1-h)}$$

$$= \lim_{h\to0}\frac{\sqrt{2-h}-\sqrt{2(1-h)}}{(1-h)^2-(1-h)} \quad \left(\frac{0}{0}\text{form}\right)$$

Use L–H Rule, weget.

$$= \lim_{h\to0}\frac{\frac{1}{2\sqrt{2-h}}(-1)-\frac{1}{2\sqrt{2(1-h)}}\times2(-1)}{2(1-h)(-1)+1}$$

$$= \lim_{h\to0}\frac{\frac{-1}{2\sqrt{2-h}}+\frac{1}{\sqrt{2(1-h)}}}{2(h-1)+1}$$

$$= \frac{-\frac{1}{2\sqrt2}+\frac{1}{\sqrt2}}{-1} = -1/2\sqrt2$$

R.H.L. $= \lim\limits_{x\to1^+} f(x) = \lim\limits_{h\to0} f(1+h)$

$$= \lim_{h\to0}\frac{\sqrt{1+h+1}-\sqrt{2(1+h)}}{(1+h)^2-(1+h)}$$

$$= \lim_{h\to0}\frac{\sqrt{2+h}-\sqrt{2(1+h)}}{(1+h)^2-(1+h)} \quad \left(\frac{0}{0}\text{form}\right)$$

Use, L–H rule, weget

$$= \lim_{h\to0}\frac{\frac{1}{2\sqrt{2+h}}-\frac{1}{2\sqrt{2(1+h)}}\times2.1}{2(1+h)-1}$$

$$= \frac{\frac{1}{2\sqrt2}-\frac{1}{\sqrt2}}{1} = -1/2\sqrt2$$

Since, L.H.L. = R.H.L. = $-1/2\sqrt2$.

∴ It has removable discontinuity.

Q.20 Which of the following function(s) defined below has/ have single point continuity.

(A) $f(x) = \begin{cases}1 & \text{if } x\in Q\\ 0 & \text{if } x\notin Q\end{cases}$

(B) $g(x) = \begin{cases}x & \text{if } x\in Q\\ 1-x & \text{if } x\notin Q\end{cases}$

(C) $h(x) = \begin{cases}x & \text{if } x\in Q\\ 0 & \text{if } x\notin Q\end{cases}$

(D) $k(x) = \begin{cases}x & \text{if } x\in Q\\ -x & \text{if } x\notin Q\end{cases}$

Sol. **[B,C, D]**

Q.21 If $f(x) = \sqrt{x}$ and g(x) = x – 1, then -

(A) fog is continuous on [0, ∞)

(B) gof is continuous on [0,∞)

(C) fog is continuous on [1, ∞)

(D) None of these

Sol. **[B,C]**

$f(x) = \sqrt{x}$ and g(x) = x – 1

(fog) (x)= f(g(x)) = $\sqrt{(x-1)}$

(fog) (x) is defined when (x–1) ≥ 0

$\Rightarrow x \ge 1$

∴ (fog) (x) is continuous in [1,∞)

(gof) (x) = g (f(x))

$= \left(\sqrt{x}-1\right)$

(gof) (x) is define when x ≥ 0.

i.e. (gof) (x) is continuous in [0,∞)

∴ Options (B) and (C) are correct answers.

Q.22 If $f(x) = \lim\limits_{n\to\infty} (\sin x)^{2n}$ then f is -

(A) continuous at $x = \dfrac{\pi}{2}$

(B) discontinuous at $x = \dfrac{\pi}{2}$

(C) discontinuous at $x = \dfrac{3\pi}{2}$

(D) discontinuous at infinite number of points

Sol. **[B,C,D]**

$f(x) = \lim\limits_{n\to\infty} (\sin x)^{2n}$

We have to check for every option

At x = π/2, it is $(1)^\infty$ type. Which is indeterminate form.

∴ f(x) is discontinuous at x = π/2

At $x = \dfrac{3\pi}{2}$, It is $(1)^\infty$ type which is also undefined form.

∴ f(x) is discontinuous x = 3π/2.

Hence, there will be infinite number of points where f(x) will be undefined.

∴ Options, (B), (C) and (D) are correct answers.

Part-C Assertion-Reason type Questions

The following questions 23 to 29 consists of two statements each, printed as Assertion and Reason. While answering these

questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Q.23 **Assertion (A) :** $f(x) = \{\tan x\} - [\tan x]$ is continuous at $x = \frac{\pi}{3}$, where [.] and {.} represent greatest integral function and fractional part function part function.

Reason (R) : If $y = f(x)$ & $y = g(x)$ are continuous at $x = a$ then $y = f(x) \pm g(x)$ are continuous at $x = a$

Sol. [A]

Q.24 **Assertion (A) :** $f(x) = |x-2| + \frac{x^2 - 5x + 6}{x - 1} + \tan x$ is continuous function within the domain of f (x).

Reason (R) : All absolute valued polynomial function, rational polynomial function, trigonometric functions are continuous within their domain.

Sol. [A]

Q.25 **Assertion (A) :** The function $f(x) = \frac{x^3}{4} - \sin \pi x + 3$ takes the value 7/3 with in interval [–2, 2].

Reason (R): A continuous function in [a, b] assumes at least once every value between its maximum and minimum value.

Sol. **[D]**

Assertion : $f(x) = \frac{x^3}{4} - \sin\pi x + 3$

$x \in [-2, 2]$

$x = 7/3 = 2.33$

since $2.33 \notin [-2, 2]$

Hence, x = 7/3 will not be taken by f(x). Assertion is false.

Reason : Reason is true because any function in [a, b] is continuous. or simply we can understand this by a string stretched between two ends. If we disturb at one point, then a wave will travel along from disturbing point to other point . As wave travel, it will pass through every point on its path at least once a time. Hence, Reason is correct answer .

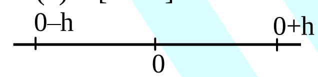
Therefore, option (D) is correct answer.

Q.26 **Assertion (A):** Function $f(x) = [\tan^2 x]$ is discontinuous at x = 0.

Reason (R): [f(x)] is discontinuous at point where f(x) takes integral values between its maximum and minimum value.

Sol. **[D]**

Assertion : $f(x) = [\tan^2 x]$



L.H.L.= $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} [\tan^2(0-h)]$

$= \lim_{h \to 0} [\tan^2 h]$

$= 0$

R.H.L. = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} [\tan^2(0+h)]$

$= \lim_{h \to 0} [\tan^2 h]$

$= 0$

$f(0) = \lim_{x \to 0} f(0) = \lim_{x \to 0} [\tan^2 x] = 0$

Since, L.H.L. = R.H.L. = f(0) = 0

Hence, f(x) is continuous at x = 0

$\therefore$ assertion is incorrect.

Reason : Reason is correct as [f(x)] will be discontinuous if f(x) will become integer.

$\therefore$ option (D) is correct answer.

Q.27 **Assertion (A):** The function defined by $f(u) = \frac{3}{2u^2 + 5u - 3}$ and $u = \frac{1}{x+2}$ is discontinuous at $x = \frac{-7}{3}, -2, 0$.

Reason (R): If f(x) and g(x) are discontinuous at x = a then fog (x) is discontinuous at x = a.

Sol. **[C]**

Assertion : $f(u) = 3/(2u^2 + 5u - 3)$

And $u = 1/(x+2)$; $x \neq -2$

Also $2u^2 + 5u - 3 \neq 0$

$u \neq \frac{-5 \pm \sqrt{25+49}}{2 \times 2}$

$\Rightarrow u \neq \frac{-5 \pm 7}{4}$

$\Rightarrow u \neq -3, 1/2$ and $x \neq -2$

$u = -3 = \frac{1}{x+2} \Rightarrow x + 2 = -\frac{1}{3} \Rightarrow x = -7/3$

$u = \frac{1}{2} = \frac{1}{x+2} \Rightarrow x = 0$

Hence, At x = –7/3, 0, –2, function will be discontinuous. Assertion is true.

Reason : If f(x) and g(x) not defined at x = a then (fog) (x) may or may not be defined at x = a. Reason is false.

∴ option (C) is correct answer.

Q.28 **Assertion (A) :** For f(x) = [x] and

$g(x) = \begin{cases} 0 & x \in I \\ x^2 & x \in R - I \end{cases}$ gof is continuous while

f o g is a discontinuous function.

Reason (R) : gof is a constant function while fog involves greatest integer function discontinuous at I – {0}.

Sol. **[A]**

Assertion : f(x) = [x]

$$g(x) = \begin{cases} 0; x \in I \\ x^2; x \in R - I \end{cases}$$

$$(gof)(x) = g(f(x)) = \begin{cases} 0; x \in I \\ [x]^2; x \in R - I \end{cases}$$

$$(fog)(x) = f(g(x)) = \begin{cases} 0; x \in I \\ [x^2]; x \in R \end{cases}$$

Since, it is given that (gof) (x) is continuous.

Then (fog) (x) will be discontinuous.

Hence, Assertion is correct.

Reason : Reason is correct and explanation of assertion as explained above.

Q.29 **Assertion :** $\lim_{x\to 0} \frac{1-\cos(1-\cos x)}{x^4}$

$$= \lim_{x\to 0} \frac{1-\cos\left\{\frac{1-\cos x}{x^2}.x^2\right\}}{x^4}$$

$$= \lim_{x\to 0} \frac{1-\cos\left(\frac{x^2}{2}\right)}{\left(\frac{x^2}{2}\right)} \times \frac{1}{4} = \frac{1}{8}$$

Reason : If $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(m)$

provided ‘ f ’ is continuous at x = m.

Sol. [D]

Part-D Column Matching type questions

Match the entry in Column I with the entry in Column II.

Q.30 **Column I**

(A) If $f(x) = \begin{cases} \sin\{x\}; x < 1 \\ \cos x + a; x \geq 1 \end{cases}$ where {.} denotes the fractional part function, such that f(x) is continuous at x = 1. if $|k| = \frac{a}{\sqrt{2}\sin\frac{(4-\pi)}{4}}$ then k is.

(B) If the function $f(x) = \frac{(1-\cos(\sin x))}{x^2}$ is continuous at x = 0, then f(0) is

(C) $f(x) = \begin{bmatrix} x, x \in Q \\ 1-x, x \notin Q \end{bmatrix}$, then the values of x at which f(x) is continuous

(D) If f(x) = x + {–x} + [x], where [x] and {x} represents integral and fractional part of x, then the values of x at which f(x) is is continuous

Column II

(P) 1

(Q) 0

(R) –1

(S) $\frac{1}{2}$

Sol. A → P,R; B → S; C → S; D → P, Q, R

Q.31 **Column I**

(A) If f(x) = 1/ (1– x), then the points at which the function fofof(x) is discontinuous

(B) $f(u) = \frac{1}{u^2 + u - 2}$, where $u = \frac{1}{x-1}$. The value of x at which ‘f’ is discontinuous

(C) f(x) = u^2, where $u = \begin{cases} x-1, x \geq 0 \\ x+1, x < 0 \end{cases}$ The number of value of x at which ‘f’ is discontinuous

(D) The number of value of x at which the function $f(x)=\frac{2x^5-8x^2+11}{x^4+4x^3+8x^2+8x+4}$ is Discontinuous

Column II

(P) $\frac{1}{2}$

(Q) 0

(R) 2

(S) 1

Sol. $A \to Q, S; B \to P, R, S;\ C \to Q; D \to Q$

Q. 32 **Column I**

(A) If P(x) = [2 cos x], $x \in [-\pi,\pi]$, then P(x)

(B) If Q(x) = [2 sin x], $x \in [-\pi, \pi]$ then Q(x)

(C) If R(x)= [2 tan x /2], $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then R(x)

(D) If $S(x)=\left[3\operatorname{cosec}\frac{x}{3}\right]$, $x \in \left[\frac{\pi}{2},2\pi\right]$,then S(x)

Column II

(P) is discontinuous at exactly 7 points

(Q) is discontinuous at exactly 4 points

(R) has non- removable discontinuities

(S) is continuous at infinitely many values

Sol. $A \to P, R, S; B \to P, R, S;$
$C \to Q, R, S; D \to R, S$

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 If $f(x) = \frac{x-1}{x+1}$, $f^1(x) = f(x)$, $f^2(x) = fof(x)$, $f^{k+1}(x) = f(f^k(x))$ where k = 1, 2, 3, ... Then find the points where the function g(x) defined as $g(x) = f^{1999}(x)$ is discontinuous.

Sol. $f(x) = \frac{x-1}{x+1}$; Given : $f^1(x) = f(x)$

$f^2(x) = fof(x) = f(f(x)) \Rightarrow f^{k+1}(x) = f(f^k(x))$.
where k = 1, 2, 3, …..

$g(x) = f^{1999}(x) = f(f^{1998}(x)) = f(f(f^{1997}(x)))$.

$f^1(x) = f(x) = \frac{x-1}{x+1}$

$f^2(x) = f(f(x)) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{x-1-x-1}{x-1+x+1} = \frac{-1}{x}$ f

$^3(x) = f(f^2(x)) = f\left(-\frac{1}{x}\right) = \frac{-\frac{1}{x}-1}{-\frac{1}{x}+1}$

$= \frac{-1-x}{-1+x} = -\left(\frac{1+x}{x-1}\right) = \frac{-1}{fx}$.

$f^4(x) = f(f^3(x)) = f\left(\frac{1+x}{1-x}\right) = \frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1} = x.$

$f^4(x) = x$

$f^5(x) = f(f^4(x)) = \frac{x-1}{x+1} = f(x)$

$f^6(x) = f(f^5(x)) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{x-1-x-1}{x-1+x+1} = \frac{-1}{x}$

$f^7(x) = f(f^6(x)) = \frac{-1/x-1}{-1/x+1} = \frac{-1-x}{-1+x}$

$= \frac{-(x+1)}{(x-1)} = \frac{-1}{f(x)}$

$f^8(x) = f(f^7(x)) = \frac{\frac{x+1}{x-1}-1}{\frac{x+1}{x-1}+1} = \frac{x+1-1+x}{x+1+1-x} = x$

$f^9(x) = f(f^8(x)) = \frac{x-1}{x+1} = f(x)$

$f^{10}(x) = f(f^9(x)) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{x-1-x-1}{x-1+x+1} = \frac{-1}{x}$

$f^{11}(x) = f(f^{10}(x)) = \frac{-\frac{1}{x}-1}{-\frac{1}{x}+1} = \frac{-(1+x)}{-1+x}$

$= -\left(\frac{1+x}{1+x}\right) = \frac{-1}{f(x)}$

$f^{12}(x) = f(f^{11}(x)) = \frac{\frac{x+1}{x-1}-1}{\frac{x+1}{x-1}+1} = \frac{x+1-1+x}{1+x+1-x} = x$

Hence, we can make conclusions that
$f^{1999}(x)$ must be $-1/f(x)$
$f^{1998}(x)$ must be $-1/x$ & f^{1997} must be f(x)

$\therefore f^{1999}(x) = f(f^{1998}(x)) = f(f(f^{1997}(x)))$

f^{1997} not defined at x = –1

f^{1998} not defined at x = 0

f^{1999} not defined at x = 1

Therefore, discontinuous points will be –1,0,1.

Q.2 If $f(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots\ldots \frac{x^2}{(1+x^2)^n} + \ldots\ldots$ then check continuity of f(x) at x = 0.

Sol. $f(x)=x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots\ldots \frac{x^2}{(1+x^2)^n} + \ldots\ldots$

$= \lim_{n\to\infty} x^2\left[1+\frac{1}{1+x^2}+\frac{1}{(1+x^2)^2}+\ldots.+\frac{1}{(1+x^2)^n}+\ldots.\right]$

$= \lim_{n\to\infty} x^2$ [It forms G.P. with first term and common ratio $\frac{1}{1+x^2}$]

$= \lim_{n\to\infty} x^2 \frac{\left[1.\left[1-\frac{1}{(1+x^2)^{n+1}}\right]\right]}{1-\frac{1}{1+x^2}}$

$= \lim_{n\to\infty} x^2 \frac{\left[1-\frac{1}{(1+x^2)^{n+1}}\right]}{\frac{x^2}{1+x^2}}$

$= \lim_{n\to\infty} \left[1-\frac{1}{(1+x^2)^{n+1}}\right] \times (1 + x^2)$

$$f(x) = \lim_{n\to\infty}\left[(1+x^2) - \frac{1}{(1+x^2)^n}\right]$$

$$= \lim_{n\to\infty}\left[(1+x^2) - \frac{1}{(1+x^2)^n}\right]$$

$f(x) = 1 + x^2$

0–h   0   0+h

Continuity at x = 0

L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} [1 + (0-h)^2] = 1$

R.H.L.$= \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} [1+(0+h)^2]$

$= 1$

$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} (1+x^2) = 1.$

Since, L.H.L. = R.H.L. = f(0) = 1

Hence, f(x) is continuous at x = 0

Q.3 If $f(x) = \frac{1-\sin x}{(\pi-2x)^4}\cos x.\ (8x^3 - \pi^3);\ x \neq \pi/2.$

Then determine $f(\pi/2)$ if f(x) is continuous at $x = \pi/2$.

Sol. $f(x) = \frac{1-\sin x}{(\pi-2x)^4}.\cos x\ (8x^3 - \pi^3)\ ;\ x \neq \pi/2$

$f(\pi/2) = \lim_{x\to\pi/2} f(x)$

$$= \lim_{x\to\pi/2} \frac{(\cos x - \sin x\cos x)(2x-\pi)(4x^2+\pi^2+2\pi x)}{(2x-\pi)^4}$$

$$= \lim_{x\to\pi/2} \frac{\left(\cos x - \frac{1}{2}\sin 2x\right)(4x^2+\pi^2+2\pi x)}{(2x-\pi)^3}$$

$\left(\frac{0}{0}\text{form}\right)$

Apply L–H Rule, weget

$$= \lim_{x\to\pi/2} \frac{(-\sin x - \cos 2x)(4x^2+\pi^2+2\pi x) + \left(\cos x - \frac{1}{2}\sin 2x\right)(8x+2\pi)}{3(2x-\pi)^2}$$

$\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to\pi/2} \frac{(-\cos x + 2\sin 2x)(4x^2+\pi^2+2\pi x) + (-\sin x - \cos 2x)(8x+2\pi) + (-\sin x - \cos 2x)(8x+2\pi) + \left(\cos x - \frac{1}{2}\sin 2x\right).8}{6(2x-\pi)}$$

$$= \lim_{x\to\pi/2} \frac{(-\cos x + 2\sin 2x)(4x^2+\pi^2+2\pi x) + (-\sin x - \cos 2x)(16x+4\pi) + \left(\cos x - \frac{1}{2}\sin 2x\right).8}{6(2x-\pi)}$$

$\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to\pi/2} \frac{(\sin x + 4\cos 2x)(4x^2+\pi^2+2\pi x) + (-\cos x + 2\sin 2x)(8x+2\pi) + (-\cos x + 2\sin 2x)(16x+4\pi) + (-\sin x - \cos 2x).16 + (-\sin x - \cos 2x).8}{12}$$

$$= \frac{(1-4)\left(4.\frac{\pi^2}{4} + \pi^2 + \pi^2\right) + 0 + 0 + (-1+1)\times 16 + (-1+1)\times 8}{12}$$

$$= \frac{(-3)(3\pi^2) + 0 + 0 + 0 + 0}{12} = -3\pi^2/4$$

Q.4 If function $f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n e^x}$ then check continuity at x = 1.

Sol. $f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n e^x}$

1–h   1   1+h

$f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n e^x} = 0$

when x < 1 as $\lim_{n\to\infty} x^n \to 0$

$= \lim_{n\to\infty} \frac{1}{\frac{1}{x^n} + e^x} = 1/e^x$ when x > 1

$$f(x) = \begin{cases} 0 & ; x < 1 \\ e^{-x} & ; x > 1 \\ \frac{1}{1+e^x} & ; x = 1 \end{cases}$$

L.H.L = $\lim\limits_{x\to1^-} f(x) = \lim\limits_{h\to0} (1-h) = 0$

R.H.L. = $\lim\limits_{x\to1^+} f(x) = \lim\limits_{h\to0} f(1+h) = \lim\limits_{h\to0} e^{-(1+h)} = e^{-1}$

Since, R.H.L $\neq$ L.H.L.

$\therefore$ f(x) is discontinuous at x = 1.

Q.5 Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and

$g(x) = \begin{cases} \dfrac{f(x)}{x-3} & ;\ x \neq 3 \\ \lambda & ;\ x = 3 \end{cases}$ then-

(a) Find all zeros of f

(b) Find the value of λ that makes g continuous at x = 3

(c) Using the value of λ, determine whether g is an even function

Sol. $f(x) = x^3 - 3x^2 - 4x + 12$

$g(x) = \begin{cases} \dfrac{f(x)}{x-3} & ;\ x \neq 3 \\ \lambda & ;\ x = 3 \end{cases}$

(A) $f(x) = x^3 - 3x^2 - 4x + 12 = 0$

Put x = 2

$f(2) = 8 - 12 - 8 + 12 = 0$

$\therefore$ (x – 2) is a factor of above equation

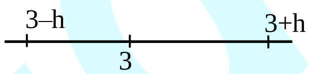
$x^2(x-2) - x(x-2) - 6(x-2) = 0$

$(x-2)(x^2 - x - 6) = 0$

$(x-2)(x+2)(x-3) = 0$

x = –2, 2, 3

(B) $g(x) = \begin{cases} \dfrac{(x-2)(x+2)(x-3)}{(x-3)} & ; x \neq 3 \\ \lambda & ; x = 3 \end{cases}$



L.H.L. = $\lim\limits_{x\to3^-} f(x) = \lim\limits_{h\to0} f(3-h)$

$= \lim\limits_{h\to0} \dfrac{(3-h-2)(3-h+2)(3-h-3)}{(3-h-3)}$

$= \lim\limits_{h\to0} (3-h-2)(3-h+2)$

= 5

Since, g(x) is continuous at x = 3 .

L.H.L. = f(3) $\Rightarrow \lambda = 5$

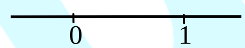
(C) $g(x) = \begin{cases} \dfrac{(x-2)(x+2)(x-3)}{(x-3)} & ; x \neq 3 \\ 5 & ; x = 3 \end{cases}$

$g(-x) = \begin{cases} \dfrac{(-x-2)(-x+2)(-x-3)}{(-x-3)} & ; x \neq 3 \\ = g(x) & \\ 5 & ; x = 3 \end{cases}$

Hence, g(x) is an even function.

Q.6 Draw the graph of the function $f(x) = x - |x - x^2|,\ -1 \le x \le 1$ and discuss the continuity or discontinuity of f in the interval $-1 \le x \le 1$.

Sol. $f(x) = x - |x - x^2|$



When x < 0

$f(x) = x + (x - x^2)$

$= 2x - x^2$

When 0 < x < 1

$f(x) = x - (x - x^2)$

$= x^2$

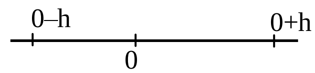
When x > 1

$f(x) = x + (x - x^2)$

$= 2x - x^2$

$f(x) = \begin{cases} 2x - x^2 ; x < 0 \\ x^2 \quad ; 0 \le x < 1 \\ 2x - x^2 ; x \ge 1 \end{cases}$

continuity at x = 0 :



L.H.L. = $\lim\limits_{x\to0^-} f(x) = \lim\limits_{h\to0} f(0-h)$

$= \lim\limits_{h\to0} (2(0-h) - (0-h)^2)$

$= \lim\limits_{h\to0} (-2h - h^2) = 0$

R.H.L. = $\lim\limits_{x\to0^+} f(x) = \lim\limits_{h\to0} f(0+h)$

$= \lim\limits_{h\to0} (0+h)^2$

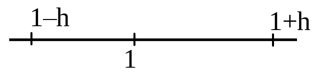
= 0

$f(0) = \lim\limits_{x\to0} f(0) = 0$

Since, L.H.L = R.H.L. = f(0) = 0

$\therefore$ f(x) is continuous at x = 0

Continuity at x = 1



L.H.L. = $\lim_{x\to1^-} f(x) = \lim_{h\to0} f(1-h)$

$= \lim_{h\to0} (1-h)^2$

$= 1$

R.H.L. = $\lim_{x\to1^+} f(x) = \lim_{h\to0} f(1+h)$

$= \lim_{h\to0} [2(1+h) - (1+h)^2]$

$= 2 - 1$

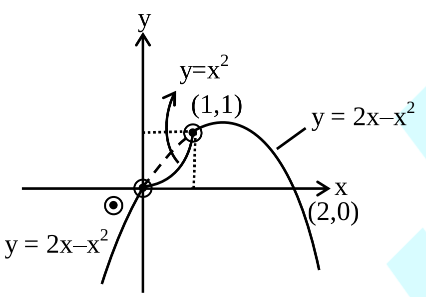
$= 1$

$f(1) = \lim_{x\to1} f(x) = 2 - 1$

$= 1.$

Since, L.H.L. = R.H.L. = f(1) = 1

$\therefore$ f(x) is continuous at x = 1.



Q.7 Discuss the continuity of the function f(x) = |x| + |x – 1| and draw its graph.

Sol. f(x) = |x| + |x – 1|

When x < 0,

since |x| = – x

$x < 0 \Rightarrow x - 1 < -1$

$|x - 1| = -(x - 1)$

$= -x + 1$

Then, $f(x) = -x - (x - 1)$

$= -x - x + 1$

$f(x) = -2x + 1$

When $0 \le x < 1$

$\Rightarrow |x| = x$ and $|x - 1| = -(x - 1) = -x + 1$

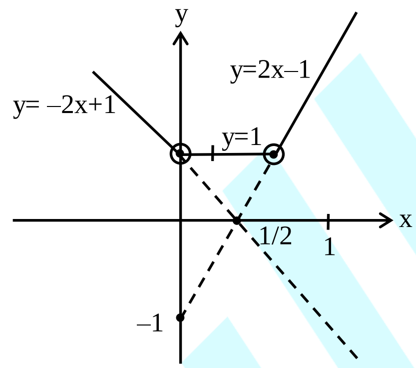
$f(x) = x - x + 1 = 1$

When $x \ge 1$,

$f(x) = x + x - 1$

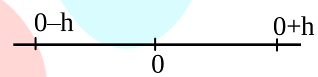
$= 2x - 1$

$$f(x) = \begin{cases} -2x+1 & ; x < 0 \\ 1 & ; 0 \le x < 1 \\ 2x-1 & ; x \ge 1 \end{cases}$$



$$\begin{cases} y = -2x+1 \\ 2x + y = 1 \\ \dfrac{x}{1/2} + y = 1 \end{cases} \quad \begin{cases} y = 2x - 1 \\ 2x - y = 1 \\ \dfrac{x}{1/2} + \dfrac{y}{-1} = 1 \end{cases}$$

Continuity at x = 0



L.H.L. = $\lim_{x\to0^-} f(x) = \lim_{h\to0} f(0-h)$

$= \lim_{h\to0} (-2(0-h)+1)$

$= 1.$

R.H.L. = $\lim_{x\to0^+} f(x) = \lim_{h\to0} f(0+h)$

$= \lim_{h\to0} 1$

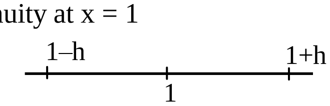
$= 1$

$f(0) = \lim_{x\to0} f(x) = 1$

Since, L.H.L. = R.H.L. = f(0) = 1.

Hence, f(x) is continuous at x = 0

Continuity at x = 1



L.H.L. = $\lim_{x\to1^-} f(x) = \lim_{h\to0} f(1-h)$

$= \lim_{h\to0} (1)$

$= 1.$

R.H.L. = $\lim_{x\to1^+} f(x) = \lim_{h\to0} f(1+h)$

$= \lim_{h\to0} (2(1+h) - 1)$

$= 2 - 1$

$= 1$

$f(1) = \lim_{x\to1} f(x) = 2 - 1$

$= 1$

Since, L.H.L. = R.H.L. = f(1) = 1.

Hence, f(x) is continuous at x = 1.

Q.8 Draw the graph of the function F(x) defined as follows:

$$f(x) = \begin{cases} x - [x]; & 2n \le x < 2n+1 \\ 1/2 \quad ; & 2n+1 \le x < 2n+2 \end{cases}$$

Where n is an integer and [x] denotes the largest integer not exceeding x. What are the points of discontinuity.

Sol. $f(x) = x - [x] \qquad 2n \le x < 2n+1$

$= 1/2 \qquad 2n+1 \le x < 2n+2$

When $0 \le x < 1 \Rightarrow [x] = 0$; y = x

When $1 \le x < 2 \Rightarrow [x] = 1$; $y = x - 1 \Rightarrow x - y = 1$

When $2 \le x < 3 \Rightarrow [x] = 2$; $y = x - 2 \Rightarrow x - y = 2$

$3 \le x < 4 \Rightarrow [x] = 3$; $y = x - 3 \Rightarrow x - y = 3$

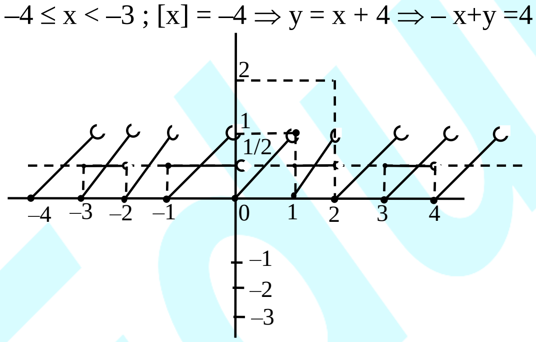
$4 \le x < 5$; $[x] = 4 \Rightarrow y = x - 4 \Rightarrow x - y = 4$

$-1 \le x < 0$; $[x] = -1 \Rightarrow y = x + 1 \Rightarrow -x + y = 1$

$-2 \le x < -1$; $[x] = -2 \Rightarrow y = x + 2 \Rightarrow -x + y = 2$

$-3 \le x < -2$; $[x] = -3 \Rightarrow y = x + 3 \Rightarrow -x + y = 3$

$-4 \le x < -3$; $[x] = -4 \Rightarrow y = x + 4 \Rightarrow -x + y = 4$



y = 1/2 , $2n + 1 \le x < 2n + 2$

i.e $1 \le x < 2$

$3 \le x < 4$

$5 \le x < 6$

$-1 \le x < 0$

$-3 \le x < -2$

$-5 \le x < -4$

Discontinuities will be at Integer points.

Q.9 Given that $f(x) = 1 - x$; $0 \le x \le 1$

$= x + 2$; $1 < x \le 2$

$= 4 - x$; $2 < x \le 4$

Determine g(x) = f [f(x)] and hence find the points of discontinuity of g, if any?

Sol. continuous at $x \in R - \{2, 3\}$

Q.10 If f(x.y) = f(x) f(y) for all x, $y \in R$ and f(x) is continuous at x =1. Prove that f(x) is continuous for all $x \in R$ except x = 0.

Sol. $f(x.y) = f(x) . f(y)$; $x, y \in R$

Given f(x) is continuous at x = 1.

$\therefore \lim_{h\to0} f(1-h) = \lim_{h\to0} f(1+h) = f(1)$

$f(1.1) = f(1). f(1) \Rightarrow f^2(1) = f(1) \Rightarrow f(1) = 1$

$\lim_{h\to0} f(1-h) = \lim_{h\to0} f(1+h) = 1.$

Let a general point, x = a ; $a \in R$

a–h a a+h

L.H.L. $= \lim_{h\to0} f(a-h) = \lim_{h\to0} f(a(1-h/a))$

$= \lim_{h\to0} f(a) \times f(1-h/a)$

$= f(a) \times \lim_{h\to0} f(1-h/a)$

(use given property) f(x.y) = f(x). f(y) = f(a)

If a = 0, $\lim_{h\to0} f(1-h/a) \to$ undefined

R.H.L. $= \lim_{h\to0} f(a+h) = \lim_{h\to0} f(a(1+h/a))$

$= \lim_{h\to0} f(a) \times f(1 + h/a)$

$= f(a) \times \lim_{h\to0} f(1 + h/a)$

$= f(a).$

Put a =0, $\lim_{h\to0} (1+h/a)$ undefined

$f(a) = \lim_{x\to a} f(x)$

Since, L.H.L. = R.H. L. = f(a).

Hence, f(x) is continuous at all $x \in R$ except o.

Q.11 If $g(x) = \lim_{m\to\infty} \dfrac{x^m f(x) + h(x) + 1}{2x^m + 3x + 3}$ is continuous at x = 1 and $g(1) = \lim_{x\to1} \{\log_e(ex)\}^{2/\log_e x}$, then find the value of 2g(1) + 2f(1) – h(1) .Assume that f(x) and h(x) are continuous at x = 1.

Sol. **[1]**

Q.12 Draw the graph of $f(x) = (-1)^{[|x|]}$ where [·] denotes the greatest integer function.

Q.13 Discuss the continuity of $f(x) = \{x + (x - [x])^2\}$

at x = 2 and x = 2.5, where {·} stands for fraction part of x and [·] is greatest integer function.

Q.14 Let f be a function satisfying

$f(x + y) = f(x)\,f(y) - \sqrt{4 - f(y)}$ and $f(x) \to 4^-$ as $x \to 0$. Discuss the continuity of f

Sol. discontinuous if $f(x) \neq 0$,
continuous if $f(x) = 0$ for $x \in R-\{0\}$

Part-B Passage based objective questions

Passage I (Question 15 to 17)

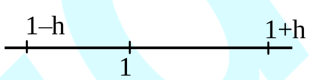
A function f(x) is said to have jump discontinuity at a point x = a if both of the limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists but not equal i.e. $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ and f(a) may be equal to either of the above limits.

The non-negative difference between L.H.L and R.H.L is called jump of the function y = f(x) at x = a. A function having a number of jumps in a given interval is called sectional or piecewise continuous function.

Q.15 If $f(x) = \begin{cases} 5 + x^2 & x < 1 \\ x - 4 & x \geq 1 \end{cases}$, then jump in the function f(x) is -

(A) 3 (B) 6
(C) 9 (D) None of these

Sol.[C] $f(x) = \begin{cases} 5 + x^2 & x < 1 \\ x - 4 & x \geq 1 \end{cases}$



L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1 - h)$

$= \lim_{h\to 0} [5 + (1 - h)^2]$

$= 6$

R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1 + h)$

$= \lim_{h\to 0} (1 + h - 4)$

$= \lim_{h\to 0} (h - 3)$

$= -3.$

Since, R.H.L. ≠ L.H.L

Hence, function discontinuous at x = 1.

Jump = Non-negative difference between L.H.L and R.H.L.

= 6 + 3 = 9

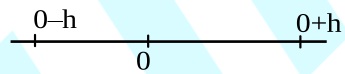
∴ Option (C) is correct Answer.

Q.16 The jump of the function at the point of discontinuity of the function $f(x) = \dfrac{1 - a^{1/x}}{1 + a^{1/x}}$ (a > 0) is -

(A) 4 (B) 2
(C) 3 (D) None of these

Sol.[B] $f(x) = \dfrac{1 - a^{1/x}}{1 + a^{1/x}}$; (a > 0)

Since, f(x) is not defined at x = 0.



L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h)$

$= \lim_{h\to 0} \dfrac{1 - a^{-1/h}}{1 + a^{-1/h}}$

$= 1$ as $\lim_{h\to 0} a^{-1/h} = a^{-\infty}$

$= 0\ (a > 0)$

R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h)$

$= \lim_{h\to 0} \dfrac{1 - a^{1/0+h}}{1 + a^{1/0+h}}$

$= \lim_{h\to 0} \dfrac{1 - a^{1/h}}{1 + a^{1/h}}$

$= \lim_{h\to 0} \dfrac{a^{-1/h} - 1}{a^{-1/h} + 1}$

$= -1.$

Since, R.H.L. ≠ L.H.L

∴ f(x) is discontinuous at x = 0

Jump = 1 + 1

= 2

∴ Option (B) is correct answer.

Q.17 If $f(x) = \begin{cases} |x + 1|\ ; & x \leq 0 \\ x\ ; & x > 0 \end{cases}$ and

$g(x) = \begin{cases} |x| + 1\ ; & x \leq 1 \\ -|x - 2|\ ; & x > 1 \end{cases}$ then the number of jumps in f(x) + g(x) at point of discontinuities are -

(A) 1 (B) 2
(C) 3 (D) None of these

Sol.[B]

Passage- II (Q. No. 18 to Q. 19)

Let f: R $\to$ R be a function defined as,

$f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ & g(x) = f(x–1)+f(x + 1),

$\forall$ x $\in$ R. Then

Q.18 The value of g(x) is

(A) $g(x) = \begin{cases} 0 & , & x \le -3 \\ 2+x & , & -3 \le x \le -1 \\ -x & , & -1 < x \le 0 \\ x & , & 0 < x \le 1 \\ 2-x & , & 1 < x \le 3 \\ 0 & , & x > 3 \end{cases}$

(B) $g(x) = \begin{cases} 0 & , & x \le -2 \\ 2+x & , & -2 \le x \le -1 \\ -x & , & -1 < x \le 0 \\ x & , & 0 < x \le 1 \\ 2-x & , & 1 < x \le 2 \\ 0 & , & x > 2 \end{cases}$

(C) $g(x) = \begin{cases} 0 & , & x \le 0 \\ 2+x & , & 0 < x < 1 \\ -x & , & 1 \le x \le 2 \\ x & , & 2 < x < 3 \\ 2-x & , & 3 \le x < 4 \\ 0 & , & 4 \le x \end{cases}$

(D) None of these

Sol. **[B]**

Q.19 The function g(x) is continuous for x $\in$

(A) R – {0, 1, 2, 3, 4}

(B) R – {–2, –1, 0, 1, 2}

(C) R

(D) None of these

Sol. **[C]**

EXERCISE # 4

Old IIT-JEE Questions

Q.1 If $f(x) = \frac{x}{2} - 1$ then on the interval $[0, \pi]$, **[IIT-1989]**

(A) tan [f(x)] and 1/f(x) are both continuous

(B) tan [f(x)] & 1/f(x) are both discontinuous

(C) tan [f(x)] and f^{-1} (x) are both continuous

(D) tan [f(x)] is continuous but 1/f(x) is not

($\forall$ [.] is greatest integer function.)

Sol.[B] We have, $f(x) = \frac{1}{2}x - 1$ for $0 \le x \le \pi$

$\therefore \quad [f(x)] = \begin{cases} -1, & 0 \le x < 2 \\ 0, & 2 \le x \le \pi \end{cases}$

$\Rightarrow \quad \tan[f(x)] = \begin{cases} \tan(-1), & 0 \le x < 2 \\ \tan 0, & 2 \le x \le \pi \end{cases}$

$\therefore \quad \lim\limits_{x \to 2^-} \tan[f(x)] = -\tan 1$

and $\lim\limits_{x \to 2^+} \tan[f(x)] = 0$

So, tan f (x) is not continuous at x = 2

Now $f(x) = \frac{1}{2}x - 1 \Rightarrow f(x) = \frac{x-2}{2}$

$\Rightarrow \quad \frac{1}{f(x)} = \frac{2}{x-2}$

clearly, f(x) is not continuous at x = 2,

tan[f (x)] and tan $\left[\frac{1}{f(x)}\right]$ are both

discontinuous at x = 2

Q.2 The function $f(x) = [x] \cos\{(2x - 1)/2\}\pi$, [·] denotes the greatest integer function, is discontinuous at **[IIT 95]**

(A) all x

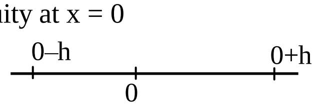
(B) all integer points

(C) no x

(D) x which is not an integer

Sol.[C] $f(x) = [x] \cos\{(2x-1)\pi/2\}$

Continuity at x = 0



L.H.L.$= \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} [0-h]\cos\{(2(0-h)-1)\pi/2\}$

$= \lim\limits_{h\to 0} (-1) \times \cos\{(-2h-1)\pi/2\}$

$= -1 \times 0 = 0.$

R.H.L. $= \lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} [0+h]\cos\{(2(0+h)-1)\pi/2\}$

$= \lim\limits_{h\to 0} [0+h]\cos\{(2h-1)\pi/2\}$

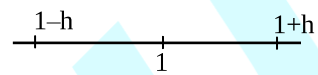
$= 0 \times \cos(-\pi/2) = 0$

$f(0) = \lim\limits_{x\to 0} f(x) = [0]\cos(-\pi/2) = 0$

L.H.L. = R.H.L. = f(0) = 0

$\therefore$ f(x) is continous at x = 0

At x = 1



L.H.L.$= \lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} [1-h]\cos\{(2(1-h)-1)\pi/2\}$

$= (0) \times \lim\limits_{h\to 0} \cos\{(1-2h)\pi/2\}$

$= 0.$

R.H.L. $= \lim\limits_{h\to 0} f(1+h) = \lim\limits_{h\to 0} [1+h]\cos\{(2(1+h)-1)\pi/2\}$

$= \lim\limits_{h\to 0} 1.\cos\{(2h+1)\pi/2\}$

$= 1.0 = 0$

$f(1) = \lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} [1]\cos\{(2-1)\pi/2\}$

$= 1 \times 0$

$= 0$

Since, L.H.L. = R.H.L. = f(1) = 0

$\therefore$ f(x) is continuous at x = 1.

We can prove that for all $x \in R$, f(x) would be continuous Hence, at No point, f(x) will not be discontinuous

$\therefore$ Option (C) is correct answer.

Q.3 Let $f(x) = [x] \sin\left(\frac{\pi}{[x+1]}\right)$, where [.] denotes the greatest integer function. The domain of f is....... and the points of discontinuity of f in the domain are......... **[IIT-1996]**

Sol. The function is not defined for those values of x for which [x + 1] = 0, In other words it means that $0 \le x + 1 < 1$ or $-1 \le x < 0$. ….(1)

Hence the function is defined outside the region given by (1). In other words for

$x \geq 0$ and $x < -1$ or $x \in]-\infty, -1[\cup [0, \infty]$

Now consider integral values of x say x = n

R.H.L. $= \lim [n + h] \sin \dfrac{\pi}{[n+1+h]}$

$= n \sin \dfrac{\pi}{(n+1)}$

L.H.L. $= \lim\limits_{h \to 0} [n - h] \sin \dfrac{\pi}{[n+1-h]}$

$= (n-1) \sin \dfrac{\pi}{n}$

$V = n \sin \dfrac{\pi}{h+1}$

Since $R \neq L = V$. Hence the given function is not continuous for integral values of n ($n \neq 0, -1$)

At x = 0, f(0) = 0, lim f(0 + h)

$= \lim\limits_{h \to 0} [h] \sin \dfrac{\pi}{[h+1]} = 0$

The function is not defined for x < 0. Hence we cannot find lim f(0 – h). Thus f(x) is continuous at x = 0. Hence the point of discontinuity are given by I – {0} where I is set of integers n except n = – 1.

Q.4 Let f(x) be a continuous function defined for $1 \leq x \leq 3$. If f(x) takes rational values for all x and f(2) = 10, then f(1.5) = ………

[IIT-1997]

Sol. Since f (x) is given continuous on the closed bounded interval [1, 3], f (x) is bounded and assumes all the values lying in the interval [m, M] where

m = min f (x) and M = max f(x)

$1 \leq x \leq 3 \qquad f(1) \leq f(x) \leq (3)$

If $m \gtrless M$, then f (x) must assume all the irrational values lying in the [m, M]. But since f (x) takes only rational values, we must have m = M i.e. f (x) must be a constant function. As

f (2) = 10, we get

$f(x) = 10 \ \forall \ x \in [1, 3]$

f [1.5] = 10

Q.5 The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at - **[IIT 99]**

(A) All integers

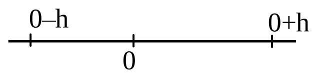
(B) All integers except 0 and 1

(C) All integers except 0

(D) All integers except 1

Sol.[D] $f(x) = [x]^2 - [x^2]$

Continuity at x = 0



L.H.L.$= \lim\limits_{h \to 0} \{[0-h]^2 - [(0-h)^2]\}$

$= \lim\limits_{h \to 0} \{1 - [h^2]\}$

= 1.

R.H.L.$= \lim\limits_{h \to 0} \{[0+h]^2 - [(0+h)^2]\}$

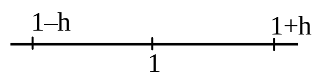
$= \lim\limits_{h \to 0} \{0 - [h^2]\}$

= 0

Since, L.H.L ≠ R.H.L

Hence, f(x) is discontinuous at x = 0.

At x = 1



L.H.L.$= \lim\limits_{h \to 0} \{[1-h]^2 - [(1-h)^2]\}$

$= \lim\limits_{h \to 0} \{0 - [(\text{Value less than} 1)^2]\}$

= 0.

R.H.L.$= \lim\limits_{h \to 0} \{[1+h]^2 - [(1+h)^2]\}$

$= \lim\limits_{h \to 0} \{1 - 1\} = 0$

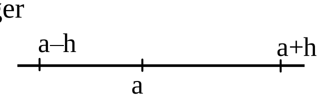
$f(1) = \lim\limits_{h \to 0} \{[1]^2 - [1^2]\}$

= {1 – 1} = 0

∴ L.H.L = R.H.L = f(1) = 0

Hence, f(x) is continuous at x = 1.

Let us take, a general case, when x=a; $a \in$ Integer



L.H.L.$= \lim\limits_{h \to 0} f(a-h) \lim\limits_{h \to 0} \{[a-h]^2 - [(a-h)^2]\}$

$= \lim\limits_{h \to 0} \{(a-1)^2 - (a^2 - 1)\}$

$= a^2 + 1 - 2a - a^2 + 1 = 2(1 - a)$

R.H.L.$= \lim\limits_{h \to 0} f(a+h) = \lim\limits_{h \to 0} \{[a+h]^2 - [(a+h)^2]\}$

$= \lim\limits_{h \to 0} \{a^2 - a^2\} = 0.$

Since, L.H.L. ≠ R.H.L.
Hence f(x) is discontinuous at all Integer points except x = 1.

Q.6 Determine the constants a, b and c for which

the function $f(x) = \begin{cases} (1+ax)^{1/x} & , x<0 \\ b & , x=0 \\ \dfrac{(x+c)^{1/3}-1}{(x+1)^{1/2}-1} & , x>0 \end{cases}$

is continuous at x = 0 . **[REE 99]**

Sol. $f(x) = \begin{cases} (1+ax)^{1/x} & , x<0 \\ b & , x=0 \\ \dfrac{(x+c)^{1/3}-1}{(x+1)^{1/2}-1} & , x>0 \end{cases}$

0–h 0 0+h

L.H.L. $= \lim_{h\to 0} f(0-h) = \lim_{h\to 0} (1+a(0-h))^{1/0-h}$

$= \lim_{h\to 0} (1-ah)^{-1/h}$

It is the type of $(1)^{\infty}$

$= e^{\lim_{h\to 0}(-ah)\times\left(\frac{-1}{h}\right)}$

$= e^{a}$

R.H.L. $= \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \dfrac{(0+h+c)^{1/3}-1}{(0+h+1)^{1/2}-1}$

$= \lim_{h\to 0} \dfrac{(h+c)^{1/3}-1}{(h+1)^{1/2}-1}$

Since, limit exists, then $\lim_{h\to 0} (h+c)^{1/3} = 1$

$\Rightarrow c = 1$

$f(0) = \lim_{x\to 0} f(x) = b$

R.H.L. $= \lim_{h\to 0} \dfrac{(h+1)^{1/3}-1}{(h+1)^{1/2}-1} \left(\frac{0}{0}\text{form}\right)$

Use L–H Rule, weget

$= \lim_{h\to 0} \dfrac{\frac{1}{3}(h+1)^{\frac{1}{3}-1}-0}{\frac{1}{2}(h+1)^{\frac{1}{2}-1}-0}$

$= \dfrac{2}{3} \lim_{h\to 0} \dfrac{(h+1)^{-2/3}}{(h+1)^{-1/2}} = 2/3$

Since, f(x) is continuous at x = 0

∴ L.H.L. = R.H.L. = f(0)

$e^{a} = \dfrac{2}{3} = b$

$\Rightarrow a = \log 2/3 \;;\; b = \dfrac{2}{3}$ and c = 1 .

Q.7 Discuss the continuity of the function

$f(x) = \dfrac{x}{1+|x|}$, |x| ≥ 1, $f(x) = \dfrac{x}{1-|x|}$, |x| < 1.

[REE 2000]

Sol. $f(x) = \dfrac{x}{1+|x|}$; |x| ≥ 1 x ≤ –1 or x ≥ 1

$= \dfrac{x}{1-|x|}$; |x| < 1 ⇒ –1 < x < 1

$= \begin{cases} \dfrac{x}{1-x} ; x \le -1 \\ \dfrac{x}{1+x} ; -1 < x < 0 \\ \dfrac{x}{1-x} ; 0 \le x < 1 \\ \dfrac{x}{1+x} ; x \ge 1 \end{cases}$

Continuity at x = –1

–1–h –1 –1+h

L.H.L. $= \lim_{h\to 0} f(-h-1)$

$= \lim_{h\to 0} \dfrac{1-h}{1-(1-h)} = \lim_{h\to 0} \dfrac{1-h}{h} = -1.$

R.H.L. $= \lim_{h\to 0} \dfrac{1-h}{1-1-h} = \lim_{h\to 0} \dfrac{-1-h}{-h} = 1.$

Since, L.H.L. ≠ R.H.L.

Hence, f(x) is discontinuous at x = –1.

Continuity at x = 0

0–h 0 0+h

L.H.L. $= \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \dfrac{0-h}{1-0-h}$

$= \lim_{h\to 0} \dfrac{-h}{1-h}$

$= 0$

R.H.L. $= \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \dfrac{0+h}{1-(0+h)}$

$= \lim_{h\to 0} \dfrac{h}{1-h}$

$= 0$

$f(0) = \lim_{h\to 0} f(x) = \lim_{h\to 0}\left(\frac{x}{1-x}\right) = 0$

since, L.H.L. = R.H.L. = f(0) = 0

Hence, f(x) is continuous at x = 0

Continuity at x = 1

1–h 1 1+h

L.H.L.$= \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} \frac{1-h}{1+1-h} = \frac{1}{2}$

R.H.L.$= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \frac{1+h}{1-(1+h)}$

$= \lim_{h\to 0} \frac{1+h}{-h} = -1$

Since, L.H.L. ≠ R.H.L.

Hence, f(x) is continuous at x = 0

f(x) is Discontinuous at x = –1, 1.

Q.8 Discuss the continuity of the function

$$f(x) = \begin{cases} \frac{e^{1/(x-1)} - 2}{e^{1/(x-1)} + 2}, & x \neq 1 \\ 1 & , x = 1 \end{cases} \quad \text{at } x = 1.$$

[REE 2001]

Sol.

$$f(x) = \begin{cases} \frac{e^{1/(x-1)} - 2}{e^{1/(x-1)} + 2}, & x \neq 1 \\ 1 & , x = 1 \end{cases}$$

1–h 1 1+h

L.H.L.$= \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} \frac{e^{1/(1-h-1)} - 2}{e^{1/1-h-1} + 2}$

$= \lim_{h\to 0} \frac{e^{-1/h} - 2}{e^{-1/h} + 2}$

$= -1$ as $\lim_{h\to 0} e^{-1/h} = e^{-\infty} = 0$.

R.H.L.$= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \frac{e^{1/(1+h-1)} - 2}{e^{1/1+h-1} + 2}$

$= \lim_{h\to 0} \frac{e^{1/h} - 2}{e^{1/h} + 2}$

$= \lim_{h\to 0} \frac{1 - 2e^{-1/h}}{1 + 2e^{-1/h}}$

$= 1$, as $\lim_{h\to 0} e^{-1/h} = e^{-\infty} = 0$

Since, R.H.L. ≠ L.H.L.

Hence, f(x) is not continuous at x = 1.

Q.9 For every integer n, let a_n and b_n be real numbers. Let function f: IR → IR be given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases},$$

for all integers n. If f is continuous, then which of the following hold(s) for all n? **[IIT-2012]**

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$

(C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

Sol.[B, D] At $x = 2n$

$x \to 2n^+$ $a_n + \sin 2n\pi = a_n$

$x \to 2n^-$ $b_n + \cos 2n\pi = b_n + 1$

For continuous $a_n = b_n + 1$

At $x = 2n + 1$

$x \to 2n + 1^+$ $b_{n+1} + \cos\pi(2n + 1) = b_{n+1} - 1$

$x \to 2n + 1^-$ $a_n + \sin \pi(2n + 1) = a_n$

for continuous $a_n = b_{n+1} - 1$

$a_n - b_{n+1} = -1$

for n = n – 1 $a_{n-1} - b_n = -1$

EXERCISE # 5

Q.1 Let f(x) be a continuous and g(x) be a discontinuous function, prove that f(x) + g(x) is a discontinuous function. **[IIT-1987]**

Sol. Given that f (x) is a continuous functions, and g(x) is a discontinuous functions, then for some arbitrary real number a, we must have

$$\lim_{x\to a} f(x) = f(a) \quad .....(1)$$

and $\lim_{x\to a} g(x) \neq g(a)$ .....(2)

Now, $\lim_{x\to a} [f(x) + g(x)]$

$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \neq f(a) + g(a)$

[Using (1) and (2)]

$\Rightarrow$ f (x) + g (x) is discontinuous.

Alternative :

Let h (x) = f (x) + g (x) be continuous.

Then, g (x) = h (x) – f (x)

Now h (x) and f (x) both are continuous functions.

$\therefore$ h (x) – f (x) must also be continuous. But it is a contradiction as given that g (x) is discontinuous. Therefore our assumption of f (x) + g (x) to be a continuous function is wrong and hence f (x) + g (x) is discontinuous

Q.2 Find the values of a and b so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b, & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x, & \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$. **[IIT-1989]**

Sol. Given that,

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & , \quad 0 \le x < \pi/4 \\ 2x\cot x + b & , \quad \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x & , \quad \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$.

$\therefore$ f (x) must be continuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$

$\therefore \lim_{x\to\pi/4^-} f(x) = f(\pi/4)$

$\Rightarrow \lim_{h\to 0} f\left(\frac{\pi}{4} - h\right) = \frac{2\pi}{4}\cot\frac{\pi}{4} + b$

$\Rightarrow \lim_{h\to 0} \left(\frac{\pi}{4} - h\right) + a\sqrt{2}\sin\left(\frac{\pi}{4} - h\right) = \frac{\pi}{2} + b$

$\Rightarrow \frac{\pi}{4} + a = \frac{\pi}{2} + b \Rightarrow a - b = \frac{\pi}{4}$ ….(1)

Also, $\lim_{x\to\pi/2^+} f(x) = f(\pi/2)$

$\Rightarrow \lim_{h\to 0} f\left(\frac{\pi}{2} + h\right) = 2 \cdot \frac{\pi}{2} \cot \frac{\pi}{2} + b$

$\Rightarrow \lim_{h\to 0} a\cos 2\left(\frac{\pi}{2} + h\right) - b\sin\left(\frac{\pi}{2} + h\right) = b$

$\Rightarrow a\cos\pi - b\sin\pi/2 = b \Rightarrow -a - b = b$

$\Rightarrow \quad a + 2b = 0$ ….(2)

Solving (1) and (2), we get, $a = \frac{\pi}{6}$ & $b = \frac{-\pi}{12}$

Q.3 Let $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4}, & x > 0 \end{cases}$

Determine the value of a, if possible, so that the function is continuous at x = 0. **[IIT-1990]**

Sol. We are given that,

$$f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & , \quad x < 0 \\ a & , \quad x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4} & , \quad x > 0 \end{cases}$$

Here L.H.L. at x = 0 is,

$= \lim_{h\to 0} \frac{1-\cos 4(0-h)}{(0-h)^2} = \lim_{h\to 0} \frac{1-\cos 4h}{h^2}$

$= \lim_{h\to 0} \frac{2\sin^2 2h}{4h^2} \cdot 4 = 8$

R.H.L. at x = 0 is,

$= \lim_{h\to 0} \frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}} - 4}$

$= \lim_{h\to 0} \frac{\sqrt{h}(\sqrt{16+\sqrt{h}} + 4)}{\sqrt{16+\sqrt{h}} - 16}$

$= \lim_{h\to 0} \sqrt{16+\sqrt{h}} + 4 = \sqrt{16} + 4 = 4 + 4 = 8$

For continuity of function f(x), we must have

L.H.L. = R.H.L. = f(0)

$\Rightarrow$ f(0) = 8 $\Rightarrow$ a = 8

Q.4 Which of the following functions are continuous on $(0, \pi)$ **[IIT-1991]**

(A) tan x

(B) $\int_0^x t \sin \frac{1}{t} dt$

(C) $\begin{cases} 1, & 0 < x \le \frac{3\pi}{4} \\ 2\sin \frac{2}{9}x, & \frac{3\pi}{4} < x < \pi \end{cases}$

(D) $\begin{cases} x\sin x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}\sin(\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$

Sol.[B,C] On $(0, \pi)$

(A) tan x = f(x)

We know tan x is discontinuous at $x = \frac{\pi}{2}$

$\therefore$ (a) is not correct

(B) $f(x) = \int_0^x t\sin\left(\frac{1}{t}\right)dt$

$\Rightarrow f'(x) = x\sin\left(\frac{1}{x}\right)$ Which exist on $(0, \pi)$

$\therefore$f (x), being differentiable, is continuous on $(0, \pi)$

$\therefore$ (B) is correct.

(C) $f(x) = \begin{cases} 1 & , \quad 0 < x \le 3\pi/4 \\ 2\sin\frac{2x}{9} & , \quad 3\pi/4 < x < \pi \end{cases}$

Clearly f(x) is continuous on $(0, \pi)$ except possible at $x = 3\pi/4$, where,

LHL = $\lim_{h\to 0} f\left(\frac{3\pi}{4} - h\right) = \lim_{x\to 0} 1 = 1$

RHL = $\lim_{h\to 0} f\left(\frac{3\pi}{4} + h\right)$

$= \lim_{x\to 0} 2\sin\frac{2}{9}\left(\frac{3\pi}{4} + h\right)$

$= \lim_{h\to 0} 2\sin\left(\frac{\pi}{6} + \frac{2h}{9}\right) = 2\sin\frac{\pi}{6} = 2 \cdot \frac{1}{2} = 1$

Also $f\left(\frac{3\pi}{4}\right) = 1$

As LHL = RHL = $f\left(\frac{3\pi}{4}\right)$

$\therefore$ f(x) is continuous on $(0, \pi)$

$\therefore$ (C) is correct.

(D) $f(x) = \begin{cases} x\sin x & , \quad 0 < x \le \pi/2 \\ \frac{\pi}{2}\sin(\pi + x) & , \quad \frac{\pi}{2} < x < \pi \end{cases}$

Here f(x) will be continuous on $(0, \pi)$ if it is continuous at $x = \frac{\pi}{2}$

At $x = \frac{\pi}{2}$

LHL = $\lim_{h\to 0} f\left(\frac{\pi}{2} - h\right)$

$= \lim_{h\to 0} \left(\frac{\pi}{2} - h\right)\sin\left(\frac{\pi}{2} - h\right) = \frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2}$

RHL = $\lim_{h\to 0} f\left(\frac{\pi}{2} + h\right)$

$= \lim_{h\to 0} \frac{\pi}{2} \sin\left(\pi + \frac{\pi}{2} + h\right)$

$= \frac{\pi}{2}\sin\left(\pi + \frac{\pi}{2}\right) = \frac{-\pi}{2}\sin\frac{\pi}{2} = -\frac{\pi}{2}$

As LHL $\ne$ RHL $\therefore$ f(x) is not continuous

Q.5 Let $f(x) = \begin{cases} \{1 + |\sin x|\}^{a/|\sin x|} & ; \quad -\frac{\pi}{6} < x < 0 \\ b & ; \quad x = 0 \\ e^{\tan 2x/\tan 3x} & ; \quad 0 < x < \frac{\pi}{6} \end{cases}$

Determine a and b such that f(x) is continuous at x = 0. **[IIT-1994]**

Sol. Given that ,

$$f(x) = \begin{cases} \{1 + |\sin x|\}^{a/|\sin x|} & ; \quad -\frac{\pi}{6} < x < 0 \\ b & ; \quad x = 0 \\ e^{\tan 2x/\tan 3x} & ; \quad 0 < x < \frac{\pi}{6} \end{cases}$$

is continuous at x = 0

$\therefore \lim_{h\to 0} f(0 - h) = f(0) = \lim_{h\to 0} f(0 + h)$

We have, $\lim_{h\to 0} f(0 - h)$

$= \lim_{h\to 0} [1 + |\sin(-h)|]^{\frac{a}{|\sin(-h)|}}$

$= \lim_{h\to 0} [1 + \sin h]^{\frac{a}{\sin h}} = e^{\lim_{h\to 0}\frac{a}{\sin h}\log(1+\sin h)} = e^a$

and f(0) = b

$\therefore$ $e^a = b$ …..(1)

Also $\lim\limits_{h\to 0} f(0 + h) = \lim\limits_{h\to 0} e^{\tan 2h/\tan 3h}$

$$= e^{\lim\limits_{h\to 0}\frac{\tan 2h}{2h}\times\frac{3h}{\tan 3h}\times\frac{2}{3}}$$

$$= e^{2/3} \qquad \ldots\ldots(2)$$

$\therefore \qquad e^{2/3} = b$

From (1) and (2) $\quad e^{a} = b = e^{2/3}$

$\Rightarrow \qquad a = 2/3 \quad$ and $\quad b = e^{2/3}$

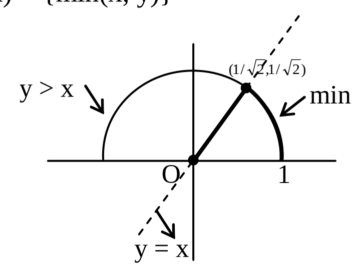
Q.6 Let $f(x) = \{\min(x, y)\}^{\{\max(y, x)\}}$

where $y = \sqrt{1-x^2}$

Discuss the continuity of f(x) in [0, 1].

Sol. $y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1$

$f(x) = \{\min(x, y)\}^{\max(y, x)}$



$$f(x) = \begin{cases} x^{\sqrt{1-x^2}}, & 0 \le x < \frac{1}{\sqrt{2}} \\ \left(\sqrt{1-x^2}\right)^{x}, & \frac{1}{\sqrt{2}} \le x < 1 \end{cases}$$

checking continuity

L.H.L = R.H.L at $x = \frac{1}{\sqrt{2}}$

Passage I (Question 7 to 9)

Let h(x) be a function defined as

$$h(x) = \frac{1}{\sqrt{b-a}} \cdot \frac{\sqrt{\frac{b-a}{a}}\sin 2x}{\sqrt{1+\left(\sqrt{\frac{b-a}{a}}\sin x\right)^2}} \cdot \sqrt{a + b\tan^2 x}$$

consider functions f(x) and g(x) defined as $f(x) = [h(x)]\left\{\frac{h(x)}{2}\right\} \cdot g(x) = \text{sgn}(h(x))$ for $x \in$ domain of h. Where $[\cdot]$ and $\{\cdot\}$ denotes greatest integer and fractional part function respectively and $f(x) = 0 = g(x)$ for $x \notin$ domain of h.

Q.7 Function h(x) is discontinuous at-

(A) $x = \frac{\pi}{2}$ (B) x = 0 (C) $x = \pi$ (D) None

Sol.[A] h (x) is not defined at $x = \pi/2$

$\therefore$ Option (A) is correct answer.

Q.8 Function f(x) is discontinuous at-

(A) $x = \frac{3\pi}{2}$ (B) $x = \pi$

(C) x = 0 (D) All of these

Sol.[D] $f(x) = [h(x)] \cdot \left\{\frac{h(x)}{2}\right\}$

Since at $x = 0, \pi, 3\pi/2$, h(x) = Integer.

But Greatest Integral function does not defined at Integer points.

$\therefore$ option (D) is correct answer.

Q.9 Function g(x) is discontinuous at -

(A) $x = \frac{3\pi}{2}$ (B) $x = \pi/2$

(C) $x = \pi$ (D) All of these

Sol.[D] $g(x) = \text{sgn}(h(x)) = \begin{cases} 1 & ; h(x) > 0 \\ -1 & ; h(x) < 0 \end{cases}$

& for $x \in$ Domain of h.

Since, L.H.L = –1 & R.H.L = + 1

Hence, for every value of domain of h (x), g(x) will be discontinuous

$\therefore$ option (D) is correct answer.

Q.10 If g : [a, b] onto [a, b] is continuous show that there is some $c \in [a, b]$ such that g(c) = c.

Sol. $[a, b] \longrightarrow [a, b]$

$g(x) = f(x) - x$

$g(a) = f(a) - a \ge a$

$g(b) = f(b) - b \le b$

Q.11 Given $f(x) = \sum\limits_{r=1}^{n} \tan\left(\frac{x}{2^r}\right) \sec\left(\frac{x}{2^{r-1}}\right)$; $r, n \in N$

$$g(x) = \lim_{n\to\infty} \frac{\lambda n\left(f(x)+\tan\frac{x}{2^n}\right) - \left(f(x)+\tan\frac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan\frac{x}{2}\right)\right]}{1+\left(f(x)+\tan\frac{x}{2^n}\right)^n}$$

$= k$ for $x = \frac{\pi}{4}$ & the domain of g(x) is $(0, \pi/2)$ where [ ] denotes the greatest integer function. Find the value of k, if possible, so that g(x) is continuous at $x = \pi/4$. Also state the points of discontinuity of g(x) in $(0, \pi/4)$, if any.

Sol. Let $\frac{x}{2^r} = \theta$

$\Rightarrow T_r = \sec 2\theta . \tan\theta = \left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)\tan\theta$

$= \tan\theta\left[\frac{2}{1-\tan^2\theta} - 1\right] \Rightarrow T_r = \tan 2\theta - \tan\theta$

$\therefore\ f(x) = \sum_{r=1}^{n} \tan\frac{x}{2^{r-1}} - \tan\frac{x}{2^r}$

f(x) =

$\left(\tan \underline{x} - \tan\frac{x}{2}\right) + \left(\tan\frac{x}{2} + \tan\frac{x}{2^3}\right) + ... +$

$\left(\tan\frac{x}{2^{n-1}} - \tan\frac{x}{2^n}\right)$

$f(x) = \tan x - \tan\frac{x}{2^n}$

$\lim_{x\to\frac{\pi}{4}^+} g(x) = \lim_{x\to\frac{\pi}{4}^+}\left(\lim_{n\to\infty}\frac{l n \tan x - (\tan x)^n[\sin(\tan(x/2))]}{1+(\tan x)^n}\right)$

$= \lim_{x\to\frac{\pi}{4}^+} 0 - \frac{[\sin(\tan x/2)]}{1+0} = 0$

$\lim_{x\to\frac{\pi}{4}^-} g(x) = \lim_{x\to\frac{\pi}{4}^-}\left(\frac{\text{in} \tan x - 0}{1+0}\right) = 0$

$$f(x) = \begin{cases} l n(\tan x) & ; \quad 0 < x < \pi/4 \\ k = 0 & ; \quad x = \frac{\pi}{4} \\ -[\sin(\tan x/2)] = 0 & ; \quad \pi/2 > x > \frac{\pi}{4} \end{cases}$$

Q.12 Let f be continuous on the interval [0, 1] to R such that f(0) = f(1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.

Sol. $f : [0, 1] \longrightarrow R$

$g(x) = f\left(x + \frac{1}{2}\right) - f(x)$

f(0) = f(1)

(i) g(0) = f(1/2) – f(0) = f(1/2) – f(1) ⊕

(ii) g (1/2) = f(1) – f(1/2) = – f(1/2) – f(1) Θ

⇒ change in sign

∴ one root ∈ [0, 1/2]

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ans.	D	B	B	D	D	B	C	B	A	C	A	C	B	C	C	C	D	C

19. True **20.** True **21.** True **22.** False **23.** $2\beta - 1$ **24.** $(2n-1)\frac{\pi}{2}, n \in I$

EXERCISE # 2

(Part-A)

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13
Ans.	A	C	B	A	A	B	C	A	C	C	C	D	B

(Part-B)

Q.No.	14	15	16	17	18	19	20	21	22
Ans.	A,B	B, D	B,D	A,B,C	B,C,D	B,D	B,C,D	B,C	B,C,D

(Part-C)

Q.No.	23	24	25	26	27	28	29
Ans.	A	A	A	D	C	A	D

(Part-D)

30. A $\rightarrow$ P, R; B $\rightarrow$ S; C $\rightarrow$ S; D $\rightarrow$ P, Q, R

31. A $\rightarrow$ Q, S; B $\rightarrow$ P, R, S; C $\rightarrow$ Q ; D $\rightarrow$ Q

32. A $\rightarrow$ P, R, S; B $\rightarrow$ P, R, S; C $\rightarrow$ Q, R, S; D $\rightarrow$ R, S

EXERCISE # 3

(1) $x = 0, 1, -1$

(2) discontinuous

(3) $-\frac{3\pi^2}{16}$

(4) discontinuous

(5) (a) 3, ± 2 (b) 5 (c) Even function

(6) f is continuous in $-1 \le x \le 1$

(7) continuous for all $x \in R$

(8) discontinuous at integers

(9) continuous at $x \in R - \{2, 3\}$

(11) 1

(14) discontinuous if $f(x) \neq 0$, continuous if $f(x) = 0$ for $x \in R-\{0\}$

(15) C **(16)** B **(17)** B **(18)** B **(19)** C

EXERCISE # 4

(1) B **(2)** C **(3)** $(-\infty, -1) \cup [0, \infty)$, I – {0} where I is the set of integer except x = –1

(4) 10 **(5)** D **(6)** $a = \log\frac{2}{3}$, $b = \frac{2}{3}$, c = 1

(7) discontinuous at |x| = 1 **(8)** f (x) is discontinuous at x = 1

(9) B, D

EXERCISE # 5

(2) $a = \frac{\pi}{6}$, $b = \frac{-\pi}{12}$ **(3)** a = 8 **(4)** B, C **(5)** $a = \frac{2}{3}$, $b = e^{2/3}$ **(6)** Continuous for $x \in [0, 1]$

(7) A **(8)** D **(9)** D **(11)** k = 0; $g(x) = \begin{cases} \lambda n(\tan x) & \text{if} \quad 0 < x < \frac{\pi}{4} \\ 0 & \text{if} \quad \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$.

Hence g(x) is continuous everywhere.