SOLVED EXAMPLES

Ex. 1 The values of x and y satisfying the equation $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$ are

Sol. $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i \quad \Rightarrow \quad (4+2i)x+(9-7i)y-3i-3=10i$

Equating real and imaginary parts, we get $2x-7y=13$ and $4x+9y=3$.

Hence $x=3$ and $y=-1$.

Ex. 2 Find the square root of $7+24i$.

Sol. Let $\sqrt{7+24i}=a+ib$

Squaring $a^2-b^2+2iab=7+24i$

Compare real & imaginary parts $a^2-b^2=7$ & $2ab=24$

By solving these two equations

We get $a=\pm 4$, $b=\pm 3$

$\sqrt{7+24i}=\pm(4+3i)$

Ex. 3 Find the value of expression $x^4-4x^3+3x^2-2x+1$ when $x=1+i$ is a factor of expression.

Sol. $x=1+i$

$\Rightarrow \quad x-1=i$

$\Rightarrow \quad (x-1)^2=-1$

$\Rightarrow \quad x^2-2x+2=0$

Now $\quad x^4-4x^3+3x^2-2x+1$

$=(x^2-2x+2)(x^2-3x-3)-4x+7$

$\therefore$ when $x=1+i$ i.e. $x^2-2x+2=0$

$x^4-4x^3+3x^2-2x+1 = 0-4(1+i)+7=-4+7-4i=3-4i$

Ex. 4 Find modulus and argument for $z=1-\sin\alpha+i\cos\alpha$, $\alpha\in(0,2\pi)$

Sol. $|z|=\sqrt{(1-\sin\alpha)^2+(\cos\alpha)^2}=\sqrt{2-2\sin\alpha}=\sqrt{2}\left|\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right|$

Case I For $\alpha\in\left(0,\frac{\pi}{2}\right)$, z will lie in I quadrant.

$$\text{amp}(z)=\tan^{-1}\frac{\cos\alpha}{1-\sin\alpha}\Rightarrow \text{amp}(z)=\tan^{-1}\frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}=\tan^{-1}\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

$\Rightarrow \quad \arg z=\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$

Since $\frac{\pi}{4}+\frac{\alpha}{2}\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

$\therefore \quad \text{amp}(z)=\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$, $|z|=\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)$

Case II at $\alpha = \frac{\pi}{2}$: $z = 0 + 0i$

$|z| = 0$

amp (z) is not defined.

Case III For $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, z will lie in IV quadrant

So $\quad \text{amp}(z) = -\tan^{-1}\tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

Since $\quad \frac{\alpha}{2} + \frac{\pi}{4} \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore \quad \text{amp}(z) = -\left(\frac{\alpha}{2} + \frac{\pi}{4} - \pi\right) = \frac{3\pi}{4} - \frac{\alpha}{2}$, $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$

Case IV at $\alpha = \frac{3\pi}{2}$: $z = 2 + 0i$

$|z| = 2$

amp (z) = 0

Case V For $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$, z will lie in I quadrant

$\arg(z) = \tan^{-1}\tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

Since $\quad \frac{\alpha}{2} + \frac{\pi}{4} \in \left(\pi, \frac{5\pi}{4}\right)$

$\therefore \quad \arg z = \frac{\alpha}{2} + \frac{\pi}{4} - \pi = \frac{\alpha}{2} - \frac{3\pi}{4}$, $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$

Ex. 5 If $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right)$ then $x_1 x_2 x_3 \ldots.. \infty$ is equal to -

Sol. $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right) = 1 \times e^{i\frac{\pi}{2^n}}$

$x_1 x_2 x_3 \ldots.. \infty$

$$= e^{i\frac{\pi}{2^1}}.e^{i\frac{\pi}{2^2}} --- e^{i\frac{\pi}{2^n}} = e^{i\left(\frac{\pi}{2} + \frac{\pi}{2^2} + ---- + \frac{\pi}{2^n}\right)}$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\ldots..\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\ldots..\right) = -1$$

$$\left(\text{as } \frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\ldots.. = \frac{\pi/2}{1 - 1/2} = \pi\right)$$

Ex. 6 If $\left|\frac{z-i}{z+i}\right| = 1$, then locus of z is -

Sol. We have, $\left|\frac{z-i}{z+i}\right| = 1 \Rightarrow \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = 1$

$\Rightarrow \frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = 1 \Rightarrow x^2+(y-1)^2 = x^2+(y+1)^2 \Rightarrow 4y = 0; y = 0$, which is x-axis

Ex. 7 Solve for z if $z^2 + |z| = 0$

Sol. Let $z = x + iy$

$\Rightarrow \quad (x+iy)^2 + \sqrt{x^2+y^2} = 0$

$\Rightarrow \quad x^2 - y^2 + \sqrt{x^2+y^2} = 0$ and $2xy = 0$

$\Rightarrow \quad x = 0$ or $y = 0$

when $x = 0 \quad -y^2 + |y| = 0$

$\Rightarrow \quad y = 0, 1, -1 \quad \Rightarrow \quad z = 0, i, -i$

when $y = 0 \quad x^2 + |x| = 0$

$\Rightarrow \quad x = 0$

$\Rightarrow \quad z = 0$

$z = 0, z = i, z = -i$

Ex. 8 If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then $\left(\frac{z_1}{z_2}\right)$ is -

Sol. Here let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), |z_1| = r_1$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2), |z_2| = r_2$

$\therefore \quad |(z_1+z_2)|^2 = |(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|^2$

$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2) = |z_1|^2 + |z_2|^2$ if $\cos(\theta_1-\theta_2) = 0$

$\therefore \quad \theta_1 - \theta_2 = \pm\frac{\pi}{2}$

$\Rightarrow \quad \text{amp}(z_1) - \text{amp}(z_2) = \pm\frac{\pi}{2}$

$\Rightarrow \quad \text{amp}\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \frac{z_1}{z_2}$ is purely imaginary

Ex. 9 The locus of the complex number z in argand plane satisfying the inequality

$$\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 \quad \left(\text{where } |z-1| \neq \frac{2}{3}\right) \text{ is -}$$

Sol. We have, $\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 = \log_{1/2}\left(\frac{1}{2}\right)$

$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2}$ $\quad$ [$\because$ $\log_a x$ is a decreasing function if $a < 1$]

$\Rightarrow 2|z-1|+8 < 3|z-1|-2$ as $|z-1| > 2/3$

$\Rightarrow |z-1| > 10$

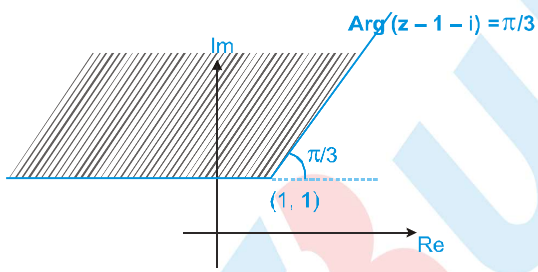
which is exterior of a circle.

Ex. 10 Sketch the region given by

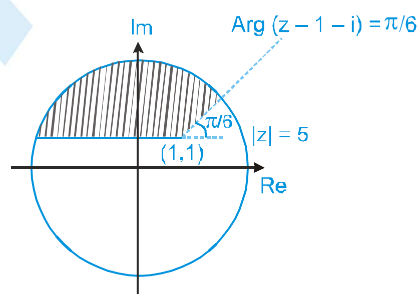
(i) Arg $(z - 1 - i) \geq \pi/3$ $\qquad$ **(ii)** $|z| \leq 5$ & Arg $(z - i - 1) > \pi/6$

Sol.

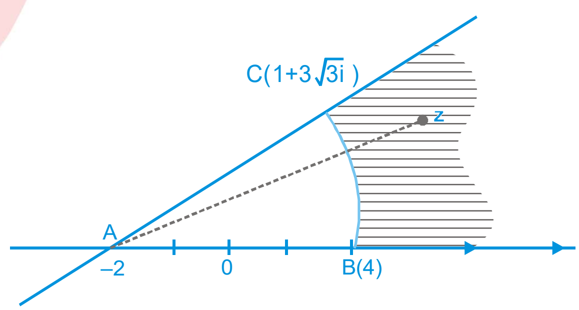
(i)



(ii)



Ex. 11 Shaded region is given by -



(A) $|z+2| \geq 6, 0 \leq \arg(z) \leq \frac{\pi}{6}$

(B) $|z+2| \geq 6, 0 \leq \arg(z) \leq \frac{\pi}{3}$

(C) $|z+2| \leq 6, 0 \leq \arg(z) \leq \frac{\pi}{2}$

(D) None of these

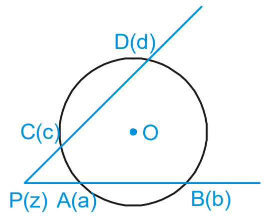
Sol. Note that AB = 6 and $1 + 3\sqrt{3}i = -2 + 3 + 3\sqrt{3}i = -2 + 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2 + 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

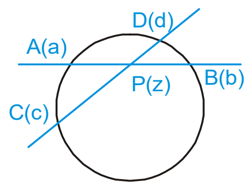
$\therefore \quad \angle BAC = \frac{\pi}{3}$

Thus, shaded region is given by $|z+2| \geq 6$ and $0 \leq \arg(z+2) \leq \frac{\pi}{3}$

Ex. 12 Two different non parallel lines cut the circle $|z| = r$ in point a, b, c, d respectively. Prove that these lines meet in the point z given by $z = \dfrac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$

Sol. Since point P, A, B are collinear





$$\therefore \quad \begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0 \qquad \Rightarrow \qquad z\left(\bar{a} - \bar{b}\right) - \bar{z}\,(a - b) + \left(a\bar{b} - \bar{a}b\right) = 0 \qquad .....(i)$$

Similarly, points P, C, D are collinear, so

$$z\left(\bar{c} - \bar{d}\right) - \bar{z}\,(c - d) + \left(c\bar{d} - \bar{c}d\right) = 0 \qquad .....(ii)$$

On applying **(i)** × (c – d) – **(ii)** (a – b), we get

$$\therefore \quad z\left(\bar{a} - \bar{b}\right)(c - d) - z\left(\bar{c} - \bar{d}\right)(a - b) = \left(c\bar{d} - \bar{c}d\right)(a - b) - \left(a\bar{b} - \bar{a}b\right)(c - d) \qquad .....(iii)$$

$$\because \quad z\bar{z} = r^2 = k \text{ (say)} \therefore \qquad \bar{a} = \frac{k}{a},\ \bar{b} = \frac{k}{b},\ \bar{c} = \frac{k}{c} \text{ etc.}$$

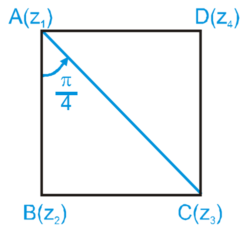
From equation **(iii)** we get

$$z\left(\frac{k}{a} - \frac{k}{b}\right)(c - d) - z\left(\frac{k}{c} - \frac{k}{d}\right)(a - b) = \left(\frac{ck}{d} - \frac{kd}{c}\right)(a - b) - \left(\frac{ak}{b} - \frac{bk}{a}\right)(c - d)$$

$$\therefore \quad z = \frac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$$

Ex. 13 If the vertices of a square ABCD are z_1, z_2, z_3 & z_4 then find z_3 & z_4 in terms of z_1 & z_2.

Sol. Using vector rotation at angle A



$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\frac{\pi}{4}}$$

$$\because \quad |z_3 - z_1| = AC \text{ and } |z_2 - z_1| = AB$$

Also $AC = \sqrt{2}\, AB$

$$\therefore \quad |z_3 - z_1| = \sqrt{2}\,|z_2 - z_1|$$

$$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\Rightarrow \quad z_3 - z_1 = (z_2 - z_1)(1 + i)$$

$$\Rightarrow \quad z_3 = z_1 + (z_2 - z_1)(1 + i)$$

Similarly $z_4 = z_2 + (1 + i)(z_1 - z_2)$

Ex. 14 If A(2 + 3i) and B(3 + 4i) are two vertices of a square ABCD (take in anticlock wise order) then find C and D.

Sol. Let affix of C and D are z_3 and z_4 respectively.

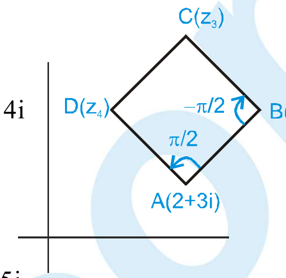
Considering $\angle DAB = 90^\circ$ and AD = AB

we get $\dfrac{z_4-(2+3i)}{(3+4i)-(2+3i)} = \dfrac{AD}{AB} e^{\frac{i\pi}{2}}$

$\Rightarrow$ $z_4 - (2 + 3i) = (1 + i)\, i$ $\Rightarrow$ $z_4 = 2 + 3i + i - 1 = 1 + 4i$

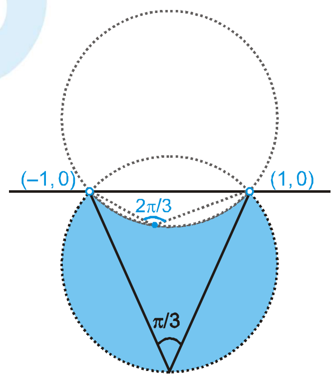
and $\dfrac{z_3-(3+4i)}{(2+3i)-(3+4i)} = \dfrac{CB}{AB} e^{-\frac{i\pi}{2}}$

$\Rightarrow$ $z_3 = 3 + 4i - (1 + i)(-i)$ $\Rightarrow$ $z_3 = 3 + 4i + i - 1 = 2 + 5i$



Ex. 15 Plot the region represented by $\dfrac{\pi}{3} \le \arg\left(\dfrac{z+1}{z-1}\right) \le \dfrac{2\pi}{3}$ in the Argand plane.

Sol. Let us take $\arg\left(\dfrac{z+1}{z-1}\right) = \dfrac{2\pi}{3}$, clearly z lies on the minor arc of the circle passing through (1, 0) and (–1, 0). Similarly, $\arg\left(\dfrac{z+1}{z-1}\right) = \dfrac{\pi}{3}$ means that 'z' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now if we take any point in the region included between two arcs say $P_1(z_1)$ we get $\dfrac{\pi}{3} \le \arg\left(\dfrac{z+1}{z-1}\right) \le \dfrac{2\pi}{3}$



Thus $\dfrac{\pi}{3} \le \arg\left(\dfrac{z+1}{z-1}\right) \le \dfrac{2\pi}{3}$ represents the shaded region (excluding points (1, 0) and (–1, 0)).

Ex. 16 If z_1, z_2 & z_3 are the affixes of three points A, B & C respectively and satisfy the condition $|z_1 - z_2| = |z_1| + |z_2|$ and $|(2 - i)\, z_1 + iz_3| = |z_1| + |(1 - i)\, z_1 + iz_3|$ then prove that Δ ABC in a right angled.

Sol. $|z_1 - z_2| = |z_1| + |z_2|$

$\Rightarrow$ z_1, z_2 and origin will be collinear and z_1, z_2 will be opposite side of origin

Similarly $|(2 - i)\, z_1 + iz_3| = |z_1| + |(1 - i)\, z_1 + iz_3|$

$\Rightarrow$ z_1 and $(1 - i)\, z_1 + iz_3 = z_4$ say, are collinear with origin and lies on same side of origin.

Let $z_4 = \lambda z_1$, λ real

then $(1 - i)\, z_1 + iz_3 = \lambda z_1$

$\Rightarrow$ $i(z_3 - z_1) = (\lambda - 1)\, z_1$ $\Rightarrow$ $\dfrac{(z_3 - z_1)}{-z_1} = (\lambda - 1)\, i$

$\Rightarrow$ $\dfrac{z_3 - z_1}{0 - z_1} = me^{i\pi/2}$, $m = \lambda - 1$ $\Rightarrow$ $z_3 - z_1$ is perpendicular to the vector $0 - z_1$.

i.e. also z_2 is on line joining origin and z_1

so we can say the triangle formed by z_1, z_2 and z_3 is right angled.

Ex. 17 If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is imaginary cube root of unity), then find the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$.

Sol. We have $x^3 - 3x^2 + 3x + 7 = 0$

$\therefore \quad (x-1)^3 + 8 = 0$

$\therefore \quad (x-1)^3 = (-2)^3$

$$\Rightarrow \left(\frac{x-1}{-2}\right)^3 = 1 \Rightarrow \frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2 \qquad \text{(cube roots of unity)}$$

$\therefore \quad x = -1, 1-2\omega, 1-2\omega^2$

Here $\quad \alpha = -1, \beta = 1-2\omega, \gamma = 1-2\omega^2$

$\therefore \quad \alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$

Then $$\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right) + \left(\frac{-2\omega}{-2\omega^2}\right) + \left(\frac{-2\omega^2}{-2}\right) = \frac{1}{\omega} + \frac{1}{\omega} + \omega^2 = \omega^2 + \omega^2 + \omega^2$$

Therefore $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = 3\omega^2$.

Ex. 18 If z is a point on the Argand plane such that $|z-1| = 1$, then $\frac{z-2}{z}$ is equal to -

Sol. Since $|z-1| = 1$,

$\therefore$ let $z - 1 = \cos\theta + i\sin\theta$

Then, $z - 2 = \cos\theta + i\sin\theta - 1$

$$= -2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \qquad \text{..... (i)}$$

and $z = 1 + \cos\theta + i\sin\theta$

$$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \qquad \text{..... (ii)}$$

From **(i)** and **(ii)**, we get $\frac{z-2}{z} = i\tan\frac{\theta}{2} = i\tan(\arg z) \left(\because \arg z = \frac{\theta}{2} \text{ from (ii)}\right)$

Ex. 19 Let a be a complex number such that $|a| < 1$ and $z_1, z_2, ... , z_n$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + ... a^k$, then show that vertices of the polygon lie within the circle $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$.

Sol. We have, $z_k = 1 + a + a^2 + ... + a^k = \frac{1-a^{k+1}}{1-a}$

$$\Rightarrow z_k - \frac{1}{1-a} = \frac{-a^{k+1}}{1-a} \Rightarrow \left|z_k - \frac{1}{1-a}\right| = \frac{|a|^{k+1}}{|1-a|} < \frac{1}{|1-a|} \qquad (\because |a| < 1)$$

$\therefore$ Vertices of the polygon $z_1, z_2, ..., z_n$ lie within the circle $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$

Ex. 20 If $z_1 = a + ib$ and $z_2 = c + id$ are complex number such that $|z_1| = |z_2| = 1$ and Re $(z_1\bar{z}_2) = 0$, then show that the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies the following

(i) $|w_1| = 1$ **(ii)** $|w_2| = 1$ **(iii)** Re $(w_1\bar{w}_2) = 0$

Sol. $a = \cos\theta$, $b = \sin\theta$

$c = \cos\phi$, $d = \sin\phi$

Re $(z_1\bar{z}_2) = 0$ $\Rightarrow$ $\theta - \phi = \frac{n\pi}{2}$ $n = \pm 1$ $\Rightarrow$ $c = \sin\theta$, $d = -\cos\theta$

$\Rightarrow$ $w_1 = \cos\theta + i\sin\theta$

$w_2 = \sin\theta - i\cos\theta$

$\Rightarrow$ $|w_1| = 1, |w_2| = 1$

$w_1\bar{w}_2 = \cos\theta\sin\theta - \sin\theta\cos\theta + i(\sin^2\theta - \cos^2\theta) = -i\cos 2\theta$

$\Rightarrow$ Re $(w_1\bar{w}_2) = 0$

Ex. 21 If $\theta \in [\pi/6, \pi/3]$, $i = 1, 2, 3, 4, 5$ and $z^4\cos\theta_1 + z^3\cos\theta_2 + z^2\cos\theta_3 + z\cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$, then show that $|z| > \frac{3}{4}$

Sol. Given that $\cos\theta_1.z^4 + \cos\theta_2.z^3 + \cos\theta_3.z^2 + \cos\theta_4.z + \cos\theta_5 = 2\sqrt{3}$

or $\left|\cos\theta_1.z^4 + \cos\theta_2.z^3 + \cos\theta_3.z^2 + \cos\theta_4.z + \cos\theta_5\right| = 2\sqrt{3}$

$2\sqrt{3} \le \left|\cos\theta_1.z^4\right| + \left|\cos\theta_2.z^3\right| + \left|\cos\theta_3.z^2\right| + \left|\cos\theta_4.z\right| + \left|\cos\theta_5\right|$

✈ $\theta_i \in [\pi/6, \pi/3]$

$\therefore$ $\frac{1}{2} \le \cos\theta_i \le \frac{\sqrt{3}}{2}$

$$2\sqrt{3} \le \frac{\sqrt{3}}{2}|z|^4 + \frac{\sqrt{3}}{2}|z|^3 + \frac{\sqrt{3}}{2}|z|^2 + \frac{\sqrt{3}}{2}|z| + \frac{\sqrt{3}}{2}$$

$\Rightarrow$ $3 \le |z|^4 + |z|^3 + |z|^2 + |z|$ $\Rightarrow$ $3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + ...\infty$

$\Rightarrow$ $3 < \frac{|z|}{1-|z|}$ $\Rightarrow$ $3 - 3|z| < |z|$

$\Rightarrow$ $4|z| > 3$ $\therefore$ $|z| > \frac{3}{4}$

Ex. 22 If z_1 and z_2 are two complex numbers and $C > 0$, then prove that $|z_1 + z_2|^2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$

Sol. We have to prove that : $|z_1 + z_2|^2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$

i.e. $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$

or $z_1\bar{z}_2 + \bar{z}_1 z_2 \le C|z_1|^2 + C^{-1}|z_2|^2$

or $C|z_1|^2 + \frac{1}{C}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \ge 0$ (using Re $(z_1\bar{z}_2) \le |z_1\bar{z}_2|$)

or $\left(\sqrt{C}|z_1| - \frac{1}{\sqrt{C}}|z_2|\right)^2 \ge 0$ which is always true.

Ex. 23 Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then show that $\frac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

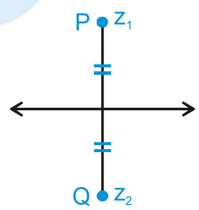
Sol. $z_1 = r(\cos\theta + i\sin\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$z_2 = r(\cos\phi + i\sin\phi), \quad -\pi < \phi < 0$

$$\Rightarrow \quad \frac{z_1 + z_2}{z_1 - z_2} = -i\cot\left(\frac{\theta - \varphi}{2}\right), \qquad -\frac{\pi}{4} < \frac{\theta - \varphi}{2} < \frac{3\pi}{4}$$

Hence purely imaginary

Ex. 24 Two given points P & Q are the reflection points w.r.t. a given straight line if the given line is the right bisector of the segment PQ. Prove that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha} z + \alpha \bar{z} + r = 0$ if and only if; $\bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.



Sol. Let $P(z_1)$ is the reflection point of $Q(z_2)$ then the perpendicular bisector of z_1 & z_2 must be the line

$$\bar{\alpha} z + \alpha \bar{z} + r = 0 \qquad \text{......(i)}$$

Now perpendicular bisector of z_1 & z_2 is, $|z - z_1| = |z - z_2|$

or $\quad (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$$-z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 = -z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2 \qquad (z\bar{z} \text{ cancels on either side})$$

or $\quad (\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} + z_1\bar{z}_1 - z_2\bar{z}_2 = 0 \qquad$ ......(ii)

Comparing (i) & (ii) $\frac{\bar{\alpha}}{\bar{z}_2 - \bar{z}_1} = \frac{\alpha}{z_2 - z_1} = \frac{r}{z_1\bar{z}_1 - z_2\bar{z}_2} = \lambda$

$\therefore \quad \bar{\alpha} = \lambda(\bar{z}_2 - \bar{z}_1) \qquad$ ......(iii)

$\alpha = \lambda(z_2 - z_1) \qquad$ ......(iv)

$r = \lambda(z_1\bar{z}_1 - z_2\bar{z}_2) \qquad$ ......(v)

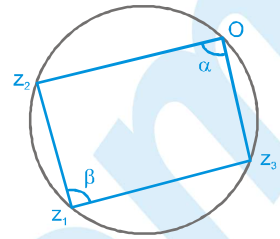
Multiplying (iii) by z_1; (iv) by $\bar{z}_2$ and adding

$$\bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$$

Note that we could also multiply (iii) by z_2 & (iv) by $\bar{z}_1$ & add to get the same result.

Ex. 25 If z_1, z_2, z_3 are complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

Sol. We have, $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$



$\Rightarrow \quad \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1} \quad \Rightarrow \quad \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3}$

$\Rightarrow \quad \frac{z_2 - z_1}{z_3 - z_1} = \frac{-z_2}{z_3} \quad \Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(\frac{-z_2}{z_3}\right)$

$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right)$

$\Rightarrow \quad$ or $\beta = \pi - \arg\frac{z_3}{z_2} = \pi - \alpha = \alpha + \beta = \pi$

Thus the sum of a pair of opposite angle of a quadrilateral is 180°. Hence, the points 0, z_1, z_2 and z_3 are the vertices of a cyclic quadrilateral i.e. lie on a circle.

Exercise # 1

[Single Correct Choice Type Questions]

1. The argument of the complex number $\sin\frac{6\pi}{5} + i\left(1+\cos\frac{6\pi}{5}\right)$ is

(A) $\frac{6\pi}{5}$ **(B)** $\frac{5\pi}{6}$ **(C)** $\frac{9\pi}{10}$ **(D)** $\frac{2\pi}{5}$

2. The principal value of the arg(z) and $|z|$ of the complex number $z = 1 + \cos\left(\frac{11\pi}{9}\right) + i\sin\left(\frac{11\pi}{9}\right)$ are respectively

(A) $\frac{11\pi}{18}, 2\cos\frac{\pi}{18}$ **(B)** $-\frac{7\pi}{18}, 2\cos\frac{7\pi}{18}$ **(C)** $\frac{2\pi}{9}, 2\cos\frac{7\pi}{18}$ **(D)** $-\frac{\pi}{9}, -2\cos\frac{\pi}{18}$

3. The inequality $|z-4| < |z-2|$ represents :

(A) $Re(z) > 0$ **(B)** $Re(z) < 0$ **(C)** $Re(z) > 2$ **(D)** $Re(z) > 3$

4. The sequence $S = i + 2i^2 + 3i^3 + \ldots\ldots$ upto 100 terms simplifies to where $i = \sqrt{-1}$ -

(A) $50(1-i)$ **(B)** $25i$ **(C)** $25(1+i)$ **(D)** $100(1-i)$

5. The region of Argand diagram defined by $|z-1| + |z+1| \le 4$ is :

(A) interior of an ellipse **(B)** exterior of a circle
(C) interior and boundary of an ellipse **(D)** none of these

6. The system of equations $\left.\begin{matrix} |z+1-i| & = 2 \\ \text{Re } z & \ge 1 \end{matrix}\right\}$, where z is a complex number has :

(A) no solution **(B)** exactly one solution
(C) two distinct solutions **(D)** infinite solution

7. If z_1, z_2, z_3 are 3 distinct complex numbers such that $\frac{3}{|z_2-z_3|} = \frac{4}{|z_3-z_1|} = \frac{5}{|z_1-z_2|}$,

then the value of $\frac{9}{z_2-z_3} + \frac{16}{z_3-z_1} + \frac{25}{z_1-z_2}$ equals

(A) 0 **(B)** 3 **(C)** 4 **(D)** 5

8. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other, for

(A) $x = n\pi$ **(B)** $x = 0$ **(C)** $x = \frac{n\pi}{2}$ **(D)** no value of x

9. Real part of $e^{e^{i\theta}}$ is -

(A) $e^{\cos\theta}[\cos(\sin\theta)]$ **(B)** $e^{\cos\theta}[\cos(\cos\theta)]$ **(C)** $e^{\sin\theta}[\sin(\cos\theta)]$ **(D)** $e^{\sin\theta}[\sin(\sin\theta)]$

10. If $z\ (\ne -1)$ is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is equal to

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5

11. Let A, B, C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then the orthocentre is represented by the complex number :

(A) $z_1 + z_2 - z_3$ **(B)** $z_2 + z_3 - z_1$ **(C)** $z_3 + z_1 - z_2$ **(D)** $z_1 + z_2 + z_3$

12. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = \alpha + i\beta$ then $2 . 5 . 10 \ldots (1 + n^2) =$

(A) $\alpha - i\beta$ **(B)** $\alpha^2 - \beta^2$ **(C)** $\alpha^2 + \beta^2$ **(D)** none of these

13. $\sin^{-1}\left\{\frac{1}{i}(z-1)\right\}$, where z is nonreal, can be the angle of a triangle if

(A) $Re(z) = 1, Im(z) = 2$ **(B)** $Re(z) = 1, 0 < Im(z) \le 1$
(C) $Re(z) + Im(z) = 0$ **(D)** none of these

14. If $z = \frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, then $\left(\frac{|z|}{amp(z)}\right)$ equals

(A) 1 **(B)** π **(C)** 3π **(D)** 4

15. If $1, \alpha_1, \alpha_2 \ldots\ldots, \alpha_{2008}$ are $(2009)^{th}$ roots of unity, then the value of $\sum_{r=1}^{2008} r(\alpha_r + \alpha_{2009-r})$ equals

(A) 2009 **(B)** 2008 **(C)** 0 **(D)** –2009

16. If $x^2 + x + 1 = 0$, then the numerical value of

$$\left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^3+\frac{1}{x^3}\right)^2 + \left(x^4+\frac{1}{x^4}\right)^2 + \ldots\ldots + \left(x^{27}+\frac{1}{x^{27}}\right)^2$$ is equal to

(A) 54 **(B)** 36 **(C)** 27 **(D)** 18

17. Let $i = \sqrt{-1}$. Define a sequence of complex number by $z_1 = 0$, $z_{n+1} = z_n^{\ 2} + i$ for $n \ge 1$. In the complex plane, how far from the origin is z_{111}?

(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** $\sqrt{100}$

18. Number of values of x (real or complex) simultaneously satisfying the system of equations
$1 + z + z^2 + z^3 + \ldots\ldots + z^{17} = 0$ and $1 + z + z^2 + z^3 + \ldots\ldots + z^{13} = 0$ is -

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

19. Let z_1 and z_2 be two non real complex cube roots of unity and $|z - z_1|^2 + |z - z_2|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter then the value of λ is

(A) 4 **(B)** 3 **(C)** 2 **(D)** $\sqrt{2}$

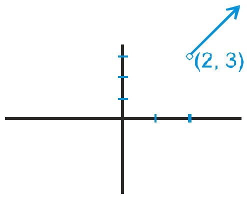
20. In G.P. the first term & common ratio are both $\frac{1}{2}\left(\sqrt{3}+i\right)$, then the absolute value of its n^{th} term is :

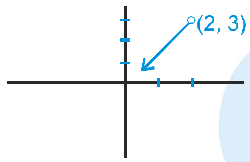
(A) 1 **(B)** 2^n **(C)** 4^n **(D)** none

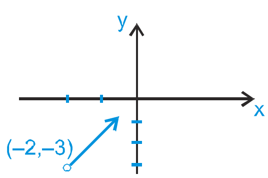
21. If P and Q are represented by the complex numbers z_1 and z_2 such that $\left|\frac{1}{z_1}+\frac{1}{z_2}\right|=\left|\frac{1}{z_1}-\frac{1}{z_2}\right|$, then the circumcentre of ΔOPQ (where O is the origin) is

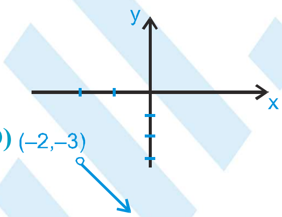
(A) $\frac{z_1-z_2}{2}$ **(B)** $\frac{z_1+z_2}{2}$ **(C)** $\frac{z_1+z_2}{3}$ **(D)** z_1+z_2

22. If Arg $(z-2-3i)=\frac{\pi}{4}$, then the locus of z is

(A)


(B)


(C)


(D)


23. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if :

(A) $z_1+z_4=z_2+z_3$ **(B)** $z_1+z_3=z_2+z_4$ **(C)** $z_1+z_2=z_3+z_4$ **(D)** none

24. The set of points on the complex plane such that z^2+z+1 is real and positive (where $z=x+iy,\ x,\ y\in R$) is-

(A) Complete real axis only

(B) Complete real axis or all points on the line $2x+1=0$

(C) Complete real axis or a line segment joining points $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ & $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ excluding both.

(D) Complete real axis or set of points lying inside the rectangle formed by the lines.

$2x+1=0$; $2x-1=0$; $2y-\sqrt{3}=0$ & $2y+\sqrt{3}=0$

25. If z_1, z_2, z_3 are vertices of an equilateral triangle inscribed in the circle $|z|=2$ and if $z_1=1+i\sqrt{3}$, then

(A) $z_2=-2,\ z_3=1+i\sqrt{3}$ **(B)** $z_2=2,\ z_3=1-i\sqrt{3}$

(C) $z_2=-2,\ z_3=1-i\sqrt{3}$ **(D)** $z_2=1-i\sqrt{3}$, $z_3=-1-i\sqrt{3}$

26. The vector $z=-4+5i$ is turned counter clockwise through an angle of 180º & stretched 1.5 times. The complex number corresponding to the newly obtained vector is :

(A) $6-\frac{15}{2}i$ **(B)** $-6+\frac{15}{2}i$ **(C)** $6+\frac{15}{2}i$ **(D)** none of these

27. If $|z| = 1$ and $|\omega - 1| = 1$ where $z, \omega \in C$, then the largest set of values of $|2z - 1|^2 + |2\omega - 1|^2$ equals

(A) [1, 9] **(B)** [2, 6] **(C)** [2, 12] **(D)** [2, 18]

28. If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = 1$, then the value of θ is

(A) $\frac{3m\pi}{n(n+1)}, m \in Z$ **(B)** $\frac{2m\pi}{n(n+1)}, m \in Z$ **(C)** $\frac{4m\pi}{n(n+1)}, m \in Z$ **(D)** $\frac{m\pi}{n(n+1)}, m \in Z$

29. Points z_1 & z_2 are adjacent vertices of a regular octagon. The vertex z_3 adjacent to z_2 $(z_3 \neq z_1)$ can be represented by -

(A) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 + z_2)$ **(B)** $z_2 + \frac{1}{\sqrt{2}}(-1 \pm i)(z_1 - z_2)$

(C) $z_2 + \frac{1}{\sqrt{2}}(-1 \pm i)(z_2 - z_1)$ **(D)** none of these

30. If $\log_{1/2}\left(\frac{|z-1| + 4}{3|z-1| - 2}\right) > 1$, then find locus of z

(A) Exterior to circle with center 1 + i0 and radius 10
(B) Interior to circle with center 1 + i0 and radius 10
(C) Circle with center 1 + i0 and radius 10
(D) None of these

31. If $A_1, A_2, \ldots\ldots, A_n$ be the vertices of an n-sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then find the value of n

(A) 5 **(B)** 7 **(C)** 8 **(D)** 9

32. If $x = a + b + c$, $y = a\alpha + b\beta + c$ and $z = a\beta + b\alpha + c$, where α and β are imaginary cube roots of unity, then xyz =

(A) $2(a^3 + b^3 + c^3)$ **(B)** $2(a^3 - b^3 - c^3)$ **(C)** $a^3 + b^3 + c^3 - 3abc$ **(D)** $a^3 - b^3 - c^3$

33. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and $\text{Arg}(z) - \text{Arg}(\omega) = \pi/2$, then $\bar{z}\,\omega$ is equal to -

(A) 1 **(B)** –1 **(C)** i **(D)** –i

34. The expression $\left(\frac{1 + i\tan\alpha}{1 - i\tan\alpha}\right)^n - \frac{1 + i\tan n\alpha}{1 - i\tan n\alpha}$ when simplified reduces to :

(A) zero **(B)** $2\sin n\alpha$ **(C)** $2\cos n\alpha$ **(D)** none

35. If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $x^5 - 1 = 0$, then find the value of $\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4}$

(where ω is imaginary cube root of unity.)

(A) ω **(B)** ω^2 **(C)** 1 **(D)** – 1

Exercise # 2 Part # I [Multiple Correct Choice Type Questions]

1. Which of the following complex numbers lies along the angle bisectors of the line -
$L_1 : z = (1+3\lambda) + i\ (1+4\lambda)$ $\qquad$ $L_2 : z = (1+3\mu) + i\,(1-4\mu)$

(A) $\frac{11}{5} + i$ **(B)** $11 + 5i$ **(C)** $1 - \frac{3i}{5}$ **(D)** $5 - 3i$

2. On the argand plane, let $\alpha = -2 + 3z$, $\beta = -2 - 3z$ & $|z| = 1$. Then the correct statement is -

(A) α moves on the circle, centre at (–2, 0) and radius 3

(B) α & β describe the same locus

(C) α & β move on different circles

(D) $\alpha - \beta$ moves on a circle concentric with $|z| = 1$

3. POQ is a straight line through the origin O . P and Q represent the complex number a + i b and c + i d respectively and OP = OQ. Then

(A) $|a + i b| = |c + i d|$ **(B)** $a + c = b + d$
(C) $\arg(a + i b) = \arg(c + i d)$ **(D)** none of these

4. The common roots of the equations $z^3 + (1 + i)z^2 + (1 + i)z + i = 0$, (where $i = \sqrt{-1}$) and $z^{1993} + z^{1994} + 1 = 0$ are - (where ω denotes the complex cube root of unity)

(A) 1 **(B)** ω **(C)** ω^2 **(D)** ω^{981}

5. If g(x) and h(x) are two polynomials such that the polynomial $P(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then -

(A) $g(1) = h(1) = 0$ **(B)** $g(1) = h(1) \neq 0$ **(C)** $g(1) = -h(1)$ **(D)** $g(1) + h(1) = 0$

6. The value of $i^n + i^{-n}$, for $i = \sqrt{-1}$ and $n \in I$ is -

(A) $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$ **(B)** $\frac{(1+i)^{2n}}{2^n} + \frac{(1-i)^{2n}}{2^n}$ **(C)** $\frac{(1+i)^{2n}}{2^n} - \frac{2^n}{(1-i)^{2n}}$ **(D)** $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$

7. The equation $|z - i| + |z + i| = k$, $k > 0$, can represent

(A) an ellipse if $k > 2$ **(B)** line segment if $k = 2$
(C) an ellipse if $k = 5$ **(D)** line segment if $k = 1$

8. If the equation $|z|(z+1)^8 = z^8|z+1|$ where $z \in C$ and $z(z+1) \neq 0$ has distinct roots $z_1, z_2, z_3, \ldots, z_n$ (where $n \in N$) then which of the following is/are true?

(A) $z_1, z_2, z_3, \ldots, z_n$ are concyclic points. **(B)** $z_1, z_2, z_3, \ldots, z_n$ are collinear points

(C) $\sum_{r=1}^{n} \text{Re}(z_r) = \frac{-7}{2}$ **(D)** $= 0$

9. If $x_r = \text{CiS}\left(\frac{\pi}{2^r}\right)$ for $1 \leq r \leq n$; r, $n \in N$ then -

(A) $\lim_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = -1$ **(B)** $\lim_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = 0$ **(C)** $\lim_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = 1$ **(D)** $\lim_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = 0$

10. If $|z_1| = |z_2| = |z_3| = 1$ and z_1, z_2, z_3 are represented by the vertices of an equilateral triangle then

(A) $z_1 + z_2 + z_3 = 0$ **(B)** $z_1z_2z_3 = 1$

(C) $z_1z_2 + z_2z_3 + z_3z_1 = 0$ **(D)** none of these

11. If S be the set of real values of x satisfying the inequality $1 - \log_2 \frac{|x+1+2i| - 2}{\sqrt{2}-1} \ge 0$, then S contains -

(A) $[-3, -1)$ **(B)** $(-1, 1]$ **(C)** $[-2, 2]$ **(D)** $[-3, 1]$

12. Let z_1, z_2 be two complex numbers represented by points on the circle $|z_1| = 1$ and $|z_2|=2$ respectively, then -

(A) $\max|2z_1+z_2| = 4$ **(B)** $\min|z_1 - z_2| = 1$ **(C)** $\left|z_2 + \frac{1}{z_1}\right| \le 3$ **(D)** none of these

13. If z is a complex number then the equation $z^2 + z|z| + |z^2| = 0$ is satisfied by (ω and ω^2 are imaginary cube roots of unity)

(A) $z = k\omega$ where $k \in R$ **(B)** $z = k\omega^2$ where k is non negative real

(C) $z = k\omega$ where k is positive real **(D)** $z = k\omega^2$ where $k \in R$.

14. If the complex numbers z_1, z_2, z_3 represents vertices of an equilateral triangle such that $|z_1|=|z_2|=|z_3|$, then which of following is correct ?

(A) $z_1 + z_2 + z_3 \neq 0$ **(B)** $\text{Re}(z_1 + z_2 + z_3) = 0$ **(C)** $\text{Im}(z_1 + z_2 + z_3) = 0$ **(D)** $z_1 + z_2 + z_3 = 0$

15. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\varphi = y + \frac{1}{y}$, then

(A) $x^n + \frac{1}{x^n} = 2\cos(n\theta)$ **(B)** $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \varphi)$

(C) $xy + \frac{1}{xy} = 2\cos(\theta + \varphi)$ **(D)** none of these

16. Value(s) of $(-i)^{1/3}$ is/are -

(A) $\frac{\sqrt{3} - i}{2}$ **(B)** $\frac{\sqrt{3} + i}{2}$ **(C)** $\frac{-\sqrt{3} - i}{2}$ **(D)** $\frac{-\sqrt{3} + i}{2}$

17. If z be a non-real complex number satisfying $|z| = 2$, then which of the following is/are true?

(A) $\arg\left(\frac{z-2}{z+2}\right) = \pm\frac{\pi}{2}$ **(B)** $\arg\left(\frac{z+1+i\sqrt{3}}{z-1+i\sqrt{3}}\right) = \frac{\pi}{6}$

(C) $|z^2 - 1| \ge 3$ **(D)** $|z^2 - 1| \le 5$

18. If α, β be any two complex numbers such that $\left|\frac{\alpha - \beta}{1 - \bar{\alpha}\beta}\right| = 1$, then which of the following may be true -

(A) $|\alpha| = 1$ **(B)** $|\beta| = 1$ **(C)** $\alpha = e^{i\theta}, \theta \in R$ **(D)** $\beta = e^{i\theta}, \theta \in R$

19. The equation $||z + i| - |z - i|| = k$ represents

(A) a hyperbola if $0 < k < 2$
(B) a pair of ray if $k > 2$
(C) a straight line if $k = 0$
(D) a pair of ray if $k = 2$

20. If amp $(z_1z_2) = 0$ and $|z_1| = |z_2| = 1$, then :-

(A) $z_1 + z_2 = 0$
(B) $z_1z_2 = 1$
(C) $z_1 = \bar{z}_2$
(D) none of these

21. If centre of square ABCD is at z=0. If affix of vertex A is z_1, centroid of triangle ABC is/are -

(A) $\frac{z_1}{3}(\cos \pi + i \sin \pi)$
(B) $4\left[\left(\cos\frac{\pi}{2}\right) - i\left(\sin\frac{\pi}{2}\right)\right]$
(C) $\frac{z_1}{3}\left[\left(\cos\frac{\pi}{2}\right) + i\left(\sin\frac{\pi}{2}\right)\right]$
(D) $\frac{z_1}{3}\left[\left(\cos\frac{\pi}{2}\right) - i\left(\sin\frac{\pi}{2}\right)\right]$

22. Let z_1, z_2, z_3 be non-zero complex numbers satisfying the equation $z^4 = iz$. Which of the following statement(s) is/are correct ?

(A) The complex number having least positive argument is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
(B) $\sum_{k=1}^{3} \text{Amp}(z_k) = \frac{\pi}{2}$
(C) Centroid of the triangle formed by z_1, z_2 and z_3 is $\left(\frac{1}{\sqrt{3}}, \frac{-1}{3}\right)$
(D) Area of triangle formed by z_1, z_2 and z_3 is $\frac{3\sqrt{3}}{2}$

23. If the vertices of an equilateral triangle are situated at z =0, z=z_1, z =z_2, then which of the following is/are true -

(A) $|z_1| = |z_2|$
(B) $|z_1 - z_2| = |z_1|$
(C) $|z_1 + z_2| = |z_1| + |z_2|$
(D) $|\arg z_1 - \arg z_2| = \pi/3$

24. If z satisfies the inequality $|z - 1 - 2i| \le 1$, then

(A) min (arg (z)) = $\tan^{-1}\left(\frac{3}{4}\right)$
(B) max (arg(z)) = $\frac{\pi}{2}$
(C) min (|z|) = $\sqrt{5} - 1$
(D) max (|z|) = $\sqrt{5} + 1$

25. Let z, ωz and z + ωz represent three vertices of ΔABC, where ω is cube root unity, then -

(A) centroid of ΔABC is $\frac{2}{3}(z + \omega z)$
(B) orthocenter of ΔABC is $\frac{2}{3}(z + \omega z)$
(C) ABC is an obtuse angled triangle
(D) ABC is an acute angled triangle

Part # II [Assertion & Reason Type Questions]

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I :** There are exactly two complex numbers which satisfy the complex equations $|z - 4 - 5i| = 4$ and $\text{Arg}(z - 3 - 4i) = \frac{\pi}{4}$ simultaneously.

Statement-II : A line cuts the circle in atmost two points.

2. Let z_1, z_2, z_3 represent vertices of a triangle.

Statement - I : $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$, when triangle is equilateral.

Statement - II : $|z_1|^2 - z_1\bar{z}_0 - \bar{z}_1 z_0 = |z_2|^2 - z_2\bar{z}_0 - \bar{z}_2 z_0 = |z_3|^2 - z_3\bar{z}_0 - \bar{z}_3 z_0$, where z_0 is circumcentre of triangle.

3. **Statement-I :** If $z = i + 2i^2 + 3i^3 + \ldots\ldots\ldots\ldots + 32i^{32}$, then $z, \bar{z}, -z$ & $-\bar{z}$ forms the vertices of square on argand plane.

Statement-II : $z, \bar{z}, -z, -\bar{z}$ are situated at the same distance from the origin on argand plane.

4. **Statement - 1 :** Roots of the equation $(1 + z)^6 + z^6 = 0$ are collinear.

Statement - II : If z_1, z_2, z_3 are in A.P. then points represented by z_1, z_2, z_3 are collinear

5. Let z_1, z_2, z_3 satisfy $\left|\frac{z+2}{z-1}\right| = 2$ and $z_0 = 2$. Consider least positive arguments wherever required.

Statement – I : $2\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{z_1 - z_0}{z_2 - z_0}\right)$.

Statement – II : z_1, z_2, z_3 satisfy $|z - z_0| = 2$.

6. Let $1, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ be the n, n^{th} roots of unity,

Statement - I : $\sin\frac{\pi}{n} \cdot \sin\frac{2\pi}{n} \cdot \sin\frac{3\pi}{n} \ldots\ldots \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

Statement - II : $(1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3)\ldots\ldots(1 - \alpha_{n-1}) = n$.

7. **Statement-I :** If $z_1 = 9 + 5i$ and $z_2 = 3 + 5i$ and if $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$ then $|z - 6 - 8i| = 3\sqrt{2}$

Statement-II : If z lies on circle having z_1 & z_2 as diameter then $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$.

8. **Statement-I :** Let z_1, z_2, z_3 be three complex numbers such that $|3z_1 + 1| = |3z_2 + 1| = |3z_3 + 1|$ and $1 + z_1 + z_2 + z_3 = 0$, then z_1, z_2, z_3 will represent vertices of an equilateral triangle on the complex plane.

Statement-II : z_1, z_2, z_3 represent vertices of an equilateral triangle if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

Exercise # 3 Part # I [Matrix Match Type Questions]

Following question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled as A, B, C and D while the statements in Column-II are labelled as p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II.

1. **Column - I** | **Column - I**

(A) If z be the complex number such that $\left|z+\frac{1}{z}\right|=2$ then minimum value of $\frac{|z|}{\tan\frac{\pi}{8}}$ is — (p) 0

(B) $|z| = 1$ & $z^{2n}+1 \neq 0$ then $\frac{z^n}{z^{2n}+1}-\frac{\overline{z}^n}{\overline{z}^{2n}+1}$ is equal to — (q) 3

(C) If $8iz^3 + 12z^2 - 18z + 27i = 0$ then $2|z| =$ — (r) 11

(D) If z_1, z_2, z_3, z_4 are the roots of equation $z^4 + z^3 + z^2 + z + 1 = 0$, then $\prod_{i=1}^{4}(z_i + 2)$ is — (s) 1

2. Let z_1 lies on $|z| = 1$ and z_2 lies on $|z| = 2$.

Column – I | **Column – II**

(A) Maximum value of $|z_1 + z_2|$ — (p) 3

(B) Minimum value of $|z_1 - z_2|$ — (q) 1

(C) Minimum value of $|2z_1 + 3z_2|$ — (r) 4

(D) Maximum value of $|z_1 - 2z_2|$ — (s) 5

3. **Column - I** | **Column - II**

(A) Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has 4 real roots $(a, b, c, d \in R)$. If $|f(-i)| = 1$ (where $i = \sqrt{-1}$), then the value of $a^2 + b^2 + c^2 + d^2$ equals — (p) 0

(q) 1

(B) If $\arg(z+3) = \frac{\pi}{6}$ and $\arg(z-3) = \frac{2\pi}{3}$, then $\tan^2(\arg z) - 2\cos(\arg z)$, is $\sum_{r=1}^{n}\text{Im}(z_r)$ — (r) 2

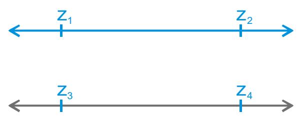
(s) 3

(C) If the points $A(z)$, $B(-z)$ and $C(z+1)$ are vertices of an equilateral triangle, then $5 + 4\,\text{Re}(z)$ equals

(D) If $z_1 = 1 + i\sqrt{3}$, $z_2 = 1 - i\sqrt{3}$ and $z_3 = 2$, then value of x satisfying $z_1^x + z_2^x = 2^x$ can be — (t) 4

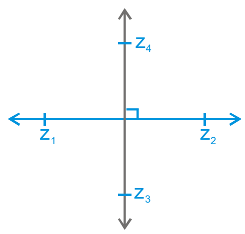
4. Match the figure in column-I with corresponding expression -

Column - I | **Column - I**

(A) 

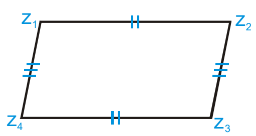
two parallel lines

(p) $\dfrac{z_4 - z_3}{z_2 - z_1} + \dfrac{\bar{z}_4 - \bar{z}_3}{\bar{z}_2 - \bar{z}_1} = 0$

(B) 

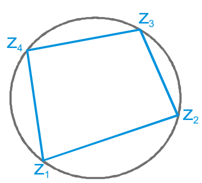
two perpendicular lines

(q) $\dfrac{z_2 - z_1}{z_4 - z_3} = \dfrac{\bar{z}_2 - \bar{z}_1}{\bar{z}_4 - \bar{z}_3}$

(C) 

a parallelogram

(r) $\dfrac{z_4 - z_1}{z_2 - z_1} \cdot \dfrac{z_2 - z_3}{z_4 - z_3} = \dfrac{\bar{z}_4 - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \cdot \dfrac{\bar{z}_2 - \bar{z}_3}{\bar{z}_4 - \bar{z}_3}$

(D) 

(s) $z_1 + z_3 = z_2 + z_4$

Part # II [Comprehension Type Questions]

Comprehension # 1

Let z be any complex number. To factorise the expression of the form $z^n - 1$, we consider the equation $z^n = 1$. This equation is solved using De moiver's theorem. Let 1, α_1, α_2,........ α_{n-1} be the roots of this equation, then $z^n - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2)........(z - \alpha_{n-1})$ This method can be generalised to factorize any expression of the form $z^n - k^n$.

for example, $z^7 + 1 = \prod_{m=0}^{6}\left(z - \text{C iS}\left(\dfrac{2m\pi}{7} + \dfrac{\pi}{7}\right)\right)$

This can be further simplified as

$$z^7 + 1 = (z + 1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right) \quad .....\text{(i)}$$

These factorisations are useful in proving different trigonometric identities e.g. in eqaution **(i)** if we put z = i, then equation **(i)** becomes

$$(1 - i) = (i + 1)\left(-2i\cos\frac{\pi}{7}\right)\left(-2i\cos\frac{3\pi}{7}\right)\left(-2i\cos\frac{5\pi}{7}\right)$$

i.e. $\cos\dfrac{\pi}{7}\cos\dfrac{3\pi}{7}\cos\dfrac{5\pi}{7} = -\dfrac{1}{8}$

1. If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z-2)(z^2 - pz + 4)(z^2 - qz + 4)$, where p > q, then the value of $p^2 - 2q$ -

(A) 8 **(B)** 4 **(C)** –4 **(D)** –8

2. By using the factorisation for $z^5 + 1$, the value of $4\sin\frac{\pi}{10}\cos\frac{\pi}{5}$ comes out to be -

(A) 4 **(B)** 1/4 **(C)** 1 **(D)** –1

3. If $(z^{2n+1} - 1) = (z-1)(z^2 - p_1 z + 1)........(z^2 - p_n z + 1)$ where $n \in N$ & $p_1, p_2 .... p_n$ are real numbers then $p_1 + p_2 + ... + p_n =$

(A) –1 **(B)** 0 **(C)** $\tan(\pi/2n)$ **(D)** none of these

Comprehension # 2

Let z_1, z_2, z_3, z_4 are three distinct complex numbers such that $|z_1| = |z_2| = |z_3| = |z_4|$, satisfying.
$|(1-d)z_1 + z_2 + z_3 + z_4| = |z_1 + (1-d)z_2 + z_3 + z_4| = |z_1 + z_2 + (1-d)z_3 + z_4|$ where $d \in R - \{0\}$.

1. Arg $(z_1 + z_2 + z_3 + z_4)$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{2}$ **(C)** π **(D)** Not defined.

2. $|z_1 + z_2 + z_3 + z_4|$ is

(A) 1 **(B)** 2 **(C)** 0 **(D)** ≥ 4

3. The point dz_1, dz_2, dz_3 lie on a circle with

(A) centre (1, 0), radius $|d|$
(B) centre (0, 0), radius $|dz_1|$
(C) centre (0, 1), radius $|dz_2|$
(D) None of these

Comprehension # 3

ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. Let the points D and M represent complex numbers 1 + i and 2 – i respecetively.

If θ is arbitary real, then $z = re^{i\theta}$, $R_1 \leq r \leq R_2$ lies in annular region formed by concentric circles $|z| = R_1$, $|z| = R_2$.

1. A possible representation of point A is

(A) $3 - \frac{i}{2}$ **(B)** $3 + \frac{i}{2}$ **(C)** $1 + \frac{3}{2}i$ **(D)** $3 - \frac{3}{2}i$

2. $e^{iz} =$

(A) $e^{-r\cos\theta}(\cos(r\cos\theta) + i\sin(r\sin\theta))$
(B) $e^{-r\cos\theta}(\sin(r\cos\theta) + i\cos(r\cos\theta))$
(C) $e^{-r\sin\theta}(\cos(r\cos\theta) + i\sin(r\cos\theta))$
(D) $e^{-r\sin\theta}(\sin(r\cos\theta) + i\cos(r\sin\theta))$

3. If z is any point on segment DM then $w = e^{iz}$ lies in annular region formed by concentric circles.

(A) $|w|_{min} = 1, |w|_{max} = 2$
(B) $|w|_{min} = \frac{1}{e}, |w|_{max} = e$
(C) $|w|_{min} = \frac{1}{e^2}, |w|_{max} = e^2$
(D) $|w|_{min} = \frac{1}{2}, |w|_{max} = 1$

Comprehension # 4

Let A, B, C be three sets of complex numbers as defined below.

$A = \{z : |z+1| \le 2 + \text{Re}(z)\}$, $B = \{z : |z-1| \ge 1\}$ and $C = \left\{z : \left|\frac{z-1}{z+1}\right| \ge 1\right\}$

1. The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is

(A) 4 **(B)** 5 **(C)** 6 **(D)** 10

2. The area of region bounded by $A \cap B \cap C$ is

(A) $2\sqrt{3}$ **(B)** $\sqrt{3}$ **(C)** $4\sqrt{3}$ **(D)** 2

3. The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude is

(A) –1 **(B)** $\frac{-3}{2}$ **(C)** $\frac{-1}{2}$ **(D)** –2

Comprehension # 5

In the figure |z| = r is circumcircle of ΔABC.D,E & F are the middle points of the sides BC, CA & AB respectively, AD produced to meet the circle at L. If ∠CAD = θ, AD = x, BD = y and altitude of ΔABC from A meet the circle |z|= r at M, z_a, z_b & z_c are affixes of vertices A, B & C respectively.

1. Area of the ΔABC is equal to -

(A) xy cos (θ + C) **(B)** (x + y) sin θ

(C) xy sin (θ + C) **(D)** $\frac{1}{2}$ xy sin (θ + C)

2. Affix of M is -

(A) $2z_be^{i2B}$ **(B)** $z_be^{i(\pi-2B)}$ **(C)** z_be^{iB} **(D)** $2z_be^{iB}$

3. Affix of L is -

(A) $z_be^{i(2A-2\theta)}$ **(B)** $2z_be^{i(2A-2\theta)}$ **(C)** $z_be^{i(A-\theta)}$ **(D)** $2z_be^{i(A-\theta)}$

Exercise # 4 [Subjective Type Questions]

1. If $x = 1 + i\sqrt{3}$; $y = 1 - i\sqrt{3}$ & $z = 2$, then prove that $x^p + y^p = z^p$ for every prime $p > 3$.

2. Interpret the following locii in $z \in C$.

(A) $1 < |z - 2i| < 3$

(B) $\text{Re}\left(\frac{z + 2i}{iz + 2}\right) \le 4 \ (z \ne 2i)$

(C) $\text{Arg}(z + i) - \text{Arg}(z - i) = \pi/2$

(D) $\text{Arg}(z - a) = \pi/3$ where $a = 3 + 4i$.

3. Find the modulus, argument and the principal argument of the complex numbers.

(A) $z = 1 + \cos\frac{18\pi}{25} + i\sin\frac{18\pi}{25}$

(B) $z = -2(\cos 30° + i\sin 30°)$

(C) $(\tan 1 - i)^2$

(D) $\dfrac{i - 1}{i\left(1 - \cos\frac{2\pi}{5}\right) + \sin\frac{2\pi}{5}}$

4. If $a_1, a_2, a_3 \ldots a_n, A_1, A_2, A_3 \ldots A_n$, k are all real numbers, then prove that

$\frac{A_1^2}{x - a_1} + \frac{A_2^2}{x - a_2} + \ldots\ldots + \frac{A_n^2}{x - a_n} = k$ has no imaginary roots.

5. For complex numbers z & ω, prove that, $|z|^2\omega - |\omega|^2 z = z - \omega$ if and only if, $z = \omega$ or $z\bar{\omega} = 1$

6. If $|z_1| = |z_2| = \ldots\ldots = |z_n| = 1$ then show that

(i) $\bar{z}_1 = \frac{1}{z_1}$

(ii) $|z_1 + z_2 + \ldots.. + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots\ldots + \frac{1}{z_n}\right|$.

And hence interpret that the centroid of polygon with 2n vertices $z_1, z_2, \ldots\ldots z_n, \frac{1}{z_1}, \frac{1}{z_2}, \ldots\ldots.. \frac{1}{z_n}$ (need not be in order) lies on real axis.

7. (A) Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z \,|\, |z| \le 2\}$ and $B = \{z \,|\, (1 - i)z + (1 + i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

(B) For all real numbers x, let the mapping $f(x) = \frac{1}{x - i}$, where $i = \sqrt{-1}$. If there exist real numbers a, b, c and d for which f(a), f(b), f(c) and f(d) form a square on the complex plane. Find the area of the square.

8. Let circles C_1 and C_2 on Argand plane be given by $|z + 1| = 3$ and $|z - 2| = 7$ respectively. If a variable circle $|z - z_0| = r$ be inside circle C_2 such that it touches C_1 externally and C_2 internally then locus of 'z_0' describes a conic E. If eccentricity of E can be written in simplest form as $\frac{p}{q}$ where $p, q \in N$, then find the value of $(p + q)$.

9. If z_1, z_2 are the roots of the equation $az^2 + bz + c = 0$, with $a, b, c > 0$; $2b^2 > 4ac > b^2$; $z_1 \in$ third quadrant ; $z_2 \in$ second quadrant in the argand's plane then, show that $\arg\left(\frac{z_1}{z_2}\right) = 2\cos^{-1}\left(\frac{b^2}{4ac}\right)^{1/2}$

10. For any two complex numbers z_1, z_2 and any two real numbers a, b show that
$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

11. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum $3 + 4i$ and the other two with product $13 + i$. Find the value of 'b'.

12. If A, B and C are the angle of a triangle $D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$ where $i = \sqrt{-1}$, then find the value of D.

13. If α is imaginary n^{th} $(n \geq 3)$ root of unity then show that $\sum_{r=1}^{n-1}(n-r)\,\alpha^r = \frac{n\alpha}{1-\alpha}$

Hence deduce that $\sum_{r=1}^{n-1}(n-r)\,\sin\frac{2r\pi}{n} = \frac{n}{2}\cot\frac{\pi}{n}$.

14. Let $A = \{a \in R |$ the equation $(1+2i)x^3 - 2(3+i)x^2 + (5-4i)x + 2a^2 = 0\}$ has at least one real root. Find the value of $\sum_{a \in A} a^2$.

15. Consider two concentric circles $S_1 : |z| = 1$ and $S_2 : |z| = 2$ on the Argand plane. A parabola is drawn through the points where 'S_1' meets the real axis and having arbitrary tangent of 'S_2' as its directrix. If the locus of the focus of drawn parabola is a conic C then find the area of the quadrilateral formed by the tangents at the ends of the latus-rectum of conic C.

16. Let z_1 and z_2 be two complex numbers such that $\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$, find $|z_1|$.

17. If O is origin and affixes of P, Q, R are respectively z, iz, z + iz. Locate the points on complex plane. If $\Delta PQR = 200$ then find **(i)** $|z|$ **(ii)** sides of quadrilateral OPRQ

18. If Z_r, r = 1, 2, 3,..... 2m, m $\in$ N are the roots of the equation $Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots\ldots + Z + 1 = 0$ then prove that $\sum_{r=1}^{2m}\frac{1}{Z_r - 1} = -m$

19. ABCD is a rhombus in the Argand plane. If the affixes of the vertices be z_1, z_2, z_3, z_4 and taken in anti-clockwise sense and $\angle CBA = \pi/3$, show that

(A) $2z_2 = z_1(1 + i\sqrt{3}) + z_3(1 - i\sqrt{3})$ & **(B)** $2z_4 = z_1(1 - i\sqrt{3}) + z_3(1 + i\sqrt{3})$

20. Find the locus of mid-point of line segment intercepted between real and imaginary axes, by the line $a\bar{z} + \bar{a}z + b = 0$, where 'b' is real parameter and 'a' is a fixed complex number such that $Re(a) \neq 0$, $Im(a) \neq 0$.

21. P is a point on the Argand plane. On the circle with OP as diameter two points Q & R are taken such that $\angle POQ = \angle QOR = \theta$. If 'O' is the origin & P, Q & R are represented by the complex numbers Z_1, Z_2 & Z_3 respectively, show that : $Z_2^2 \cos 2\theta = Z_1 . Z_3 \cos^2\theta$.

22. A polynomial $f(z)$ when divided by $(z - w)$ leaves remainder $2 + i\sqrt{3}$ and when divided by $(z - w^2)$ leaves remainder $2 - i\sqrt{3}$. If the remainder obtained when $f(z)$ is divided by $z^2 + z + 1$ is $az + b$ (where w is a non- real cube root of unity and $a, b \in R^+$), then find the value of $(a + b)$.

23. The points A, B, C depict the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. Show that : $(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$

24. Let z_1, z_2, z_3 are three pair wise distinct complex numbers and t_1, t_2, t_3 are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1z_1 + t_2z_2 + t_3z_3$ lies inside a triangle with vertices z_1, z_2, z_3 or on its boundary.

25. Let $A \equiv z_1$; $B \equiv z_2$; $C \equiv z_3$ are three complex numbers denoting the vertices of an acute angled triangle. If the origin 'O' is the orthocentre of the triangle, then prove that $z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$.

26. If $a = e^{i\alpha}$, $b = e^{i\beta}$, $c = e^{i\gamma}$ and $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then prove the following

(i) $a + b + c = 0$

(ii) $ab + bc + ca = 0$

(iii) $a^2 + b^2 + c^2 = 0$

(iv) $\Sigma \cos 2\alpha = 0 = \Sigma \sin 2\alpha$

(v) $\Sigma \sin^2\alpha = \Sigma \cos^2\alpha = 3/2$

27. **(A)** If ω is an imaginary cube root of unity then prove that :

$(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$..... to 2n factors $= 2^{2n}$.

(B) If ω is a complex cube root of unity, find the value of ;

$(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$..... to n factors.

28. Let z_i $(i = 1, 2, 3, 4)$ represent the vertices of a square all of which lie on the sides of the triangle with vertices (0,0), (2,1) and (3, 0). If z_1 and z_2 are purely real, then area of triangle formed by z_3, z_4 and origin is $\frac{m}{n}$ (where m and n are in their lowest form). Find the value of $(m + n)$.

29. The points A, B, C represent the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. Show that $(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$.

30. Evaluate : $\sum_{p=1}^{32}(3p + 2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$.

Exercise # 5 — Part # I [Previous Year Questions] [AIEEE/JEE-MAIN]

1. The inequality $|z-4| < |z-2|$ represents the following region **[AIEEE-2002]**

(1) $\text{Re}(z) > 0$ (2) $\text{Re}(z) < 0$ (3) $\text{Re}(z) > 2$ (4) none of these

2. Let z and ω are two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z + \arg \omega = \pi$, then z equal to **[AIEEE-2002]**

(1) ω (2) $-\omega$ (3) $\bar{\omega}$ (4) $-\bar{\omega}$

3. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex, Further, assume that the origin z_3, z_1 and z_2 form an equilateral triangle. then- **[AIEEE-2003]**

(1) $a^2 = b$ (2) $a^2 = 2b$ (3) $a^2 = 3b$ (4) $a^2 = 4b$

4. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and $\text{Arg}(z) - \text{Arg}(\omega) = \pi/2$, then $\bar{z}\,\omega$ is equal to **[AIEEE-2003]**

(1) 1 (2) –1 (3) i (4) –i

5. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then **[AIEEE-2003]**

(1) x = 4n, where n is any positive integer
(2) x = 2n, where n is any positive integer
(3) x = 4n + 1, where n is any positive integer
(4) x = 2n + 1, where n is any positive integer

6. Let z, w be complex numbers such that $\bar{z} + i\,\bar{w} = 0$ and $\arg zw = \pi$. Then arg z equals **[AIEEE-2004]**

(1) $\pi/4$ (2) $\pi/2$ (3) $3\pi/4$ (4) $5\pi/4$

7. If $|z^2 - 1| = |z|^2 + 1$, then z lies on **[AIEEE-2004]**

(1) the real axis (2) the imaginary axis (3) a circle (4) an ellipse

8. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\frac{\left(\frac{x}{p} + \frac{y}{q}\right)}{(p^2 + q^2)}$ is equal to- **[AIEEE-2004]**

(1) 1 (2) –1 (3) 2 (4) –2

9. If z_1 and z_2 are two non zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$ then $\arg z_1 - \arg z_2$ is equal to- **[AIEEE-2005]**

(1) $-\pi$ (2) $\frac{\pi}{2}$ (3) $-\frac{\pi}{2}$ (4) 0

10. If $w = \frac{z}{z - \frac{1}{3}i}$ and $|w| = 1$ then z lies on **[AIEEE-2005]**

(1) a circle (2) an ellipse (3) a parabola (4) a straight line

11. If $|z + 4| \leq 3$, then the maximum value of $|z + 1|$ is- **[AIEEE-2007]**

(1) 4 (2) 10 (3) 6 (4) 0

12. The conjugate of a complex number is $\frac{1}{i-1}$, then that complex number is- **[AIEEE-2008]**

(1) $\frac{-1}{i-1}$ (2) $\frac{1}{i+1}$ (3) $\frac{-1}{i+1}$ (4) $\frac{1}{i-1}$

13. If $\left|Z - \frac{4}{Z}\right| = 2$, then the maximum value of |Z| is equal to :- **[AIEEE-2009]**

(1) 2 (2) $2+\sqrt{2}$ (3) $\sqrt{3}+1$ (4) $\sqrt{5}+1$

14. The number of complex numbers z such that |z – 1| = |z + 1| = |z – i| equals :- **[AIEEE-2010]**

(1) 0 (2) 1 (3) 2 (4) ∞

15. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re z = 1, then it is necessary that :- **[AIEEE-2011]**

(1) $|\beta| = 1$ (2) $\beta \in (1, \infty)$ (3) $\beta \in (0, 1)$ (4) $\beta \in (-1, 0)$

16. If ω(≠1) is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals :- **[AIEEE-2011]**

(1) (1, 0) (2) (–1, 1) (3) (0, 1) (4) (1, 1)

17. If z ≠ 1 and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies : **[AIEEE-2012]**

(1) on the imaginary axis.
(2) either on the real axis or on a circle passing through the origin.
(3) on a circle with centre at the origin.
(4) either on the real axis or on a circle not passing through the origin.

18. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals **[JEE (Main)-2013]**

(1) –θ (2) $\frac{\pi}{2} - \theta$ (3) θ (4) π – θ

19. If z is a complex number such that |z| ≥ 2, then the minimum value of $\left|z + \frac{1}{2}\right|$: **[JEE (Main)-2014]**

(1) is equal to $\frac{5}{2}$
(2) lies in the interval (1, 2)
(3) is strictly greater than $\frac{5}{2}$
(4) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

20. A complex number z is said to be unimodular if |z| = 1. Suppose z_1 and z_2 are complex number such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a : **[JEE (Main)-2015]**

(1) circle of radius 2.
(2) circle of radius $\sqrt{2}$
(3) straight line parallel to x-axis
(4) straight line parallel to y-axis

21. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary is : **[JEE (Main)-2016]**

(1) $\frac{\pi}{6}$ (2) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (3) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (4) $\frac{\pi}{3}$

Part # II [Previous Year Questions][IIT-JEE ADVANCED]

1. **(A)** If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$ then $|z_1 + z_2 + z_3|$ is -

(A) equal to 1 **(B)** less than 1 **(C)** greater than 3 **(D)** equal to 3

(B) If arg (z) < 0, then arg (–z) – arg (z) = **[JEE 2000]**

(A) π **(B)** $-\pi$ **(C)** $-\frac{\pi}{2}$ **(D)** $\frac{\pi}{2}$

2. **(A)** The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is -

(A) of area zero **(B)** right-angled isosceles **(C)** equilateral **(D)** obtuse-angled isosceles

(B) Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin. Then n must be of the form

(A) 4k + 1 **(B)** 4k + 2 **(C)** 4k + 3 **(D)** 4k **[JEE 2001]**

3. **(A)** Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is - **[JEE 2002]**

(A) 3ω **(B)** $3\omega(\omega - 1)$ **(C)** $3\omega^2$ **(D)** $3\omega(1 - \omega)$

(B) For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is **[JEE 2002]**

(A) 0 **(B)** 2 **(C)** 7 **(D)** 17

(C) Let a complex number α, $\alpha \neq 1$, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$ where p,q are distinct primes. Show that either $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + .. + \alpha^{q-1} = 0$, but not both together. **[JEE 2002]**

4. If $|z| = 1$ and $\omega = \frac{z-1}{z+1}$ (where $z \neq -1$), then Re (w) equals – **[JEE 2003]**

(A) 0 **(B)** $-\frac{1}{|z+1|^2}$ **(C)** $\left|\frac{z}{z+1}\right| \cdot \frac{1}{|z+1|^2}$ **(D)** $\frac{\sqrt{2}}{|z+1|^2}$

5. If z_1 and z_2 are two complex numbers such that $|z_1| < 1$ and $|z_2| > 1$ then show that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$ **[JEE 2003]**

6. Show that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_i| < 2$ for i = 1, 2,.......n. **[JEE 2003]**

7. The least positive value of 'n' for which $(1 + \omega^2)^n = (1 + \omega^4)^n$, where ω is a non real cube root of unity is -

(A) 2 **(B)** 3 **(C)** 6 **(D)** 4 **[JEE 2004]**

8. Find the centre and radius formed by all the points represented by z = x + i y satisfying the relation $\frac{|z-\alpha|}{|z-\beta|} = K \ (K \neq 1)$ where α & β are constant complex numbers, given by $\alpha = \alpha_1 + i\alpha_2$ & $\beta = \beta_1 + i\beta_2$ **[JEE 2004]**

9. If a, b, c are integers not all equal and ω is cube root of unity ($\omega \neq 1$) then the minimum value of $|a + b\omega + c\omega^2|$ is - **[JEE 2005]**

(A) 0 **(B)** 1 **(C)** $\frac{\sqrt{3}}{2}$ **(D)** $\frac{1}{2}$

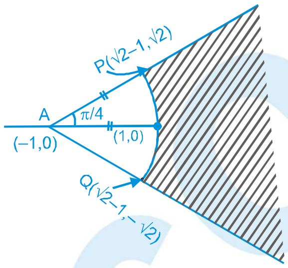
10. Area of shaded region belongs to - **[JEE 2005]**

(A) $z : |z + 1| > 2, |\arg(z + 1)| < \pi/4$

(B) $z : |z - 1| > 2, |\arg(z - 1)| < \pi/4$

(C) $z : |z + 1| < 2, |\arg(z + 1)| < \pi/2$

(D) $z : |z - 1| < 2, |\arg(z - 1)| < \pi/2$



11. If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}i$. Find the other vertices of square. **[JEE 2005]**

12. If $w = \alpha + i\beta$ where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\frac{w - \bar{w}z}{1 - z}$ is purely real, then the set of values of z is - **[JEE 2006]**

(A) $\{z : |z| = 1\}$ **(B)** $\{z : z = \bar{z}\}$ **(C)** $\{z : z \neq 1\}$ **(D)** $\{z : |z| = 1, z \neq 1\}$

13. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is : **[JEE 2007]**

(A) $3e^{i\pi/4} + 4i$ **(B)** $(3 - 4i)e^{i\pi/4}$ **(C)** $(4 + 3i)e^{i\pi/4}$ **(D)** $(3 + 4i)e^{i\pi/4}$

14. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1 - z^2}$ lie on : **[JEE 2007]**

(A) a line not passing through the origin **(B)** $|z| = \sqrt{2}$

(C) the x-axis **(D)** the y-axis

Comprehension (for 15 to 17)

Let A, B, C be three sets of complex numbers as defined below **[JEE 2008]**

$$A = \{z : \text{Im } z \geq 1\}$$
$$B = \{z : |z - 2 - i| = 3\}$$
$$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$$

15. The number of elements in the set $A \cap B \cap C$ is -

(A) 0 **(B)** 1 **(C)** 2 **(D)** ∞

16. Let z be any point in $A \cap B \cap C$. Then $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between -

(A) 25 and 29 **(B)** 30 and 34 **(C)** 35 and 39 **(D)** 40 and 44

17. Let z be any point in $A \cap B \cap C$ and let ω be any point satisfying $|\omega - 2 - i| < 3$. Then, $|z| - |\omega| + 3$ lies between -

(A) –6 and 3 **(B)** –3 and 6 **(C)** –6 and 6 **(D)** –3 and 9

18. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by - **[JEE 2008]**

(A) $6 + 7i$ **(B)** $-7 + 6i$ **(C)** $7 + 6i$ **(D)** $-6 + 7i$

19. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is - **[JEE 2009]**

(A) $\frac{1}{\sin 2°}$ **(B)** $\frac{1}{3\sin 2°}$ **(C)** $\frac{1}{2\sin 2°}$ **(D)** $\frac{1}{4\sin 2°}$

20. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is - **[JEE 2009]**

(A) 48 **(B)** 32 **(C)** 40 **(D)** 80

21. Match the conics in Column I with the statements/ expressions in Column II. **[JEE 2009]**

	Column I		Column II
(A)	Circle	**(p)**	The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(B)	Parabola		
(C)	Ellipse	**(q)**	Points z in the complex plane satisfying $\lvert z+2\rvert - \lvert z-2\rvert = \pm 3$
(D)	Hyperbola	**(r)**	Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y = \frac{2t}{1+t^2}$
		(s)	The eccentricity of the conic lies in the interval $1 \le x < \infty$
		(t)	Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \lvert z\rvert^2 + 1$

22. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then **[JEE 2010]**

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

23. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to **[JEE 2010]**

24. Match the statements in Column-I with those in Column-II. **[JEE 2010]**

[**Note :** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column I		Column II	
(A)	The set of points z satisfying $\|z-i\|z\|\| = \|z+i\|z\|\|$ is contained in or equal to	**(p)**	an ellipse with eccentricity $\frac{4}{5}$
(B)	The set of points z satisfying $\|z+4\| + \|z-4\| = 10$ is contained in or equal to	**(q)**	the set of points z satisfying Im z = 0
(C)	If $\|w\| = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	**(t)**	the set of points z satisfying $\|\text{Im } z\| \le 1$
(D)	If $\|w\| = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	**(s)**	the set of points z satisfying $\|\text{Re } z\| \le 2$
		(t)	the set of points z satisfying $\|z\| \le 3$

25. **Comprehension (3 questions together)**

Let a,b and c be three real numbers satisfying

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \text{...(E)}$$

(i) If the point P(a,b,c), with reference to (E), lies on the plane 2x + y + z = 1, then the value of 7a+b+c is

(A) 0 **(B)** 12 **(C)** 7 **(D)** 6

(ii) Let ω be a solution of $x^3 - 1 = 0$ with Im(ω) > 0. If a = 2 with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to -

(A) –2 **(B)** 2 **(C)** 3 **(D)** –3

(iii) Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is -

(A) 6 **(B)** 7 **(C)** $\frac{6}{7}$ **(D)** ∞ **[JEE 2011]**

26. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is **[JEE 2011]**

27. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$a + b + c = x$

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z.$

Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[JEE 2011]**

28. Match the statements given in Column I with the values given in Column II

	Column I		Column II
(A)	If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	**(p)**	$\frac{\pi}{6}$
(B)	If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	**(q)**	$\frac{2\pi}{3}$
(C)	The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	**(r)**	$\frac{\pi}{3}$
(D)	The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for $|z| = 1, z \neq 1$ is given by	**(s)**	π
		(t)	$\frac{\pi}{2}$

[JEE 2011]

29. Match the statements given in Column I with the intervals/union of intervals given in Column II

	Column I		Column II
(A)	The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number}, |z| = 1, z \neq \pm 1\right\}$ is	**(p)**	$(-\infty, -1) \cup (1, \infty)$
(B)	The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	**(q)**	$(-\infty, 0) \cup (0, \infty)$
(C)	If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is	**(r)**	$[2, \infty)$
(D)	If $f(x) = x^{3/2}(3x - 10), x \ge 0$, then $f(x)$ is increasing in	**(s)**	$(-\infty, -1] \cup [1, \infty)$
		(t)	$(-\infty, 0] \cup [2, \infty)$

[JEE 2011]

30. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a cannot take the value - **[JEE 2012]**

(A) -1 **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{4}$

31. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$ **[JEE Ad. 2013]**

(A) $\frac{1}{\sqrt{2}}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{\sqrt{7}}$ **(D)** $\frac{1}{3}$

32. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n = **[JEE Ad.]**

(A) 57 **(B)** 55 **(C)** 58 **(D)** 56

33. Let $w=\frac{\sqrt{3}+i}{2}$ and $P=\{w^n : n=1,2,3,.....\}$. Further $H_1=\left\{z\in C : \operatorname{Re} z>\frac{1}{2}\right\}$ and $H_2=\left\{z\in C : \operatorname{Re} z<\frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P\cap H_1$, $z_2\in P\cap H_2$ and O represents the origin, then $\angle z_1Oz_2=$ **[JEE-Ad. 2013]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Paragraph for Question 34 and 35

Let $S=S_1\cap S_2\cap S_3$, where $S_1=\{z\in C : |z|<4\}$, $S_2=\left\{z\in C : \operatorname{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right]>0\right\}$ and $S_3=\{z\in C : \operatorname{Re} z>0\}$.

34. $\min\limits_{z\in S}|1-3i-z|=$ **[JEE Ad. 2013]**

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

35. Area of S = **[JEE Ad. 2013]**

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

36. Let $z_k=\cos\left(\frac{2k\pi}{10}\right)+i\sin\left(\frac{2k\pi}{10}\right)$; $k=1,2,.....,9$. **[JEE Ad. 2014]**

	List - I		List - II
(p)	For each z_k there exists a z_j such $z_k . z_j=1$	(1)	True
(q)	There exists a $k\in\{1,2,.....,9\}$ such that $z_1 . z=z_k$ has no solution z in the set of complex numbers	(2)	False
(r)	$\frac{\|1-z_1\|\|1-z_2\|.......\|1-z_9\|}{10}$ equals	(3)	1
(s)	$1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	(4)	2

Codes :

	p	q	r	s
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

37. For any integer k, let $\alpha_k = \left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expresion

$$\frac{\sum_{k=1}^{12}\left|\alpha_{k+1} - \alpha_k\right|}{\sum_{k=1}^{3}\left|\alpha_{4k-1} - \alpha_{4k-2}\right|}$$ is **[JEE Ad. 2015]**

38. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s $\in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2.

Then the total number of ordered pairs (r, s) for which $P^2 = -1$ is **[JEE Ad. 2016]**

39. Let a, b $\in$ R and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in C : z = \frac{1}{a+ibt'} t \in R, t \neq 0\right\}$, where

$i = \sqrt{-1}$. if $z = x + iy$ and $z \in S$, then (x,y) lies on

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(C) the x-axis for $a \neq 0, b = 0$

(D) the y-axis for $a = 0, b \neq 0$ **[JEE Ad. 2016]**

MOCK TEST

SECTION - I : STRAIGHT OBJECTIVE TYPE

1. If 'p' and 'q' are distinct prime numbers, than the number of distinct imaginary numbers which are p^{th} as well as q^{th} roots of unity are -

(A) $\min^m(p, q)$ (B) $\max^m(p, q)$ (C) 1 (D) zero

2. Number of solution of the equation $z^3 + \frac{3(\bar{z})^2}{|z|} = 0$ where z is a complex number is

(A) 2 (B) 3 (C) 6 (D) 5

3. If 1, α_1, α_2, α_3........ and α_8 are nine, ninth roots of unity (taken in counter-clockwise sequence) then $|(2-\alpha_1)(2-\alpha_3)(2-\alpha_5)(2-\alpha_7)|$ is equal to

(A) $\sqrt{255}$ (B) $\sqrt{511}$ (C) $\sqrt{1023}$ (D) 15

4. The point of intersection the curves $\arg(z-i+2) = \frac{\pi}{6}$ & $\arg(z+4-3i) = -\frac{\pi}{4}$ is given by

(A) $(-2+i)$ (B) $2-i$ (C) $2+i$ (D) none of these

5. If $|z_2 + iz_1| = |z_1| + |z_2|$ and $|z_1| = 3$ & $|z_2| = 4$ then area of Δ ABC, if affix of A, B & C are (z_1), (z_2) and $\left(\frac{z_2 - iz_1}{1-i}\right)$ respectively, is

(A) $\frac{5}{2}$ (B) 0 (C) $\frac{25}{2}$ (D) $\frac{25}{4}$

6. The principal argument of the complex number $\frac{(1+i)^5(1+\sqrt{3}\,i)^2}{-2i(-\sqrt{3}+i)}$ is

(A) $\frac{19\pi}{12}$ (B) $-\frac{7\pi}{12}$ (C) $-\frac{5\pi}{12}$ (D) $\frac{5\pi}{12}$

7. Image of the point, whose affix is $\frac{2-i}{3+i}$, in the line $(1+i)z + (1-i)\bar{z} = 0$ is the point whose affix is

(A) $\frac{1+i}{2}$ (B) $\frac{1-i}{2}$ (C) $\frac{-1+i}{2}$ (D) $-\frac{1+i}{2}$

8. If a complex number z satisfies $|2z+10+10i| \le 5\sqrt{3} - 5$, then the least principal argument of z is

(A) $-\frac{11\pi}{12}$ (B) $-\frac{2\pi}{3}$ (C) $-\frac{5\pi}{6}$ (D) $-\frac{3\pi}{4}$

9. If t and c are two complex numbers such that $|t| \ne |c|$, $|t| = 1$ and $z = \frac{at+b}{t-c}$, $z = x + iy$. Locus of z is (where a, b are complex numbers)

(A) line segment (B) straight line (C) circle (D) none

10. S_1 : Let z_k (k = 0, 1, 2, 3, 4, 5, 6) be the roots of the equation $(z + 1)^7 + (z)^7 = 0$ then $\sum_{k=0}^{6} \text{Re}(z_k)$ is equal to $-\frac{7}{2}$

S_2 : If α, β, γ and a, b, c are complex numbers such that $\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c} = 1 + i$ and $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma} = 0$, then the value of $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}$ is equal to – 1

S_3 : If $z_1, z_2, ... z_6$ are six roots of the equation $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0$ then the value of $\prod_{i=1}^{6}(z_i + 1)$ is equal to 4

S_4 : Number of solutions of the equation $z^3 = \bar{z}\, i|z|$ are 5

(A) TTFT **(B)** TFFT **(C)** FFTF **(D)** TTFF

SECTION - II : MULTIPLE CORRECT ANSWER TYPE

11. If n is the smallest positve integer for which $(a + ib)^n = (a - ib)^n$ where a > 0 & b > 0 then the numerical value of b/a is :

(A) $\tan\frac{\pi}{3}$ **(B)** $\sqrt{3}$ **(C)** 3 **(D)** $\frac{1}{\sqrt{3}}$

12. If z is a complex number satisfying |z – i Re (z) | = |z – Im (z)| then z lies on

(A) y = x **(B)** y = – x **(C)** y = x + 1 **(D)** y = – x + 1

13. If $z_1 = 5 + 12i$ and $|z_2| = 4$ then

(A) maximum $(|z_1 + iz_2|) = 17$

(B) minimum $(|z_1 + (1 + i)z_2|) = 13 - 9\sqrt{2}$

(C) minimum $\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{4}$

(D) maximum $\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3}$

14. If α, β be the roots of the equation $\mu^2 - 2\mu + 2 = 0$ and if $\cot\theta = x + 1$, then $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta}$ is equal to

(A) $\frac{\sin n\theta}{\sin^n\theta}$ **(B)** $\frac{\cos n\theta}{\cos^n\theta}$ **(C)** $\frac{\sin n\theta}{\cos^n\theta}$ **(D)** $\frac{\text{cosec}^n\theta}{\text{cosec}\, n\theta}$

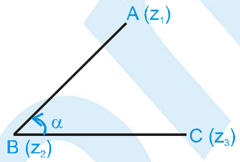
15. If z_1 lies on |z| = 1 and z_2 lies on |z| = 2, then

(A) $3 \le |z_1 - 2z_2| \le 5$ **(B)** $1 \le |z_1 + z_2| \le 3$ **(C)** $|z_1 - 3z_2| \ge 5$ **(D)** $|z_1 - z_2| \ge 1$

SECTION - III : ASSERTION AND REASON TYPE

16. **Statement - I :** If $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of an equilateral triangle ABC, then $\arg\left(\frac{z_2+z_3-2z_1}{z_3-z_2}\right)=\frac{\pi}{4}$

Statement - II : If $\angle B=\alpha$, then $\frac{z_1-z_2}{z_3-z_2}=\frac{AB}{BC}e^{i\alpha}$ or $\arg\left(\frac{z_1-z_2}{z_3-z_2}\right)=\alpha$



(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

17. **Statement - I :** If $x+\frac{1}{x}=1$ and $p=x^{4000}+\frac{1}{x^{4000}}$ and q be the digit at unit place in the number $2^{2^n}+1$, $n\in N$ and $n>1$, then the value of $p+q=8$.

Statement - II : ω,ω^2 are the roots of $x+\frac{1}{x}=-1$, then $x^2+\frac{1}{x^2}=-1$, $x^3+\frac{1}{x^3}=2$

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

18. **Statement - I :** If z_1, z_2, z_3 are complex number representing the points A, B, C such that $\frac{2}{z_1}=\frac{1}{z_2}+\frac{1}{z_3}$. Then circle through A, B, C passes through origin.

Statement - II : If $2z_2=z_1+z_3$ then z_1, z_2, z_3 are collinear.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

19. **Statement - I :** $3+ix^2y$ and x^2+y+4i are complex conjugate numbers, then $x^2+y^2=4$.

Statement - II : If sum and product of two complex numbers is real then they are conjugate complex number.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

20. **Statement - I :** If $|z|<\sqrt{2}-1$, then $|z^2+2z\cos\alpha|<1$

Statement - II : $|z_1+z_2|\le|z_1|+|z_2|$ also $|\cos\alpha|\le 1$.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I.
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

SECTION - IV : MATRIX - MATCH TYPE

21. **Column - I** / **Column - II**

(A) Locus of the point z satisfying the equation $Re(z^2) = Re(z + \bar{z})$ — **(p)** A parabola

(B) Locus of the point z satisfying the equation $|z - z_1| + |z - z_2| = \lambda, \lambda \in R^+$ and $\lambda \nless |z_1 - z_2|$ — **(q)** A straight line

(C) Locus of the point z satisfying the equation $\left|\frac{2z - i}{z + 1}\right| = m$ where $i = \sqrt{-1}$ and $m \in R^+$ — **(r)** An ellipse

(D) If $|\bar{z}| = 25$ then the points representing the complex number $-1 + 75\bar{z}$ will be on a — **(s)** A rectangular hyperbola

(t) A circle

22. If z_1, z_2, z_3, z_4 are the roots of the equation $z^4 + z^3 + z^2 + z + 1 = 0$ then

Column – I	Column – II
(A) $\left|\sum_{i=1}^{4} z_i^4\right|$ is equal to	**(p)** 0
(B) $\sum_{i=1}^{4} z_i^5$ is equal to	**(q)** 4
(C) $\prod_{i=1}^{4} (z_i + 2)$ is equal to	**(r)** 1
(D) least value of $[|z_1 + z_2|]$ is (Where [ ] represents greatest integer function)	**(s)** 11
	(t) $\left|4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right|$

SECTION - V : COMPREHENSION TYPE

23. **Read the following comprehension carefully and answer the questions.**

The complex slope of a line passing through two points represented by complex numbers z_1 and z_2 is defined by $\frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$ and we shall denote by ω. If z_0 is complex number and c is a real number, then $\bar{z}_0 z + z_0 \bar{z} + c = 0$ represents a straight line. Its complex slope is $-\frac{z_0}{\bar{z}_0}$. Now consider two lines

$\alpha\bar{z} + \bar{\alpha}z + i\beta = 0$ ...**(i)** and $a\bar{z} + \bar{a}z + b = 0$ ...**(ii)**

where α, β and a, b are complex constants and let their complex slopes be denoted by ω_1 and ω_2 respectively

1. If the lines are inclined at an angle of 120° to each other, then

(A) $\omega_2\bar{\omega}_1 = \omega_1\bar{\omega}_1$ **(B)** $\omega_2\bar{\omega}_1^2 = \omega_1\bar{\omega}_2^2$ **(C)** $\omega_1^2 = \omega_2^2$ **(D)** $\omega_1 + 2\omega_2 = 0$

2. Which of the following must be true

(A) a must be pure imaginary **(B)** β must be pure imaginary

(C) a must be real **(D)** b must be imaginary

3. If line **(i)** makes an angle of 45° with real axis, then $(1+i)\left(-\frac{2\alpha}{\bar{\alpha}}\right)$ is

(A) $2\sqrt{2}$ **(B)** $2\sqrt{2}\,i$ **(C)** $2(1-i)$ **(D)** $-2(1+i)$

24. **Read the following comprehension carefully and answer the questions.**

Let $(1+x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$. For sum of series $C_0 + C_1 + C_2 + ...$, put $x = 1$. For sum of series $C_0 + C_2 + C_4 + C_6 + ...$, or $C_1 + C_3 + C_5 + ...$ add or substract equations obtained by putting $x = 1$ and $x = -1$.

For sum of series $C_0 + C_3 + C_6 + ...$ or $C_1 + C_4 + C_7 + ...$ or $C_2 + C_5 + C_8 + ...$ we subsitute $x = 1$, $x = \omega$, $x = \omega^2$ and add or manupulate results.

Similarly, if suffixes differ by 'p' then we substitute p^{th} roots of unity and add.

1. $C_0 + C_3 + C_6 + C_9 + ... =$

(A) $\frac{1}{3}\left[2^n - 2\cos\frac{n\pi}{3}\right]$ **(B)** $\frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$ **(C)** $\frac{1}{3}\left[2^n - 2\sin\frac{n\pi}{3}\right]$ **(D)** $\frac{1}{3}\left[2^n + 2\sin\frac{n\pi}{3}\right]$

2. $C_1 + C_5 + C_9 + ... =$

(A) $\frac{1}{4}\left[2^n - 2^{n/2}2\cos\frac{n\pi}{4}\right]$ **(B)** $\frac{1}{4}\left[2^n + 2^{n/2}2\cos\frac{n\pi}{4}\right]$

(C) $\frac{1}{4}\left[2^n - 2^{n/2}2\sin\frac{n\pi}{4}\right]$ **(D)** $\frac{1}{4}\left[2^n + 2^{n/2}2\sin\frac{n\pi}{4}\right]$

3. $C_2 + C_6 + C_{10} + ... =$

(A) $\frac{1}{4}\left[2^n - 2^{n/2}\,2.\cos\frac{n\pi}{4}\right]$ **(B)** $\frac{1}{4}\left[2^n + 2^{n/2}\,2.\cos\frac{n\pi}{4}\right]$

(C) $\frac{1}{4}\left[2^n - 2^{n/2}\,2.\sin\frac{n\pi}{4}\right]$ **(D)** $\frac{1}{4}\left[2^n + 2^{n/2}\,2.\sin\frac{n\pi}{4}\right]$

25. **Read the following comprehension carefully and answer the questions.**

Consider ΔABC in Argand plane. Let A(0), B(1) and C(1 + i) be its vertices and M be the mid point of CA. Let z be a variable complex number in the plane. Let u be another variable complex number defined as $u = z^2 + 1$

1. Locus of u, when z is on BM, is

(A) Circle **(B)** Parabola **(C)** Ellipse **(D)** Hyperbola

2. Axis of locus of u, when z is on BM, is

(A) real – axis **(B)** Imaginary – axis **(C)** $z + \bar{z} = 2$ **(D)** $z - \bar{z} = 2i$

3. Directrix of locus of u, when z is on BM, is

(A) real– axis **(B)** imaginary – axis **(C)** $z + \bar{z} = 2$ **(D)** $z - \bar{z} = 2i$

SECTION - VI : INTEGER TYPE

26. If $\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\left(\sec^{-1}\frac{1}{x}+\sin^{-1}x\right)$ $x \neq 0, -1 \le x \le 1$, then find the number of positive integers less than 20 satisfying above equation.

27. Let $f_p(\alpha) = e^{\frac{i\alpha}{p^2}}, e^{\frac{2i\alpha}{p^2}} ......e^{\frac{i\alpha}{p}}$ $p \in N$ (where $i = \sqrt{-1}$, then find the value of $\left|\lim_{n\to\infty} f_n(\pi)\right|$

28. If $|z| = \min(|z-1|, |z+1|\}$, then find the value of $|z + \bar{z}|$.

29. If z is a complex number and the minimum value of $|z| + |z-1| + |2z-3|$ is λ and if $y = 2[x] + 3 = 3[x-\lambda]$, then find the value of $[x + y]$ (where [.] denotes the greatest integer function}

30. If $\alpha = e^{\frac{2\pi i}{7}}$ and $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$, then the value of $\sum_{r=0}^{6} f(\alpha^r x) = n(A_0 + A_n x^n + A_{2n} x^{2n})$ then find the value of n.

ANSWER KEY

EXERCISE - 1

1. C **2.** B **3.** D **4.** A **5.** C **6.** B **7.** A **8.** D **9.** A **10.** A **11.** D **12.** C **13.** B
14. D **15.** D **16.** A **17.** B **18.** A **19.** B **20.** A **21.** B **22.** A **23.** B **24.** B **25.** C **26.** A
27. D **28.** C **29.** B **30.** A **31.** B **32.** C **33.** D **34.** A **35.** A

EXERCISE - 2 : PART # I

1. AC **2.** ABD **3.** AB **4.** BC **5.** ACD **6.** BD **7.** ABC **8.** BCD **9.** AD
10. AC **11.** AB **12.** ABC **13.** BC **14.** BCD **15.** ABC **16.** AC **17.** ACD **18.** ABCD
19. ACD **20.** BC **21.** CD **22.** AB **23.** ABD **24.** ABCD **25.** AC

PART - II

1. D **2.** B **3.** B **4.** B **5.** A **6.** A **7.** C **8.** B

EXERCISE - 3 : PART # I

1. $A \rightarrow s$ $B \rightarrow p$ $C \rightarrow q$ $D \rightarrow r$ **2.** $A \rightarrow p$ $B \rightarrow q$ $C \rightarrow$, $D \rightarrow s$ **3.** $A \rightarrow p$ $B \rightarrow r$ $C \rightarrow t$ $D \rightarrow q,s$
4. $A \rightarrow q$ $B \rightarrow p$ $C \rightarrow q,s$ $D \rightarrow r$

PART - II

Comprehension # 1 : 1. A **2.** C **3.** A **Comprehension # 2 : 1.** D **2.** C **3.** B
Comprehension # 3 : 1. A **2.** C **3.** B **Comprehension # 4 : 1.** C **2.** A **3.** B
Comprehension # 5 : 1. C **2.** B **3.** A

EXERCISE - 5 : PART # I

1. 4 **2.** 4 **3.** 3 **4.** 4 **5.** 1 **6.** 3 **7.** 2 **8.** 4 **9.** 4 **10.** 4 **11.** 3 **12.** 3 **13.** 4
14. 2 **15.** B **16.** 4 **17.** 2 **18.** 3 **19.** 2 **20.** 1 **21.** 3

PART - II

1. **(A)** A **(B)** A **2.** **(A)** C, **(B)** D **3.** **(A)** B **(B)** B **4.** A **7.** B

8. $\frac{\alpha - k^2\beta}{1-k^2}$ & $\left|\frac{1}{k^2-1}\right|\sqrt{|\alpha - k^2\beta|^2 - (k^2|\beta|^2 - |\alpha|^2)(k^2-1)}$ **9.** B **10.** A

11. $(-\sqrt{3}\ i)$, $(1-\sqrt{3})+i$ and $(1+\sqrt{3})-i$ **12.** D **13.** D **14.** D **15.** B **16.** C **17.** D **18.** D **19.** D

20. A **21.** $A \rightarrow p$ $B \rightarrow s,t$ $C \rightarrow r$ $D \rightarrow q,s$ **22.** A, C, D **23.** 1

24. **(A)** $\rightarrow q,r$ **(B)** $\rightarrow p$ **(C)** $\rightarrow p,s,t$ **(D)** $\rightarrow q,r,s,t$ **25.** **(i)** D, **(ii)** A, **(iii)** B **26.** 5 **27.** 3

28. **(A)** $\rightarrow q$ **(B)** $\rightarrow p$ **(C)** $\rightarrow s$ **(D)** $\rightarrow t$ **29.** **(A)** $\rightarrow s$ **(B)** $\rightarrow t$ **(C)** $\rightarrow r$ **(D)** $\rightarrow r$ **30.** D

31. C **32.** BCD **33.** CD **34.** C **35.** B **36.** C **37.** 4 **38.** 1 **39.** ACD

MOCK TEST

1. D	**2.** D	**3.** B	**4.** D	**5.** D	**6.** C	**7.** C	**8.** C	**9.** C
10. B	**11.** AB	**12.** AB	**13.** AD	**14.** AD	**15.** ABCD	**16.** D	**17.** D	**18.** B
19. D	**20.** A	**21.** $A \to s$ $B \to q,r$ $C \to a,t$ $D \to t$			**22.** $A \to r$ $B \to q,t$ $C \to s$ $D \to p$			
23. **1.** B	**2.** B	**3.** C	**24.** **1.** B	**2.** D	**3.** A	**25.** **1.** B	**2.** C	**3.** D
26. 4	**27.** 1	**28.** 1	**29.** 30	**30.** 7				