COMPLEX NUMBER

EXERCISE # 1

Question based on **Power of iota**

Q.1 The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = -1$ is -

(A) 1 (B) 2 (C) 3 (D) 4

Sol. **[B]**

$\left(\frac{1+i}{1-i}\right)^n = \left[\frac{(1+i)}{(1-i)} \times \frac{(1+i)}{(i+i)}\right]^n$

$= \left[\frac{(1+i)^2}{1+1}\right]^n$

$= \left[\frac{1-1+2i}{2}\right]^n$

$= (i)^n$

Smallest positive integer must be 2 so that $(i)^2 = -1$

∴ Option (B) is correct answer.

Q.2 The value of $\sum_{n=0}^{100} i^{n!}$ equals (where $i = \sqrt{-1}$)

(A) –1 (B) i (C) 2i + 95 (D) 97 + i

Sol. **[C]**

$\sum_{n=0}^{100} i^{n!}$

$= i^{0!} + i^{1!} + i^{2!} + i^{3!} + i^{4!} + \ldots + i^{100!}$

$= i + i + i^2 + i^6 + i^{4!} + \ldots + i^{100!}$

$= 2i - 2 + (1 + 1 + \ldots 97 \text{ times})$

$= 2i + 95$

Q.3 The value of the sum $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals -

(A) i (B) i – 1 (C) –i (D) 0

Sol. **[B]**

Question based on **Representation of complex number**

Q.4 $\left(\frac{-1-i}{\sqrt{2}}\right)^{100}$ equals-

(A) 1 (B) i (C) – i (D) – 1

Sol. **[D]**

$\left(\frac{-1-i}{\sqrt{2}}\right)^{100} = \left(\frac{1+i}{\sqrt{2}}\right)^{100}$

$= \left[\frac{(1+i)^2}{2}\right]^{50} = i^{50} = i^2 = -1$

Q.5 If $z^2 / (z - 1)$ is always real, then z, can lie on-

(A) real axis (B) a parabola

(C) imaginary axis (D) None of these

Sol. **[A]**

$\Theta \ \frac{(x+iy)^2}{(x-1+iy)} \times \frac{((x-1)-iy)}{((x-1)-iy)}$

$= \frac{(x^2 - y^2 + 2i\,xy)[(x-1)-iy]}{(x-1)^2 + y^2}$

always real so

$2xy(x-1) - y(x^2 - y^2) = 0$

$\Rightarrow y(2x^2 - 2x - x^2 + y^2) = 0$

$\Rightarrow y(x^2 - 2x + y^2) = 0$

$\Rightarrow y = 0$ real axis

Q.6 If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is-

(A) equal to 1 (B) less than 1

(C) greater than 3 (D) equal to 3

Sol. **[A]**

Given : $|z_1| = |z_2| = |z_3|$

$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$

$|z_1| = 1 = z_1 \bar{z}_1$

$|z_2| = 1 = z_2 \bar{z}_2$

$|z_3| = 1 = z_3 \bar{z}_3$

$\Rightarrow \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$

$\Rightarrow \left|\frac{\bar{z}_1}{z_1\bar{z}_1}+\frac{\bar{z}_2}{z_2\bar{z}_2}+\frac{\bar{z}_3}{z_3\bar{z}_3}\right| = 1$

$\Rightarrow \left|\frac{\bar{z}_1}{|z_1|^2}+\frac{\bar{z}_2}{|z_2|^2}+\frac{\bar{z}_3}{|z_3|^2}\right| = 1$

$\Rightarrow |\bar{z}_1+\bar{z}_2+\bar{z}_3| = 1$

$\Rightarrow |\overline{z_1+z_2+z_3}| = 1$

$\Rightarrow |z_1 + z_2 + z_3|$

$\therefore$ Option (A) is correct answer.

Q.7 $\left(\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n =$

(A) $\cos n\theta + i\sin n\theta$ (B) $\sin n\theta + i\cos n\theta$

(C) $\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}$ (D) $\cos n\theta$

Sol. **[A]**

$$\frac{(1+\cos\theta+i\sin\theta)^n}{(1+\cos\theta-i\sin\theta)^n} = \left(\frac{2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)^n$$

$$= \frac{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^n}{\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)^n} = \frac{e^{i\frac{n\theta}{2}}}{e^{-\frac{in\theta}{2}}}$$

$= e^{in\theta}$

$= \cos n\theta + i\sin n\theta$

Q.8 If $(x + iy)^{1/5} = a + ib$, and $u = x/a - y/b$, then

(A) $a - b$ is a factor of u

(B) $a + b$ is a factor of x

(C) $a + ib$ is a factor of y

(D) $a - ib$ is a factor of a

Sol. **[A]**

$(x + iy)^{1/5} = (a + ib) \Rightarrow (x + iy) + (a + ib)^5$

$x = {}^5C_0a^5 - {}^5C_2a^3b^2 + {}^5C_4ab^4$

$y = {}^5C_1a^4b - {}^5C_3a^2b^3 + {}^5C_5b^5$

$u = \frac{x}{a} - \frac{y}{b} = ({}^5C_0a^4 - {}^5C_2a^2b^2 + {}^5C_4b^4)$

$\qquad - ({}^5C_1a^4 - {}^5C_3a^2b^2 + {}^5C_5b^4)$

$u = 4(b^4 - a^4) = -4(a^2 + b^2)(a - b)(a + b)$

Question based on

Algebraic operation of a complex number

Q.9 z_1, z_2, z_3, z_4 be the vertices A, B, C, D respectively of a square on the argand diagram taken in anticlockwise direction then

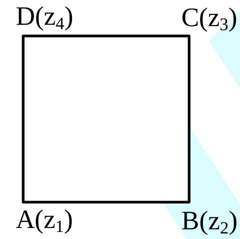
(A) $2z_2 = (1 + i)z_1 + (1 - i)z_3$

(B) $2z_4 = (1 - i)z_1 + (1 + i)z_3$

(C) $2z_2 = (1 + i)z_1 - (1 - i)z_3$

(D) $2z_4 = (1 - i)z_1 - (1 + i)z_3$

Sol. **[A, B]**



$\because \frac{z_1+z_3}{2} = \frac{z_2+z_4}{2}$

$\Rightarrow z_1 + z_3 = z_2 + z_4$ ....(i)

ABCD is a square so

$\overrightarrow{AD} = \overrightarrow{AB}\, e^{i\pi/2}$

$\Rightarrow (z_4 - z_1) = (z_2 - z_1)i$ ...(ii)

From (i)

$\Rightarrow z_4 = z_1 + (z_1 + z_3 - z_4 - z_1)i$

$\Rightarrow z_4(1 + i) = z_1 + iz_3$

$\Rightarrow 2z_4 = z_1(1 - i) + z_3(1 + i)$

Again from (i) and (ii)

$(z_1 + z_3 - z_2 - z_1) = z_2i - z_1i$

$\Rightarrow z_2(1 + i) = z_1i + z_3$

$\Rightarrow 2z_2 = z_1(1 + i) + z_3(1 - i)$

Q.10 Let the complex numbers z_1, z_2, z_3 represents vertices of an equilateral triangle. If z_0 be the circumcentre of the triangle then $z_1^2 + z_2^2 + z_3^2 =$

(A) z_0^2 (B) $2z_0^2$

(C) $3z_0^2$ (D) $9z_0^2$

Sol. **[C]**

$\because$ triangle is equilateral so

circumcenter $= \frac{z_1+z_2+z_3}{3} = z_0$

$\Rightarrow z_1 + z_2 + z_3 = 3z_0$

$\Rightarrow (z_1 + z_2 + z_3)^2 = 9z_0^2$

$\Rightarrow z_1^2+z_2^2+z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1) = 9z_0^2$

$\because z_1^2+z_2^2+z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$\Rightarrow 3(z_1^2+z_2^2+z_3^2) = 9z_0^2$

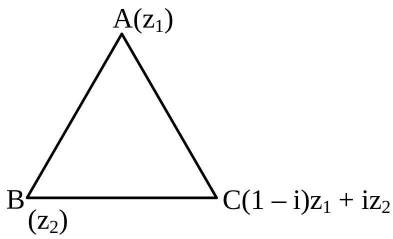
$\Rightarrow z_1^2+z_2^2+z_3^2 = 3z_0^2$

Q.11 The point z_1, z_2 and $(1 - i)z_1 + iz_2$ of a complex plane are the vertices of a triangle which is -

(A) right angled isosceles

(B) equilateral
(C) isosceles
(D) scalene

Sol. **[A]**



AB = $(z_2 - z_1)$
BC = $(1 - i)(z_1 - z_2)$
CA = $i(z_2 - z_1)$
Clearly $|AB| = |CA|$
angle between AB & CA is 90°.

Q.12 If $(3 + i)(z + \bar{z}) - (2 + i)(z - \bar{z}) + 14i = 0$, then $\bar{z}z$ is equal to-

(A) 10 (B) 8 (C) –9 (D) –10

Sol. **[A]**
$(3 + i)(z + \bar{z}) - (2 + i)(z - \bar{z}) + 14i = 0$
Let $z = x + iy$
$\Rightarrow (3 + i)(2x) - (2 + i)(2iy) + 14i = 0$
$\Rightarrow (3 + i)x - (2 + i)iy + 7i = 0$
$\Rightarrow (3x + y) + (x - 2y + 7)i = 0$
$\Rightarrow 3x + y = 0$ and $x - 2y + 7 = 0$
Solving we get $x = -1, y = 3$
$z\bar{z} = x^2 + y^2 = 1 + 9 = 10$

Q.13 If arg (z) < 0, then arg (– z) – arg (z) =

(A) π (B) $-\pi$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

Sol. **[A]** Given Arg (z) < 0
i.e. Arg (z) = – θ
Then = $r[(\cos(-\theta) + i\sin(-\theta)]$
$z = r[\cos\theta - i\sin\theta]$
$-z = -r[\cos\theta - i\sin\theta]$
$-z = -r\cos\theta + i\,r\sin\theta$
$-z = r[-\cos\theta + i\sin\theta]$
$-z = r[\cos(\pi - \theta) + i\sin(\pi - \theta)]$
$\therefore$ Arg $(-z) = \pi - \theta$
$\therefore$ Arg $(-z)$ – Arg $(z) = \pi - \theta - (-\theta)$
$= \pi - \theta + \theta$
$= \pi$
$\therefore$ Option (A) is correct answer.

Question based on
De Moiver's Theorem & roots of unity

Q.14 If 1, $\alpha_1, \alpha_2, \ldots\ldots \alpha_{n-1}$ are the n^{th} roots of unity, then $(1 + \alpha_1)(1 + \alpha_2) \ldots\ldots(1 + \alpha_{n-1})$ is equal to (when n is even)-

(A) n – 1 (B) n
(C) 0 (D) None of these

Sol. **[C]**
$x^n - 1 = 0$
$\Rightarrow x = (1)^{1/n}$
Given 1, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots\alpha_{n-1}$ are the nth roots of unity, then.
$(x^n - 1) = (x - 1)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)\ldots$
$\ldots\ldots\ldots\ldots\ldots (x - \alpha_{n-1})$
Put x = – 1, we get
$[(-1)^n - 1] = (-1 - 1)(-1 - \alpha_1)(-1 - \alpha_2)(-1 - \alpha_3)$
$(-1 - \alpha_4)\ldots\ldots\ldots(-1 - \alpha_{n-1})$
Since, n is even
$\Rightarrow (-2)(-1 - \alpha_1)(-1 - \alpha_2)(-1 - \alpha_3)(-1 - \alpha_4)\ldots$
$\ldots\ldots(-1 - \alpha_{n-1}) = 0$
$(1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3)(1 + \alpha_4)\ldots(1 + \alpha_{n-1}) = 0$
$\therefore$ Option (C) is correct answer.

Q.15 If $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_8$ are the 8, 8^{th} roots of unity then -

(A) $(\alpha_1)^3 + (\alpha_2)^3 + (\alpha_3)^3 + \ldots + (\alpha_8)^3 = 8$
(B) $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_8 = 1$
(C) $(\alpha_1)^{50} + (\alpha_2)^{50} \ldots + (\alpha_8)^{50} = 0$
(D) $(\alpha_1)^{16} + (\alpha_2)^{16} \ldots + (\alpha_8)^{16} = 0$

Sol. **[C]**
$x^8 - 1 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)\ldots\ldots$
$\ldots\ldots\ldots (x - \alpha_8)$
$\alpha_1^8 = 1; \alpha_2^8 = 1; \alpha_3^8 = 1; \alpha_4^8 = 1; \ldots\ldots\ldots; \alpha_8^8 = 1$
We have to check for every option as follows.
$(\alpha_1)^3 + (\alpha_2)^3 + (\alpha_3)^3 + (\alpha_4)^3 + \ldots + (\alpha_8)^3$
$= (\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_8)^3 - 6\Sigma\,\alpha_1\alpha_2\alpha_3 - 3\Sigma\alpha_1\alpha_2$
$= 0 - 0 - 0$
$= 0$
$\therefore$ Option (A) is not correct answer.
For B : Sum of roots, $\alpha_1 + \alpha_2 + \alpha_3 + \ldots.\alpha_8 = 0$
$\therefore$ Option (B) is not correct answer.
For C : $(\alpha_1)^{50} + (\alpha_2)^{50} + (\alpha_3)^{50} + (\alpha_4)^{50} + \ldots + (\alpha_8)^{50}$
$= (\alpha_1^8)^6\,\alpha_1^2 + (\alpha_2^8)^6\,\alpha_2^2 + (\alpha_3^8)^6\,\alpha_3^2 + (\alpha_4^8)^6\,\alpha_4^2$
$+ \ldots.. + (\alpha_8^8)^6\,\alpha_8^2$
$= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \ldots\ldots\alpha_8^2$
$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots\ldots\ldots.. + \alpha_8)^2 - 2\Sigma\alpha_1\alpha_2$
$= 0 - 0$
$= 0$
$\therefore$ Option (C) is correct answer.

For D : $(\alpha_1)^{16} + (\alpha_2)^{16} + (\alpha_3)^{16} + (\alpha_4)^{16} + \ldots\ldots + (\alpha_8)^{16}$

$= (\alpha_1^{8})^2 + (\alpha_2^{8})^2 + (\alpha_3^{8})^2 + (\alpha_4^{8})^2 + \ldots\ldots + (\alpha_8^{8})^2$

$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

$= 8$

Since, $\alpha_1^8 = 1;\ \alpha_2^8 = 1 \ldots\ldots\ldots ;\alpha_8^8 = 1$

$\therefore$ Option (D) is not correct answer.

Q.16 If $1, \omega, \omega^2, \ldots\ldots \omega^{n-1}$ are the n^{th} roots of unity, then $(2-\omega)(2-\omega^2)\ldots\ldots(2-\omega^{n-1})$ equals -

(A) $2^n - 1$

(B) ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$

(C) $[{}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots. + {}^{2n+1}C_n]^{1/2} - 1$

(D) none of these

Sol. **[A,B,C]**

Let equation $x^n - 1 = 0$ have n roots as

$(x^n - 1) = (x-1)(x-\omega)(x-\omega^2)(x-\omega^3) \ldots. \ldots. (x-\omega^{n-1})$

$\Rightarrow \dfrac{x^n - 1}{x-1} = (x-\omega)(x-\omega^2)(x-\omega^3) \ldots. (x-\omega^{n-1})$

Put x = 2, we get

$\Rightarrow \dfrac{2^n - 1}{2-1} = 2^n - 1$

$= (2-\omega)(2-\omega^2)(2-\omega^3) \ldots\ldots\ldots (2-\omega^{n-1})$

Also, $(1+1)^n = 2^n$

$= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$

$\Rightarrow {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots. + {}^nC_n = 2^n - 1$

Also, $(1+1)^{2n+1} = {}^{(2n+1)}C_0 + {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2 + {}^{(2n+1)}C_3 + \ldots\ldots\ldots + {}^{(2n+1)}C_n + {}^{2n+1}C_{n+1} + \ldots + {}^{(2n+1)}C_{2n+1}$

$\Rightarrow 2^{2n+1} = 2 \times ({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots.. + {}^{(2n+1)}C_n)$

$\Rightarrow \dfrac{2^{2n+1}}{2} = {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2 + \ldots + {}^{2n+1}C_n$

$\Rightarrow {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2 + \ldots + {}^{(2n+1)}C_n = 2^{2n}$

$\therefore ({}^{(2n+1)}C_1 + {}^{(2n+1)}C_1 + \ldots\ldots\ldots + {}^{(2n+1)}C_n)^{1/2} - 1 = (2^{2n})^{1/2} - 1$

$\therefore$ Option (A), (B) and (C) are correct answer.

Q.17 If $z = \dfrac{\sqrt{3} - i}{2}$, then $(i^{101} + z^{101})^{103}$ equals-

(A) iz (B) z

(C) $\bar{z}$ (D) None of these

Sol. **[B]**

$z = \dfrac{\sqrt{3} - i}{2} = -i\left(\dfrac{1 + i\sqrt{3}}{2}\right) = i\,\omega^2$

$\Theta\ (i^{101} + z^{101})^{103} = [\,i^{101} + i^{101}\omega^{202}]^{103}$

$= (i + i\omega)^{103} = [-i\omega^2]^{103} = -i^3\,\omega^{206} = i\omega^2 = z$

Q.18 $(-64)^{1/4}$ equals-

(A) $\pm 2(1 + i)$ (B) $\pm 2(1 - i)$

(C) $\pm 2(1 \pm i)$ (D) None of these

Sol. **[C]**

$(-64)^{1/4} = [(-64)^{1/2}]^{1/2} = (\pm 8i)^{1/2}$

$= [4(\pm 2i)]^{1/2} = [4(1 \pm 2i - 1)]^{1/2}$

$= [4(1 \pm i)^2]^{1/2} = \pm 2(1 \pm i)$

Q.19 If α is non real and $\alpha = \sqrt[5]{1}$ then the value of $2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is equal to-

(A) 4 (B) 2

(C) 1 (D) none of these

Sol. **[A]**

$2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$

$= 2^{|1+\alpha+\alpha^2+\alpha^3-\alpha^4|}$ $(\Theta\ \alpha^5 = 1)$

$= 2^{|-2\alpha^4|}$ $[\Theta\ 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0]$

$= 2^{2\left|\frac{1}{\alpha}\right|}$

$= 4$ $(\Theta\ |\alpha| = 1)$

Q.20 The value of $\sum_{k=1}^{6}\left(\sin\dfrac{2\pi k}{7} - i\cos\dfrac{2\pi k}{7}\right)$ is-

(A) –1 (B) 0

(C) –i (D) i

Sol. **[D]**

$-i\sum_{k=1}^{6}\cos\dfrac{2\pi k}{7} + i\sin\dfrac{2\pi k}{7}$

$-i\left[\left\{\sum_{k=0}^{6}\cos\dfrac{2\pi k}{7} + i\sin\dfrac{2\pi k}{7}\right\} - \left(\cos\dfrac{2\pi k}{7} + i\sin\dfrac{2\pi k}{7}\right)_{k=0}\right]$

sum of all seven values of seventh root of unity = 0

$-i\{0 - (\cos 0 + i\sin 0)\} = -i\{-1 + 0\} = i$

Q.21 The value of the expression $1.(2-\omega).(2-\omega^2) + 2.(3-\omega)(3-\omega^2) + \ldots\ldots + (n-1)(n-\omega)(n-\omega^2)$, where ω is an imaginary cube root of unity is-

(A) $\left(\dfrac{n(n+1)}{2}\right)^2$ (B) $\left(\dfrac{n(n+1)}{2}\right)^2 - n$

(C) $\left(\frac{n(n+1)}{2}\right)^2 + n$ (D) None of above

Sol. **[B]**

General term $(r-1)(r-\omega)(r-\omega^2)$

$(r-1)\{r^2 - r\omega^2 - r\omega + \omega^3\}$

$(r-1)\{r^2 - r(\omega+\omega^2) + \omega^3\}$

$(r-1)(1+r+r^2) = \lambda^3 - 1$

$$\sum_{r=1}^{n}(r^3-1) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} 1 = \left[\frac{n(n+1)}{2}\right]^2 - n$$

Q.22 If ω ($\neq 1$) is a cube root of unity and $(1+\omega)^7 = A + B\omega$, then A & B are respectively the numbers-

(A) 0, 1 (B) 1, 1

(C) 1, 0 (D) –1, 1

Sol. **[B]**

ω is a cube roots of unity

i.e. $\omega = \frac{-1\pm i\sqrt{3}}{2}$ but $\omega \neq 1$

Let $\omega = \frac{-1\pm i\sqrt{3}}{2} \Rightarrow \omega^2 = \frac{-1-i\sqrt{3}}{2}$

$1 + \omega + \omega^2 = 0 \Rightarrow 1 + \omega = -\omega^2$

$\Rightarrow (1+\omega)^7 = (-\omega^2)^7$

$= -\omega^{14}$

$= -\omega^{12} \times \omega^2$

$= -\omega^2$

$= -\frac{-1-i\sqrt{3}}{2}$

$= \frac{1+i\sqrt{3}}{2}$

$= \frac{1+i\sqrt{3}}{2} + \frac{1}{2} - \frac{1}{2}$

$= 1 + \frac{-1+i\sqrt{3}}{2}$

$= 1 + \omega$

$= A + \omega.B$

$\Rightarrow$ A = 1 and B = 1

$\therefore$ Option (B) is correct answer.

Question based on Geometrical figures in complex plane

Q.23 If $z = z_1 + \frac{1}{z_1}$ and z_1 is any point on a fixed circle with the centre at the origin then z lies on

(A) straight line (B) circle

(C) ellipse (D) hyperbola

Sol. **[C]**

Let $z_1 = r\cos\theta + ir\sin\theta$

$z = z_1 + \frac{1}{z_1} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$

$z = x + i$

$x + iy = \cos\theta\left(r + \frac{1}{r}\right) + i\sin\theta\left(r - \frac{1}{r}\right)$

$x = \cos\theta\left(r + \frac{1}{r}\right)$

$y = \sin\theta\left(r - \frac{1}{r}\right)$

$$\frac{x^2}{\left(r+\frac{1}{r}\right)^2} + \frac{y^2}{\left(r-\frac{1}{r}\right)^2} = 1$$

Q.24 If $z \in C$ & $\log_{|z+i|}|3+4i| < \log_{|z+i|}|5+12i|$, then z lies-

(A) inside a circle passing through the origin

(B) z lies outside a circle passing through the origin

(C) z lies inside the circle $|z+i| < 5$

(D) z lies outside the curve $z = -i + e^{i\theta}$, $\theta \in R$.

Sol. **[D]**

Given, $z \in C$

$\log_{|z+i|}|3+4i| < \log_{|z+i|}|5+12i|$

$\Rightarrow$ above inequality holds only if

$|z+i| > 1$

$\Rightarrow |z+i| > |e^{i\theta}|$

$\Rightarrow z + i > e^{i\theta}$

$\Rightarrow z > -i + e^{i\theta}$

i.e. z lies outside the curve $z = -i + e^{i\theta}$

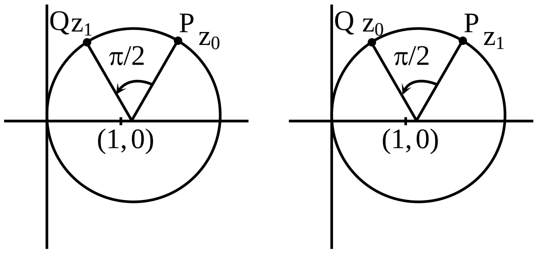
$\therefore$ Option (D) is correct answer.

Q.25 If z_0, z_1 represent points P, Q on the locus $|z-1| = 1$ and the line segment PQ subtends an angle $\pi/2$ at the point z = 1 then z_1 is equal to-

(A) $1 + i(z_0 - 1)$ (B) $\frac{i}{z_0 - 1}$

(C) $1 - i(z_0 + 1)$ (D) $i(z_0 - 1)$

Sol. **[A, C]**



$(z_1 - 1) = (z_0 - 1)\, e^{i\pi/2}$
$z_1 - 1 = (z_0 - 1)\, i$
$z_1 = 1 + i\,(z_0 - 1)$
$(z_0 - 1) = (z_1 - 1)\, e^{i\pi/2}$
$(z_0 - 1) = (z_1 - 1)\, i$
$z_1 - 1 = -i\,(z_0 - 1)$
$z_1 = 1 - i\,(z_0 - 1)$

Q.26 Locus of the point z satisfying the equation $|iz - 1| + |z - i| = 2$ is-

(A) straight line segment
(B) a circle
(C) an ellipse
(D) a pair of straight lines

Sol. **[A]** $|\, iz - 1\,| + |\, z - i\,| = 2$
$|\, iz + i^2\,| + |\, z - i\,| = 2$
$\Rightarrow |\, i\,|\,|\, z + i\,| + |\, z - i\,| = 2$
$\Rightarrow |\, z + i\,| + |\, z - i\,| = 2$
Let z = x + iy
$\Rightarrow |\, x + iy + i\,| + |\, x + iy - i\,| = 2$
$\Rightarrow |\, x + i(y + 1)\,| + |\, x + i(y - 1)\,| = 2$
$\Rightarrow \sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-1)^2} = 2$
$\Rightarrow \sqrt{x^2 + (y+1)^2} = 2 - \sqrt{x^2 + (y-1)^2}$
Squaring both sides, we have
$\Rightarrow x^2 + (y+1)^2 = 4 + x^2 + (y-1)^2 - 4\sqrt{x^2 + (y-1)^2}$
$\Rightarrow y^2 + 1 + 2y = 4 + y^2 + 1 - 2y - 4\sqrt{x^2 + (y-1)^2}$
$\Rightarrow 4y = 4 - 4\sqrt{x^2 + (y-1)^2}$
$\Rightarrow (y - 1) = -\sqrt{x^2 + (y-1)^2}$
$\Rightarrow (y - 1)^2 = x^2 + (y - 1)^2$
$\Rightarrow x^2 = 0$
$\Rightarrow x = 0$
Which is a straight line
$\therefore$ Option (A) is correct answer.

Q.27 The region of argand plane defined by $|z - 1| + |z + 1| \le 4$ is-

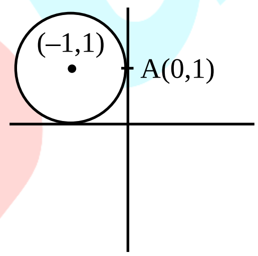
(A) interior of an ellipse
(B) exterior of a circle
(C) interior and to boundary of an ellipse
(D) None of these

Sol. **[C]**
$|\, z - 1\,| + |\, z + 1\,| \le 4$
Let P = x + iy, A = 1 + i0 and B = –1 + i0
then PA + PB ≤ 4 (constant)
Clearly locus of z is interior and boundary of an ellipse.

Q.28 Among the complex numbers z, satisfying the condition $|z + 1 - i| \le 1$, the number having the least positive argument is-

(A) 1 – i
(B) –1 + i
(C) –i
(D) None of these

Sol. **[D]**
$|z + 1 - i| \le 1$
$|x + iy + 1 - i| \le 1$
$(x + 1)^2 + (y - 1)^2 \le 1$



point A(0, 1) = i having least positive argument

Q.29 The vector z = –4 + 5i turned counter stretched 1.5 times. The complex number corresponding to the newly obtained vector is-

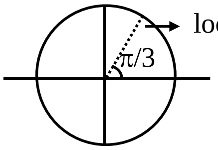
(A) $6 - \frac{15}{2}\, i$
(B) $-6 + \frac{15}{2}\, i$
(C) $6 + \frac{15}{2}\, i$
(D) none of these

Sol. **[A]**
$z_1 = \frac{3}{2}\, z\, e^{i\pi}$
$= \frac{3}{2}\,(-4 + 5i)\,(\cos \pi + i \sin \pi)$
$= -\frac{3}{2}\,(-4 + 5i)$
$= 6 - \frac{15}{2}\, i$

Fill in the blanks type questions

Q.30 The set of points in an Argand diagram which satisfy both $|z| \le 4$ and $\arg(z) = \pi/3$ is ......

Sol. $|z| \le 4$ locus of circle



$\arg(z) = \dfrac{\pi}{3}$ equation of straight line

radius of circle

Q.31 If m and x are two real numbers, then $e^{2mi\cot^{-1}x}\left(\dfrac{xi+1}{xi-1}\right)^m$ is equal to –

Sol. $e^{2mi\cot^{-1}x}\left(\dfrac{1+xi}{-1+xi}\right)^m$

$= e^{2mi\cot^{-1}x}\dfrac{(i)^m(x-i)^m}{(i)^m(x+i)^m}$

$= e^{2mi\cot^{-1}x}\dfrac{\left(\dfrac{x}{1+x^2}-\dfrac{i}{1+x^2}\right)^m}{\left(\dfrac{x}{1+x^2}+\dfrac{i}{1+x^2}\right)^m}$

$= e^{2mi\cot^{-1}x}\dfrac{(\cos(\cot^{-1}x)-i\sin(\cot^{-1}x))^m}{(\cos\cot^{-1}x+i\sin\cot^{-1}x)^m}$

$= e^{2mi\cot^{-1}x}$ (cos (m $\cot^{-1}$ x) – i sin (m $\cot^{-1}$ x) ×

(cos (m $\cot^{-1}$ x) – i sin (m $\cos^{-1}$ x)

$= e^{2mi\cot^{-1}x}$ [$\cos^2$(m $\cot^{-1}$x) – $\sin^2$ (m $\cot^{-1}$x)
– i2 sin (m $\cot^{-1}$ x) cos (m $\cot^{-1}$ x)]

$= e^{2mi\cot^{-1}x}$ [cos (2m $\cot^{-1}$ x) – i sin (2m $\cot^{-1}$x)]

$= e^{2mi\cot^{-1}x} \times e^{-2mi\cot^{-1}x}$

= 1

Q.32 For any two complex numbers z_1, z_2 and any real number a and b,
$|az_1 - bz_2|^2 + |bz_1 + az_2|^2$ = ...

Sol. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

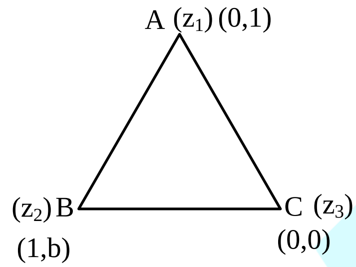
$(az_1 - bz_2)\,(a\bar{z}_1 - b\bar{z}_2) + (bz_1 + az_2)\,(b\bar{z}_1 + a\bar{z}_2)$

$a^2|z_1|^2 - abz_1\bar{z}_2 - ab\bar{z}_1 z_2 + b^2|z_2|^2 + b^2|z_1|^2 + abz_1\bar{z}_2 + ab\bar{z}_1 z_2 + a^2|z_2|^2$

$\Rightarrow (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

Q.33 If α, β, γ are the numbers between 0 & 1 such that points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then a = ... & b = ...

Sol.



$(AB)^2 = (AC)^2$ ....(1) $\qquad (AC)^2 = (AB)^2$

$a^2 + 1 = b^2 + 1 \qquad a^2 + 1 = (a-1)^2 + (b-1)^2$

$a = b \qquad a = b$

$a^2 - 4a + 1 = 0$

same sign $\qquad a = 2 \pm \sqrt{3}$

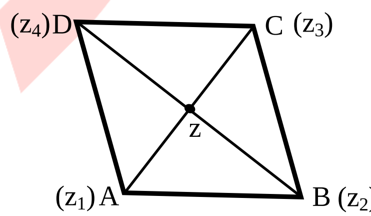
$a = 2 - \sqrt{3} \qquad a = 2 + \sqrt{3}$

$b = 2 - \sqrt{3} \qquad a = 2 - \sqrt{3}$

$(0 < a < 1)$

Q.34 ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. If the points D and M represent the complex numbers 1+i and 1–i respectively, then A represents the complex number........... or........

Sol.



Let point A = z_1

$\dfrac{(z_4 - z)}{(z_4 - z)} = \dfrac{(z_1 - z)}{(z_1 - z)} e^{\pm i\pi/2}$

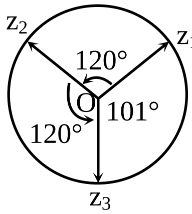
$2(z_4 - z) = (z_1 - z)(\pm i)$

$z_4 = (1 + i)$ put $z = 3 - i/2$

$z = 1 - i \qquad z = 1 - (3/2)\,i$

Q.35 Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$ then z_2 = ..., z_3 =…

Sol.



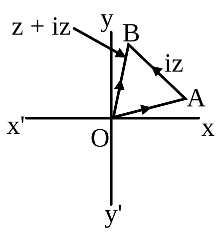
$z_2 = z_1 e^{i\frac{2\pi}{3}} = (1 + i\sqrt{3})\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$= \frac{(i\sqrt{3})^2 - 1}{2} = \frac{-3-1}{2} = -2$

$z_3 = z_1\, e^{i\frac{4\pi}{3}} = 1 - i\sqrt{3}$

Q.36 The area of the triangle on the Argand diagram formed by the complex numbers z, iz and z + iz is …………..

Sol. We have ; $iz = ze^{i\pi/2}$. This implies that iz is the vector obtained by rotating vector z in anticlockwise direction through 90º. Therefore, OA ⊥ AB. So,



Area of ΔOAB = $\frac{1}{2}$ OA×OB

$= \frac{1}{2}|z|\,|iz| = \frac{1}{2}|z|^2.$

➢ True or false type questions

Q.37 If $z = \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$, then principal value of argument z is $\frac{\pi}{2}$.

Sol. $z = \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$

$= \frac{(\sqrt{5+12i} + \sqrt{5-12i})^2}{(5+12i) - (5-12i)}$

$= \frac{5+12i+5-12i+2\sqrt{25+144}}{24i}$

$= \frac{36}{24i} = -\frac{3}{2}i$

Principal arg (z) = $-\frac{\pi}{2}$

Q.38 If $\log_{1/2}\frac{|z-1|+4}{3|z-1|-2} > 1$ then locus of z is exterior to circle with center 1 + i0 & radius 10.

Sol. $\log_{1/2}\frac{|z-1|+4}{3|z-1|-2} > 1$

$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2}$

$\Rightarrow 2|z-1| + 8 < 3|z-1| - 2$

$\Rightarrow |z-1| > 10$

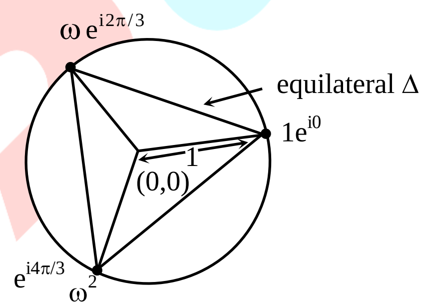
⇒ z lies exterior of the circle with center 1 + i0 and radius 10

Q.39 The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle.

Sol. $1 = e^{i0}$

$\omega = e^{i2\pi/3}$

$\omega^2 = e^{i4\pi/3}$



centre of circle is (0, 0)

Radius is '1'

Q.40 Let P(x) and Q(x) be two polynomials. Suppose that $f(x) = P(x^3) + xQ(x^3)$ is divisible by $x^2 + x + 1$, then f(x) is divisible by (x – 1).

Sol. Given : $f(x) = P(x^3) + x.Q(x^3)$

Since, f(x) is divisible by $(x^2 + x + 1)$ i.e.

It must be divisible by (x – 1). Because $(x^2 + x + 1)$ is a factor of $(x^3 - 1) = (x - 1)(x^2 + x + 1)$

Hence, Both P(x) and Q(x) must be divisible by (x – 1)

EXERCISE # 2

Part-A **Only single correct answer type questions**

Q.1 The value of $\sum_{k=0}^{n} \sin\frac{2\pi k}{n}$ is -

(A) 1 (B) –1 (C) 0 (D) k

Sol.[C] Sum of n, n^{th} roots of unity is zero.

$\Rightarrow \sum_{k=1}^{n}\left[\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right] = 0$

$\Rightarrow \sum_{k=1}^{n}\sin\frac{2\pi k}{n} = 0$

$\Rightarrow \sum_{k=0}^{n}\sin\frac{2\pi k}{n} = 0$

Q.2 If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$, where p, q, r are the moduli of u, v, w then

(A) arg (w / v) = arg $\left(\frac{w-u}{v-u}\right)^2$

(B) arg (w / v) = arg $\left(\frac{w+u}{v-u}\right)^2$

(C) arg (w / v) = arg $\left(\frac{w-u}{v+u}\right)^2$

(D) arg (w / v) = arg $\left(\frac{w+u}{v+u}\right)^2$

Sol. [A] arg (w / v) = arg $\left(\frac{w-u}{v-u}\right)^2$

Q.3 If P, P′ represent the complex number z_1 and its additive inverse respectively then the complex equation of the circle with PP′ as a diameter is

(A) $\frac{z}{z_1} = \overline{\left(\frac{z_1}{z}\right)}$ (B) $z\bar{z} + z_1\bar{z}_1 = 0$

(C) $z\bar{z}_1 + \bar{z}z_1 = 0$ (D) None of these

Sol. [A] $|z|^2 = |z_1|^2$

$z\bar{z} = z_1\bar{z}_1$

$\frac{z}{z_1} = \frac{\bar{z}_1}{\bar{z}}$

$\Rightarrow \frac{z}{z_1} = \overline{\left(\frac{z_1}{z}\right)}$

Q.4 Let z_1, z_2, be two complex numbers represented by points on the circle |z| = 1 and |z| = 2 respectively then

(A) max $|2z_1 + z_2| = 4$ (B) max $|z_1 - z_2| = 1$

(C) $\left|z_2 + \frac{1}{z_1}\right| \ge 3$ (D) None of these

Sol. $|2z_1 + z_2| \le |2z_1| + |z_2|$

$\le 2|z_1| + |z_2|$

$\le 2(1) + (2)$

$\Rightarrow$ Max. $|2z_1 + z_2| \le 4$

Q.5 If $\left|\frac{1-\bar{z}_1 z_2}{z_1 - z_2}\right| = 1$ then

(A) either $|z_1| = 0$ or $|z_2| = 0$

(B) $|z_1| = 1$, $|z_2| = 0$

(C) $|z_1| = |z_2| = 0$

(D) either $|z_1| = 1$ or $|z_2| = 1$

Sol. [D] $\left|\frac{1-\bar{z}_1 z_2}{z_1 - z_2}\right| = 1$

$\Rightarrow |1 - \bar{z}_1 z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow (1 - \bar{z}_1 z_2)(1 - z_1\bar{z}_2) = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow 1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_1\bar{z}_1 . z_2\bar{z}_2$

$= z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$

$\Rightarrow |z_1|^2 + |z_2|^2 - |z_1|^2 |z_2|^2 - 1 = 0$

$\Rightarrow (|z_1|^2 - 1)(|z_2|^2 - 1) = 0$

$\Rightarrow$ either $|z_1| = 1$ or $|z_2| = 1$

Q.6 If |z| = 1 and z is non-real then $z = \frac{c+i}{c-i}$ where, c is real . The value of c is

(A) $\tan\left[\frac{1}{2}\arg(z)\right]$ (B) $\cot\left[\frac{1}{2}\arg(z)\right]$

(C) tan [arg (z)] (D) cot [arg (z)]

Sol. [B] $|z| = 1 \Rightarrow$ Let $z = e^{i\theta}$

$\Rightarrow \frac{e^{i\theta}}{1} = \frac{c+i}{c-i}$

$\Rightarrow$ (using componendo and dividendo)

$\frac{2c}{2i} = \left(\frac{e^{i\theta}+1}{e^{i\theta}-1}\right)$

$\Rightarrow \frac{c}{i} = \frac{(\cos\theta+1)+i\sin\theta}{(\cos\theta-1)+i\sin\theta}$

$\frac{2\cos^2\frac{\theta}{2}+i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-2\sin^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$\frac{c}{i} = \frac{2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}.i\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]}$

$\Rightarrow c = \cot\frac{\theta}{2}$

Q.7 Let ω, ω^2 be complex cube roots of unity. Then the determinant

$\Delta = \begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$ is divisible by

(A) x (B) x^5

(C) x^7 (D) x^6

Sol. $\Delta = \begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$

use $C_1 \to C_1 + C_2 + C_3$

$\Delta = \begin{vmatrix} x & \omega & \omega^2 \\ x & x+\omega^2 & 1 \\ x & 1 & x+\omega \end{vmatrix}$

$= x . \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$= x . \begin{vmatrix} 0 & x & x \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$

$= x^2 \begin{vmatrix} 0 & 1 & 1 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$

$= x^2 . [-\{(x+\omega)\omega - \omega^2\} + \{\omega - \omega^2(x+\omega^2)\}]$

$= x^2[-\omega x - \omega^2 + \omega^2 + \omega - \omega^2 x - \omega^4]$

$= x^3$

Q.8 If α is a complex constant such that $\alpha z^2 + z + \overline{\alpha} = 0$ has a real root then

(A) $\alpha + \overline{\alpha} = 1$ (B) $\alpha + \overline{\alpha} = 0$

(C) $\alpha - \overline{\alpha} = -1$ (D) None of these

Sol. [A] If z is real, then $z = \overline{z}$.

equation is

$\alpha z^2 + z + \overline{\alpha} = 0$ ...(i)

$\overline{\alpha}\,\overline{z}^2 + \overline{z} + \alpha = 0$ (taking conjugate)

$\Rightarrow \overline{\alpha} z^2 + z + \alpha = 0$ ....(ii)

$\{\because z = \overline{z}\}$

equation (i) & (ii) will have one root in common

$(\alpha^2 - \overline{\alpha}^2)^2 = (\alpha - \overline{\alpha})(\alpha - \overline{\alpha})$

$\Rightarrow (\alpha + \overline{\alpha})^2 = 1$

$\alpha + \overline{\alpha} = \pm 1$

Q.9 If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$ then the area of the triangle whose vertices are z_1, z_2, z_3 is-

(A) $\frac{3\sqrt{3}}{4}$ (B) $\frac{\sqrt{3}}{4}$

(C) 1 (D) None of these

Sol. **[A]**

$|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$

$\Rightarrow z_1\overline{z}_1 = z_2\overline{z}_2 = z_3\overline{z}_3 = 1$

$\Rightarrow z_1 = \frac{1}{\overline{z}_1}$; $z_2 = \frac{1}{\overline{z}_2}$; $z_3 = \frac{1}{\overline{z}_3}$

$\Rightarrow z_1 + z_2 + z_3 = \frac{1}{\overline{z}_1} + \frac{1}{\overline{z}_2} + \frac{1}{\overline{z}_3} = 0$

$\Rightarrow \frac{1}{\overline{z}_1} + \frac{1}{\overline{z}_2} + \frac{1}{\overline{z}_3} = 0$

Taking conjugate of whole complex number, we get

$\Rightarrow \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$

$\Rightarrow z_1z_2 + z_2z_3 + z_3z_1 = 0$

Also, from $z_1 + z_2 + z_3 = 0$

Squaring both sides, we get

$(z_1 + z_2 + z_3)^2 = 0$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1) = 0$

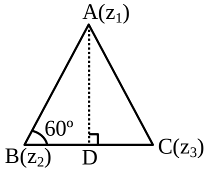
$\Rightarrow z_1^2 + z_2^2 + z_3^2 + 0 = 0$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = 0$

$\Rightarrow$ It is an equilateral triangle.

Area, A = $\frac{1}{2}$ AD.BC

$= \frac{1}{2} \sin 60^\circ$ AB.BC



$= \frac{1}{2} \times \frac{\sqrt{3}}{2} \times (AB)^2$ ($\because$ |AB| = |BC|)

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} \times |z_1 - z_2|^2$

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} \times (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} \times (z_1\bar{z}_1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2)$

$= \frac{\sqrt{3}}{4} \times (|z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_2\bar{z}_1))$

$= \frac{\sqrt{3}}{4} \times (|z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2|\cos 120^\circ)$

$= \frac{\sqrt{3}}{4} \times (1 + 1 - 2 \times 1 \times 1 (-\frac{1}{2}))$

$= \frac{\sqrt{3}}{4} \times (1 + 1 + 1) = \frac{3\sqrt{3}}{4}$ square unit.

Q.10 Let S denote the set of complex numbers z such that $\log_{1/3}(\log_{1/2}(|z|^2 + 4|z| + 3)] < 0$, then S is contained in-

(A) (0, 1)

(B) {z | Re (z) > 0}

(C) |z| Re (z) > 3}

(D) None of these

Sol. **[D]**

$\log_{1/3}[\log_{1/2}(|z|^2 + 4|z| + 3)] < 0$

$\Rightarrow \log_{1/2}(|z|^2 + 4|z| + 3)] < 1$

$\Rightarrow |z|^2 + 4|z| + 3 < 1/2$

$\Rightarrow |z|^2 + 4|z| + 5/2 < 0$

$\Rightarrow 2|z|^2 + 8|z| + 5 < 0$

We can solve this quadratic equation as.

$|z| = \frac{-8 \pm \sqrt{64 - 40}}{2 \times 2}$

$|z| = \frac{-8 \pm \sqrt{24}}{4}$

$|z| = \frac{-8 \pm 2\sqrt{6}}{4}$

$|z| = \frac{-4 \pm \sqrt{6}}{2}$ = –ve

Which is not possible.

Q.11 $\sin^{-1}\left\{\frac{1}{i}(z-1)\right\}$, where z non-real, can be the angle of a triangle if -

(A) Re(z) = 1, Im (z) = 2

(B) Re(z) = 1, 0 < Im (z) ≤ 1

(C) Re(z) + Im(z) = 0

(D) None of these

Sol.[B] $\frac{1}{i}(z-1)$ must be real and less than or equal to 1.

$\Rightarrow \frac{1}{i}(z - 1) = \frac{\bar{z} - 1}{-i}$

$\Rightarrow z - 1 = 1 - \bar{z}$

$\Rightarrow z + \bar{z} = 2$

$\Rightarrow$ Re(z) = 1

Now let z = 1 + iy

$\Rightarrow \frac{1}{i}(z - 1) = \frac{1}{i}(iy)$

$= y$

for $\sin^{-1}y$ to exist and angle of triangle.

$0 < y \le 1$

$\Rightarrow 0 < \text{Im}(z) \le 1$

Q.12 If ω is the imaginary cube root of unity then the three points with complex numbers Z_1, Z_2 and $(-\omega Z_1 - \omega^2 Z_2)$ on the complex plane are-

(A) the vertices of a right triangle

(B) the vertices of an isosceles triangle which is not right

(C) the vertices of an equilateral triangle

(D) collinear

Sol. **[C]**

$A(z_1), B(z_2), C(-\omega z_1 - \omega^2 z_2)$

$AB = |z_1 - z_2|$

$BC = |-\omega z_1 - (1 + \omega^2)z_2|$

$= |-\omega|\ |z_1 - z_2| = |z_1 - z_2|$

$AC = |(1 + \omega)z_1 + \omega^2 z_2|$

$= |-\omega^2|\ |z_1 - z_2| = |z_1 - z_2|$

$\Rightarrow AB = BC = AC$

$\Rightarrow$ equilateral triangle

Q.13 If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ & $\text{Re}(z_1 \bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies -

(A) $|w_1| = 1$ (B) $|w_2| = 1$

(C) $\text{Re}(w_1 \bar{w}_2) = 0$ (D) None of these

Sol. **[A, B, C]**

$z_1 = a + ib$

$z_2 = c + id$

given $|z_1| = |z_2| = 1$

$\Rightarrow a^2 + b^2 = 1$ ....(i)

$c^2 + d^2 = 1$ ....(ii)

also $\text{Re}(z_1 \bar{z}_2) = 0$

$\Rightarrow ac + bd = 0$

$\Rightarrow a^2c^2 = b^2d^2$

$\Rightarrow a^2c^2 = [1 - c^2]\,[1 - a^2]$

$\Rightarrow a^2c^2 = 1 - a^2 - c^2 + a^2c^2$

$\Rightarrow a^2 + c^2 = 1$

$\Rightarrow |\omega_1| = 1$

similarly $|\omega_2| = 1$

Now $\text{Re}(\omega_1 \bar{\omega}_2) = ab + cd$

from (i), $a^2 + b^2 = 1$

$\Rightarrow a^2b^2 = 1 - a^2 - b^2 + a^2b^2$

$\Rightarrow a^2b^2 = (1 - a^2)(1 - b^2)$

$\Rightarrow a^2b^2 = c^2.d^2$

$\Rightarrow ab = \pm cd$

$\Rightarrow ab + cd = 0$

Q.14 If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, then the two triangles-

(A) have the same area (B) are similar

(C) are congruent (D) none of these

Sol. **[B]**

$$c = \frac{(1-r)a + rb}{(1-r) + r}$$

$\Rightarrow$ a, b, c are in straight line.

Similarly u, v, w are in straight line.

Q.15 Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1 + z_2}{z_1 - z_2}$ may be

(A) zero (B) real and positive

(C) real and negative (D) purely imaginary

Sol. **[A, C]**

$z_1 \neq z_2$ and $|z_1| = |z_2|$

$R_e(z_1) > 0$ and $I_m(z_2) < 0$

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{z_1 + z_2}{z_1 - z_2} \times \frac{\overline{(z_1 + z_2)}}{\overline{(z_1 - z_2)}}$$

$$= \frac{(z_1 + z_2) \times (\bar{z}_1 - \bar{z}_2)}{|z_1 - z_2|^2}$$

$$= \frac{z_1\bar{z}_1 + z_2\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_2}{|z_1 - z_2|^2}$$

$$= \frac{(z_2\bar{z}_1 - z_1\bar{z}_2)}{|z_1 - z_2|^2}$$

Which is either zero or $I_m(z_2\bar{z}_1)$

Since $I_m(z_2) < 0$

$\therefore$ Option (A) and (C) are correct answer.

Part-B One or more than one correct answer type questions

Q.16 If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$ then $|9z_1z_2 + 4z_1z_3 + z_3z_2|$ is equal to

(A) 6 (B) 36 (C) 216 (D) None

Sol. **[A]**

$|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$

$|z_1 + z_2 + z_3| = 1$

$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$

$$\Rightarrow \left|\frac{1}{z_1} + \frac{4}{z_2} + \frac{9}{z_3}\right| = 1 \quad \{\Theta\ z\bar{z} = |z|^2\}$$

$$= \frac{|9z_1z_2 + 4z_1z_3 + z_3z_2|}{|z_1z_2z_3|} = 1$$

$\Rightarrow |9z_1z_2 + 4z_1z_3 + z_3z_2| = (1)(2)(3) = 6$

Q.17 The points representing the complex number z for which $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3}$ lie on -

(A) a circle (B) a straight line
(C) an ellipse (D) a parabola

Sol. **[A]**

$\text{Arg}\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3}$

Let z = x + iy

$\frac{z-2}{z+2} = \frac{x+iy-2}{x+iy+2}$

$= \frac{(x-2)+iy}{(x+2)+iy}$

Multiplying by [(x + 2) – iy] in numerator and denominator, we get

$= \frac{(x-2)+iy}{(x+2)+iy} \times \frac{(x+2)-iy}{(x+2)-iy}$

$= \frac{(x^2-4)+iy(x+2)-iy(x-2)+y^2}{(x+2)^2+y^2}$

$= \frac{(x^2+y^2-4)+ixy+2iy-ixy+2iy}{(x+2)^2+y^2}$

$= \frac{(x^2+y^2-4)+4iy}{(x+2)^2+y^2}$

Hence, $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3}$

$\Rightarrow \frac{4y}{x^2+y^2-4} = \tan\frac{\pi}{3}$

$\Rightarrow \frac{4y}{x^2+y^2-4} = \sqrt{3}$

$\Rightarrow x^2 + y^2 - 4 = \frac{4}{\sqrt{3}}y$

$\Rightarrow x^2 + y^2 - \frac{4}{\sqrt{3}}y - 4 = 0$

Which a circle

$\therefore$ Option (A) is correct answer.

Q.18 If $x_r = \text{CiS}\left(\frac{\pi}{2^r}\right)$ for $1 \le r \le nr$, $n \in N$ then

(A) $\lim_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = 1$

(B) $\lim_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = 0$

(C) $\lim_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = 1$

(D) $\lim_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = 0$

Sol. **[A, D]**

$\lim_{n\to\infty} \prod_{r=1}^{n} x_r = \lim_{n\to\infty} \prod_{r=1}^{n} \text{cis}\left(\frac{\pi}{2^r}\right)$

$= \text{cis}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.....\right)\pi$

$= \text{cis}\,\pi$

$= \cos\pi + i\sin\pi = -1$

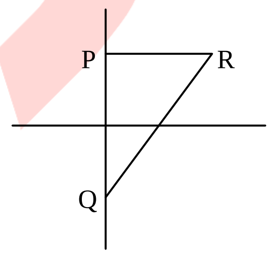
Q.19 The equation $||z + i| - |z - i|| = k$ represents -

(A) a hyperbola if $0 < k < 2$
(B) a pair of ray if $k > 2$
(C) a straight line if $k = 0$
(D) a pair of ray if $k = 2$

Sol. **[A, C, D]**

$||z + i| - |z - i|| = k$

$\Rightarrow |QR - PR| = k$



(i) If $0 < k < 2$

by the definition, the locus of R is hyperbola.

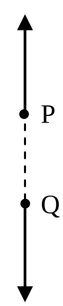
(ii) If k = 0

$\Rightarrow$ QR = PR

$\Rightarrow$ Locus is perpendicular bisector of line joining P and Q.

(iii) If k = 2

The locus is as shown



That is pair of a ray.

Q.20 If z satisfies the inequality $|z - 1 - 2i| \le 1$, then

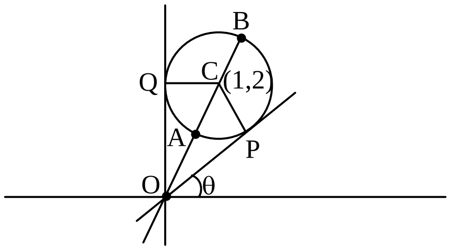
(A) min (arg (z)) = $\tan^{-1}\left(\frac{3}{4}\right)$

(B) max (arg (z)) = $\frac{\pi}{2}$

(C) min (|z|) = $\sqrt{5} - 1$

(D) max (|z|) = $\sqrt{5} + 1$

Sol. [A, B, C, D] $|z - (1 + 2i)| \le 1$



$\Rightarrow$ The inside region of a circle with centre (1, 2) and radius '1'.

Let line OP is y = mx.

$\Rightarrow \frac{m-2}{\sqrt{1+m^2}} = 1$

$m = \frac{3}{4}$

$\Rightarrow$ Min (Arg (z)) = $\frac{3}{4}$

and Max. (Arg (z)) = $\frac{\pi}{2}$

also min. |z| = OA = OC – r

and max. |z| = OB = OC + r

Q.21 The reflection of the complex number $\frac{2-i}{3+i}$, (where $i = \sqrt{-1}$) in the straight line $z(1 + i) = \bar{z}(i - 1)$ is –

(A) $\frac{-1-i}{2}$ (B) $\frac{-1+i}{2}$ (C) $\frac{i(i+1)}{2}$ (D) $\frac{-1}{1+i}$

Sol. **[B, C, D]**

$\frac{2-i}{3+i} = \frac{(2-i)(3-i)}{10} = \frac{5}{10} - \frac{5}{10}i$

$= \frac{1}{2}(1 - i)$

So point is $\left(\frac{1}{2}, -\frac{1}{2}\right)$

Line is $z(1 + i) = \bar{z}(i - 1)$

$(x + iy)(1 + i) = (x - iy)(i - 1)$

$\Rightarrow x + xi + iy - y = xi - x + y + iy$

$\Rightarrow y = x$

so image is $\left(-\frac{1}{2}, \frac{1}{2}\right)$

$= \frac{1}{2}(-1 + i)$

Q.22 If z_1, z_2, z_3, z_4 be the vertices of a parallelogram taken in anticlockwise direction, and $|z_1 - z_2| = |z_1 - z_4|$ then-

(A) $\sum_{r=1}^{4} (-1)^r z_r = 0$ (B) $z_1 + z_2 - z_3 - z_4 = 0$

(C) arg $\frac{z_4 - z_2}{z_3 - z_1} = \frac{\pi}{2}$ (D) none of these

Sol. **[A, C]**

$\sum_{r=1}^{4} (-1)^r z_r = 0$; arg $\frac{z_4 - z_2}{z_3 - z_1} = \frac{\pi}{2}$

Part-C Assertion-Reason type questions

The following questions 23 to 25 consists of two statements each, printed as Statement-1 and Statement-2. While answering these questions you are to choose any one of the following five responses.

(A) If both Statement-1 and Statement-2 are true & the Statement-2 is correct explanation of the Statement-1.

(B) If both Statement-1 and Statement-2 are true but Statement-2 is not correct explanation of the Statement-1.

(C) If Statement-1 is true but the Statement-2 is false.

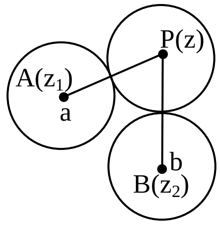
(D) If Statement-1 is false but Statement-2 is true.

(E) If Statement-1 & Statement-2 are false.

Q.23 **Statement-1** : The locus of the centre of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$ externally (z, z_1 and z_2 are complex numbers) will be hyperbola.

Statement-2: $|z - z_1| - |z - z_2| < |z_2 - z_1|$ $\Rightarrow$ z lies on hyperbola.

Sol. **[A]**



$|z - z_1| - |z - z_2| = k < |z_2 - z_1|$

PA – PB = k

⇒ definition of hyperbola.

Q.24 **Statement-1** : If $\left|\dfrac{z\,z_1 - z_2}{z\,z_1 + z_2}\right| = k,\ (z_1,\ z_2 \neq 0)$ then locus of z is a circle.

Statement-2 : As $\left|\dfrac{z - z_1}{z - z_2}\right| = \lambda$, represents a circle if $\lambda \notin \{0, 1\}$.

Sol. **[D]** $\left|\dfrac{z\,z_1 - z_2}{z\,z_1 + z_2}\right| = k$

$$\Rightarrow \left|\dfrac{z - \dfrac{z_2}{z_1}}{z + \dfrac{z_2}{z_1}}\right| = k$$

will represent circle if $k \neq 0, 1$

similarly the statement 2.

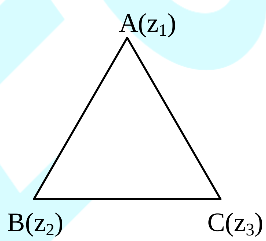
Q.25 Let z_1, z_2, z_3 represent vertices of a triangle.

Statement 1 : $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_2 - z_1} = 0$, when triangle is equilateral.

Statement-2 : $|z_1|^2 - z_1\bar{z}_0 - \bar{z}_1 z_0 = |z_2|^2 - z_2\bar{z}_0 - \bar{z}_2 z_0 = |z_3|^2 - z_3\bar{z}_0 - \bar{z}_3 z_0$, where z_0 is circumcentre of triangle.

Sol. **[D]** $z_2 - z_1 = (z_2 - z_3)e^{i\pi/3}$

$$z_2 - z_1 = (z_2 - z_3)\left[\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$$



$$\frac{z_2}{2} - z_1 + \frac{z_3}{2} = (z_2 - z_3)\left(\frac{i\sqrt{3}}{2}\right)$$

$$\Rightarrow (z_2 + z_3 - 2z_1) = i\sqrt{3}\,(z_2 - z_3)$$

squaring

$$z_2^2 + z_3^2 + 4z_1^2 + 2z_2z_3 - 4z_1z_2 - 4z_1z_3 = -3[z_2^2 + z_3^2 - 2z_2z_3]$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$\Rightarrow (z_1 - z_2)(z_2 - z_3) + (z_2 - z_3)(z_3 - z_1) + (z_3 - z_1)(z_1 - z_2) = 0$$

$$\Rightarrow \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

Now, If z_0 is the circumcentre,

$|z_1 - z_0|^2 = |z_2 - z_0|^2$

$(z_1 - z_0)(\bar{z}_1 - \bar{z}_0) = (z_2 - z_0)(\bar{z}_2 - \bar{z}_0)$

$\Rightarrow |z_1|^2 - z_1\bar{z}_0 - z_0\bar{z}_1 = |z_2|^2 - z_0\bar{z}_0\ \bar{z}_2 z_0$

Part-D Column Matching type questions

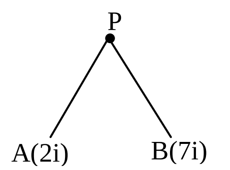
Q.26 Match the column

Column-I	**Column-II**
(A) If $|z - 2i| + |z - 7i| = k$, then locus of z is	(P) ellipse if $k > 5$
(B) If $|z - 1| + |z - 6| = k$, then locus of z is	(Q) hyperbola if $0 < k < 5$
(C) If $|z - 3| - |z - 4i| = k$, then locus of z is	(R) hyperbola if $k > 5$
(D) If $|z - (2 + 4i)|$	(S) straight line if $k = 5$
$= \dfrac{k}{50}\,|a\bar{z} + \bar{a}z + b|$,	(T) nothing if $k < 5$
where $a = 3 + 4i$, then locus of z is	(W) pair of rays if $k = 5$

Sol. **A → P, S, T; B → P, S, T; C → Q, W; D → R**

(A) $|z - 2i| + |z - 7i| = k$

⇒ PA + PB = k and AB = 5



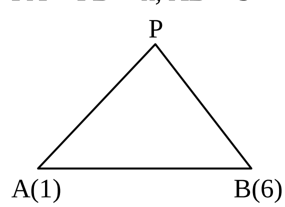
so locus will be ellipse if $k > 5$

straight line if $k = 5$

No locus if $k < 5$

(B) $|z - 1| + |z - 6| = k$

PA + PB = k, AB = 5

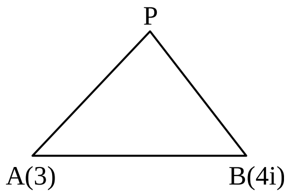


Similarly as part (A).

(C) $|z - 3| - |z - 4i| = k$

so PA – PB = k

and AB = 5



If $0 < k < 5$
locus will be hyperbola and pair of rays if k = 5
(Also refer Q.19)

(D) $|z - (2 + 4i)| = \frac{k}{50}|a\bar{z} + \bar{a}z + b|$

$\Rightarrow |z - (2 + 4i)| = \frac{k}{50}[\,|6x + 8y + b|\,]$

$\sqrt{(x-2)^2 + (y-4)^2} = \frac{k}{5}\left|\frac{6x+8y+b}{10}\right|$

for ellipse $0 < \frac{k}{5} < 1$

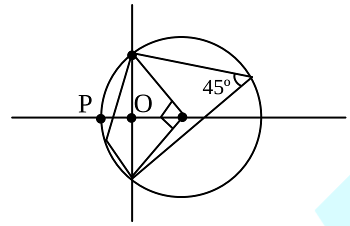
and hyperbola if $k > 5$

Q.27 **Column-I** **Column-II**

Column-I	Column-II
(A) If $\arg\left(\frac{z-i}{z+i}\right) = \frac{3\pi}{4}$, then |z| is always less than	(P) $\frac{\pi}{4}$
(B) If $\arg(z) = \frac{\pi}{4}$, then $\arg(z) - \arg\left(\frac{4}{z}\right)$ is equal to	(Q) 0
(C) z_1 and z_2 are two complex numbers satisfying $|z + 2| + |z - 2| = 4$ and $|z| = 1$, then $z_1 + z_2$ is equal to	(R) 1
(D) Area of the region bounded by $|z| \le 1$ and $-\frac{\pi}{4} \le \arg(z) \le \frac{\pi}{4}$ is	(S) $\frac{\pi}{2}$

Sol. **A → R; B → S; C → Q; D → P**

(A) $\text{Arg}\left(\frac{z-i}{z+i}\right) = \frac{3\pi}{4}$



minor arc of circle with centre (1, 0) and radius $\sqrt{2}$.

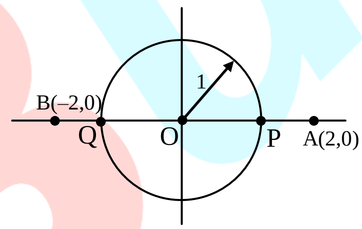
$\Rightarrow$ Max. $|z| = OP = 1$

(B) $\text{Arg}(z) - \text{Arg}\left(\frac{4}{z}\right)$

$= 2\text{Arg}(z) - \text{Arg}(4)$

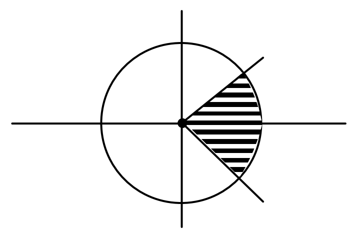
$= 2\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{2}$

(C) $|z + 2| + |z - 2| = 9$



$z_1 = 1, z_2 = -1$
corresponding to point P and Q.

(D) Area $= \frac{1}{4}[\pi(1)^2] = \frac{\pi}{4}$



Q.28 **Column-I** **Column-II**

Column-I	Column-II
(A) If ω_1, ω_2 be complex cube roots of unity, then $\omega_1^4 + \omega_2^4$ is equal to	(P) $-\frac{1}{\omega_1\omega_2}$
(B) If $\omega \neq 1$ be nth roots of unity, then $\omega + \omega^2 + \omega^3 + ... + \omega^{n-1}$ is equal to	(Q) –1
(C) If z_1 and z_2 be two n^{th} roots of unity, then	(R) $\frac{2\pi}{n}$

$\arg\left(\frac{z_1}{z_2}\right)$ is a multiple of

(D) If 1, $\omega, \omega^2 \ldots \omega^{n-1}$ are n^{th} roots of unity then the value of $(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1})$ is equal to — (S) n

Sol. **A → P, Q; B → P, Q; C → R; D → S**

(A) $\omega_1^4 + \omega_2^4$

$= \omega_1^3.\omega_1 + \omega_2^3.\omega_2$

$= \omega_1 + \omega_2 = \omega + \omega^2 = -1$

(B) $\omega.\left[\frac{\omega^{n-1}-1}{\omega-1}\right] = \frac{\omega^n - \omega}{\omega-1}$

$= \frac{1-\omega}{\omega-1} = -1$

(C) $z_1 = e^{i\left(\frac{2\pi}{n}\right)}$ and all roots are in G.P. with common ratio $\left(\frac{2\pi}{n}\right)$

(D) $x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$

$\Rightarrow \frac{x^n-1}{x-1} = (x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$

taking by limit $x \to 1$

$= n$

Q.29

Column-I	Column-II
(A) The value of $\sin\frac{\pi}{900}\left\{\sum_{r=1}^{10}(r-\omega)(r-\omega^2)\right\} =$	(P) $1 + i^2$
(B) If roots of $t^2 + t + 1 = 0$ be α, β, then $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1}$ is equal to	(Q) 4
(C) If $\left(\frac{1+\cos\theta + i\sin\theta}{\sin\theta + i(1+\cos\theta)}\right)^4 = \cos n\theta - i\sin n\theta$, then n is equal to	(R) 1
(D) If $z = 3 + \lambda + i\sqrt{5-\lambda^2}$, where $\lambda \in (-\sqrt{5}, \sqrt{5})$ is arbitrary real, then locus of z is circle $(x-h)^2 + y^2 = r^2$, where $2h - r^2$ is equal to	(S) 0

Sol. **A → R; B → P, S; C → Q; D → R**

(A) $\sin\left\{\frac{\pi}{900}.\left[\sum_{r=1}^{10}(r^2+r+1)\right]\right\}$

$= \sin\left\{\frac{\pi}{900}.\left[\frac{10\times11\times21}{6} + \frac{10\times11}{2} + 10\right]\right\}$

$= \sin\left(\frac{\pi}{900}\times 450\right) = 1$

(B) $\alpha = \omega, \beta = \omega^2$

$\Rightarrow \alpha^4 + \beta^4 + \frac{1}{\alpha\beta}$

$= \omega^4 + \omega^8 + \frac{1}{\omega^3}$

$= \omega + \omega^2 + 1 = 0$

(C) $\left(\frac{1 + 2\cos^2\frac{\theta}{2} - 1 + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + i\left(2\cos^2\frac{\theta}{2} - 1 + 1\right)}\right)^4$

$= \left(\frac{e^{i\frac{\theta}{2}}}{ie^{-i\frac{\theta}{2}}}\right)^4$

$= \cos n\theta + i\sin n\theta$ (n = 4)

(D) $z = (3+\lambda) + i(\sqrt{5-\lambda^2})$

Let $z = x + iy$

$\Rightarrow x = 3 + \lambda$

$y = \sqrt{5-\lambda^2}$

$y^2 - 5 = -\lambda^2$

$y^2 - 5 = -(x-3)^2$

$\Rightarrow (x-3)^2 + y^2 = 5$

So $h = 3$, $r = \sqrt{5}$

$2h - r^2 = 1$

Q.30

Column-I	Column-II
(A) If $\lvert z_1 + z_2\rvert = \lvert z_1 - z_2\rvert$, then	(P) $\arg z_1 = \arg z_2$
(B) If $\lvert z_1 + z_2\rvert = \lvert z_1\rvert + \lvert z_2\rvert$, then	(Q) $\arg\frac{z_1}{z_2} = \pm\frac{\pi}{2}$
(C) If $\lvert z_1 - z_2\rvert = \lvert\lvert z_1\rvert - \lvert z_2\rvert\rvert$, then	(R) $\frac{z_1}{z_2}$ is purely real
(D) If $\lvert z_1 - z_2\rvert = \lvert z_1\rvert + \lvert z_2\rvert$, then	(S) $\frac{z_1}{z_2}$ is purely

imaginary

Sol. **A → Q, S; B → P, R; C → P, R; D → S**

(A) $\left|\frac{z_1+z_2}{z_1-z_2}\right| = 1$

Let $\frac{z_1+z_2}{z_1-z_2} = \frac{e^{i\theta}}{1}$

$\Rightarrow \frac{z_1}{z_2} = \frac{e^{i\theta}+1}{e^{i\theta}-1} = \frac{(\cos\theta+1)+i\sin\theta}{(\cos\theta-1)+i\sin\theta}$

$= \frac{2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-2\sin^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$\frac{z_1}{z_2} = -i\cot\frac{\theta}{2}$

(B) Let $z_1 = r_1[\cos\theta_1 + i\sin\theta_1]$

$z_2 = r_2[\cos\theta_2 + i\sin\theta_2]$

So $|z_1 + z_2|^2 = (|z_1| + |z_2|)^2$

$(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$

$= r_1^2 + r_2^2 + 2r_1r_2$

$\Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\theta_1 = \theta_2$

(C) $(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$= (r_1 - r_2)^2$

$\Rightarrow \cos(\theta_1 - \theta_2) = 1$

(D) $(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$= (r_1 + r_2)^2$

$\Rightarrow \cos(\theta_1 - \theta_2) = -1$

$\theta_1 - \theta_2 = \pi$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{(\theta_1-\theta_2)} = \frac{r_2}{r_1} e^{i\pi}$

$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}[-1] \rightarrow$ Real

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 For $z \in C$, Prove that

(i) $Re(z) = 0 \Leftrightarrow z = -\bar{z}$

(ii) $Im\left(z - \frac{1}{z}\right)^4 = 0$ if $|z| = 1$.

Sol. For $z \in C$

(i) Let $z = x + iy \Rightarrow \bar{z} = x - iy$

$z - \bar{z} = x + iy - (x - iy)$

$= x + iy - x + iy = 2iy$

$\Rightarrow Re(z) = 0$ if $z = \bar{z}$

(ii) $|z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar{z} = 1 \Rightarrow z = 1/z$

$\left(z - \frac{1}{z}\right)^4 = (z - \bar{z})^4$

$= (x + iy - x + iy)^4 = (2iy)^4 = 16y^4$

$\Rightarrow Im\left(z - \frac{1}{z}\right)^4 = 0$ if $|z| = 1$

Q.2 Evaluate :

$$\sum_{p=1}^{32}(3p+2)\left\{\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right\}^p.$$

Sol. $\sum_{p=1}^{32}(3p+2)\left\{\sum_{q=1}^{10}\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right\}^p$

$$\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)$$

$$= \sum_{q=1}^{10}(-1)\left(i\cos\frac{2q\pi}{11} - \sin\frac{2q\pi}{11}\right)$$

$$= \sum_{q=1}^{10}(-i)\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right)$$

$$= \sum_{q=1}^{10}(-i)\, e^{i\frac{2\pi q}{11}} = -i\sum_{q=1}^{10} e^{i\frac{2\pi q}{11}}$$

$$= -i\sum_{p=1}^{10}\left(-1 + 1 + e^{i\frac{2q\pi}{11}}\right)$$

$$= -\left[-1 + 1 + e^{i\frac{2\pi}{11}} + e^{i\frac{4\pi}{11}} + e^{i\frac{6\pi}{11}} + \ldots + e^{i\frac{20\pi}{11}}\right]$$

$$= -i\left[-1 + \frac{1.1 - e^{\frac{i2\pi}{11}\times 11}}{1 - e^{\frac{i2\pi}{11}}}\right] = -i\left[-1 + \frac{1 - 1 + i.0}{1 - e^{\frac{i2\pi}{11}}}\right] = i$$

$$= \left\{\sum_{q=1}^{10}\left(\sin\frac{2\pi q}{11} - i\cos\frac{2q\pi}{11}\right)\right\}^p = (i)^p$$

$$\sum_{p=1}^{32}(3p+2)(i)^p = 3\sum_{p=1}^{32} p(i)^p + 2\sum_{p=1}^{32}(i)^p$$

$$\sum_{p=1}^{32}(i)^p = i + i^2 + i^3 + i^4 + i^5 + \ldots + i^{32}$$

$$= i\,\frac{(1 - i^{32})}{1 - i} = i\,\frac{1 - ((i)^2)^{16}}{1 - i}$$

$$= i\,\frac{1 - (-1)^{16}}{1 - i} = i\,\frac{1 - 1}{1 - i} = 0$$

$$\sum_{p=1}^{32} p\,(i)^p = 1.i + 2.i^2 + 3.\,i^3 + 4.i^4 + 5.i^5 + \ldots + 32.(i)^{32}$$

It forms A.P. – G.P.

Multiplying by i and subtracting, we have

$$\sum_{p=1}^{32} p(i)^p = 1.i + 2.i^2 + 3.i^3 + 4.i^4 + 5.i^5 + \ldots + 32.i^{32}$$

$$i\sum_{p=1}^{32} p(i)^p = i^2 + 2i^3 + 3i^4 + 4i^5 + 5.i^6 + \ldots + 31.i^{32} + 32.i^{33}$$

$$\sum_{p=1}^{32} p(i)^p \times (1 - i) = i + i^2 + i^3 + i^4 + i^5 + \ldots + i^{32} \ldots 32.i^{33}$$

$$= i\left(\frac{1 - i^{32}}{1 - i}\right) - 32\, i.i^{32} = 0 - 32\,i$$

$$\sum_{p=1}^{32} p(i)^p \times (1 - i) = -32\,i$$

$$\Rightarrow \sum_{p=1}^{32} p(i)^p = \frac{-32i}{(1-i)} \times \frac{(1+i)}{(1+i)}$$

$$= \frac{-32i(1+i)}{2} = -16(i-1) = 16(1-i)$$

$$\Rightarrow \sum_{p=1}^{32} (3p+2)\left\{\sum_{q=1}^{10}\left(\sin\frac{2\pi q}{11} - i\cos\frac{2\pi q}{11}\right)\right\}^p$$

$= 16(1-i) \times 3 + 2 \times 0 = 48(1-i)$ Ans.

Q.3 Prove that

(i) $\cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1$

(ii) $\dfrac{\sin 6\theta}{\sin\theta} = 32c^5 - 32c^3 + 6c$ where $c = \cos\theta$

Sol. (i) $\cos 6\theta = 2\cos^2 3\theta - 1$

$= 2(4\cos^3\theta - 3\cos\theta)^2 - 1$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

$= 32c^6 - 48c^4 + 18c^2 - 1$

(ii) $\dfrac{\sin 6\theta}{\sin\theta} = \dfrac{2\sin 3\theta\cos 3\theta}{\sin\theta}$

$$= \frac{2(3\sin\theta - 4\sin^3\theta)(4\cos^3\theta - 3\cos\theta)}{\sin\theta}$$

$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

$= 32c^5 - 32c^3 + 6c$

Q.4 Interpret the following focii in $z \in C$

(i) $1 < |z - 2i| < 3$

(ii) $\text{Arg}(z + i) - \text{Arg}(z - i) = \pi/2$

(iii) $\text{Arg}(z - a) = \pi/3$, where $a = 3 + 4i$

(iv) $\text{Arg}\left(\dfrac{1+z}{1-z}\right) = \dfrac{\pi}{3}$

Sol.

(i) $1 < |z - 2i| < 3$

Let $z = x + iy$

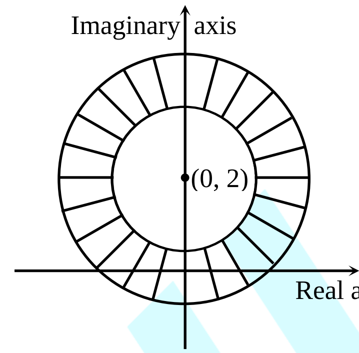
$z - 2i = x + iy - 2i$

$= x + i(y-2)$

$= x + i(y-2)$

$\Rightarrow |z - 2i| = \sqrt{x^2 + (y-2)^2}$

$\Rightarrow 1 < \sqrt{x^2 + (y-2)^2} < 3$



(ii) $\text{Arg}(z + i) - \text{Arg}(z - i) = \pi/2$

$\Rightarrow \text{Arg}\,\dfrac{z+i}{z-i} = \pi/2$

$\Rightarrow \dfrac{z+i}{z-i} = \dfrac{x+iy+i}{x+iy-i} = \dfrac{x+i(y+1)}{x+i(y-1)}$

Multiplying by $x - i(y-1)$ in numerator and denominator as

$$\frac{z+i}{z-i} = \frac{x+i(y+1)}{x+i(y-1)} \times \frac{x-i(y-1)}{x-i(y-1)}$$

$$= \frac{x^2 + ix(y+1) - ix(y-1) + y^2 - 1}{x^2 + (y-1)^2}$$

$$= \frac{(x^2+y^2-1) + i(xy + x - xy + x)}{x^2 + (y-1)^2}$$

$$= \frac{(x^2+y^2-1) + i \times 2x}{x^2+(y-1)^2}$$

$$\text{Arg}\left(\frac{z+i}{z-i}\right) = \text{Arg}\,\frac{(x^2+y^2-1)+i\times 2x}{x^2+(y-1)^2} = \pi/2$$

$\Rightarrow \dfrac{2x}{x^2+y^2-1} = \tan\pi/2 = \dfrac{1}{0}$

$\Rightarrow x^2 + y^2 - 1 = 0$

$\Rightarrow x^2 + y^2 = 1$

$\Rightarrow |z| = 1$

which is a circle of centre (0, 0) and radius 1.

(iii) $\text{Arg}(z - a) = \pi/3$, where $a = 3 + 4i$

$\text{Arg}(x + iy - 3 - 4i) = \pi/3$

$\text{Arg}[(x-3) + i(y-4)] = \pi/3$

$\Rightarrow \tan^{-1}\dfrac{y-4}{x-3} = \pi/3$

$\Rightarrow \dfrac{y-4}{x-3} = \tan\pi/3 = \sqrt{3}$

$\Rightarrow \sqrt{3}x - 3\sqrt{3} = y - 4$

$\Rightarrow \sqrt{3}x - y = 3\sqrt{3} - 4$

Which is a straight line.

(iv) Arg $\frac{1+z}{1-z} = \pi/3$

$$\frac{1+z}{1-z} = \frac{1+x+iy}{1-x-iy} = \frac{(1+x)+iy}{(1-x)-iy}$$

Multiplying by (1 – x) + iy in numerator and denominator as :

$$\frac{1+z}{1-z} = \frac{(1+x)+iy}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$$

$$= \frac{(1-x^2)+iy(1-x)+iy(1+x)-y^2}{(1-x)^2+y^2}$$

$$= \frac{(1-x^2-y^2)+i(y-yx+y+yx)}{(1-x)^2+y^2}$$

$$\frac{1+z}{1-z} = \frac{(1-x^2-y^2)+2iy}{(1-x)^2+y^2}$$

$$\Rightarrow \text{Arg}\,\frac{1+z}{1-z} = \pi/3 = \tan^{-1}\frac{2y}{1-x^2-y^2}$$

$$\Rightarrow \frac{2y}{1-x^2-y^2} = \sqrt{3}$$

$$\Rightarrow 1 - x^2 - y^2 = \frac{2}{\sqrt{3}}y$$

$$\Rightarrow x^2 + y^2 + \frac{2}{\sqrt{3}}y - 1 = 0$$

$$\Rightarrow x^2 + \left(y+\frac{1}{\sqrt{3}}\right)^2 - \frac{1}{3} - 1 = 0$$

$$\Rightarrow x^2 + \left(y+\frac{1}{\sqrt{3}}\right)^2 = 4/3$$

Which is a circle of centre $\left(0,-\frac{1}{\sqrt{3}}\right)$ & radius $= \frac{2}{\sqrt{3}}$

Q.5 The roots z_1, z_2, z_3 of the equation $x^3 + 3ax^2 + 3bx + c = 0$ in which a, b, c are complex numbers corresponding to the points A,B,C on the gaussian plane, find the centroid of triangle ABC and show that it will be equilateral if $a^2 = b$.

Sol. $x^3 + 3a\,x^2 + 3bx + c = 0$

$\Rightarrow x^3 + 3ax^2 + 3bx + c = (x - z_1)(x - z_2)(x - z_3)$

sum of roots, $z_1 + z_2 + z_3 = -3a$

$\Sigma z_1 z_2 = z_1 z_2 + z_2 z_3 + z_3 z_1 = 3b$

$z_1 z_2 z_3 = -c$

$$D\left[\frac{z_2+z_3}{2}\right]$$

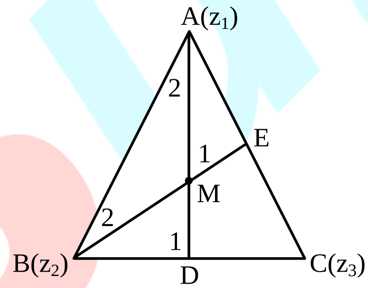
$$M\left[\frac{z_1.1+2.\frac{z_2+z_3}{2}}{1+2}\right]$$

$$M\left[\frac{z_1+z_2+z_3}{3}\right]$$

Hence, centroid of ΔABC = $\frac{z_1+z_2+z_3}{3}$

$= \frac{-3a}{3}$

$= -a$



In case of equilateral triangle,

$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3 z_1$

we know, that

$(z_1 + z_2 + z_3)^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_2z_3 + z_3z_1)$

$\Rightarrow (z_1 + z_2 + z_3)^2 - 3(z_1 z_2 + z_2 z_3 + z_3 z_1)$

$= (z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1)$

$= 0$

$\Rightarrow (-3a)^2 - 3 \times 3b = 0$

$\Rightarrow a^2 = b$ Hence Proved

Q.6 If z_1, z_2, z_3 are complex numbers such that at least one is not real & $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$ show that z_1, z_2, z_3, lie on a circle passing through the origin.

Sol. Given : $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$

Taking modulus both sides, we get

$$\left|\frac{2}{z_1}\right| = \left|\frac{1}{z_2} + \frac{1}{z_3}\right|$$

$$\Rightarrow \left|\frac{1}{z_2}+\frac{1}{z_3}\right| \geq \left|\frac{1}{z_2}\right| - \left|\frac{1}{z_3}\right| \;\&\; \left|\frac{1}{z_2}+\frac{1}{z_3}\right| \leq \left|\frac{1}{z_2}\right| + \left|\frac{1}{z_3}\right|$$

Let $z_1 = |z_1|\, e^{i\alpha} = r_1\, e^{i\alpha} \Rightarrow |z_1| = r_1$

$z_2 = |z_2|\, e^{i\beta} = r_2\, e^{i\beta} \Rightarrow |z_2| = r_2$

$z_3 = |z_3|\, e^{ir} = r_3\, e^{ir} \Rightarrow |z_3| = r_3$

$$\Rightarrow \frac{2}{|z_1|} \geq \frac{1}{|z_2|} - \frac{1}{|z_3|} \Rightarrow \frac{2}{r_1} \geq \frac{1}{r_2} - \frac{1}{r_3}$$

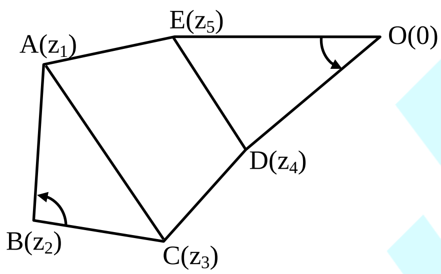
$$\Rightarrow \frac{2}{|z_1|} \leq \frac{1}{|z_2|} + \frac{1}{|z_3|} \Rightarrow \frac{2}{r_1} \leq \frac{1}{r_2} - \frac{1}{r_3}$$

It is only possible when $r_1 = r_2 = r_3 \Rightarrow |z_1| = |z_2| = |z_3|$

i.e. z_1, z_2, z_3 all lie on a circle passing through origin.

Q.7 A,B,C,D and E are the points on the complex plane representing the complex numbers z_1, z_2, z_3, z_4 & z_5 respectively. If $(z_3 - z_2)z_4 = (z_1 - z_2)\, z_5$, then prove that the triangles ABC and DOE are similar, 'O' being the origin.

Sol.



Applying rotation theorem at point B

$$\frac{z_1 - z_2}{z_3 - z_2} = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\alpha_1} \quad \ldots (1)$$

Also, rotation theorem at O,

$$\frac{z_4}{z_5} = \frac{|z_4|}{|z_5|} e^{i\alpha_2} \quad \ldots (2)$$

Since, we are given

$$(z_3 - z_2)\, z_4 = (z_1 - z_2)\, z_5 \Rightarrow \frac{z_1 - z_2}{z_3 - z_2} = \frac{z_4}{z_5}$$

Taking modulus of both sides, we have

$$\left|\frac{z_1 - z_2}{z_3 - z_2}\right| = \left|\frac{z_4}{z_5}\right|$$

from (1) and (2), compare above expression we have,

$$\alpha_1 = \alpha_2$$

$\Rightarrow$ Triangles ΔABC and ΔDOE are similar.

Q.8 Among the complex numbers z satisfying the condition $\left|z+3-\sqrt{3}i\right| = \sqrt{3}$, find the number having the least positive argument.

Sol. Let $z = x + iy$

$\Rightarrow |z + 3 - i\sqrt{3}| = \sqrt{3}$

$\Rightarrow |x + iy + 3 - i\sqrt{3}| = \sqrt{3}$

$\Rightarrow |(x + 3) + i(y - \sqrt{3})| = \sqrt{3}$

$\Rightarrow (x + 3)^2 + (y - \sqrt{3})^2 = 3$

$\Rightarrow |y| = \sqrt{3} + \sqrt{3-(x+3)^2}$

$\Rightarrow \text{Arg } z = \theta \text{ (let)} = \text{Tan}^{-1}\left|\frac{y}{x}\right|$

$$\Rightarrow \theta = \text{Tan}^{-1}\frac{\sqrt{3}+\sqrt{3-(x+3)^2}}{x}$$

Differentiating w.r.t. x, we get

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{\sqrt{3}+\sqrt{3-(x+3)^2}}{x}\right)^2} \times \frac{\frac{(-2(x+3))\times x}{2\sqrt{3-(x+3)^2}} - \frac{1}{(\sqrt{3}+\sqrt{3-(x+3)^2}}}{x^2}$$

For maximum of minimum value of θ, Put $\frac{d\theta}{dx} = 0$

$$\Rightarrow +\frac{x(x+3)}{\sqrt{3-(x+3)^2}} = -\left(\sqrt{3}+\sqrt{3-(x+3)^2}\right)$$

$$\Rightarrow x(x+3) = -\sqrt{3}\left(\sqrt{3-(x+3)^2}\right) - 3 + (x+3)^2$$

$$\Rightarrow (x+3)\,[x - x - 3] = -\sqrt{3}\left(\sqrt{3-(x+3)^2}\right) - 3$$

$$\Rightarrow -3\,(x+3) = -\sqrt{3}\left(\sqrt{3-(x+3)^2}\right) - 3$$

$$\Rightarrow x + 3 = \frac{\sqrt{3-(x+3)^2}}{\sqrt{3}} + 1$$

$$\Rightarrow (x+2)\sqrt{3} = \sqrt{3-(x+3)^2}$$

Squaring both sides, we have

$\Rightarrow 3(x + 2)^2 = 3 - (x + 3)^2$

$\Rightarrow 3x^2 + 12 + 12x + x^2 + 9 + 6x = 3$

$\Rightarrow 4x^2 + 18x + 18 = 0$

$\Rightarrow 2x^2 + 9x + 9 = 0$

$$\Rightarrow x = \frac{-9 \pm \sqrt{81-72}}{2\times 2} = \frac{-9\pm 3}{4}$$

$\Rightarrow x = -3, -3/2$

$\Rightarrow |y| = \sqrt{3} + \sqrt{3-(x+3)^2}$

when $x = -3 \Rightarrow |y| = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$

$z_1 = -3 + 2\sqrt{3}\, i$

$\text{Arg}(z_1) = \theta_1 = \tan^{-1}\left|\frac{2\sqrt{3}}{-3}\right|$

$= \tan^{-1}\left|\frac{2}{\sqrt{3}}\right|$

when $x = -3/2$

$\Rightarrow |y| = \sqrt{3} + \sqrt{3-(-3/2+3)^2}$

$= \sqrt{3} + \sqrt{3-9/4}$

$= \sqrt{3} + \sqrt{3}/2 = 3\sqrt{3}/2$

$\therefore z_2 = 3/2 + i3\sqrt{3}/2$

$\text{Arg}(z_2) = \theta_2 = \tan^{-1}\left|\frac{\frac{3\sqrt{3}}{2}}{\frac{-3}{2}}\right|$

$= \tan^{-1}|-\sqrt{3}|$

$= \pi/3$

$\therefore$ Required ans is $z_2 = -3/2 + i\,3\sqrt{3}/2$ because, $\theta_1 > \theta_2$

Q.9 Prove geometrically and analytically the following inequality $\left|\frac{z}{|z|}-1\right| \le |\arg z|$

Sol. Let $z = r.e^{i\theta}$

$\Rightarrow \left|\frac{z}{|z|}-1\right| = |e^{i\theta} - 1|$

$= |(\cos\theta - 1) + i\sin\theta|$

$= \sqrt{(\cos\theta-1)^2+\sin^2\theta}$

$= \sqrt{2(1-\cos\theta)}$

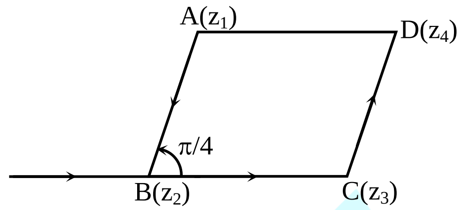
$= 2\left|\sin\frac{\theta}{2}\right| \le 2\left|\frac{\theta}{2}\right|$

$= |\theta|$

$= |\text{Arg}(z)|$

Q.10 z_1, z_2 are two vertices of a parallelogram whose angle at z_2 is $\frac{\pi}{4}$. Find the complex number z_3 representing the vertex adjacent to z_2 if z_1, z_2, z_3 are in anticlockwise sense and the side joining z_3, z_2 is half the length of the side joining z_1, z_2.

Sol. Given, $|z_3 - z_2| = \frac{1}{2}|z_2 - z_1|$



Applying conimethod or rotation theorem at B, we get

$$\frac{z_1-z_2}{z_3-z_2} = \left|\frac{z_1-z_2}{z_3-z_2}\right| e^{i\pi/4}$$

Since, $|z_3 - z_2| = \frac{1}{2}|z_2 - z_2|$

$\Rightarrow \frac{z_1-z_2}{z_3-z_2} = 2e^{i\pi/4}$

$\Rightarrow (z_3 - z_2) = \frac{(z_1-z_2)}{2} \times e^{-i\pi/4}$

$\Rightarrow z_3 = z_2 + \frac{1}{2}(z_1 - z_2) \times (\cos\pi/4 - i\sin\pi/4)$

$\Rightarrow z_3 = z_2 + \frac{1}{2}(z_1 - z_2) \times \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$\Rightarrow z_3 = z_2 + \frac{(1-i)}{2\sqrt{2}} z_1 - \frac{1}{2\sqrt{2}}(1-i) z_2$

$= \frac{(1-i)}{2\sqrt{2}} z_1 + \left(1 - \frac{1}{2\sqrt{2}} + \frac{i}{2\sqrt{2}}\right) z_2$

$\Rightarrow z_3 = \frac{(1-i)}{2\sqrt{2}} z_1 + \left(\frac{(2\sqrt{2}-1)}{2\sqrt{2}} + \frac{i}{2\sqrt{2}}\right) z_2$ Ans.

Q.11 If $\cos\alpha + \cos\beta + \cos\gamma = 0$, $\sin\alpha + \sin\beta + \sin\gamma = 0$ then prove that:

(a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

(b) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Sol. Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$

$\Rightarrow a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i0 = 0$

$\Theta\ a + b + c = 0$

$\Rightarrow a^3 + b^3 + c^3 = 3abc$

$\Rightarrow (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$\Rightarrow \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Again $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = (\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma)$

$= 0 - i0 = 0$

$\Rightarrow ab + bc + ca = 0$

Now $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow 0 = a^2 + b^2 + c^2 + 0$

$\Rightarrow a^2 + b^2 + c^2 = 0$

$\Rightarrow (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

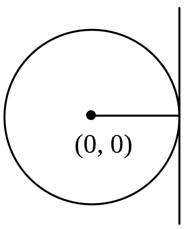
$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ and

$\sin 2\alpha + 2\beta + \sin 2\gamma = 0$

Q.12 Show that the equation of the tangent to the circle $|z| = r$ at z_1 is $\bar{z}_1 z + z_1\bar{z} = 2r^2$.

Sol. Let $z = x + iy$

$z_1 = x_1 + iy_1$



equation of tangent is : $xx_1 + yy_1 = r^2$

$$= \left(\frac{z+\bar{z}}{2}\right)\left(\frac{z_1+\bar{z}_1}{2}\right) + \left(\frac{z-\bar{z}}{2i}\right)\left(\frac{z_1-\bar{z}_1}{2i}\right) = r^2$$

$\Rightarrow \bar{z}_1 z + z_1\bar{z} = 2r^2$

Q.13 z_1, z_2 and z_3 are three non-collinear complex numbers and z is a variable complex number satisfying $\frac{1}{z-z_1} + \frac{1}{z-z_2} + \frac{1}{z-z_3} = 0$, prove that z always lies inside the triangle formed by z_1, z_2 and z_3.

Sol. Given

$$\frac{1}{z-z_1} + \frac{1}{z-z_2} + \frac{1}{z-z_3} = 0$$

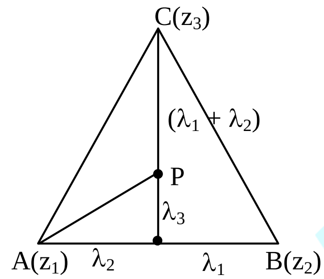
$$\Rightarrow \frac{\bar{z}-\bar{z}_1}{|z-z_1|^2} + \frac{\bar{z}-\bar{z}_2}{|z-z_2|^2} + \frac{\bar{z}-\bar{z}_3}{|z-z_3|^2} = 0$$

$\Rightarrow \lambda_1(\bar{z} - \bar{z}_1) + \lambda_2(\bar{z} - \bar{z}_2) + \lambda_3(\bar{z} - \bar{z}_3) = 0$

where $\lambda_1, \lambda_2, \lambda_3 \in R^+$

$$\Rightarrow \bar{z} = \frac{\lambda_1\bar{z}_1 + \lambda_2\bar{z}_2 + \lambda_3\bar{z}_3}{\lambda_1+\lambda_2+\lambda_3}$$

$$\Rightarrow z = \frac{\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3}{\lambda_1+\lambda_2+\lambda_3}$$



$$z = \frac{(\lambda_1+\lambda_2).\left(\frac{\lambda_1 z_1 + \lambda_2 z_2}{\lambda_1+\lambda_2}\right) + \lambda_3 z_3}{\lambda_1+\lambda_2+\lambda_3}$$

which is corresponding to point 'P' inside the triangle.

Q.14 Find the point in Argand plane which is equidistance from roots of $(z + 1)^4 = 16z^4$.

Sol. $(z + 1)^4 = 16z^4$

$\left(\frac{z+1}{z}\right) = 2(1)^{1/4}$

But $(1)^{1/4} = \pm 1, \pm i$

so roots of the equation are

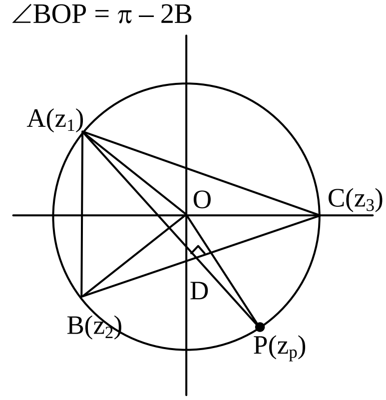
$z = 1, -\frac{1}{3}, -\frac{(1+2i)}{5}, -\left(\frac{1-2i}{5}\right)$

The point equidistant from all points is the circumcentre $\left(\frac{1}{3}, 0\right)$.

Q.15 A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively on the complex plane and the circumcentre of the triangle ABC lies at the origin. If the altitude of the triangle through the vertex A meets the circumcircle again at P, then prove that P represents the complex number $-z_2z_3/z_1$.

Sol. (Refer Q.23)

$\angle BOP = \pi - 2B$



so $z_p = z_2 \cdot e^{i(\pi - 2B)}$ ....(i)

also $\angle AOC = 2B$

$\Rightarrow z_1 = z_3\, e^{i(2B)}$ ....(ii)

Multiply equation (i) and (ii)

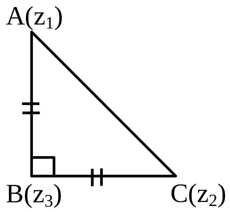
$z_1 z_p = z_2 z_3 e^{i\pi}$

$\Rightarrow z_p = -\dfrac{z_2 z_3}{z_1}$

Q.16 Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Sol. Since, Δ is right angled isosceles Δ.

$\therefore$ Rotating z_2 about z_3 in anticlock wise direction through an angle of $\pi/2$, we get



$\dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{|z_2 - z_3|}{|z_1 - z_3|} e^{i\pi/2}$ where $|z_2 - z_3| = |z_1 - z_3|$

$\Rightarrow (z_2 - z_3) = i(z_1 - z_3)$

squaring both sides we get,

$(z_2 - z_3)^2 = -(z_1 - z_3)^2$

$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -z_1^2 - z_3^2 + 2z_1z_3$

$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_3^2 - 2z_1z_2$

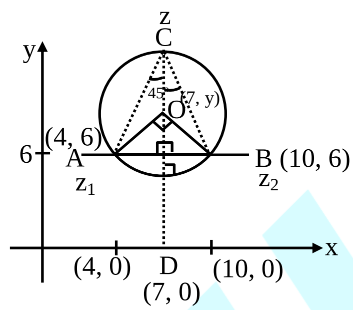
$\Rightarrow (z_1 - z_2)^2 = 2\{(z_1z_3 - z_3^2) + (z_2z_3 - z_1z_2)\}$

$\Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Q.17 Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $\dfrac{(z - z_1)}{(z - z_2)}$ is $\dfrac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

Sol. As $z_1 = 10 + 6i$, $z_2 = 4 + 6i$

and $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \dfrac{\pi}{4}$ represents locus of z is a circle shown as



As from the figure centre is (7, y) and $\angle AOB = 90^\circ$ clearly OC = 9.

$\Rightarrow$ OD = 6 + 3 = 9

$\therefore$ Centre = (7, 9) and radius = $\dfrac{6}{\sqrt{2}} = 3\sqrt{2}$

$\Rightarrow$ Equation of circle : $|z - (7 + 9i)| = 3\sqrt{2}$

Q.18 (a) If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$

(b) If $|z| \le 1$, $|w| \le 1$, show that

$|z - w|^2 \le (|z| - |w|)^2 + (\text{Arg } z - \text{Arg } w)^2$

Sol. (A) $iz^3 + z^2 - z + i = 0$

$\Rightarrow iz^3 + z^2 + i^2 z + i = 0$

$\Rightarrow z^2(iz + 1) + i(iz + 1) = 0$

$\Rightarrow (iz + 1)(z^2 + i) = 0$

$\Rightarrow iz + 1 = 0$ or $z^2 + i = 0$

$\Rightarrow z = -1/i$ or $z^2 = -1$

$\Rightarrow z = -i^2/i = i$ or $z^2 = -i$

$\Rightarrow |z|^2 = 1$

Hence, Proved

(B) If $|z| \le 1$, $|w| \le 1$

Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$

$w = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow |z - w|^2 = |r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2)|^2$

$= |(r_1\cos\theta_1 - r_2\cos\theta_2)|^2 + i|(r_1\sin\theta_1 + r_2\sin\theta_2)|^2$

$= r_1^2 + r_2^2 - 2r_1r_2 + 2r_1r_2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1r_2 \times 2\sin^2\left(\dfrac{\theta_1 - \theta_2}{2}\right)$

use, $\sin\theta \le \theta$ for small θ.

$= (r_1 - r_2)^2 + 2r_1r_2 \times 2 \times \left(\dfrac{\theta_1 - \theta_2}{2}\right)^2$

$= (r_1 - r_2)^2 + 4r_1r_2 \times (\theta_1 - \theta_2)^2/4$

$= (r_1 - r_2)^2 + r_1r_2 \times (\theta_1 - \theta_2)^2$

$= (|z| - |w|)^2 + |z|\,.\,|w|\,(\text{Arg } z - \text{Arg } w)^2$

$\le (|z| - |w|)^2 + (\text{Arg } z - \text{Arg } w)^2$

$\Rightarrow |z - w|^2 \le (|z| - |w|)^2 + (\text{Arg } z - \text{Arg } w)^2$

(Since, $|z| \le 1$ and $|w| \le 1$

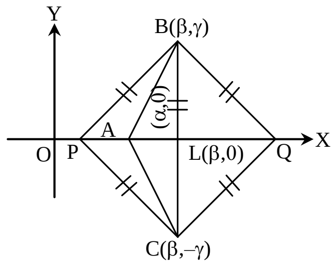
Arg z = θ_1 and Arg w = θ_2

Hence, Proved

Part-B Passage based objective questions

Passage - I (Q. 19 to 21)

A cubic equation f(x) = 0 has one real and two complex roots α, β + iγ and β – iγ respectively as shown, then



Q.19 The distance PL is -

(A) $\sqrt{3}\ |\beta|$ (B) $\sqrt{3}\ |\alpha|$

(C) $\sqrt{3}\ |\gamma|$ (D) None of these

Sol. **[C]** Diagonals of Rhombus, PCQB will intersect at right angle and point of intersection will be mid point of both diagonals PQ and BC.

∴ In triangle, ΔBLQ,

$BQ^2 = BL^2 + LQ^2$

$\Rightarrow \left(\frac{BC}{2}\right)^2 + (PL)^2 = BQ^2$

$\Rightarrow r^2 + (PL)^2 = 4r$

$\Rightarrow PL\ \sqrt{3}\ |r|$

∴ Option (C) is correct answer.

Q.20 The roots of the derived equation f ′(x) are complex if -

(A) 'A' falls inside one of the two equilateral Δ′s described on BC.

(B) 'A' falls outside one of the two equilateral Δ's described on BC.

(C) 'A' forms an equilateral triangle with BC

(D) None of these

Sol. **[A]** f(x) = (x – α) (x – β – ir) (x – β + ir)

differentiating w.r.t., we get

f ' (x) = (x – β – ir) (x – β + ir) + (x – α) (x – β + ir) + (x – α) (x – β – ir) = 0

Roots will be complex i.e. point A must be inside one of the two equilateral triangles described on BC.

∴ Option (A) is correct answer.

Q.21 The roots of the derived equation f(x) are real and distinct if:-

(A) 'A' falls outside one of the two equilateral triangle with BC

(B) 'A' falls inside one of the two equilateral triangle with BC

(C) 'A' lies anywhere

(D) None of these

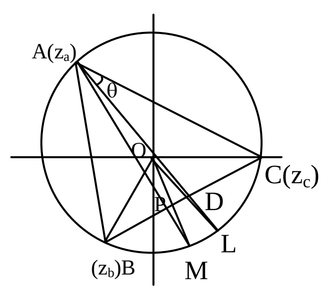
Sol. **[D]**

f(x) = 0 will have real and distinct roots if A must be out side both the triangles described on BC.

∴ Option (D) is correct answer.

Passage II (Q. 22 to 24)

In the figure |z| = r is circumcircle of ΔABC. D, E & F are the middle points of the sides BC, CA & AB respectively, AD produced to meet the circle at L. If ∠CAD = θ, AD = x, BD = y and altitude of ΔABC from A meet the circle |z| = r at M, z_a, z_b & z_c are affixes of vertices A, B & C respectively Then



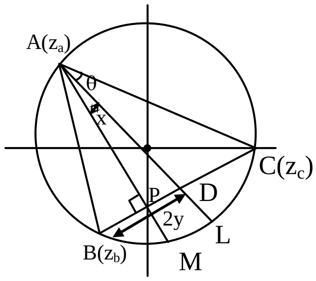
Q.22 Area of the ΔABC is equal to

(A) xy cos (θ + C) (B) (x + y) sin θ

(C) xy sin (θ + C) (D) $\frac{1}{2}$ xy sin(θ + C)

Sol.[C] ∠CAP = 90 – C

⇒ ∠DAP = (90 – C – θ)



so Area = $\frac{1}{2} \times BC \times AP$

$= \frac{1}{2} \times (2y) \cdot (x \cos (90^o - C - \theta))$

$= xy \sin (C + \theta)$

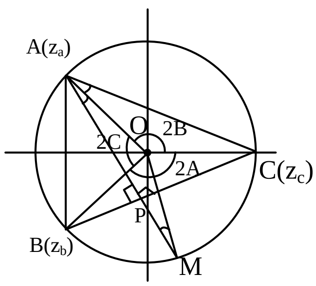
Q.23 Affix of M is -

(A) $2z_b e^{i2B}$ (B) $z_b e^{i(\pi - 2B)}$

(C) $z_b e^{iB}$ (D) $2z_b e^{Ib}$

Sol. [B] $\angle AOB = 2C$

$\angle OAC = \frac{1}{2} [\pi - 2B]$



$\angle PAC = 90^o - C$

So, $\angle OAP = \angle OMP = \angle PAC - \angle OAC$

$= 90^o - C - \frac{1}{2} [180^o - 2B]$

Now from ΔOAM

$= 2\left[90^o - C - \frac{1}{2}[180^o - 2B]\right] + 2C + \angle BOM$

$= 180^o$

$\Rightarrow \angle BOM = (180^o - 2B)$

so $z_M = z_b e^{i(\pi - 2B)}$

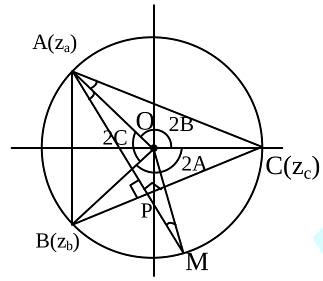
Q.24 Affix of L is

(A) $z_b e^{i(2A - 2\theta)}$ (B) $2z_b e^{i(2A - 2\theta)}$

(C) $z_b e^{i(A - \theta)}$ (D) $2z_b e^{i(A - \theta)}$

Sol. [B] $\angle AOB = 2C$

$\angle OAC = \frac{1}{2} [\pi - 2B]$



$\angle PAC = 90^o - C$

So, $\angle OAP = \angle OMP = \angle PAC - \angle OAC$

$= 90^o - C - \frac{1}{2} [180^o - 2B]$

Now from ΔOAM

$= 2\left[90^o - C - \frac{1}{2}[180^o - 2B]\right] + 2C + \angle BOM$

$= 180^o$

$\Rightarrow \angle BOM = (180^o - 2B)$

so $z_M = z_b e^{i(\pi - 2B)}$

Passage III (Q. 25 to 28)

A regular heptagon (seven sides) is inscribed in a circle of radius 1. Let A_1A_2 .... A_7 be its vertices, G_1 is centroid of $\Delta A_1A_2A_5$ and G_2 be centroid of $\Delta A_3A_6A_7$. P is centroid of ΔOG_1G_2, where O is centre of circum scribing circle.

Q.25 $\angle POA_1$ is equal to-

(A) $\frac{\pi}{7}$ (B) $\frac{2\pi}{7}$ (C) $\frac{5\pi}{7}$ (D) $\frac{6\pi}{7}$

Sol.[A] Let $A_k = e^{i\left(\frac{2\pi}{7}\right)k}$

Where k = 1, 2, ………..7

So point 'P' is :

$$\frac{\frac{A_1 + A_2 + A_5}{3} + \frac{O}{3} + \frac{A_3 + A_6 + A_7}{3}}{3}$$

$$= \left(\frac{O - A_4}{9}\right)$$

$$= \frac{1}{9}\left(-e^{i\left(\frac{8\pi}{7}\right)}\right) = \frac{-e^{i\left(\frac{8\pi}{7}\right)}}{9} = \frac{e^{i\left(\frac{\pi}{7}\right)}}{9}$$

$$\angle POA_1 = \frac{2\pi}{7} - \frac{\pi}{7} = \frac{\pi}{7}$$

Q.26 OP is equal to-

(A) $\frac{10}{9}$ (B) $\frac{8}{9}$ (C) $\frac{1}{9}$ (D) 1

Sol.[C] Let $A_k = e^{i\left(\frac{2\pi}{7}\right)k}$

Where k = 1, 2, ...7

So point 'P' is :

$$\frac{\frac{A_1 + A_2 + A_5}{3} + \frac{O}{3} + \frac{A_3 + A_6 + A_7}{3}}{3}$$

$$= \left(\frac{O - A_4}{9}\right)$$

$$= \frac{1}{9}\left(-e^{i\left(\frac{8\pi}{7}\right)}\right) = \frac{-e^{i\left(\frac{8\pi}{7}\right)}}{9} = \frac{e^{i\left(\frac{\pi}{7}\right)}}{9}$$

$$OP = \left|\frac{e^{i(\pi/7)}}{9}\right| = \frac{1}{9}$$

Q.27 G_3 lies on segment OA_4 such that centroid of triangle $G_1G_2G_3$ is O, then -

(A) $3OG_3 = OA_4$ (B) $3OG_2 = A_4G_3$

(C) $2OG_3 = OA_4$ (D) $OG_3 = G_3A_4$

Sol.[A] Let $A_k = e^{i\left(\frac{2\pi}{7}\right)k}$

Where k = 1, 2, ...7

So point 'P' is :

$$\frac{\frac{A_1 + A_2 + A_5}{3} + \frac{O}{3} + \frac{A_3 + A_6 + A_7}{3}}{3}$$

$$= \left(\frac{O - A_4}{9}\right)$$

$$= \frac{1}{9}\left(-e^{i\left(\frac{8\pi}{7}\right)}\right) = \frac{-e^{i\left(\frac{8\pi}{7}\right)}}{9} = \frac{e^{i\left(\frac{\pi}{7}\right)}}{9}$$

Let $G_3 = re^{i\left(\frac{8\pi}{7}\right)}$

$$\text{So } \frac{\frac{A_{+}A_2 + A_5}{3} + \frac{A_3 + A_6 + A_7}{3} + G_3}{3} = 0$$

$$\frac{-e^{i\left(\frac{8\pi}{7}\right)}}{3} + G_3 = 0$$

$\Rightarrow$ $|G_3| = \frac{1}{3}$

$\Rightarrow$ $3\,OG_3 = OA_4$

Q.28 PA_1 is equal to-

(A) $\frac{1}{9}\sqrt{\left(82 - 18\cos\frac{\pi}{7}\right)}$

(B) $\frac{1}{9}\sqrt{\left(82 + 18\cos\frac{\pi}{7}\right)}$

(C) $\frac{1}{9}\sqrt{\left(82 - 18\sin\frac{\pi}{7}\right)}$

(D) None of these

Sol.[A] Let $A_k = e^{i\left(\frac{2\pi}{7}\right)k}$

Where k = 1, 2, ...7

So point 'P' is :

$$\frac{\frac{A_1 + A_2 + A_5}{3} + \frac{O}{3} + \frac{A_3 + A_6 + A_7}{3}}{3}$$

$$= \left(\frac{O - A_4}{9}\right)$$

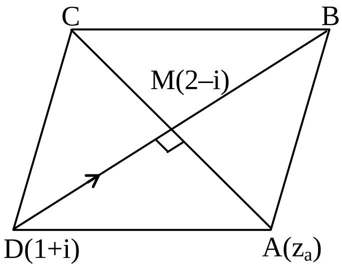
$$= \frac{1}{9}\left(-e^{i\left(\frac{8\pi}{7}\right)}\right) = \frac{-e^{i\left(\frac{8\pi}{7}\right)}}{9} = \frac{e^{i\left(\frac{\pi}{7}\right)}}{9}$$

$$PA_1 = \left|e^{i\left(\frac{2\pi}{7}\right)} - \frac{e^{i\left(\frac{\pi}{7}\right)}}{9}\right|$$

$$= \frac{|9e^{i\left(\frac{\pi}{7}\right)} - 1|}{9}$$

$$= \frac{\left|\left(9\cos\frac{\pi}{7} - 1\right) + 9i\sin\frac{\pi}{7}\right|}{9}$$

$$= \frac{1}{9}\cdot\sqrt{82 - 18\cos\frac{\pi}{7}}$$



Passage IV (Q. 29 to 31)

ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. Let the points D and M represent complex numbers 1 + i and 2 – i respectively. If θ is arbitrary real then $z = re^{i\theta}$, $R_1 \le r \le R_2$ lies in annular region formed by concentric circle $|z| = R_1$, $|z| = R_2$.

Q.29 A possible representation of point A is -

(A) $3 - \frac{i}{2}$ (B) $3 + \frac{i}{2}$

(C) $1 + \frac{3}{2}i$ (D) $3 - \frac{3}{2}\ i$

Sol.[A] By Rotation

$$\frac{z_a - (2-i)}{1} = \frac{(1+i)-(2-i)}{2}\ e^{\pm i\frac{\pi}{2}}$$

$$z_a - (2-i) = \left(\frac{-1+2i}{2}\right)(\pm i)$$

$$\Rightarrow\ z_a = 1 - \frac{3}{2}i \quad \text{or } 3 - \frac{i}{2}$$

Q.30 $e^{iz} =$

(A) $e^{-r\cos\theta}$ (cos (r cos θ) + i sin (r sin θ))

(B) $e^{-r\cos\theta}$ (sin (r cos θ) + i cos (r sin θ))

(C) $e^{-r\sin\theta}$ (cos (r cos θ) + i sin (r cos θ))

(D) $e^{-r\cos\theta}$ (sin (r cos θ) + i cos (r sin θ))

Sol.[C] $e^{i[r\cos\theta + r\, i\sin\theta]}$

$= e^{-r\sin\theta}\ e^{i(r\cos\theta)}$

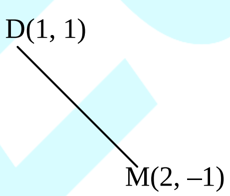
$= e^{-r\sin\theta}\ [\cos(r\cos\theta) + i\sin(r\cos\theta)]$

Q.31 If z is any point on segment DM then $w = e^{iz}$ lies in annular region formed by concentric circles -

(A) $|w| = 1$, $|w| = 2$ (B) $|w| = \frac{1}{e}$, $|w| = e$

(C) $|w| = \frac{1}{e^2}$, $|w| = e^2$ (D) $|w| = \frac{1}{2}$, $|w| = 1$

Sol.[B]



$w = e^{iz} = e^{-r\sin\theta}$ [cos (r cos θ) + i sin(r cos θ)]

$|w| = e^{-r\sin\theta}$

z lies on DM

So $-1 \le r\sin\theta \le 1$

$\Rightarrow -1 \le -r\sin\theta \le 1$

So $e^{-1} \le |w| \le e^{1}$

$\frac{1}{e} \le |w| \le e$

EXERCISE # 4

Old IIT-JEE questions

Q.1 The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangles which is **[IIT-2001]**

(A) of area zero

(B) right angled isosceles

(C) equilateral

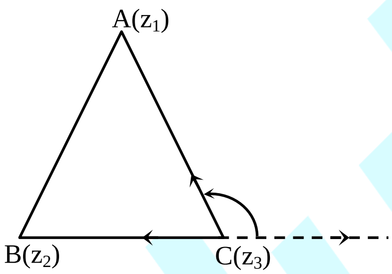
(D) obtuse angled isosceles

Sol. **[C]** $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$

$\Rightarrow \frac{z_1 - z_3}{z_3 - z_2} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Applying rotation theorem, we have

$\frac{z_1 - z_3}{z_3 - z_2} = \left|\frac{z_1 - z_3}{z_3 - z_2}\right| e^{i2\pi/3}$



$\frac{z_1 - z_3}{z_3 - z_2} + \frac{-1}{2} + \frac{i\sqrt{3}}{2}$

$\therefore$ Option (C) is correct answer.

Q.2 If z_1 and z_2 be the nth roots of unity which subtend right angle at the origin. Then n must be of the form **[IIT-2001]**

(A) 4 k + 1 (B) 4k + 2

(C) 4k + 3 (D) 4k

Sol. **[D]**

$x^n - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$ …… $(x - \alpha_{n-1})$

$\Rightarrow |z_1| = 1$ and $|z_2| = 1$

$\Rightarrow \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{i\pi/2} = e^{i\pi/2} = i$

$\Rightarrow \left(\frac{z_1}{z_2}\right)^n = (i)^n$

$\Rightarrow \frac{z_1^n}{z_2^n} = (i)^n = 1 = (i)^{4K}$

$\Rightarrow n = 4k$

$\therefore$ Option (D) is correct answer.

Q.3 For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is- **[IIT-2002]**

(A) 0 (B) 2 (C) 7 (D) 17

Sol. **[B]**

Given : $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$

use, $|z_1 - z_2| \ge |z_1| - |z_2|$

$\Rightarrow |z_1 - (z_2 - 3 - 4i) - (3 + 4i)|$

$\ge |z_1| - |z_2 - 3 - 4i| - |3 + 4i|$

$\ge 12 - 5 - 5$

≥ 2

$\therefore$ Option (B) is correct answer.

Q.4 Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is - **[IIT-2002]**

(A) 3ω (B) $3\omega(\omega - 1)$

(C) $3\omega^2$ (D) $3\omega(1 - \omega)$

Sol.[B] $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$

$= 1(\omega^2 - \omega^4) - 1(\omega - \omega^2) + 1(\omega^2 - \omega)$

$= \omega^2 - \omega - \omega + \omega^2 + \omega^2 = \omega$

$= 3\omega^2 - 3\omega$

$= 3\omega(\omega - 1)$

$\therefore$ Option (B) is correct answer.

Q.5 Let a complex number α be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$, where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \ldots.. + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{q-1} = 0$, but not both together. **[IIT- 2002]**

Sol. $z^{p+q} - z^p - z^q + 1 = 0$

$\Rightarrow z^p . z^q - z^p - z^q + 1 = 0$

$\Rightarrow z^p (z^q - 1) - 1(z^q - 1) = 0$

$\Rightarrow (z^p - 1)(z^q - 1) = 0$

$\Rightarrow$ Either $z^p - 1 = 0$ or $z^q - 1 = 0$

when $z^p - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \ldots (z - \alpha_{p-1})$

where $1, \alpha_1, \alpha_2, \ldots, \alpha_{p-1}$ are root of $z^q - 1$

$\therefore$ sum of root $1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{p-1} = 0$

when $z^q - 1 = 0$

$\Rightarrow z^q - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)(z - \alpha_4) \ldots (z - \alpha_{q-1})$

$\therefore$ Sum of roots $= 1 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{q-1} = 0$

$\Rightarrow 1 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{q-1} = 0$

Hence, Proved.

Q.6 If $|z| = 1$, $z \neq -1$ and $w = \frac{z-1}{z+1}$ then real part of w = ? **[IIT- Scr-2003]**

(A) $\frac{-1}{|z+1|^2}$ (B) $\frac{1}{|z+1|^2}$

(C) $\frac{2}{|z+1|^2}$ (D) 0

Sol. **[D]** Given : $|z| = 1$, $z \neq 1$

$\omega = \frac{z-1}{z+1}$

Multiplying by $(\bar{z} + 1)$ in numerator and denominator, we have

$\omega = \frac{z-1}{z+1} \times \frac{\bar{z}+1}{\bar{z}+1}$

$\omega = \frac{(z-1)\times(\bar{z}+1)}{(z+1)(\bar{z}+1)} = \frac{z\bar{z} - \bar{z} + z - 1}{|z+1|^2}$

$\Rightarrow \omega = \frac{|z|^2 + z - \bar{z} - 1}{|z+1|^2} = \frac{1 + z - \bar{z} - 1}{|z+1|^2}$

$\Rightarrow \omega = \frac{z - \bar{z}}{|z+1|^2}$

Since, $z - \bar{z}$ = Purely imaginary

$\therefore$ Real part of $\omega = 0$

$\therefore$ Option (D) is correct answer.

Q.7 Let z_1 & z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$. Prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$. **[IIT-2003]**

Sol. $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$

squaring both sides, we have

$\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right|^2 < 1$

$\Rightarrow |1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$

$\Rightarrow (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow (1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 . z_2\bar{z}_2) < (z_1\bar{z}_1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2)$

$\Rightarrow (1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + |z_1|^2 |z_2|^2) < (|z_1|^2 + |z_2|^2 - z_2\bar{z}_1 - z_1\bar{z}_2)$

$\Rightarrow 1 + |z_1|^2 . |z_2|^2 < |z_1|^2 + |z_2|^2$

$\Rightarrow 1 + |z_1|^2 |z_2|^2 - |z_1|^2 - |z_2|^2 < 0$

$\Rightarrow |z_1|^2 (|z_2|^2 - 1) - 1(|z_2|^2 - 1) < 0$

$\Rightarrow (|z_2|^2 - 1)(|z_1|^2 - 1) < 0$

For $|z_2|^2 - 1 < 0$ and $|z_1|^2 - 1 > 0$

$\Rightarrow$ Either $|z_1|^2 - 1 > 0$ and $|z_2|^2 - 1 < 0$

or

$|z_2|^2 - 1 < 0$ and $|z_1|^2 - 1 > 0$

$\Rightarrow$ Either $|z_1|^2 > 1$ and $|z_2|^2 < 1$

or

$|z_2|^2 < 1 < |z_1|^2$

$\Rightarrow$ Either $|z_1| < 1 < |z_2|$

or

$|z_2| < 1 < |z_1|$

Hence, Proved.

Q.8 Let a_i, i = 1, 2, 3, ... are complex numbers such that $|a_r| < 2$, then prove that there is no complex number such that $|z| < 1/3$ and $\sum_{r=1}^{n} a_r z^r = 1$. **[IIT-2003]**

Sol. Given, a_i ; i = 1, 2, 3, ……….

$|a_r| < 2$

$\sum_{r=1}^{n} a_r . z^r = 1$

$\Rightarrow a_1 z + a_2 z^2 + a_3 z^3 + a_4 . z^4 + a_4 z^4 + a_5 . z^5 + \ldots + a_r . z^r + \ldots + a_n z^n = 1$

Taking modulus of both sides, we get.

$\Rightarrow |a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_r z^r + \ldots + a_n z^n| = 1$

Using property $|z_1 + z_2| \geq |z_1| + |z_2|$

$\Rightarrow |a_1 z| + |a_2 z^2| + |a_3 z^3| + |a_4 z^4| + |a_5 z^5| + \ldots + |a_r z^r| + \ldots + |a_n z^n| \le 1$

Since, $|a_r| < 2$

$\Rightarrow 2(|z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + |z|^6 + \ldots + |z|^r + \ldots + |z|^n) \le 1$

$\Rightarrow |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + \ldots + |z|^r + \ldots + |z|^n \le 1/2$

It forms a G.P. with common ratio $|z|$

$$\Rightarrow \frac{|z|(|z|^n - 1)}{|z| - 1} \le \frac{1}{2}$$

$$\Rightarrow \frac{|z|^{n+1} - |z|}{|z| - 1} \le \frac{1}{2}$$

$$\Rightarrow |z|^{n+1} - |z| \le \frac{1}{2}|z| - \frac{1}{2}$$

$$\Rightarrow |z|^{n+1} - \frac{3}{2}|z| \le -\frac{1}{2}$$

$$\Rightarrow -\frac{3}{2}|z| + |z|^{n+1} \le -\frac{1}{2}$$

$$\Rightarrow \frac{3}{2}|z| - |z|^{n+1} \ge \frac{1}{2}$$

$$\Rightarrow |z| - \frac{2}{3}|z|^{n+1} > \frac{1}{2} \times \frac{2}{3}$$

$$\Rightarrow |z| \ge \frac{1}{3} + \frac{2}{3}|z|^{n+1}$$

i.e. there will be no complex number if $|z| < 1/3$

and $\sum_{r=1}^{n} a_r . z^r = 1$

Q.9 If ω is cube root of unity ($\omega \ne 1$) then the least value of n, where n is positive integer such that $(1+\omega^2)^n = (1+\omega^4)^n$ is- **[IIT-Scr-2004]**

(A) 2 (B) 3 (C) 5 (D) 6

Sol. **[B]**

Since, ω is cube root of unity

$$\therefore \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$\Rightarrow 1 + \omega + \omega^2 = 0$

$\Rightarrow (1+\omega^2)^n = (1+\omega^4)^n$

$= (1 + \omega.\omega^3)^n$

$= (1+\omega)^n$

Let $n = 2 \Rightarrow (1+\omega^2)^2 = (1+\omega)^2$

$\Rightarrow 1 + \omega^4 + 2\omega^2 = 1 + \omega^2 + 2\omega$

$\Rightarrow 1 + \omega + \omega^2 + \omega^2 = 1 + \omega + \omega^2 + \omega$

$\Rightarrow \omega^2 = \omega$ which is not true.

Take $n = 3 \Rightarrow (1+\omega^2)^3 = (1+\omega)^3$

$\Rightarrow (1+\omega^2).(1+\omega^2)^2 = (1+\omega).(1+\omega)^2$

$\Rightarrow (1+\omega^2).(1+\omega^2)^2 = (1+\omega).(1+\omega)^2$

$\Rightarrow (1+\omega^2).\omega^2 = (1+\omega)\,\omega$

$\Rightarrow \omega^2 + \omega^4 = \omega + \omega^2$

$\Rightarrow \omega^2 + \omega.\omega^3 = \omega + \omega^2$

$\Rightarrow \omega^2 + \omega = \omega + \omega^2 \Rightarrow$ L.H.S. = R.H.S.

$\therefore$ Least value of n is 3

$\therefore$ Option (B) is correct answer

Q.10 Find the centre and radius of the circle formed by all the points represented by $z = x + iy$ satisfying the relation $\left|\frac{z-\alpha}{z-\beta}\right| = k$; $(k \ne 1)$ where α and β are constant complex number given by $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$. **[IIT - 2004]**

Sol. $\left|\frac{z-\alpha}{z-\beta}\right| = k$; $(k \ne 1)$

Squaring both sides, we get

$|z - a|^2 = k^2 |z - \beta|^2$

$\Rightarrow (z-\alpha)(\bar{z} - \bar{\alpha}) = k^2 (z-\beta)(\bar{z} - \bar{\beta})$

$\Rightarrow z\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha}$

$= k^2 (z\bar{z} - \beta\bar{z} - z\bar{\beta} + \beta\bar{\beta}$

$\Rightarrow |z|^2 - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha} = k^2|z|^2 - k^2\beta\bar{z} - k^2\bar{\beta}z - k^2\beta\bar{\beta}$

$\Rightarrow |z|^2 (k^2 - 1) + z(\bar{\alpha} - k^2\bar{\beta}) + \bar{z}(\alpha - k^2\beta)$

$+ (k^2\beta\bar{\beta} - \alpha\bar{\alpha}) = 0$

$\Rightarrow |z|^2 (k^2 - 1) + z(\bar{\alpha} - k^2\bar{\beta}) + \bar{z}(\alpha - k^2\beta) + (k^2\beta\bar{\beta} - \alpha\bar{\alpha})$

$$\Rightarrow |z|^2 + z\left(\frac{\bar{\alpha} - k^2\bar{\beta}}{k^2-1}\right) + \left(\frac{\alpha - k^2\beta}{k^2-1}\right) + \left(\frac{k^2\beta\bar{\beta} - \alpha\bar{\alpha}}{k^2-1}\right) = 0$$

We can compare above equation with general. Equation of circle

$|z|^2 + a\bar{z} + \bar{a}z + b = 0$

Centre : $(-a)$

Radius $= \sqrt{|a|^2 - b}$

$= \sqrt{a\bar{a} - b}$

Centre : $\left(\dfrac{k^2\beta - \alpha}{k^2 - 1}\right)$; $k \neq 1$

Radius = $\sqrt{\dfrac{(k^2\beta-\alpha)(k^2\bar{\beta}-\bar{\alpha})}{(k^2-1)^2} - \dfrac{(k^2\beta\bar{\beta}-\alpha\bar{\alpha})}{(k^2-1)}}$

$= \dfrac{\sqrt{(k^2\beta-\alpha)(k^2\bar{\beta}-\bar{\alpha})-(k^2-1)(k^2\beta\bar{\beta}-\alpha\bar{\alpha}}}{(k^2-1)^2}$

$= \dfrac{\sqrt{k^4\beta\bar{\beta}-\alpha\bar{\beta}k^2-k^2\beta\bar{\alpha}+\alpha\bar{\alpha}-(k^4\beta\bar{\beta}-k^2\alpha\bar{\alpha}-k^2\beta\bar{\beta}+\alpha\bar{\alpha})}}{(k^2-1)}$

$= \dfrac{\sqrt{k^4\beta\bar{\beta}-\alpha\bar{\beta}k^2-k^2\beta\bar{\alpha}+\alpha\bar{\alpha}-k^4\beta\bar{\beta}+k^2\alpha\bar{\alpha}+k^2\beta\bar{\beta}-\alpha\bar{\alpha}}}{(k^2-1)}$

$= \dfrac{\sqrt{k^2(\alpha\bar{\alpha}+\beta\bar{\beta}-\alpha\bar{\beta}-\beta\bar{\alpha}}}{(k^2-1)}$

$= \dfrac{k}{(k^2-1)} \sqrt{\alpha\bar{\alpha}+\beta\bar{\beta}-\alpha\bar{\beta}-\beta\bar{\alpha}}$

$= \dfrac{k}{(k^2-1)} \sqrt{\alpha(\bar{\alpha}-\bar{\beta})-\beta(\bar{\alpha}-\bar{\beta})}$

$= \dfrac{k}{(k^2-1)} \sqrt{(\alpha-\beta)(\bar{\alpha}-\bar{\beta})}$

$= \dfrac{k}{(k^2-1)} \sqrt{|\alpha-\beta|^2}$

$= \dfrac{k}{(k^2-1)} |\alpha - \beta|$; $k \neq 1$

Hence, require answer is –

Centre = $\dfrac{k^2\beta - \alpha}{k^2 - 1}$; $k \neq 1$

Radius = $\dfrac{k}{k^2-1}$ $|\alpha - \beta|$; $k \neq 1$

Q.11 a, b, c are variable integers not all equal and $\omega \neq 1$, ω is cube root of unity, then minimum value of $x = |a + b\omega + c\omega^2|$ is - **[IIT-Scr-2005]**

(A) 0 (B) 1 (C) 2 (D) 3

Sol. **[B]**

Given : a, b, c are variable integers not all equal

ω is cube root of unity, $\omega \neq 1$

Take, $\omega = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$

$\omega^2 = -\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}$

$\therefore X^2 = |a + b\omega + c\omega^2|^2$

$= (a + b\omega + c\omega^2)(a + b\bar{\omega} + c\bar{\omega}^2)$

$= a^2 + ab\omega + ac\omega^2 + ab\bar{\omega} + b^2\omega\bar{\omega} +$

$bc\omega^2.\bar{\omega} + ac\bar{\omega}^2 + bc\omega\bar{\omega}^2 + c^2(\omega\bar{\omega})^2$

$= a^2 + b^2|\omega|^2 + c^2|\omega|^4 + ab(\omega + \bar{\omega})$

$+ bc(\omega^2\bar{\omega} + \omega\bar{\omega}^2) + ac(\omega^2 + \bar{\omega}^2)$

Since, $\omega = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2} \Rightarrow \bar{\omega} = -\dfrac{i\sqrt{3}}{2}$

$\omega^2 = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2} \Rightarrow \bar{\omega} = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$

$\omega + \bar{\omega} = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2} - \dfrac{1}{2} - \dfrac{i\sqrt{3}}{2} = -1$

$\omega^2 + \bar{\omega}^2 = -\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2} - \dfrac{1}{2} + \dfrac{i\sqrt{3}}{2} = -1$

$\omega^2.\bar{\omega} + \omega.\bar{\omega}^2$

$= \left(-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)\left(-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)$

$+ \left(-\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right)\left(-\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right)$

$= \left(-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)^2 + \left(-\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right)^2$

$= \dfrac{1}{4} - \dfrac{3}{4} + \dfrac{i2\sqrt{3}}{4} + \dfrac{1}{4} - \dfrac{3}{4} - \dfrac{2i\sqrt{3}}{4}$

$= \dfrac{1}{2} - \dfrac{3}{2} = -1$

$\therefore X^2 = a^2 + b^2 + c^2 - ab - bc - ac$

$= \dfrac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$

Take a = b such that $b \neq c$ and $a \neq c$

$X^2 = \dfrac{1}{2}\{0 + (a-c)^2 + (c-a)^2\}$

$X^2 = \dfrac{1}{2}\{2 \times (a-c)^2\}$

$X^2 = (a-c)^2$

Since, (Difference of integer numbers)$^2 \geq 1$

i.e. $(a-c)^2 \geq 1$

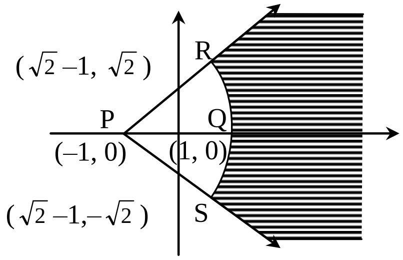
$\Rightarrow X^2 \geq 1$

$\Rightarrow$ Minimum value of $X^2 = 1$

$\Rightarrow$ Minimum value of $|a + b\omega + c\omega^2|$ is 1.

$\therefore$ Option (B) is correct answer.

Q.12 Four points P (–1, 0), Q (1, 0), R ($\sqrt{2}$ –1, $\sqrt{2}$), S ($\sqrt{2}$ –1, –$\sqrt{2}$) are given on a complex plane then equation of the locus of the shaded region excluding the boundaries **[IIT-Scr-2005]**



(A) $|z + 1| > 2$ & $|\arg(z + 1)| < \frac{\pi}{4}$

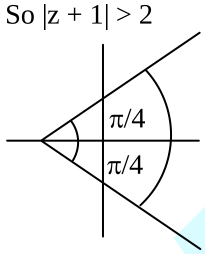
(B) $|z + 1| > 2$ & $|\arg(z + 1)| < \frac{\pi}{2}$

(C) $|z – 1| > 2$ & $|\arg(z – 1)| < \frac{\pi}{4}$

(D) $|z –1| > 2$ & $|\arg(z – 1)| < \frac{\pi}{2}$

Sol. **[A]** Shaded region is 'the outside region of circle $|z + 1| = 2$

So $|z + 1| > 2$



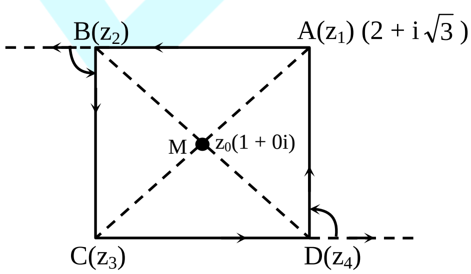
Also $\frac{-\pi}{4} < \text{Arg}(z + 1) < \frac{\pi}{4}$

$\Rightarrow |\text{Arg}(z + 1)| < \frac{\pi}{4}$

Q.13 A square having one vertex as $2 + \sqrt{3}\,i$ is circumscribed on the circle $|z – 1| = 2$. Then find the other vertices of square. **[IIT 2005]**

Sol. Given $|z – 1| = 2$

M is the mid point of diagonals AC and BD.



$\therefore z_0 = \frac{z_1 + z_3}{2} = 1 + 0i.$

$\Rightarrow z_3 = 2 – z_1$

$z_3 = 2 – 2 – i\sqrt{3}$

$z_3 = –\sqrt{3}$

Applying rotaion theorem at point D

$\Rightarrow \frac{z_1 - z_4}{z_4 - z_3} = \frac{|z_1 - z_4|}{|z_4 - z_3|} e^{i\pi/2}$

$\Rightarrow \frac{z_1 - z_4}{z_4 - z_3} = \cos \pi/2 + + i \sin \pi/2$

$\Rightarrow \frac{z_1 - z_4}{z_4 - z_3} = i$

$\Rightarrow z_1 – z_4 = i\, z_4 – i\, z_3$

$\Rightarrow z_1 + iz_3 = z_4 (1 + i)$

$\Rightarrow z_4 = \frac{z_1 + iz_3}{1+i}$

$\Rightarrow z_4 = \frac{2 + i\sqrt{3} + i(-i\sqrt{3})}{1+i}$

$\Rightarrow z_4 = \frac{2 + i\sqrt{3} + \sqrt{3}}{(1+i)} \times \frac{(1-i)}{(1-i)}$

$\Rightarrow z_4 = \frac{(2 + \sqrt{3} + i\sqrt{3})(1-i)}{2}$

$\Rightarrow z_4 = \frac{2 + \sqrt{3} + i\sqrt{3} - 2i - i\sqrt{3} + \sqrt{3}}{2}$

$\Rightarrow z_4 = \frac{2 + 2\sqrt{3} - 2i}{2}$

$\Rightarrow z_4 = 1 + \sqrt{3} – i$

Also, coni method or rotation theorem at point B.

$\frac{z_3 - z_2}{z_2 - z_1} = \frac{|z_3 - z_2|}{|z_2 - z_1|} e^{i\pi/2}$

$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = \cos \pi/2 + i \sin \pi/2$

$\Rightarrow z_3 – z_2 = i (z_2 – z_1)$

$\Rightarrow z_3 – z_2 = iz_2 – iz_1$

$\Rightarrow z_3 + iz_1 = z_2 (1 + i)$

$\Rightarrow z_2 = \frac{z_3 + iz_1}{1+i}$

$z_2 = \frac{z_3 + iz_1}{(1+i)} \times \frac{(1-i)}{(1-i)}$

$$z_2 = \frac{z_3 + iz_1 - iz_3 + z_1}{2}$$

$$z_2 = \frac{-i\sqrt{3} + i(2 + i\sqrt{3}) - i(1 - i\sqrt{3}) + 2 + i\sqrt{3}}{2}$$

$$\Rightarrow z_2 = \frac{2 - 2\sqrt{3} + 2i}{2}$$

$$\Rightarrow z_2 = 1 - \sqrt{3} + i$$

Hence, required veritus are

$z_2 = (1 - \sqrt{3}) + i$

$z_3 = -i\sqrt{3}$

$z_4 = (1 + \sqrt{3}) - i$ Ans

Q.14 If $w = \alpha + i\beta$ where $\beta \neq 0$ and $z \neq 1$ satisfies the condition that $\left(\frac{w - \bar{w}z}{1 - z}\right)$ is purely real, then set of z is – **[IIT-2006]**

(A) $\{z : z = \bar{z}\}$ (B) $\{z : |z| = 1, z \neq 1\}$

(C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1\}$

Sol. **[B]** $\frac{w - \bar{w}z}{1 - z}$ is purely real

$\Rightarrow \quad \frac{w - \bar{w}z}{1 - z} = \frac{\bar{w} - w\bar{z}}{1 - \bar{z}}$

$\Rightarrow \quad (w - \bar{w}z)(1 - \bar{z}) = (1 - z)(\bar{w} - w\bar{z})$

$\Rightarrow \quad (w - \bar{w})(|z|^2 - 1) = 0$

As $\quad w \neq \bar{w}$

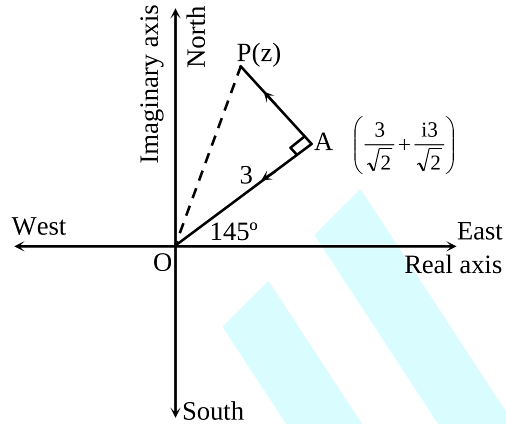
$|z| = 1$ and $z \neq 1$

Q.15 A man walks a distance of 3 units from the origin towards the north-east (N 45º E) direction. From there, he walks a distance of 4 units towards the north-west (N 45º W) direction to reach a point P. Then the position of P in the Argand plane is- **[IIT- 2007]**

(A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)\ e^{i\pi/4}$

(C) $(4 + 3i)\ e^{i\pi/4}$ (D) $(3 + 4i)\ e^{i\pi/4}$

Sol. **[D]**



$z_A = |z_A|\ e^{i\pi/4}$

$z_A = 3e^{i\pi/4}$

Applying Rotation theorem at A, we have

$$\frac{0 - 3e^{i\pi/4}}{z - 3e^{i\pi 4}} = \frac{3}{4}\ e^{i\pi/2}$$

$$\Rightarrow z - 3\ e^{i\pi/4} = -3 \times \frac{4}{3} \times e^{-i\pi/2} \times e^{i\pi/4}$$

$$\Rightarrow z - 3\ e^{i\pi/4} = +4 \times i\ e^{i\pi/4}$$

$$\Rightarrow z = (3 + 4i)\ e^{i\pi/4}$$

$\therefore$ Option (D) is the correct answer.

Q.16 If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1 - z^2}$ lie on - **[IIT- 2007]**

(A) a line not passing through the origin

(B) $|z| = \sqrt{2}$

(C) the x–axis

(D) the y–axis

Sol. **[D]**

Given $|z| = 1$ and $z \neq \pm 1$

$$\frac{z}{1 - z^2} = \frac{z}{(1 - z)(1 + z)} \times \frac{(1 - \bar{z})(1 + \bar{z})}{(1 - \bar{z})(1 + \bar{z})}$$

$$= \frac{z(1 - \bar{z} + \bar{z} - \bar{z}\,\bar{z})}{|1 - z|^2 |1 + z|^2}$$

$$= \frac{z(1 - \bar{z}\,\bar{z})}{|1 - z|^2 |1 + z|^2}$$

$$= \frac{(z - z\bar{z}.\bar{z})}{|1 - z|^2 |1 + z|^2}$$

$$= \frac{(z-\bar{z}|z|^2)}{|1-z|^2|1+z|^2}$$

$$= \frac{(z-\bar{z})}{|1-z|^2|1+z|^2} = \text{Purely Imaginary}$$

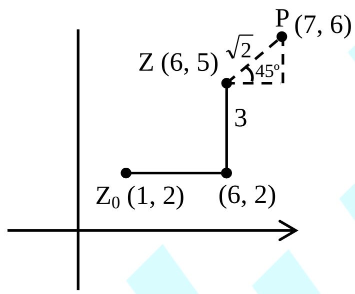
$\therefore$ Option (D) is the correct answer.

Q.17 A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particles moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by - **[IIT- 2008]**

(A) 6 + 7i (B) – 7 + 6i

(C) 7 + 6i (D) – 6 + 7i

Sol. **[D]**



In diagram you can see P point is given by complex No. 7 + 6i and now it is rotated by $\frac{\pi}{2}$ angle in anticlockwise, since so it will be i (7 + 6i) $\Rightarrow z_2 = -6 + 7i$.

Paragraph for Question Nos. 18 to 22

Let A, B, C be three sets of complex numbers as defined below

$A = \{z : \text{Im}z \geq 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{ z : \text{Re}((1-i)z) = \sqrt{2} \}$ **[IIT 2008]**

Q.18 The number of elements in the set $A \cap B \cap C$ is- **[IIT 2008]**

(A) 0 (B) 1 (C) 2 (D) ∞

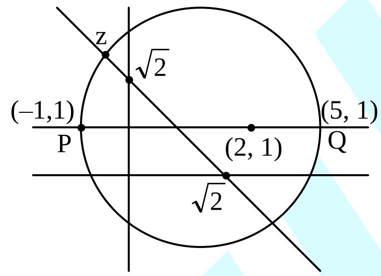
Sol. **[B]**

Given that,

$A : y \geq 1$

$B : (x-2)^2 + (y-1)^2 = 9$

$C : x + y = \sqrt{2}$



There is only one common point which satisfies to all the three given curves

Q.19 Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between **[IIT 2008]**

(A) 25 and 29 (B) 30 and 34

(C) 35 and 39 (D) 40 and 44

Sol. **[C]**

Let $z_1 = (-1, 1)$ and $z_2 = (5, 1)$.

$|z + 1 - i|^2 + |z - 5 - i|^2 = |z_1 - z_2|^2 = (6)^2 = 36$.

Q.20 Let Z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. **[IIT 2008]**

Then $|z| - |w| + 3$ lies between

(A) –6 and 3 (B) –3 and 6

(C) –6 and 6 (D) –3 and 9

Sol. **[D]**

We know that,

$$||z| - |w|| \leq |z - w|$$

$\Rightarrow -3 < |z| - |w| + 3 < 9$.

Q.21 Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is - **[IIT-2009]**

(A) $\frac{1}{\sin 2°}$ (B) $\frac{1}{3\sin 2°}$

(C) $\frac{1}{2\sin 2°}$ (D) $\frac{1}{4\sin 2°}$

Sol. **[D]**

$Z = \cos\theta + i\sin\theta$

$Z^{2m-1} = \cos(2m-1)\theta + i\sin(2m-1)\theta$

$\sum \text{Im } Z^{2m-1} = \sin\theta + \sin 3\theta + \ldots. + \sin 29\theta$

$$= \frac{\sin\left(15.\frac{2\theta}{2}\right)}{\sin\left(\frac{2\theta}{2}\right)} \sin\left(\theta + (15-1)\frac{2\theta}{2}\right)$$

$$= \frac{(\sin(15\theta))^2}{\sin\theta} = \frac{1}{4\sin 2^\circ} \;\forall\; \theta = 2^\circ$$

Q.22 Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is - **[IIT-2009]**

(A) 48 (B) 32 (C) 40 (D) 80

Sol. **[A]**

Let $Z = x + iy$

$\therefore$ $(x^2 + y^2)(x^2 - y^2 - 2ixy) + (x^2 + y^2)(x^2 - y^2 + 2ixy) = 350$

$\Rightarrow$ $2(x^2 + y^2)(x^2 - y^2) = 350$

$\Rightarrow$ $(x^2 + y^2)(x^2 - y^2) = 175$

$\therefore$ $x^2 + y^2 = 25$

$x^2 - y^2 = 7$

On solving

$x = \pm 4$

$y = \pm 3$

$\therefore$ Area of rectangle = 8 × 6 = 48

Q.23 **[IIT-2009]**

Column – I	**Column – II**
(A) Circle	(P) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(B) Parabola	(Q) Points z in the complex plane satisfying $\lvert z + 2\rvert - \lvert z - 2\rvert = \pm 3$
(C) Ellipse	(R) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(D) Hyperbola	(S) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(T) Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \lvert z\rvert^2 + 1$

Sol. **A → P; B → S,T; C → R; D → Q, S;**

(P) $\frac{1}{\sqrt{h^2 + k^2}} = 2$

$\sqrt{h^2 + k^2} = \frac{1}{2}$

$x^2 + y^2 = \frac{1}{4}$ Circle

(Q) $\lvert\, \lvert z + 2\rvert - \lvert z - 2\rvert \,\rvert = 3$

hyperbola having focus (–2, 0) and (2, 0)

(R) $\frac{x}{\sqrt{3}} = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

$\frac{x^2}{3} + y^2 = 1$ ellipse

(S) Parabola, hyperbola

(T) Let $z = x + iy$

$(x + 1)^2 - y^2 = x^2 + y^2 + 1$

$2x - y^2 = y^2$

$y^2 = x$ parabola

Q.24 Let z_1 and z_2 be two distinct complex numbers let $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a nonzero complex number w, then **[IIT 2010]**

(A) $\lvert z - z_1\rvert + \lvert z - z_2\rvert = \lvert z_1 - z_2\rvert$

(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

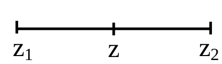
(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

Sol. **[A,C,D]**

$t = \frac{z - z_1}{z_2 - z_1}$

So, $\frac{z - z_1}{z_2 - z_1} = t\,e^{io} \;\forall\; t \in (0, 1)$

So Geometrically



So option A, C, D are true.

Q.25 Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.

Then the number of distinct complex number z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to

[IIT 2010]

Sol. **[1]**

On solving the determinant

It become $z^3 = 0$

So no. of solutions = 1

Q.26 **[Note :** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z]

[IIT-2010]

Column – I	**Column – II**
(A) The set of points z satisfying $\|z - i\|z\|\| = \|z + i\|z\|\|$ is contained in or equal to	(P) an ellipse with eccentricity 4/5
(B) The set of points z satisfying $\|z + 4\| + \|z - 4\| = 10$ is contained in or equal to	(Q) the set of points z satisfying Im z = 0
(C) If $\|w\| = 2$, then the satisfying set of points $z = w - \frac{1}{w}$ is contained in or equal to	(R) the set of points z $\|\text{Im } z\| \le 1$
(D) If $\|w\| = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(S) the set of points z satisfying $\|\text{Re } z\| \le 2$
	(T) the set of points z satisfying $\|z\| \le 3$

Sol. **A → Q, R ;** **B → P ;**
C → P, S, T ; **D → Q, R, S, T**

(A) Let $|Z| = r \ \forall\ r \in R$

$\left|\frac{Z - ir}{Z + ir}\right| = 1$ Which is the equation of line of perpendicular bisector of y = r & y = – r that is y = 0

(B) $|Z + 4| + |Z - 4| = 10$

it will represent on ellipse

having foci (–4, 0), (4, 0)

so its equation will be $\frac{x^2}{25} + \frac{y^2}{9} = 1$

whose eccentricity is 4/5

(C) Let $w = 2e^{i\theta}$.

$z = \frac{3}{2}\cos + \frac{5}{2} i \sin\theta$

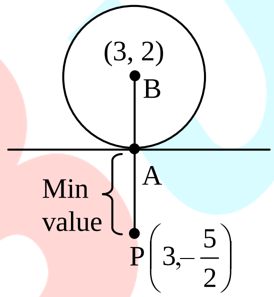
(D) Let $w = e^{i\theta}$

$Z = e^{i\theta} + e^{-i\theta}$

$= 2\cos\theta$.

Q.27 If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is **[IIT 2011]**

Sol.



So, Min of $|2z - 6 + 5i|$ = PA

$= \text{Min } 2\left|z - 3 + \frac{5i}{2}\right| = 2 \times \frac{5}{2} = 5$

Q.28 Let $\omega = e^{i2\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that $a + b + c = x$, $a + b\omega + c\omega^2 = y$, $a + b\omega^2 + c\omega = z$. Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[IIT 2011]**

Sol. wrong question if $\omega = e^{i2\pi/3}$ then ans. is 3. If $\omega = e^{i\pi/3}$ then no integral solution is possible.

Q.29 Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value

[IIT 2012]

(A) –1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Sol. [D] As a is real

So $a = \bar{a}$

$\Rightarrow z^2 + z + 1 = \bar{z}^2 + \bar{z} + 1$

$\Rightarrow (z - \bar{z})(z + \bar{z} + 1) = 0$

As z is imaginary

So $z - \bar{z} \neq 0$

$\Rightarrow z + \bar{z} + 1 = 0$

$\Rightarrow z + \bar{z} = -1 \;\; \forall\, z = x + iy$

$x = \dfrac{-1}{2}$

so $a = (x + iy)^2 + (x + iy) + 1$

$\quad = (x^2 + x + 1 - y^2) + (2x + 1)yi \;\; \forall\, x = -\dfrac{1}{2}$

$a = \dfrac{3}{4} - y^2$

so $a < \dfrac{3}{4} \;\; \forall\, y^2 > 0$

So $a \neq \dfrac{3}{4}$

EXERCISE # 5

Q.1 Let z & w be two non-zero complex numbers such that |z| = | w | and Arg z + Arg w = π, then z equal: **[IIT- 1995]**

(A) w (B) – w

(C) $\bar{w}$ (D) $-\bar{w}$

Sol. **[D]**

Given | z | = | ω |

and Arg z + Arg ω = π

Let $\omega = re^{i\theta} \Rightarrow |\omega| = r$

$\Rightarrow z = |z|\ e^{i(\pi-\theta)}$

$\Rightarrow z = r.e^{i\pi}\ e^{-i\theta}$

$\Rightarrow z = (r.e^{-i\theta})\ e^{i\pi}$

$\Rightarrow z = (-1)\ (r.e^{-i\theta})$

$\Rightarrow z = -r.e^{-i\theta}$

$\Rightarrow z = -\ \bar{\omega}$

∴ Option (D) is correct answer.

Q.2 Let z & w be two complex numbers such that |z| ≤ 1, |w| ≤ 1 and |z + i w| = |z – i$\bar{w}$| = 2, then z equals : **[IIT- 1995]**

(A) 1 or i (B) i or – i

(C) 1 or –1 (D) i or –1

Sol. **[C]**

Given : | z | ≤ 1, | ω | ≤ 1

$|z + i\omega| = |z - i\bar{\omega}| = 2$

$\Rightarrow |z + i\omega| = |z - i\bar{\omega}|$

$\Rightarrow |z + i\omega|^2 = |z - i\bar{\omega}|^2$

$\Rightarrow (z + i\omega)\ (z - i\bar{\omega}) = (z - i\bar{\omega})\ (\bar{z} + i\omega)$

$\Rightarrow z\bar{z} + i\omega\bar{z} - iz\bar{\omega} + \omega\bar{\omega} = \bar{z} - i\bar{\omega}\bar{z} + i\omega z + \omega\bar{\omega}$

$\Rightarrow i\omega\bar{z} - iz\bar{\omega} + i\bar{\omega}\bar{z} - i\omega z - 0$

$\Rightarrow i\,[\bar{z}\,(\omega + \bar{\omega}) - z\,(\omega + \bar{\omega})] = 0$

$\Rightarrow (\bar{z} - z)\ (\omega + \bar{\omega}) = 0$

$\Rightarrow$ Either $z = \bar{z}$ or $\omega = -\ \bar{\omega}$

or $z = \bar{z}$ and $\omega = -\ \bar{\omega}$

$\Rightarrow |z + i\omega| = 2$

Squaring both sides, we get

$\Rightarrow |z + i\omega|^2 = 4$

$\Rightarrow (z + i\omega)\ (\bar{z} - i\bar{\omega}) = 4$

$\Rightarrow z\bar{z} + i\omega\bar{z} - i\bar{\omega}z + \omega\bar{\omega} = 4$

(as $|z|^2 \le 1$, $|\omega|^2 \le 1$)

$\Rightarrow i(\omega z + \omega z) = 2$

$\Rightarrow 2i\ \omega z = 2$

$\Rightarrow i\omega z = 1$

Since, $z = \bar{z} \Rightarrow z - \bar{z} = 0$

$\Rightarrow$ Imp (z) = 0

$\Rightarrow$ z is purely real

$\Rightarrow z = x + i.0 = x$

$\Rightarrow |z|^2 \le 1$

$\Rightarrow x^2 \le 1$

$\Rightarrow -1 < x < 1$

$\Rightarrow \omega = -\ \bar{\omega}$

$\Rightarrow \omega + \bar{\omega} = 0$

$\Rightarrow$ Re(ω) = 0

$\Rightarrow$ ω is purely imaginary

∴ Option (C) is correct answer.

Q.3 For positive integers n_1, n_2 the value of the expression ;

$(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$,

where $i = \sqrt{-1}$, is a real number if : **[IIT- 1996]**

(A) $n_1 = n_2 + 1$ (B) $n_1 = n_2 - 1$

(C) $n_1 = n_2$ (D) $n_1 > 0$, $n_2 > 0$

Sol. **[D]**

$(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$

$= (1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$

We have to first deal with

$(1+i)^{n_1} + (1-i)^{n_1}$

$$= 2^{n_1/2}\left[(1/\sqrt{2} + i/\sqrt{2})^{n_1} + (1/\sqrt{2} - i\sqrt{2})^{n_1}\right]$$

$$= 2^{n_1/2}\left[\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{n_1} + \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)^{n_1}\right]$$

$$= 2^{n_1/2}\left[\cos\frac{n_1\pi}{4} + i\sin\frac{n_1\pi}{4} + \cos\frac{7\pi n_1}{4} + i\sin\frac{7\pi n_1}{4}\right]$$

$$= 2^{n_1/2}\left[\cos\frac{n_1\pi}{4} + i\sin\frac{n_1\pi}{4} + \cos\left(2\pi - \frac{\pi}{4}\right)n_1 + i\sin\left(2\pi - \frac{\pi}{4}\right)n_1\right]$$

$$= 2^{n_1/2}.\ 2\cos\frac{n_1\pi}{4}$$

Also, $(1+i)^{n_2} + (1-i)^{n_2}$

$= 2^{n_2/2}\,[(\cos \pi/4 + i \sin \pi/4)^{n_2} + \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)^{n_2}$

$= 2^{n_2/2}\left[\cos\frac{n_2\pi}{4} + i\sin\frac{n_2\pi}{4} + \cos\frac{7\pi}{4}n_2 + i\sin\frac{7\pi}{4}n_2\right]$

$= 2^{n_2/2}.2 \cos n_2\, \pi/4$

Then for real value of

$[(1 + i)^{n_1} + (1 - i)^{n_1} + (1 + i)^{n_2} + (i - i)^{n_2}]$,

$n_1 > 0$ & $n_2 > 0$

∴ Option (D) is correct answer.

Q.4 Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$. **[IIT - 1996]**

Sol. Given $\bar{z} = i z^2$

Let $z = x + iy \Rightarrow \bar{z} = x - iy$

We have to solve above equation

$\bar{z} = iz^2$

$\Rightarrow x - iy = i\,(x + iy)^2$

$\Rightarrow x - iy = i\,(x^2 - y^2 + 2\,ixy)$

$\Rightarrow x - iy = i(x^2 - y^2) - 2xy$

Comparing real and imaginary roots

$\Rightarrow x = -\,2xy$ and $-\,y = x^2 - y^2$

$\Rightarrow x + 2xy = 0$ and $x^2 - y^2 + y = 0$

$\Rightarrow x\,(1 + 2y)$ and $x^2 - y^2 + y = 0$

$\Rightarrow$ Either $x = 0$ or $y = -\,1/2$

When $x = 0 \Rightarrow 0 - y^2 + y = 0$

$\Rightarrow y\,(1 - y) = 0$

$\Rightarrow y = 0, 1$

∴ z = i (Non-zero)

When $y = -\,1/2 \Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0$

$\Rightarrow x^2 - 3/4 = 0$

$\Rightarrow x^2 = 3/4$

$\Rightarrow x = \pm\sqrt{3}/2$

$\Rightarrow z = \sqrt{3}/2 - i/2$ and $-\sqrt{3}/2 - i/2$

Hence, required non-zero complex numbers are i, $\sqrt{3}/2 - i/2$ and $-\sqrt{3}/2 - i/2$.

Q.5 Prove that $\sum_{k-1}^{n-1}(n-k)\cos\frac{2k\pi}{n} = -\frac{n}{2}$ where $n \ge 3$ is an integer. **[IIT- 1997]**

Q.6 For complex numbers z and w, prove that $|z|^2\, w - |\,w\,|^2\, z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$. **[IIT - 1999]**

Sol. $|\,z\,|^2 . \omega - |\,\omega\,|^2 . z = z - \omega$

$\Rightarrow z\bar{z}.\omega - \omega\bar{\omega}.z = z - \omega$

$\Rightarrow z(\bar{z}.\omega - 1) - \omega(\bar{\omega}z - 1) = 0$ …(1)

Also, $|z|^2 .\omega - |\omega|^2.z = z - \omega$

$\Rightarrow |z|^2\,\omega + \omega = |\omega|^2.z + z$

$\Rightarrow \omega\,(|\,z\,|^2 + 1) = z(|\omega|^2 + 1)$

$\Rightarrow \frac{z}{\omega} = \frac{|\,z\,|^2 + 1}{|\,\omega\,|^2 + 1}$ = Purely real

$\Rightarrow I_m\left(\frac{z}{\omega}\right) = 0$

$\Rightarrow \frac{z}{\omega} - \frac{\bar{z}}{\bar{\omega}} = 0$

$\Rightarrow \frac{z}{\omega} - \frac{\bar{z}}{\bar{\omega}}$

$\Rightarrow z.\bar{\omega} = \bar{z}.\omega$

from (1)

$\Rightarrow z(z\bar{\omega} - 1) - \omega(z - 1) = 0$

$\Rightarrow (z\bar{\omega} - 1)\,(z - \omega) = 0$

$\Rightarrow$ Either $z = \omega$ or $z\bar{\omega} = 1$

Hence, Proved.

Q.7 If a_1, a_2, a_3 .... a_n, A_1, A_2, A_3 .... A_n, k are all real numbers, then prove that the equation $\frac{A_1^2}{x - a_1} + \frac{A_2^2}{x - a_2} + ... + \frac{A_n^2}{x - a_n} = k$ has no imaginary roots.

Sol. By theory

Q.8 Find the number of solution of equation $|z|^5 - 2z\,|z|^3 = (\bar{z})^2\,|z|^3 - 1\ \forall\ z = x + iy\ \forall\ x, y \in R\ \&\ x \ne 1$.

Sol. one solution.

Q.9 Show that the product

$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]....$$

$$....\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right]$$

is equal to $\left(1 - \frac{1}{2^{2^n}}\right)(1 + i)$ where $n \ge 2$.

Sol. Using formula.

Q.10 If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\,3/2$ then prove that -

(a) $\Sigma\cos 2\alpha = 0 = \Sigma \sin 2\alpha$

(b) $\Sigma \sin(\alpha + \beta) = 0 = \Sigma \cos(\alpha + \beta)$

(c) $\Sigma \sin^2\alpha = \Sigma \cos^2\alpha = 3/2$

(d) $\Sigma \sin 3\alpha = 3 \sin (\alpha + \beta + \gamma)$

(e) $\Sigma \cos 3\alpha = 3 \cos (\alpha + \beta + \gamma)$

(f) $\cos^3 (\theta + \alpha) + \cos^3 (\theta + \beta) + \cos^3(\theta + \gamma) = 3 \cos (\theta + \alpha). \cos (\theta + \beta). \cos (\theta + \gamma)$ where $\theta \in R$.

Sol. Using formula.

Q.11 Prove that with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p, p′, q, q′ are all real.

(i) If the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$

(ii) If the equation has two equal roots then $p^2 - p'^2 = 4q$ & $pp' = 2q'$

State whether these equal roots are real or complex.

Sol. Using formula.

Q.12 If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ $(a, b, c, d \in R)$ has 4 non real roots, two with sum $3 + 4i$ & the other two with product $13 + i$. Find the value of 'b'.

Sol. 51

Q.13 If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer. Find N.

Sol. 4

Q.14 Find the roots of the equation $z^n = (z + 1)^n$ and show that the points which represent them are collinear on the complex plane. Hence show that these roots are also the roots of the equation $\left(2\sin\frac{m\pi}{n}\right)^2 \bar{z}^2 + \left(2\sin\frac{m\pi}{n}\right)^2 \bar{z} + 1 = 0$.

Sol.

Q.15 If ω is the fifth root of 2 and $x = \omega + \omega^2$, prove that $x^5 = 10x^2 + 10x + 6$.

Sol.

Q.16 (a) Without expanding the determinant at any stage, find $K \in R$ such that

$$\begin{vmatrix} 4i & 8+i & 4+3i \\ -8+i & 16i & i \\ -4+Ki & i & 8i \end{vmatrix}$$

has purely imaginary value.

(b) If A, B and C are the angles of a triangle

$$D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$$

where $i = \sqrt{-1}$ then find the value of D.

Sol. (a) k = 3

(b) – 4

Q.17 Find all real values of the parameter a for which the equation $(a - 1)z^4 - 4z^2 + a + 2 = 0$ has only pure imaginary roots.

Sol. [–3, –2]

Q.18 Show that the locus formed by z in the equation $z^3 + iz = 1$ never crosses the coordinate axes in the Argand's plane. Further show that $|z| = \sqrt{\frac{-\operatorname{Im}(z)}{2\operatorname{Re}(z)\operatorname{Im}(z)+1}}$.

Q.19 Dividing f(z) by z – i, we get the remainder i and dividing it by z + i, we get the remainder 1 + i. Find the remainder upon the division of f(z) by $z^2 + 1$.

Sol. $\frac{iz}{2} + \frac{1}{2} + i$

Q.20 If Z_r, r = 1, 2, 3, …. 2m, $m \in N$ are the roots of the equation $Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots + Z + 1 = 0$ then prove that $\sum_{r=1}^{2m} \frac{1}{Z_r - 1} = -m$.

Q.21 C is the complex number, $f : C \to R$ is defined by $f(z) = |z^3 - z + 2|$. Find the maximum value of f(z), if $|z| = 1$.

Sol. $|f(z)|$ is maximum when $z = \omega$, where ω is the cube root of unity and $|f(z)| = \sqrt{13}$

Q.22 Let $f(x) = \log_{\cos 3x} (\cos 2ix)$ if $x \neq 0$ and $f(0) = K$ (where $i = \sqrt{-1}$) is continuous at $x = 0$ then find the value of K.

Sol. $K = -4/9$

Q.23 (i) Let C_r's denotes the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$. If the integers

$a_n = C_0 + C_3 + C_6 + C_9 + \ldots\ldots\ldots$
$b_n = C_1 + C_4 + C_7 + C_{10} + \ldots\ldots\ldots$
and $c_n = C_2 + C_5 + C_8 + C_{11} + \ldots\ldots$, then prove that

(a) $a_n^3 + b_n^3 + c_n^3 - 3a_n b_n c_n = 2^n$,

(b) $(a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - a_n)^2 = 2$.

(ii) Prove the identity:

$(C_0 - C_2 + C_4 - C_6 + \ldots..)^2 + (C_1 - C_3 + C_5 - C_7 + \ldots..)^2 = 2^n$

Sol.

Q.24 Prove that

(a) $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \ldots\ldots ..+ {}^nC_n \cos (n + 1) x = 2^n.\cos^n \frac{x}{2}.\cos\left(\frac{n+2}{2}\right)x$

(b) $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \ldots.. \ldots.+ {}^nC_n \sin (n + 1) x = 2^n.\cos^n \frac{x}{2}.\sin\left(\frac{n+2}{2}\right)x$

Q.25 Let $A = \{a \in R|$ the equation $(1 + 2i)x^3 - 2(3 + i)x^2 + (5 - 4i)x + 2a^2 = 0\}$has at least one real root. Find the value of $\sum_{a \in A} a^2$.

Sol. 18

Q.26 P is a point on the Argand diagram. On the circle with OP as diameter two points Q & R are taken such that $\angle POQ = \angle QOR = \theta$. If 'O' is the origin and P, Q & R are represented by the complex numbers Z_1, Z_2 & Z_3 respectively, show that : $Z_2^2. \cos 2\theta = Z_1. Z_3 \cos^2\theta$.

Q.27 For $x \in (0, \pi/2)$ and $\sin x = \frac{1}{3}$, if $\sum_{n=0}^{\infty} \frac{\sin (nx)}{3^n} = \frac{a + b\sqrt{b}}{c}$ then find the value of $(a + b + c)$, where a, b, c are positive integers.

(You may use the fact that $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$)

Sol. 41

Q.28 If the expression $z^5 - 32$ can be factorized into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z - 2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

Sol. 4

Q.29 (a) Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z \mid |z| \leq 2\}$ and $B = \{z \mid (1 - i)z + (1 + i)\bar{z} \geq 4\}$. Find the area of the region $A \cap B$.

(b) For all real numbers x, let the mapping $f(x) = \frac{1}{x - i}$, where $i = \sqrt{-1}$. If there exist real number a, b, c and d for which f(a), f(b), f(c) and f(d) form a square on the complex plane. Find the area of the square.

Sol. (a) $\pi - 2$; (b) 1/2

Q.30 If z_1, z_2 are the roots of the equation $az^2 + bz + c = 0$, with a, b, c > 0; $2b^2 > 4ac > b^2$; $z_1 \in$ third quadrant; $z_2 \in$ second quadrant in the argand's plane then, show that

$$\arg\left(\frac{z_1}{z_2}\right) = 2\cos^{-1}\left(\frac{b^2}{4ac}\right)^{1/2}$$

Q.31 Find the set of points on the argand plane for which the real part of the complex number $(1 + i)z^2$ is positive where $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$.

Sol. Required set is constituted by the angles without their boundaries, whose sides are the straight lines $y = (\sqrt{2} - 1)x$ and $y + (\sqrt{2} + 1)x = 0$ containing the x-axis.

Q.32 Let $z = 18 + 26i$ where $z_0 = x_0 + iy_0$ ($x_0, y_0 \in R$) is the cube root of z having least positive argument. Find the value of $x_0y_0(x_0 + y_0)$.

Sol. 12

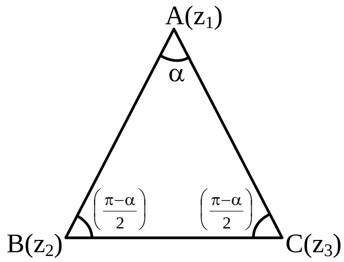
Q.33 Resolve $z^5 + 1$ into linear and quadratic factors with real coefficients. Deduce that $4 \sin \frac{\pi}{10} \cos \frac{\pi}{5} = 1$.

Sol. $(z + 1)(z^2 - 2z \cos 36° + 1)(z^2 - 2z \cos 108° + 1)$

Q.34 The points A,B,C depict the complex numbers z_1, z_2, z_3 respectively on a complex plane, and the angle B and C of the triangle ABC are each equal to (1/2) $(\pi - \alpha)$. Show that :

$(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2 \alpha / 2$

Sol. Applying rotation theorem at B and C respectively, we have



$$\frac{z_1 - z_2}{z_3 - z_2} = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\left(\frac{\pi-\alpha}{2}\right)}$$

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{AB}{BC} e^{i\left(\frac{\pi-\alpha}{2}\right)} \quad \ldots(1)$$

$$\frac{z_2 - z_3}{z_1 - z_3} = \left|\frac{z_2 - z_3}{z_1 - z_3}\right| e^{i\left(\frac{\pi-\alpha}{2}\right)}$$

$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{BC}{AC} e^{i\left(\frac{\pi-\alpha}{2}\right)} \quad \ldots(2)$$

Dividing equation (1) by (2), we get

$$\frac{z_1 - z_2}{z_3 - z_2} \times \frac{(z_1 - z_3)}{z_3 - z_2} = \frac{AB}{BC} \times \frac{AC}{BC} \times 1$$

$$-\frac{(z_1 - z_2)(z_1 - z_3)}{(z_2 - z_3)^2} = \left(\frac{AB}{BC}\right)^2 \text{ (Since, } |AB| = |AC|\text{)}$$

$$\Rightarrow \frac{(z_1 - z_2)(z_1 - z_3)}{(z_2 - z_3)^2} = \left(\frac{AB}{BC}\right)^2 \quad \ldots (3)$$

from sine rule, we get

$$\frac{\sin\alpha}{BC} = \frac{\sin\left(\frac{\pi - \alpha}{2}\right)}{AB} = \frac{\sin(\pi - \alpha)/2}{AC}$$

$$\Rightarrow \frac{2\sin\alpha/2.\cos\alpha/2}{BC} = \frac{\cos\alpha/2}{AB}$$

$\Rightarrow 2\sin\alpha/2 = BC/AB$

$\Rightarrow$ Put value of BC/AB = 2 sin $\alpha/2$ in equation (3), we get

$$\frac{(z_3 - z_1)(z_1 - z_2)}{(z_2 - z_3)^2} = \frac{1}{(2\sin\alpha/2)^2} = \frac{1}{4\sin 2\alpha/2}$$

$\Rightarrow (z_2 - z_3)^2 = 4\sin^2\alpha/2\,(z_3 - z_1)(z_1 - z_2)$

Hence Proved

Q.35 Let z_1 & z_2 be roots of the equation $z^2 + pz + q = 0$ where the coefficients p and q may be complex numbers. Let A and B represents z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and OA = OB, where O is the origin prove that $p^2 = 4q\cos^2(\alpha/2)$

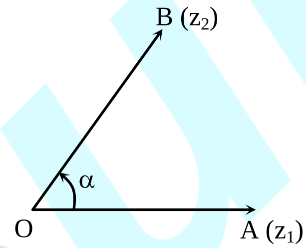
Sol. $z^2 + pz + q = 0$

$$\Rightarrow z = \frac{-p \pm \sqrt{p^2 - 4q}}{2 \times 1}$$

$\Rightarrow$ Let roots be

$$z_1 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

$$z_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$



$|OA| = |OB|$

$\Rightarrow |z_1| = |z_2|$

Apply coni method or rotation method, we have

$$\frac{z_2}{z_1} = \frac{|z_2|}{|z_1|} e^{i\alpha} = \cos\alpha + i\sin\alpha$$

$$\frac{z_2}{z_1} = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \frac{\dfrac{-p + \sqrt{p^2 - 4q}}{2}}{\dfrac{-p - \sqrt{p^2 - 4q}}{2}} = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \frac{-p + \sqrt{p^2 - 4q}}{-p - \sqrt{p^2 - 4q}} = \cos\alpha + i\sin\alpha$$

$\Rightarrow -p + \sqrt{p^2 - 4q} = -p(\cos\alpha + i\sin\alpha)$
$\qquad - \sqrt{p^2 - 4q}\,(\cos\alpha + i\sin\alpha)$

$\Rightarrow p(\cos\alpha + i\sin\alpha)$
$\qquad = -\sqrt{p^2 - 4q}\,(1 + \cos\alpha + i\sin\alpha)$

$$\Rightarrow \frac{p}{\sqrt{p^2 - 4q}} = \frac{1 + \cos\alpha + i\sin\alpha}{1 - \cos\alpha - i\sin\alpha}$$

$$\Rightarrow \frac{p}{\sqrt{p^2 - 4q}} = \frac{2\cos^2\alpha/2 + i2\sin\alpha/2\cos\alpha/2}{2\sin^2\alpha/2 - i2\sin\alpha/2\cos\alpha/2}$$

(Θ using $\cos\alpha = 2\cos^2\alpha/2 - 1 = 1 - 2\sin^2\alpha/2$)

$$\Rightarrow \frac{p}{\sqrt{p^2-4q}} = \frac{2\cos\alpha/2(\cos\alpha/2+i\sin\alpha/2)}{2\sin\alpha/2(\sin\alpha/2-i\cos\alpha/2)} \times \frac{i}{i}$$

$$= \frac{i\cot\alpha/2(\cos\alpha/2+i\sin\alpha/2)}{(\cos\alpha/2+i\sin\alpha/2)} = i\cot\frac{\alpha}{2}$$

$$\Rightarrow \frac{p}{\sqrt{p^2-4q}} = i\cot\alpha/2$$

Squaring both sides, we get

$$\Rightarrow \frac{p^2}{(p^2-4q)} = i^2\cot^2\alpha/2 = -\cot^2\alpha/2$$

$$\Rightarrow p^2 = -p^2\cot^2\alpha/2 + 4q\cot^2\alpha/2$$

$$\Rightarrow p^2(1+\cot^2\alpha/2) = 4q\cot^2\alpha/2$$

$$\Rightarrow p^2 \times \mathrm{cosec}^2\alpha/2 = 4q\cot^2\alpha/2$$

$$\Rightarrow p^2 \times \frac{1}{\sin^2\alpha/2} = 4q \times \frac{\cos^2\alpha/2}{\sin^2\alpha/2}$$

$$\Rightarrow p^2 = 4q\cos^2\alpha/2$$

Hence, Proved.

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	B	C	B	D	A	A	A	A	A,B	C	A	A	A	C	C
Q.No.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	
Ans.	A,B,C	C	B	A	D	B	B	C	D	A,C	A	C	D	A	

30. a radius of a circle **31.** 1 **32.** $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$ **33.** $2 - \sqrt{3}$, $2 - \sqrt{3}$

34. $3 - \frac{i}{2}$ or $1 - \frac{3}{2}i$ **35.** $-2, 1 - i\sqrt{3}$ **36.** $\frac{|z|^2}{2}$ **37.** False **38.** True **39.** True

40. True

EXERCISE # 2

(PART- A)

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	C	A	A	A	D	B	A	A	A	D	D	C	A,B,C	B	D

(PART- B)

Q.No.	16	17	18	19	20	21	22
Ans.	A	A	A,D	A,C,D	A,B,C,D	B,C,D	A,C

(PART- C)

Q.No.	23	24	25
Ans.	A	D	D

(PART- D)

26. A → P, S, T; B → P, S, T; C → Q, W; D → R

27. A → R; B → S; C → Q; D → P

28. A → P, Q; B → P, Q; C → R; D → S

29. A → R; B → P, S; C → Q; D → R

30. A → Q, S; B → P, R; C → P, R; D → S

EXERCISE # 3

2. 48 (1 – i) **5.** $\frac{1}{3}\Sigma z_1 = -a$ **8.** $z = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

10. $\frac{1-i}{2\sqrt{2}}z_1 + \frac{(2\sqrt{2}-1)+i(1+\sqrt{2})}{2\sqrt{2}}z_2$, 14 **14.** $\left(\frac{1}{3}, 0\right)$ **19.** (C) **20.** (A) **21.** (A) **22.**(C)

23. (B) **24.** (A) **25.** (A) **26.** (C) **27.** (A) **28.** (A) **29.** (A) **30.** (C) **31.**(B)

EXERCISE # 4

1. (C) **2.** (D) **3.** (B) **4.** (B) **6.** (D) **9.** (B)

10. Centre of the circle = – $z_0 = \frac{\alpha - \beta k^2}{1-k^2}$, Radius of the circle = r = $\left|\frac{k\ (\alpha-\beta)}{1-k^2}\right|$ **11.** (B) **12.** (A)

13. (0, $-\sqrt{3}$), (1 $-\sqrt{3}$, 1), (1 + $\sqrt{3}$, – 1) **14.** (B) **15.** (D) **16.** (D) **17.** (D) **18.** (B)

19. (C) **20.** (D) **21.** (D) **22.** (A) **23.** A $\to$ P; B $\to$ S, T; C $\to$ R; D $\to$ Q, S

24. (A, C, D) **25.** 1 **26.** A $\to$ Q,R; B $\to$ P; C $\to$ P,S,T; D $\to$ Q, R, S, T

27. 5 **28.** 3 **29.** D

EXERCISE # 5

1. (D) **2.** (C) **3.** (D) **4.** $\frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}, i$ **8.** one solution **12.** 51 **13.** 4

16. (a) K = 3, **(b)** –4 **17.** [–3, –2] **19.** $\frac{iz}{2} + \frac{1}{2} + i$

21. |f(z)| is maximum when z = ω, where ω is the cube root of unity and |f(z)| = $\sqrt{13}$

22. K = – 4/9 **25.** 18 **27.** 41 **28.** 4 **29. (a)** π – 2; **(b)** 1/2

31. Required set is constituted by the angles without their boundaries, whose sides are the straight lines y = ($\sqrt{2}$ –1) x and y + ($\sqrt{2}$ +1) x = 0 containing the x-axis.

32. 12 **33.** (z + 1) (z^2 – 2z cos 36° + 1) (z^2 – 2z cos 108° + 1)