CIRCLE

EXERCISE # 1

Question based on **Different forms of the equation of circle**

Q.1 The equation of the circle which touches the axis of y at the origin and passes through (3, 4) is-

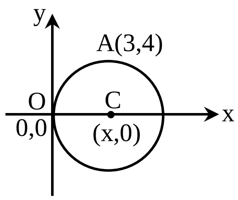
(A) $2(x^2 + y^2) - \frac{45}{3}x = 0$

(B) $3(x^2 + y^2) - 25x = 0$

(C) $4(x^2 + y^2) - 25x = 0$

(D) None of these

Sol. **[B]**



Clearly centre lies on x-axis. Let it be (x, 0)

$\Rightarrow OC = AC \Rightarrow OC^2 = AC^2 \Rightarrow x^2 = (x-3)^2 + 16$

$\Rightarrow x^2 = x^2 - 6x + 9 + 16 \Rightarrow 6x = 25 \Rightarrow x = \frac{25}{6}$

$\therefore$ Its centre is $\left(\frac{25}{6}, 0\right)$ & radius $\frac{25}{6}$

$\therefore$ Its equation is

$$\left(x - \frac{25}{6}\right)^2 + (y-0)^2 = \left(\frac{25}{6}\right)^2$$

$\Rightarrow x^2 + y^2 - \frac{25}{3}x + \left(\frac{25}{6}\right)^2 = \left(\frac{25}{6}\right)^2$

$\Rightarrow x^2 + y^2 - \frac{25}{3}x = 0 \Rightarrow 3(x^2 + y^2) - 25x = 0$

Q.2 The equation of a circle passing through (3, –6) and touching both the axes is -

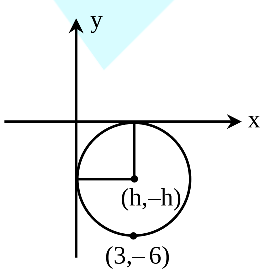
(A) $x^2 + y^2 - 6x + 6y + 8 = 0$

(B) $x^2 + y^2 + 6x - 6y + 9 = 0$

(C) $x^2 + y^2 + 30x - 30y + 225 = 0$

(D) $x^2 + y^2 - 30x + 30y + 225 = 0$

Sol. **[D]**



Since circle tonching both axes, therefore according to question its centre will be (h, –h) and its radius equal to h.

Also

$h = \sqrt{(h-3)^2 + (6-h)^2}$

$\Rightarrow h^2 = h^2 - 6h + 9 + 36 + h^2 - 12h$

$\Rightarrow h^2 - 18h + 45 = 0$

$\Rightarrow h^2 - 15h - 3h + 45 = 0$

$\Rightarrow h(h-15) - 3(h-15) = 0$

$\Rightarrow (h-3)(h-15) = 0$

$\Rightarrow h = 3, 15$

$\therefore$ When h = 3, centre (3, – 3) and radius 3 and when h = 15, centre (15, – 15) and radius 15

$\therefore$ Required equation of circle is

$(x-3)^2 + (y+3)^2 = 3^2$ or $(x-15)^2 + (y+15)^2 = 15^2$

$\Rightarrow x^2 + y^2 - 6x + 6y + 9 = 0$ or $x^2 + y^2 - 30x + 30y + 225 = 0$

from options $x^2 + y^2 - 30x + 30y + 225 = 0$

Q.3 The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$, and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. The radius of the circle with AB as diameter is

(A) $\sqrt{a^2 + b^2 + p^2 + q^2}$ (B) $\sqrt{a^2 + p^2}$

(C) $\sqrt{b^2 + q^2}$ (D) None of these

Sol. **[A]**

Let $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ since x_1, x_2 are roots of $x^2 + 2ax - b^2 = 0$ there fore

$x_1 + x_2 = -2a$

$x_1x_2 = -b^2$

Also y_1, y_2 are roots of $x^2 + 2px - q^2 = 0$

$\therefore y_1 + y_2 = -2p$

$y_1y_2 = -q^2$

$\therefore AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{\{(x_1 + x_2)^2 - 4x_1x_2\} + \{(y_1 + y_2)^2 - 4y_1y_2\}}$

$= \sqrt{4a^2 + 4b^2 + 4p^2 + 4q^2}$

$= 2\sqrt{a^2 + b^2 + p^2 + q^2}$

$\therefore$ Radius $= \frac{\text{diameter}}{2}$

$= \dfrac{2\sqrt{a^2+b^2+p^2+q^2}}{2} = \sqrt{a^2+b^2+p^2+q^2}$

Q.4 The parametric equations of the circle $x^2 + (y + \beta)^2 = a^2$ is -

(A) $x = a\cos\theta,\ y = a\sin\theta + \beta$

(B) $x = a\cos\theta,\ y = a\sin\theta - \beta$

(C) $x = a\sin\theta,\ y = a\cos\theta - \beta$

(D) None of these

Sol. [B] $x^2 + (y + \beta)^2 = a^2$

parametric equationm will be

$x = a\cos\theta$ and $y + \beta = a\sin\theta$

$\Rightarrow x = a\cos\theta$ and $y = a\sin\theta - \beta$

Q.5 The equation to the circle which passes through the points (1, – 2) and (4, –3) and which has its centre on the straight line 3x + 4y = 7 is –

(A) $15x^2 + 15y^2 + 94x + 18y + 55 = 0$

(B) $15x^2 + 15y^2 - 94x - 18y + 55 = 0$

(C) $15x^2 + 15y^2 - 94x + 18y + 55 = 0$

(D) $15x^2 + 15y^2 + 94x - 18y - 55 = 0$

Sol. **[C]**

Let equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

its centre (– g, – f) lies on 3x + 4y = 7

$\Rightarrow 3g + 4f = -7$ ……..(1)

it passes (1, – 2) therefore

$1 + 4 + 2g - 4f + c = 0$

$\Rightarrow 2g - 4f + c = -5$ …….(2)

Also passes (4, – 3) Hence

$16 + 9 + 8g - 6f + c = 0$

$\Rightarrow 8g - 6f + c = -25$ ……(3)

from (2) & (3) $\Rightarrow$ (2) – (3) gives

$-6g + 2f = 20$

$3g - f = -10$ …….(4)

from (1) & (4) $5f = 3$

$f = 3/5$

$\therefore\ 3g = \dfrac{3}{5} - 10 = -\dfrac{47}{5}$

$\Rightarrow g = -\dfrac{47}{15}$

from (2) $2\left(-\dfrac{47}{15}\right) - 4\left(\dfrac{3}{5}\right) + c = -5$

$\Rightarrow -\dfrac{94}{15} - \dfrac{12}{5} + 5 + c = 0$

$\Rightarrow -94 - 36 + 75 + 15c = 0$

$15c = 130 - 75$

$15c = 55$

$c = \dfrac{55}{15} = \dfrac{11}{3}$

$\therefore$ Required equation of circle is

$x^2 + y^2 + 2\left(-\dfrac{47}{15}\right)x + 2\left(\dfrac{3}{5}\right)y + \dfrac{11}{3} = 0$

$\Rightarrow 15x^2 + 15y^2 - 94x + 18y + 55 = 0$

$\Rightarrow 15x^2 + 15y^2 - 94x + 18y + 55 = 0$

Question based on Position of a point

Q.6 If $x^2 + y^2 - 6x + 8y - 11 = 0$ is a given circle and (0, 0), (1, 8) are two points, then -

(A) Both the points are inside the circle

(B) Both the points are outside the circle

(C) One point is on the circle another is outside the circle

(D) One point is inside and another is outside the circle.

Sol. **[D]**

$x^2 + y^2 - 6x + 8y - 11 = 0$ ……..(i)

put (0, 0) in (i) we get

$-11 < 0$

put (1, 8) in (i) we get

$1 + 64 - 6 + 64 - 11 > 0$

Hence, we can say that one point is inside the circle and another is out side the circle

Question based on Line and circle

Q.7 The equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which make equal intercepts on the positive coordinate axes, is

(A) $x + y = 2$ (B) $x + y = 2\sqrt{2}$

(C) $x + y = 4$ (D) $x + y = 8$

Sol. **[B]**

Let equation of tangent is

$\dfrac{x}{a} + \dfrac{y}{a} = 1$

$\Rightarrow x + y = a$

centre of circle (– 2, 2)

radius = $\sqrt{4 + 4 - 4} = 2$

$\therefore$ perpendicular drawn from centre (– 2, 2) to x + y = a is equal to its radius

$\Rightarrow x + y - a = 0$

$\Rightarrow \dfrac{|-2 + 2 - a|}{\sqrt{2}} = 2$

$\Rightarrow a = 2\sqrt{2}$

$\therefore$ equation of tangent is

$x + y = 2\sqrt{2}$

Q.8 Length of intercept made by line x + y = 2 on the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ is -

(A) $2\sqrt{23}$ (B) $\sqrt{23}$

(C) $\sqrt{46}$ (D) $4\sqrt{23}$

Sol. **[C]**

Length of intercept = $2\sqrt{a^2 - p^2}$

where a is radius of circle

and p is length of perpendicular

$p = \frac{2+3-2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

and $a = \sqrt{4+9+3} = 4$

$\therefore$ length of intercept = $2\sqrt{16 - \frac{9}{2}}$

$= 2\sqrt{\frac{32-9}{2}} = 2\sqrt{\frac{23}{2}}$

$= \sqrt{2}.\sqrt{23} = \sqrt{46}$

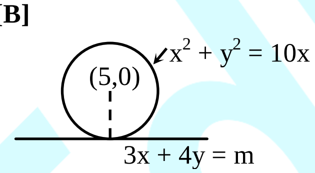
Question based on **Equation of tangent**

Q.9 If the line 3x + 4y = m touches the circle $x^2 + y^2 = 10x$, then m is equal to -

(A) – 40, 10 (B) 40, – 10

(C) 40, 10 (D) –40, –10

Sol. **[B]**



$\therefore$ perpendicular drawn from its centre = its radius

Radius of circle = $\sqrt{(5)^2} = 5$

Length of $\perp^r$ = $\frac{|15+0-m|}{\sqrt{9+16}} = 5$

$\Rightarrow |15 - m| = 25 \Rightarrow 15 - m = \pm 25$

$\Rightarrow m = -10, 40 \Rightarrow m = 40, -10$

Q.10 The value of p so that the straight line $x \cos \alpha + y \sin \alpha - p = 0$ may touch the circle $x^2 + y^2 - 2ax \cos \alpha - 2by \sin \alpha - a^2 \sin^2 \alpha = 0$ is -

(A) $a\cos^2\alpha + b\sin^2\alpha - \sqrt{a^2 + b^2 \sin^2 \alpha}$

(B) $a\cos^2\alpha - b\sin^2\alpha - \sqrt{a^2 + b^2 \sin^2 \alpha}$

(C) $a\cos^2\alpha + b\sin^2\alpha - \sqrt{a^2 - b^2 \sin^2 \alpha}$

(D) None of these

Sol. **[B]**

$x \cos \alpha + y \sin \alpha - P = 0$ touches the circle

$x^2 + y^2 - 2ax \cos \alpha - 2by \sin \alpha - a^2 \sin^2\alpha = 0$

$\therefore$ Centre of circle = $(a \cos \alpha, b \sin \alpha)$

Radius of circle

$= \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha + a^2 \sin^2 \alpha}$

$= \sqrt{a^2 + b^2 \sin^2 \alpha}$

Perpendicular drawn from centre to line is equal to the radius

i.e. $\frac{a\cos^2 \alpha + b\sin^2 \alpha - P}{\sqrt{\sin^2 \alpha + \cos^2 \alpha}} = \sqrt{a^2 + b^2 \sin^2 \alpha}$

$\Rightarrow a \cos^2\alpha + b \sin^2\alpha - P = \sqrt{a^2 + b^2 \sin^2 \alpha}$

$\Rightarrow P = a \cos^2\alpha + b \sin^2\alpha - \sqrt{a^2 + b^2 \sin^2 \alpha}$

Question based on **Equation of normal**

Q.11 The equation of the normal of the circle $2x^2 + 2y^2 - 2x - 5y - 7 = 0$ passing through the point (1, 1) is

(A) x + 2y – 3 = 0 (B) 2x + y – 3 = 0

(C) 2x + 3y – 5 = 0 (D) None of these

Sol. **[A]**

$2x^2 + 2y^2 - 2x - 5y - 7 = 0$

$\Rightarrow x^2 + y^2 - x - \frac{5}{2}y - \frac{7}{2} = 0$

Since normal of the circle passes its centre.

$\therefore$ centre of circle is $\left(\frac{1}{2}, \frac{5}{4}\right)$

since normal passes $\left(\frac{1}{2}, \frac{5}{4}\right)$ and (1, 1)

slope of normal = $\frac{\frac{5}{4} - 1}{\frac{1}{2} - 1} = \frac{\frac{1}{4}}{-\frac{1}{2}} = \frac{1}{4} \times \frac{-2}{1} = -\frac{1}{2}$

equation of normal

$y - 1 = -\frac{1}{2}(x - 1)$

$2y - 2 = -x + 1$

$x + 2y - 3 = 0$

Question based on Length of tangent and pair of tangent to a circle

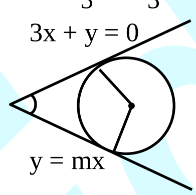
Q.12 If 3x + y = 0 is a tangent to the circle with centre at the point (2, –1), then the equation of the other tangent to the circle from the origin is -
(A) x – 3y = 0 (B) x + 3y = 0
(C) 3x – y = 0 (D) 2x + y = 0

Sol. **[A]**
Let other tangent is y = mx ⇒ mx – y = 0
Also 3x + y = 0 is a tangent of the circle with centre (2, – 1)
Length of perpendicular drawn from centre to tangent 3x + y = 0 given by $\frac{6-1}{\sqrt{9+1}} = \frac{5}{\sqrt{10}}$
and length of perpendicular drawn from centre to mx – y = 0 is given by

$$\frac{2m+1}{\sqrt{m^2+1}} = \frac{5}{\sqrt{10}} = \sqrt{\frac{5}{2}}$$

$$\Rightarrow \frac{(2m+1)^2}{m^2+1} = \frac{5}{2}$$

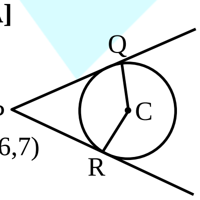
⇒ $5m^2 + 5 = 2(4m^2 + 4m + 1)$
⇒ $5m^2 + 5 = 8m^2 + 8m + 2$
⇒ $3m^2 + 8m – 3 = 0$
⇒ $3m^2 + 9m – m – 3 = 0$
⇒ 3m (m + 3) – 1 (m + 3) = 0
⇒ (m + 3) (3m – 1) = 0
⇒ m = –3, + $\frac{1}{3}$
when m = – 3 , – 3x – y = 0 ⇒ 3x + y = 0
when m = $\frac{1}{3}$, $\frac{1}{3}$ x – y = 0



⇒ x – 3y = 0 ⇒ x – 3y = 0

Q.13 From a point P(16, 7), tangent PQ and PR are drawn to the circle $x^2 + y^2 – 2x – 4y – 20 = 0$. If C be the centre then area of the quadrilateral PQCR will be-
(A) 75 (B) 150 (C) 15 (D) None

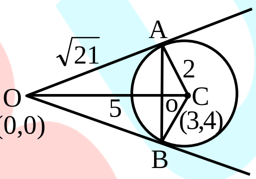
Sol. **[A]**



Area of quadrilateral PQCR will be given by
$r\sqrt{s_1}$ where r is radius of circle.
radius r = CQ = $\sqrt{1+4+20}$ = 5
length of tangent PQ = $\sqrt{s_1}$
= $\sqrt{256+49-32-28-20}$
= 15
∴ Area = $r\sqrt{s_1}$ = 5 × 15 = 75 sq. unit

Q.14 If OA and OB be the tangents to the circle $x^2 + y^2 – 6x – 8y + 21 = 0$ drawn from the origin O, then AB =
(A) 11 (B) $\frac{4}{5}\sqrt{21}$
(C) $\sqrt{\frac{17}{3}}$ (D) None

Sol. **[B]**



$xx_1 + yy_1 + g(n + m) + f(y + y_1) + c = 0$
0 + 0 – 3x – 4y + 21 = 0
3x + 4y – 21 = 0
$\frac{9+16-21}{5} = \frac{4}{5}$
$AO' = \sqrt{AC^2 - O^1C^2}$
$= \sqrt{4-\frac{16}{25}}$
$= \sqrt{\frac{100-16}{25}} = \sqrt{\frac{84}{5}} = \frac{2}{5}\sqrt{21}$
∴ AB = 2 AO′
= 2 × $\frac{2}{5}\sqrt{21}$ = $\frac{4}{5}\sqrt{21}$

Question based on Chord of contact

Q.15 The equation of the chord of contact of the circle $x^2 + y^2 + 4x + 6y – 12 = 0$ with respect to the point (2, –3) is -
(A) 4x = 17 (B) 4y = 17
(C) 4x + y = 17 (D) None of these

Sol. **[A]**
Chord of contact T = 0
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow 2x - 3y + 2(x + 2) + 3(y + (-3)) + (-12) = 0$

$\Rightarrow 2x - 3y + 2x + 4 + 3y - 9 - 12 = 0$

$\Rightarrow 4x - 17 = 0$

$\Rightarrow 4x - 17 = 0$

$\Rightarrow 4x = 17$

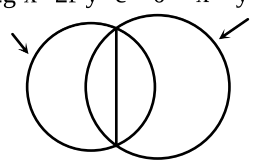
Q.16 If the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ then the length of the common chord of these two circles is -

(A) $2\sqrt{g^2 + f^2 + c}$ (B) $2\sqrt{g^2 + f^2 - c}$

(C) $2\sqrt{g'^2 + f'^2 - c'}$ (D) $2\sqrt{g'^2 + f'^2 + c'}$

Sol. **[C]**

$x^2+y^2+2g^1x+2f'y+c^1=0$ $x^2+y^2+2gx+2fy+c=0$



Required length of commgh chord will be the diameter of second circle.

$\therefore$ Radius of second circle is

$$\sqrt{g'^2 + f'^2 - c'}$$

$\therefore$ diameter = 2 × Radius

$$= 2\sqrt{g'^2 + f'^2 - c'}$$

Question based on Director circle, pole and polar

Q.17 The pole of the straight line $9x + y - 28 = 0$ with respect to the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$ is-

(A) (3, 1) (B) (1, 3) (C) (3, –1) (D) (–3, 1)

Sol. **[C]**

Given st. line is $9x + y - 28 = 0$ …..(1)

Given circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$

$\Rightarrow x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y - \frac{7}{2} = 0$

Let pole is (x_1, y_1) then polar w.r.t. circle is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow xx_1 + yy_1 - \frac{3}{4}(x + x_1) + \frac{5}{4}(y + y_1) - \frac{7}{2} = 0$

$\Rightarrow x\left(x_1 - \frac{3}{4}\right) + y\left(y_1 + \frac{5}{4}\right) - \frac{3}{4}x_1 + \frac{5}{4}y_1 - \frac{7}{2} = 0$

$\Rightarrow 4x\left(x_1 - \frac{3}{4}\right) + 4y\left(y_1 + \frac{5}{4}\right) - (3x_1 - 5y_1 + 14) = 0$

compare with (1) we get

$4\left(x_1 - \frac{3}{4}\right) = 9$

$\Rightarrow 4x_1 = 3 + 9 \Rightarrow 4x_1 = 12$

$\Rightarrow x_1 = 3$ and $4\left(y_1 + \frac{5}{4}\right) = 1$

$\Rightarrow 4y_1 = -4 \Rightarrow y_1 = -1 \Rightarrow (3, -1)$

Q.18 The polar of the point (5, –1/2) with respect to the circle $(x - 2)^2 + y^2 = 4$ is-

(A) $5x - 10y + 2 = 0$ (B) $6x - y - 20 = 0$

(C) $10x - y - 10 = 0$ (D) $x - 10y - 2 = 0$

Sol. **[B]**

$(x - 2)^2 + y^2 = 4$

$x^2 + y^2 - 4x = 0$

the polar of $\left(5, -\frac{1}{2}\right)$ w.r.t. the circle

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow 5x - \frac{y}{2} - 2(x + 5) + 0 + 0 = 0$

$\Rightarrow 5x - \frac{y}{2} - 2x - 10 = 0$

$\Rightarrow 3x - \frac{y}{2} - 10 = 0$

$\Rightarrow 6x - y - 20 = 0$

Q.19 The tangents drawn from origin to the circle $x^2 + y^2 - 2ax - 2by + b^2 = 0$ are perpendicular to each other, if -

(A) $a - b = 1$ (B) $a + b = 1$

(C) $a^2 - b^2 = 0$ (D) $a^2 + b^2 = 1$

Sol. **[C]**

$SS_1 = T^2$

$(x^2 + y^2 - 2ax - 2by + b^2)(b^2) = (-ax - by + b^2)^2$

$\Rightarrow b^2(x^2 + y^2 - 2ax - 2by + b^2) = a^2x^2 + b^2y^2 + b^4 + 2abxy - 2b^3y - 2ab^2x$

$\Rightarrow (a^2 - b^2)x^2 + \ldots\ldots.$

Tangents are perpendicular

$\therefore$ coeff. of x^2 + coeff. of $y^2 = 0$

$\Rightarrow a^2 - b^2 + 0 = 0 \Rightarrow a^2 - b^2 = 0$

Q.20 The polar of a point with respect to the circle $x^2 + y^2 - 2\lambda x + c^2 = 0$ -

(A) passes through $(\lambda, 0)$

(B) passes through a fixed point if $\lambda = 2$

(C) passes through a fixed point for all values of λ

(D) is a tangent to the circle.

Sol. **[C]**

Let point is (x_1, y_1) and its polar w.r.t. $x^2 + y^2 - 2\lambda x + c = 0$ is given by

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c^1 = 0$

$\Rightarrow xx_1 + yy_1 - \lambda(x + x_1) + 0 + c^2 = 0$

$\Rightarrow x(x_1 - \lambda) - \lambda x_1 + yy_1 + c^2 = 0$

Clearly, it passes through fixed point for all values of λ.

Q.21 If the polar of (p, q) with respect to the circle $x^2 + y^2 = a^2$ touches the circle $(x - h)^2 + (y - k)^2 = r^2$, then $(hp + kq - a^2)^2 =$

(A) $r^2 (p^2 + q^2)$ (B) $r(p^2 + q^2)$

(C) $r^2 (p^2 - q^2)$ (D) None of these

Sol. **[A]**

Polar of (p, q) w.r.t. $x^2 + y^2 = a^2$ is given by

$px + qy = a^2$ …….(1)

This touches $(x - h)^2 + (y - k)^2 = r^2$

$\therefore$ perpendicular drawn from centre (h, k) to (1) will be equal to radius r.

$$\therefore \frac{ph + qk - a^2}{\sqrt{p^2 + q^2}} = r$$

$\Rightarrow hp + kq - a^2 = r\sqrt{p^2 + q^2}$

$\Rightarrow (hp + kq - a^2)^2 = r^2(p^2 + q^2)$

Question based on

Chord with mid point

Q.22 The equation of the chord of $x^2 + y^2 - 6x + 8y = 0$ bisected at the point (5, –3) is -

(A) $2x + y = 7$ (B) $2x - y = 7$

(C) $x + 2y = 7$ (D) $x - 2y = 7$

Sol. **[A]**

Use $T = S_1$

$x.5 + y(-3) - 3(x + 5) + 4(y - 3) = 25 + 9 - 30 - 24$

$\Rightarrow 5x - 3y - 3x - 15 + 4y - 12 + 20 = 0$

$\Rightarrow (5x - 3x) + (4y - 3y) - 7 = 0$

$\Rightarrow 2x + y - 7 = 0 \Rightarrow 2x + y = 7$

Q.23 The middle point of the chord intercepted on line $\lambda x + my + n = 0$ by the circle $x^2 + y^2 = a^2$ is -

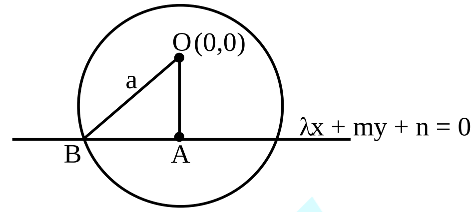
(A) $\left(\frac{\lambda n}{\lambda^2 + m^2}, \frac{mn}{\lambda^2 + m^2}\right)$

(B) $\left(\frac{-\lambda n}{\lambda^2 + m^2}, \frac{mn}{\lambda^2 + m^2}\right)$

(C) $\left(\frac{-\lambda n}{\lambda^2 + m^2}, \frac{-mn}{\lambda^2 + m^2}\right)$

(D) None of these

Sol. **[C]**



Given live $\lambda x + my + n = 0$ ……..(1)

equation of line OA which is $\perp^r$ to $\lambda n + my + n = 0$ is given by

$mx - \lambda y + k = 0$

Since it passes (0, 0)

$\therefore$ k = 0

$\therefore$ $mx - \lambda y = 0$ ……(2)

from (1) & (2)

$\lambda x + my + n = 0$ ……(1)

$mx - \lambda y = 0$ ……(2)

$m\lambda x + m^2y + mm = 0$ ……(3)

$m\lambda x - \lambda^2y = 0$ ……(4)

– +

$(m^2 + \lambda^2)\, y = -mn$

$$y = \frac{-mn}{\lambda^2 + m^2}$$

Similarly, we get

$$x = \frac{-\lambda n}{\lambda^2 + m^2}$$

$$\therefore \left(\frac{-\lambda n}{\lambda^2 + m^2}, \frac{-mn}{\lambda^2 + m^2}\right)$$

Question based on

The diameter of a circle

Q.24 The equation of the diameter of the circle $x^2 + y^2 - 2x + 4y = 0$ passing through the origin is-

(A) $x + 2y = 0$ (B) $x - 2y = 0$

(C) $2x + y = 0$ (D) $2x - y = 0$

Sol. **[C]**

Since diameter of a circle passes its centre.

Centre of circle is (1, – 2)

$\therefore$ equation of normal which passes origin

$y = mx$

where m is slope $= \frac{2}{-1} = -2$

$\therefore$ $y = -2x \Rightarrow 2x + y = 0$

Question based on

Position of two circle

Q.25 Circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x - 4y + 3 = 0$

(A) touch each other internally
(B) touch each other externally
(C) intersect each other
(D) do not intersect

Sol. **[C]**

$S_1 : x^2 + y^2 = 4 \Rightarrow C_1 : (0, 0)\ r_1 = 2$

$S_2 : x^2 + y^2 - 2x - 4y + 3 = 0$

$\Rightarrow C_2 : (1, 2),\ r_2 = \sqrt{2}$

$C_1C_2 = \sqrt{1+4} = \sqrt{5} = 2.2$

$r_1 + r_2 = 2 + \sqrt{2} = 3.14$

$r_1 - r_2 = 2 - \sqrt{2} = 0.6$

$r_1 - r_2 < C_1C_2 < r_1 + r_2$

They intersect each other

Q.26 The point of intersection of common transverse tangents of two circles $x^2 + y^2 - 24x + 2y + 120 = 0$ and $x^2 + y^2 + 20x - 6y - 116 = 0$ is -

(A) (13,0) (B) (13/2, 0)
(C) (13,2) (D) None of these

Sol. **[B]**

Given

$S_1 : x^2 + y^2 - 24x + 2y + 120 = 0$

$S_2 : x^2 + y^2 + 20x - 6y - 116 = 0$

For $S_1 \Rightarrow C_1 = (12, -1),\ r_1 = \sqrt{144+1-120} = 5$

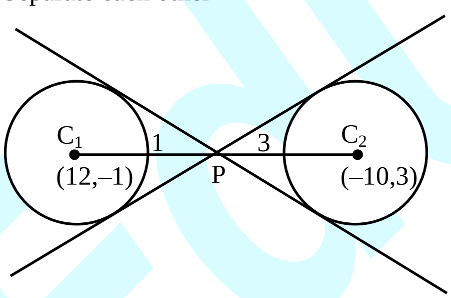
For $S_2 \Rightarrow C_2 = (-10, 3),\ r_2 = \sqrt{100+9+116} = 15$

$C_1C_2 = \sqrt{484+16} = 10\sqrt{5} \simeq 22$

$r_1 + r_2 = 20$

$\therefore\ c_1c_2 > r_1 + r_2$

Separate each other



Required point P will be

$$\left(\frac{1(-10)+3(12)}{1+3}, \frac{1(3)+3(-1)}{1+3}\right)$$

$$= \left(\frac{26}{4}, 0\right) = \left(\frac{13}{2}, 0\right)$$

Question based on
Angle of intersection of two circles

Q.27 Tangents are drawn from a point of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$. The angle between these tangents is-

(A) α (B) 2α (C) $\alpha/2$ (D) $\pi/2$

Sol. **[B]**

$x^2 + y^2 + 2gx + 3fy + c = 0$

$x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$

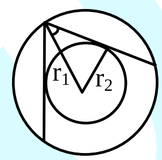
Given circles are concentric

$r_1 = \sqrt{g^2 + f^2 - c}$

$r_2 = \sqrt{(g^2 + f^2) - (g^2 + f^2)\cos^2\alpha - c\sin^2\alpha}$

$r_2 = \sqrt{(g^2 + f^2)\sin^2\alpha - c\sin^2\alpha}$

$= \sqrt{g^2 + f^2 - c}\ \sin\alpha$



Required angle $\theta = 2\sin^{-1}\frac{r_2}{r_1}$

$\theta = 2\sin^{-1}\frac{\sqrt{g^2 + f^2 - c}\sin\alpha}{\sqrt{g^2 + f^2 - c}}$

$\theta = 2\sin^{-1}\sin\alpha$

$\theta = 2\alpha$

Q.28 The equation of the circle passing through the origin & cutting the circles $x^2 + y^2 - 4x + 6y + 10 = 0$ and $x^2 + y^2 + 12y + 6 = 0$ at right angles is -

(A) $2(x^2 + y^2) - 7x + 2y = 0$
(B) $2(x^2 + y^2) - 7x - 2y = 0$
(C) $2(x^2 + y^2) + 7x - 2y = 0$
(D) $2(x^2 + y^2) + 7x + 2y = 0$

Sol. **[A]**

Let equation of circle is given by

$x^2 + y^2 + 2gx + 2fy + c = 0$ ……(1)

it cuts the circle $x^2 + y^2 - 4x + 6y + 10 = 0$ and

Also $x^2 + y^2 + 12x + 6 = 0$ at right angles.

$\therefore\ 2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$\Rightarrow 2g.(-2) + 2f(3) = c + 10$

$\Rightarrow -4g + 6f = c + 10$ …... (2)

Also

$2g(6) + 2f(0) = c + 6$

$\Rightarrow\ 12g = c + 6$ ……(3)

Since (1) passes through (0, 0)

$\therefore\ c = 0$

$\therefore$ from (2) & (3)

$-4g + 6f = 10$

and $12g = 6$

$\Rightarrow g = \frac{1}{2}$

$\therefore (-4)\left(\frac{1}{2}\right) + 6f = 10$

$\Rightarrow 6f = 12$

$\Rightarrow f = 2$

$\therefore$ Reqd. circle is $x^2 + y^2 + 2 \cdot \frac{1}{2} x + 2 \cdot 2 \cdot y = 0$

$\Rightarrow x^2 + y^2 + x + 4y = 0$

Q.29 The circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ intersect at an angle of

(A) $\pi/6$ (B) $\pi/4$

(C) $\pi/3$ (D) $\pi/2$

Sol. **[D]**

$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

Here $r_1 = \frac{1}{\sqrt{2}}$, $r_2 = \frac{1}{\sqrt{2}}$, $d = \sqrt{1} = 1$

$\cos\theta = \frac{\frac{1}{2} + \frac{1}{2} - 1}{2 \cdot \frac{1}{2}}$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$

Question based on

Radical axis & Radical centre

Q.30 The equation of the circle and its chord are respectively $x^2 + y^2 = a^2$ and $x \cos\alpha + y \sin\alpha = p$. The equation of the circle of which this chord is a diameter is -

(A) $x^2 + y^2 - 2px\cos\alpha - 2py\sin\alpha + 2p^2 - a^2 = 0$

(B) $x^2 + y^2 - 2px\cos\alpha - 2py\sin\alpha + p^2 - a^2 = 0$

(C) $x^2 + y^2 + 2px\cos\alpha + 2py\sin\alpha + 2p^2 - a^2 = 0$

(D) None of these

Sol. **[A]**

Required equation of circle is given by

$S + \lambda L = 0$

$\Rightarrow (x^2 + y^2 - a^2) + \lambda(x\cos\alpha + y\sin\alpha - p) = 0$ ……(1)

$\Rightarrow x^2 + y^2 + x\lambda\cos\alpha + y\lambda\sin\alpha - \lambda p - a^2 = 0$

its centre is $\left(-\frac{\lambda}{2}\cos\alpha, -\frac{\lambda}{2}\sin\alpha\right)$

it must lie on $x\cos\alpha + y\sin\alpha - p = 0$

so $\frac{-\lambda}{2}\cos^2\alpha - \frac{\lambda}{2}\sin^2\alpha - p = 0$

$\Rightarrow \frac{-\lambda}{2}(\cos^2\alpha + \sin^2\alpha) - p = 0$

$\Rightarrow \frac{-\lambda}{2} = p$

$\Rightarrow \lambda = -2p$

put $\lambda = -2p$ in (1), we get

$(x^2 + y^2 - a^2) - 2p(x\cos\alpha + y\sin\alpha - p) = 0$

$\Rightarrow x^2 + y^2 - a^2 - 2px\cos\alpha - 2py\sin\alpha + 2p^2 = 0$

$\Rightarrow x^2 + y^2 - 2px\cos\alpha - 2py\sin\alpha + 2p^2 - a^2 = 0$

Q.31 The radical axis of two circles-

(A) always intersects both the circles

(B) intersects only one circle

(C) bisects the line joining their centres

(D) bisects every common tangent to those circles

Sol. **[D]**

The radical axis of two circles bisects every common tangent to these circles

Q.32 The equation of the circle which pass through (2a, 0), whose radical axis in relation to the circle $x^2 + y^2 = a^2$ is $x = \frac{1}{2}a$ is -

(A) $x^2 + y^2 + 2ax = 0$ (B) $x^2 + y^2 - 2ax = 0$

(C) $x^2 + y^2 + ax = 0$ (D) $x^2 + y^2 - ax = 0$

Sol. **[B]**

$S + \lambda L = 0$

$(x^2 + y^2 - a^2) + \lambda(2x - a) = 0$

Since it passes through (2a, 0)

$\therefore 4a^2 - a^2 + \lambda(4a - a) = 0$

$\Rightarrow 3a^2 + 3a\lambda = 0 \Rightarrow \lambda = -\frac{3a^2}{3a} = -a$

$\therefore$ Reqd. circle is

$(x^2 + y^2 - a^2) - a(2x - a) = 0$

$\Rightarrow x^2 + y^2 - a^2 - 2xa + a^2 = 0 \Rightarrow x^2 + y^2 - 2ax = 0$

Q.33 The radical centre of the following set of circles $x^2 + y^2 - 16x + 60 = 0$, $x^2 + y^2 - 12x + 27 = 0$, and $x^2 + y^2 - 12y + 8 = 0$ is -

(A) $\left(\frac{33}{4}, \frac{20}{3}\right)$ (B) $\left(\frac{20}{3}, \frac{33}{4}\right)$

(C) $\left(\frac{31}{4}, \frac{20}{3}\right)$ (D) None of these

Sol. **[A]**

Radical centre will be the common point of

$S_1 - S_2 = 0$ & $S_2 - S_3 = 0$

$\therefore$ $S_1 - S_2 = 0$

$\Rightarrow -4x + 33 = 0$

$\Rightarrow$ $x = \frac{33}{4}$ …….(1)

Also $S_2 - S_3 = 0$

$\Rightarrow -12x + 12y + 19 = 0$

$\Rightarrow 12x - 12y - 19 = 0$ ……(2)

from (1) & (2)

$12 . \frac{33}{4} - 12y - 19 = 0$

$\Rightarrow$ $99 - 12y - 19 = 0$

$\Rightarrow$ $12y = 80$

$\Rightarrow$ $y = \frac{80}{12} = \frac{20}{3}$

$\left(\frac{33}{4}, \frac{20}{3}\right)$

Question based on Family of circles

Q.34 The equation of the circle passing through the origin and through the points of intersection of circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x - 4y + 4 = 0$ is-

(A) $x^2 + y^2 - x - 2y = 0$

(B) $x^2 + y^2 - 2x - 4y = 0$

(C) $x^2 + y^2 + x + 2y = 0$

(D) $x^2 + y^2 + 2x + 4y = 0$

Sol. **[A]**

Reqd. equation of circle

$S + \lambda S' = 0$

$\Rightarrow (x^2 + y^2 - 4) + \lambda(x^2 + y^2 - 2x - 4y + 4) = 0$ ……(1)

Since it passes through origin therefore,

$\Rightarrow -4 + \lambda (4) = 0$

$\Rightarrow \lambda = 1$

$\therefore$ from (1), we get

$\Rightarrow x^2 + y^2 - 4 + x^2 + y^2 - 2x - 4y + 4 = 0$

$\Rightarrow 2x^2 + 2y^2 - 2x - 4y = 0$

$\Rightarrow x^2 + y^2 - x - 2y = 0$

Q.35 A circle passes through the point of intersection of circles $x^2 + y^2 - 6x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 4y - 6 = 0$ and its centre lies on the line $y = x$. Its equation will be -

(A) $7(x^2 + y^2) - 10x - 10y - 12 = 0$

(B) $7(x^2 + y^2) - 10x - 10y - 1 = 0$

(C) $x^2 + y^2 - 10x - 10y - 12 = 0$

(D) None of these

Sol. **[A]**

Reqd. equation of circle

$S + \lambda s^1 = 0$

$(x^2 + y^2 - 6x + 2y + 4) + \lambda(x^2 + y^2 + 2x - 4y - 6) = 0$ ……(1)

$\Rightarrow x^2 (1 + \lambda) + y^2 (1 + \lambda) + x(2\lambda - 6) + y (2 - 4\lambda) + 4 - 6\lambda = 0$

$\Rightarrow x^2 + y^2 + \frac{(2\lambda - 6)}{1+\lambda}x + \frac{(2-4\lambda)}{1+\lambda}y + \frac{4-6\lambda}{1+\lambda} = 0$

Its centre lies on $y = x$

$\Rightarrow \frac{1-2\lambda}{1+\lambda} = \frac{\lambda - 3}{1+\lambda} \Rightarrow 3\lambda = 4$

$\lambda = \frac{4}{3}$

$\therefore$ from (1), we get

$(x^2 + y^2 - 6x + 2y + 4) + \frac{4}{3}(x^2 + y^2 + 2x - 4y - 6) = 0$

$\Rightarrow 3x^2 + 3y^2 - 18x + 6y + 12 + 4x^2 + 4y^2 + 6x - 16y - 24 = 0$

$\Rightarrow 7x^2 + 7y^2 - 10x - 10y - 12 = 0$

$\Rightarrow 7(x^2 + y^2) - 10x - 10y - 12 = 0$

Q.36 Equation of the circle whose radius is 5 and which touches externally the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5) is

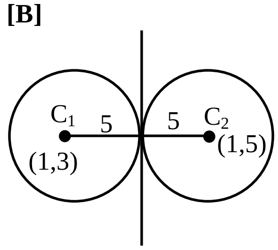
(A) $(x - 9)^2 + (y - 6)^2 = 5^2$

(B) $(x - 9)^2 + (y - 8)^2 = 5^2$

(C) $(x - 7)^2 + (y - 3)^2 = 5^2$

(D) None of these

Sol. **[B]**



P(5, 5) is the mid point of $C_1 C_2$

$\therefore$ $\frac{1+h}{2} = 5$, $\frac{2+k}{2} = 5$

$\Rightarrow h = 9$, $k = 8$

(9, 8)

$\therefore$ centre of reqd. circle is (9, 8), radius = 5

$\therefore$ Reqd. circle equation

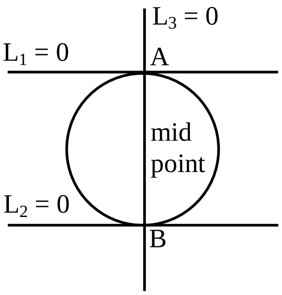
$(x - 9)^2 + (y - 8)^2 = 5^2$

$\Rightarrow (x - 9)^2 + (y - 8)^2 = 5^2$

True or false type questions

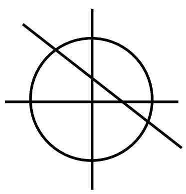
Q.37 If $L_1 = 0$ and $L_2 = 0$ are two parallel lines touching a circle and $L_3 = 0$ is a centre lines meeting $L_1 = 0$ & $L_2 = 0$ at A and B then mid point of A and B will be centre of circle.

Sol. **[True]**



Q.38 Line $x + y - 1 = 0$ intersects circle $x^2 + y^2 = 4$ at A and B then equation of circle passing through A, B and C (2, 1) is $2x^2 + 2y^2 - x - y - 7 = 0$.

Sol. **[True]**



Equation of the circle passing through the intersection point of

$$S \equiv x^2 + y^2 - 4 = 0$$

and $\quad L \equiv x + y - 1 = 0$

can be written as

$$S + \lambda L = 0 \text{ where } x \in R$$

i.e. $(x^2 + y^2 - 4) + \lambda (x + y - 1) = 0 \quad \ldots\ldots\ldots(*)$

$\because$ (2, 1) lies on the required circle, it satisfies equation (*)

$$\therefore \; 1 + \lambda 2 = 0$$

$$\lambda = -\frac{1}{2}$$

$\therefore$ required circle is

$2x^2 + 2y^2 - x - y - 7 = 0$

Q.39 Equation of circle touching line $2x + y = 3$ at (1, 1) and also passing through point (2, 3) is $x^2 + y^2 - \frac{9}{2}x - \frac{13}{4}y + \frac{23}{4} = 0$.

Sol. **[True]**

Equation of the circle touching the line $2x + y = 3$ at (1, 1) can be written as

$(x - 1)^2 + (y - 1)^2 + \lambda (2x + y - 3) = 0, \lambda \in R$

$\because$ the required circle has (2, 3) on it,

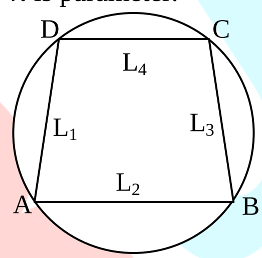
$$1 + 4 + \lambda (4 + 3 - 3) = 0$$

$$\therefore \; \lambda = -\frac{5}{4}$$

$\therefore$ required circle is

$$x^2 + y^2 - \frac{9}{2}x - \frac{13}{4}y + \frac{23}{4} = 0$$

Q.40 The equation of curve of 2nd degree circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ is given by $L_1L_3 + \lambda L_2L_4 = 0$, λ is parameter.



Sol.

Let ABCD is a quad. having sides

$(L_1, L_2, L_3, L_4) = 0$

then, equation of circumcircle of ABCD is which is equation of curve of 2^{nd} degree circumscribing quad. is

$$L_1 L_3 + \lambda(L_2L_4) = 0$$

λ can be obtained by using

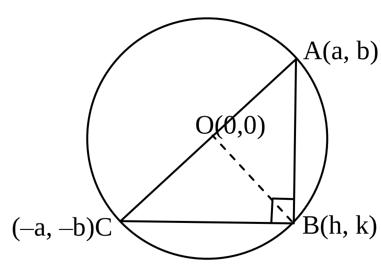
coefficient of x^2 = coefficient of y^2

coefficient of xy = 0

➢ Fill in the blanks type questions

Q.41 An isosceles right angle triangle is inscribed in the circle $x^2 + y^2 = r^2$. If the co-ordinate of an end of the hypotenuse are (a, b) then co-ordinates of the vertex are ....

Sol.



ΔABC is right angled

$x^2 + y^2 = r^2$

AC is diameter

Let third side be B(h, k)

OA = OB

$\sqrt{a^2+b^2} = \sqrt{h^2+k^2}$

$h^2 + k^2 = a^2 + b^2$ .....(1)

join OB

OB = OC

$\angle OAB = \angle OBA = 45^o$.

Hence $\angle AOB = 90^o$.

$OB \perp OA$

$m_1m_2 = -1$

$\frac{k}{h} \times \frac{b}{a} = -1$

$k = -\frac{ah}{b}$ ....(2)

Put (2) and (1)

$h^2 + \frac{a^2h^2}{b^2} = a^2 + b^2$

$h^2 = b^2$

$h = \pm b$

$k = \mu a$

Q.42 The length of common chord of the circles $(x-1)^2 + (y+1)^2 = c^2$ and $(x+1)^2 + (y-1)^2 = c^2$ is ....

Sol. $S_1 : (x-1)^2 + (y+1)^2 = c^2$

$\Rightarrow S_1 : x^2 + y^2 - 2x + 2y + 2 - c^2 = 0$

$S_2 : (x+1)^2 + (y-1)^2 = c^2$

$\Rightarrow S_2 : (x+1)^2 + (y-1)^2 = c^2$

$x^2 + y^2 + 2x - 2y + 2 - c^2 = 0$

$\therefore$ common chord $S_1 - S_2 = 0$

$\Rightarrow$ common chord $S_1 - S_2 = 0$

$\Rightarrow -4x + 4y = 0$

$\Rightarrow x - y = 0$

Length of perpendicular from (1,–1) to x – y = 0

$P = \frac{1+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$ & r = c

$\therefore$ length of chord $= 2\sqrt{r^2 - p^2}$

$= 2\sqrt{c^2-2}$

$= 2\sqrt{c^2-2}$

EXERCISE # 2

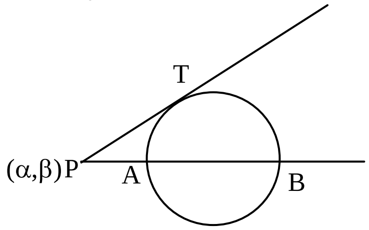
Part-A **Only single correct answer type questions**

Q.1 If a line is drawn through a fixed point P(α, β) to cut the circle $x^2 + y^2 = a^2$ at A and B, then PA . PB =

(A) $|\alpha^2 + \beta^2|$ (B) $|\alpha^2 + \beta^2 - a^2|$

(C) $|a^2|$ (D) $|\alpha^2 + \beta^2 + a^2|$

Sol. **[B]** Let $S = x^2 + y^2 - a^2 = 0$



$\therefore$ PA.PB = PT^2 where PT is length of tangent

$\therefore$ PA.PB = $\left(\sqrt{s_1}\right)^2$

$\therefore$ PA.PB = s_1

$\therefore$ PA.PB = $\alpha^2 + \beta^2 - a^2$

Q.2 α, β and γ are the parametric angles of three points P, Q and R respectively on the circle $x^2 + y^2 = 1$ and A is the point (–1, 0). If the lengths of the chords AP, AQ and AR are in G.P., then $\cos\frac{\alpha}{2}$, $\cos\frac{\beta}{2}$ and $\cos\frac{\gamma}{2}$ are in -

(where α, β, γ ∈ (0, π))

(A) A.P. (B) G.P. (C) H.P. (D) None

Sol. **[B]**

Coordinates of P, Q, R are (cosα, sinα), (cosβ, sinβ) and (cosγ, sinγ) respecively.

Also, A ≡ (–1, 0)

$\therefore$ AP = $\sqrt{(1+\cos\alpha)^2 + \sin^2\alpha}$

= 2 cos α/2

AQ = $\sqrt{(1+\cos\beta)^2 + \sin^2\beta}$

= 2 cos β/2

AR = $\sqrt{(1+\cos\gamma)^2 + \sin^2\gamma}$

= 2 cos γ/2

$\therefore$ AP, AQ, AP are in G.P. then

$\cos\frac{\alpha}{2}$, $\cos\frac{\beta}{2}$, $\cos\frac{\gamma}{2}$ are also in G.P.

Q.3 Let φ (x , y) = 0 be the equation of a circle. If φ (0, λ) = 0 has equal roots λ = 2, 2 and φ (λ, 0) = 0 has roots λ = $\frac{4}{5}$, 5 then the centre of the circle is –

(A) (2, 29/10) (B) (29/10, 2)

(C) (– 2, 29/10) (D) None of these

Sol. **[B]**

Let $\phi(x, y) \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore \phi(0, \lambda) = 0 + \lambda^2 + 0 + 2f\lambda + c = 0$

have equal roots.

then $2 + 2 = -\frac{2f}{1}$ and $2.2 = \frac{c}{1}$

$\therefore$ f = –2 & c = 4

and $\phi(l, 0) \equiv \lambda^2 + 0 + 2g\lambda + 0 + c = 0$

$\therefore \lambda^2 + 2g\lambda + 4 = 0$

Here c = 4 $\therefore \lambda^2 + 2g\lambda + 4 = 0$

have roots $\frac{4}{5}$, 5

$\therefore \frac{4}{5} + 5 = -2g \Rightarrow g = -\frac{29}{10}$

$\therefore$ centre ≡ (–g, – f)

$\equiv \left(\frac{29}{10}, 2\right)$

Q.4 The equation of the circle of radius 2√2 whose centre lies on the line x – y = 0 and which touches the line x + y = 4, and whose centre's coordinates satisfy the inequality x + y > 4 is

(A) $x^2 + y^2 - 8x - 8y + 24 = 0$

(B) $x^2 + y^2 = 8$

(C) $x^2 + y^2 - 8x - 8y = 24$

(D) None of these

Sol. **[A]**

Let centre ≡ (t, t)

$\therefore$ radius = $\left|\frac{t+t-4}{\sqrt{2}}\right| = 2\sqrt{2}$

so, |2t – 4| = 4

$\therefore$ 2t – 4 = ± 4

i.e. t = 4, 0

$\therefore$ centre = (4, 4); (0, 0)

$\therefore$ satisfying x + y > 4

$\therefore$ centre (4, 4), radius $2\sqrt{2}$

$\therefore$ equation of circle

$(x-4)^2 + (y-4)^2 = (2\sqrt{2})^2$

$\Rightarrow x^2 + y^2 - 8x - 8y + 16 + 16 = 8$

$\Rightarrow x^2 + y^2 - 8x - 8y + 24 = 0$

Q.5 The values of p for which the power of a point (2, 5) is negative with respect to circle $x^2 + y^2 - 8x - 12y + p = 0$ which neither touches the axis nor cuts them are

(A) $p \in (1, 2)$ (B) $p \in (13, 27)$

(C) $p \in (36, 47)$ (D) $p \in (49, 52)$

Sol. **[C]**

We have $x^2 + y^2 - 8x + 12y + p = 0$

Then centre and radius of the circle are (4,6) and $\sqrt{52-p}$ respectively.

$\therefore$ circle neither cuts nor touches any one of the axes of co-ordinate of centre > radius then

x coordinate of centre > radius

i.e. $4 > \sqrt{52-p}$

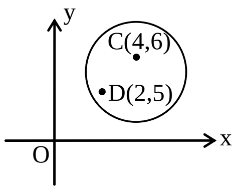
$\Rightarrow p > 36$ ....(i)

and y-co-ordinate of centre > radius

$6 > \sqrt{52-p}$

$\Rightarrow p > 16$ ....(ii)

$\therefore$



$\therefore$ D is interior point of the circle then

CD < radius $\sqrt{5} < \sqrt{52-p}$

$\Rightarrow p < 47$ ....(iii)

From (i), (ii), (iii) we obtain

$36 < p < 47$

$\therefore p \in (36, 47)$

Q.6 C has two values C_1 and C_2 for which $y = 2x + C$ touches a circle $x^2 + y^2 - 4x - 4y - 5 = 0$ then $C_1 + C_2$ is equal to-

(A) $2\sqrt{65}$ (B) 4 (C) – 4 (D) $-2\sqrt{65}$

Sol. **[C]**

y = 2x + C touches the circle

$x^2 + y^2 - 4x - 4y - 5 = 0$

$x^2 + (2x + C)^2 - 4x - 4(2x + C) - 5 = 0$

$\therefore x^2 + 4x^2 + C^2 + 4xC - 4x - 8x - 4C - 5 = 0$

$\Rightarrow x^2 + 4x^2 + C^2 + 4xC - 4x - 8x - 4C - 5 = 0$

$\Rightarrow 5x^2 + x(4C - 12) - (4C + 5 - C^2) = 0$

$\therefore D = 0$

$(4C - 12)^2 - 4.5(C^2 - 4C - 5) = 0$

$\Rightarrow 16C^2 + 144 - 96C - 20C^2 + 80C + 100 = 0$

$\Rightarrow -4C^2 - 16C + 244 = 0$

$\Rightarrow C^2 + 4C - 61 = 0$

$\therefore C_1$ & C_2 are roots of the equation

$\therefore C_1 + C_2 = -4$

Q.7 For what number of real values of λ line $\lambda x + \frac{y}{\lambda} + 1 = 0$ touches circle $x^2 + y^2 - 2x - 2y + 1 = 0$

(A) One (B) two (C) zero (D) four

Sol. **[C]**

Given circle is

$x^2 + y^2 - 2x - 2y + 1 = 0$

$\Rightarrow (x-1)^2 + (y-1)^2 = 1$

Whose centre is (1, 1) and radius = 1

If $\lambda x + \frac{y}{\lambda} + 1 = 0$ touches the circle, then length of perpendicular drawn from its centre to the line must be equal to its radius.

$$\Rightarrow \frac{\lambda + \frac{1}{\lambda} + 1}{\sqrt{\lambda^2 + \frac{1}{\lambda^2}}} = 1$$

$$\Rightarrow \left(\lambda + \frac{1}{\lambda}\right) + 1 = \sqrt{\lambda^2 + \frac{1}{\lambda^2}}$$

on squaring

$$\Rightarrow \left(\lambda + \frac{1}{\lambda}\right)^2 + 1 + 2\left(\lambda + \frac{1}{\lambda}\right) = \lambda^2 + \frac{1}{\lambda^2}$$

$$\Rightarrow \lambda^2 + \frac{1}{\lambda^2} + 2 + 1 + 2\left(\frac{\lambda^2 + 1}{\lambda}\right) = \lambda^2 + \frac{1}{\lambda^2}$$

$\Rightarrow 3\lambda + 2\lambda^2 + 2 = 0$

$\Rightarrow 2\lambda^2 + 3\lambda + 2 = 0$

its D < 0 since $b^2 - 4ac = 9 - 4 \times 2 \times 2 = -7 < 0$

$\therefore$ There is no real value of λ possible

Q.8 The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + \alpha = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + \beta = 0$ is

(A) $\sqrt{\beta - \alpha}$ (B) $\sqrt{\alpha - \beta}$

(C) $\sqrt{\alpha\beta}$ (D) $\sqrt{\alpha / \beta}$

Sol. **[A]**

Let (x_1, y_1) is the point on the given circle

$x^2 + y^2 + 2gx + 2fy + \alpha = 0$

$\therefore (x_1, y_1)$ satisfies the circle

$\therefore x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + \alpha = 0$

$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = -\alpha$ ......(i)

$\therefore$ Length of tangent from (x_1,y_1) to the circle $x^2 + y^2 + 2gx + 2fy + \beta = 0$

is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + \beta}$

$= \sqrt{-\alpha + \beta} = \sqrt{\beta - \alpha}$

Q.9 If the chord of contact of the pair of tangents from P to the circle $x^2 + y^2 = a^2$ which touches the circle $x^2 + y^2 - 2ax = 0$, then the locus of P is -

(A) $y^2 = a(a - 2x)$ (B) $x^2 + y^2 = (x + a)^2$

(C) $x^2 = a(a - 2y)$ (D) None of these

Sol. **[A]**

Let P(h, k) be the point of intersection of the tangents at the extrmities of the chord AB of the circle $x^2 + y^2 = a^2$. since AB is the chord of contact of the tangents from P to this circle, its equation is $hx + ky = a^2$. If this line touches the circle $x^2 + y^2 - 2ax = 0$, then

$$\frac{h.a + k.0 - a^2}{\sqrt{h^2 + k^2}} = \pm a$$

$\Rightarrow (h-a)^2 = h^2 + k^2$

$\therefore$ locus of (h, k) is

$(x - a)^2 = x^2 + y^2$ or $y^2 = a(a - 2x)$

Q.10 The lengths of the tangents from any point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$ to the two circles $5x^2 + 5y^2 - 24x + 32y + 75 = 0$ and $5x^2 + 5y^2 - 48x + 64y + 300 = 0$ are in the ratio

(A) 1 : 2 (B) 2 : 3 (C) 3 : 4 (D) None

Sol. **[A]**

Since (0, 0) lies on the circle

$15x^2 + 15y^2 - 48x + 64y = 0$

$\therefore$ length of tangent from (0, 0) to first circle is

$\lambda_1 = \sqrt{75}$

and lengths of tangent from (0,0) to the second circle

$\lambda_2 = \sqrt{300}$

$\therefore \frac{\lambda_1}{\lambda_2} = \frac{\sqrt{75}}{\sqrt{300}}$

$\therefore \frac{\lambda_1}{\lambda_2} = \frac{5\sqrt{3}}{10\sqrt{3}}$

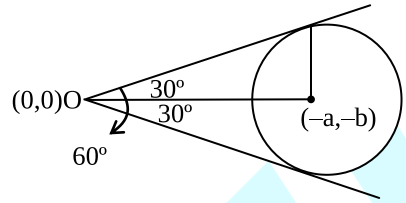
$= \frac{1}{2}$

$\therefore \lambda_1 : \lambda_2 = 1 : 2$

Q.11 If circle $x^2 + y^2 + 2ax + 2by + 5 = 0$ subtends angle 60º at origin then (a, b) lies on circle whose radius is equal to -

(A) $\sqrt{20}$ (B) $\sqrt{\frac{15}{2}}$ (C) $\sqrt{\frac{20}{3}}$ (D) $\sqrt{\frac{5}{2}}$

Sol. **[C]**



We know that $\tan\frac{\theta}{2} = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{s_1}}$

$\therefore \tan 30º = \frac{\sqrt{a^2 + b^2 - 5}}{\sqrt{5}}$

$\therefore \frac{1}{\sqrt{3}} = \frac{\sqrt{a^2 + b^2 - 5}}{\sqrt{5}}$

$\therefore$ on squaring

$\frac{1}{3} = \frac{a^2 + b^2 - 5}{5} \Rightarrow a^2 + b^2 - 5 = \frac{5}{3}$

$\Rightarrow a^2 + b^2 = 5 + \frac{5}{3} \Rightarrow a^2 + b^2 = \frac{20}{3}$

$\therefore$ (a, b) lies on circle $x^2 + y^2 = r^2$

$\therefore r^2 = \frac{20}{3}$

$\therefore r = \sqrt{\frac{20}{3}}$

Q.12 Locus of mid point of chord AB of a circle $x^2 + y^2 = 4$ which subtends 90º angle at (1, 0) is-

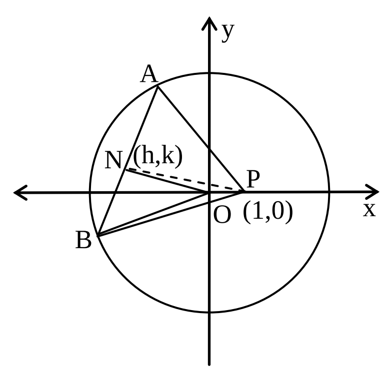
(A) $x^2 + y^2 + y + 3 = 0$

(B) $x^2 + y^2 - x - 3/2 = 0$

(C) $x^2 + y^2 - y - 3/2 = 0$

(D) none of these

Sol. **[B]**



Let N (h, k) be the middle point of chord AB which subtends 90º angle at (1, 0)

since $\angle$ APB = 90º

$\therefore$ NA = NB = NP

(since distance of the vertices from mid-point of the hypotenuse are equal)

or $(NA)^2 = (NB)^2 = (h-1)^2 + (k-0)^2$ ....(i)

But also $\angle BNO = 90^o$

$\therefore (OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow -(NB)^2 = (ON)^2 - (OB)^2$

$\Rightarrow -[(h-1)^2 + (k-0)^2] = (h^2 + k^2) - 4$

or $2(h^2 + k^2) - 2h + 1 - 4 = 0$

or $h^2 + k^2 - h - 3/2 = 0$

$\therefore$ locus is

$x^2 + y^2 - x - 3/2 = 0$

Q.13 The locus of a point such that the tangents drawn from it to the circle $x^2 + y^2 - 6x - 8y = 0$ are perpendicular to each other is

(A) $x^2 + y^2 - 6x - 8y - 25 = 0$

(B) $x^2 + y^2 + 6x - 8y - 5 = 0$

(C) $x^2 + y^2 - 6x + 8y - 5 = 0$

(D) $x^2 + y^2 - 6x - 8y + 25 = 0$

Sol. **[A]**

Given circle is

$(x-3)^2 + (y-4)^2 = 25$

since locus of point of intersection of two perpendicular tangents is director circle, then its equation is

$(x-3)^2 + (y-4)^2 = 50$

$\Rightarrow x^2 + y^2 - 6x - 8y - 25 = 0$

Q.14 Two lines $\lambda_1 x + m_1 y + n_1 = 0$ and $\lambda_2 x + m_2 y + n_2 = 0$ are conjugate lines with respect to the circle $x^2 + y^2 = a^2$ if -

(A) $\lambda_1\lambda_2 + m_1m_2 = n_1n_2$

(B) $\lambda_1\lambda_2 + m_1m_2 + n_1n_2 = 0$

(C) $a^2(\lambda_1\lambda_2 + m_1m_2) = n_1n_2$

(D) $\lambda_1\lambda_2 + m_1m_2 = a^2 n_1n_2$

Sol. **[C]**

Since $\lambda_1 x + m_1 y + n_1 = 0$ and $\lambda_2 x + m_2 y + n_2 = 0$ are conjugate lines.

let (x_1, y_1) be the polar then its polar is

$xx_1 + yy_1 = a^2 \Rightarrow xx_1 + yy_1 - a^2 = 0$

and $\lambda_1 x + m_1 y + n_1 = 0$

$$\frac{x_1}{\lambda_1} = \frac{y_1}{m_1} = \frac{-a^2}{n_1}$$

$$\therefore x_1 = -\frac{a^2\lambda_1}{n_1};\ y_1 = \frac{-a^2 m_1^2}{n_1}$$

$\therefore (x_1, y_1) \equiv \left(\frac{-a^2\lambda_1}{n_1}, \frac{-a^2 m_1}{n_1}\right)$ lies on $\lambda_2 x + m_2 y + n_2 = 0$

$$\Rightarrow \lambda_2\left(-\frac{a^2\lambda_1}{n_1}\right) + m_2\left(-\frac{a^2 m_1}{n_1}\right) + n_2 = 0$$

$$\Rightarrow -\frac{a^2\lambda_1\lambda_2}{n_1} - \frac{a^2 m_1 m_2}{n_1} + n_2 = 0$$

$\Rightarrow -a^2\lambda_1\lambda_2 - a^2 m_1m_2 + n_1n_2 = 0$

$\Rightarrow -a^2(\lambda_1\lambda_2 + m_1m_2) + n_1n_2) = 0$

$\Rightarrow -a^2(\lambda_1\lambda_2 + m_1m_2) = -n_1n_2$

$\Rightarrow a^2(\lambda_1\lambda_2 + m_1m_2) = n_1n_2$

Q.15 The Locus of the middle point of chords of the circle $x^2 + y^2 = a^2$ which passes through the fixed point (h, k) is .

(A) $x^2 + y^2 - hx - ky = 0$

(B) $x^2 + y^2 + hx + ky = 0$

(C) $x^2 + y^2 - 2hx - 2ky = 0$

(D) None of these

Sol. **[A]**

Let mid-point of circle is (x_1, y_1)

$\therefore T = S_1$

$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$

$\Rightarrow xx_1 + yy_1 = x_1^2 + y_1^2 \Rightarrow x_1^2 + y_1^2 - xx_1 - yy_1 = 0$

since it passes (h, k)

$\therefore x_1^2 + y_1^2 - hx_1 - ky_1 = 0$

$\therefore$ locus of (x_1, y_1) is given by

$x^2 + y^2 - hx - ky = 0$

Q.16 If the circle $x^2 + y^2 + 4x + 22y + c = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - d = 0$, then c + d =

(A) 60 (B) 50 (C) 40 (D) 56

Sol. **[B]**

$S_1 \equiv x^2 + y^2 + 4x + 22y + c = 0$

$S_2 \equiv x^2 + y^2 - 2x + 8y - d = 0$

$\therefore S_1 - S_2 = 0$

$\Rightarrow 6x + 14y + (c + d) = 0$

since it passes the centre of S_2 i.e. (1, – 4)

$\Rightarrow 6 - 56 + (c + d) = 0$

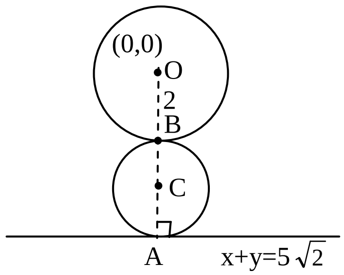
$\Rightarrow c + d = 50$

Q.17 The equation of a circle is $x^2 + y^2 = 4$. The centre of the smallest circle touching this circle and the line $x + y = 5\sqrt{2}$ has the coordinates

(A) $\left(\frac{7}{2\sqrt{2}}, \frac{7}{2\sqrt{2}}\right)$ (B) $\left(\frac{3}{2}, \frac{3}{2}\right)$

(C) $\left(-\frac{7}{2\sqrt{2}}, -\frac{7}{2\sqrt{2}}\right)$ (D) None of these

Sol. **[A]**



Here OB = radius = 2

The distance of (0, 0) from $x + y = 5\sqrt{2}$ is 5

$\therefore$ The radius of the smallest circle = $\frac{5-2}{2} = \frac{3}{2}$

and OC = $2 + \frac{3}{2} = \frac{7}{2}$

The slope of OA = 1 = $\tan\theta$

$\therefore \cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$

$\therefore$ C = (0 + OC. $\cos\theta$, 0+ OC $\sin\theta$)

$= \left(0+\frac{7}{2}.\frac{1}{\sqrt{2}}, 0+\frac{7}{2}.\frac{1}{\sqrt{2}}\right)$

$= \left(\frac{7}{2\sqrt{2}}, \frac{7}{2\sqrt{2}}\right)$

Q.18 The circle $x^2 + y^2 - 2px = a^2 - p^2$ lies within the circle $x^2 + y^2 - 2qx = b^2 - q^2$ if

(A) $p^2 + q^2 - a^2 - b^2 < 2(pq - ab)$

(B) $p^2 + q^2 - a^2 - b^2 > 2(pq - ab)$

(C) $p^2 + q^2 - a^2 - b^2 < 2(pq + ab)$

(D) None of these

Sol. **[A]**

$S_1 \equiv x^2 + y^2 - 2px - a^2 + p^2 = 0$ ....(i)

$S_2 \equiv x^2 + y^2 - 2qx - b^2 + q^2 = 0$ ....(ii)

$C_1 : (p, 0)$ $r_1 : \sqrt{p^2 + a^2 - p^2} = a$

$C_2 : (q, 0)$ $r_2 : \sqrt{q^2 + b^2 - q^2} = b$

$C_1C_2 = \sqrt{(p-q)^2} = p - q$

$r_1 - r_2 = a - b = a - b$

since one circle lies within the other circle

$\Rightarrow C_1C_2 < |r_1 - r_2| \Rightarrow (C_1C_2)^2 < |r_1 - r_2|^2$

$\Rightarrow (p - q)^2 < (a - b)^2$

$\Rightarrow p^2 + q^2 - 2pq < a^2 + b^2 - 2ab$

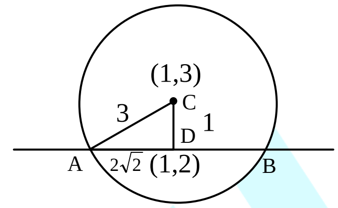
$\Rightarrow p^2 + q^2 - a^2 - b^2 < 2pq - 2ab$

$\Rightarrow p^2 + q^2 - a^2 - b^2 < 2(pq - ab)$

Q.19 For the circle $x^2 + y^2 - 2x - 6y + 1 = 0$ the chord of minimum length and passing through (1, 2) is of length-

(A) $2\sqrt{2}$ (B) $4\sqrt{2}$ (C) $6\sqrt{2}$ (D) $8\sqrt{2}$

Sol. **[B]**



Minimum length of chord = AB

= 2AD

= $2.2\sqrt{2}$

= $4\sqrt{2}$

Q.20 The equation of a chord of the circle $x^2 + y^2 + 4x - 6y = 0$ is given by $x + 2y = 0$. The equation of the circle described on this chord as diameter is

(A) $5(x^2 + y^2) + 28x + 14y = 0$

(B) $5(x^2 + y^2) + 28x - 14y = 0$

(C) $x^2 + y^2 + 28x - 14y = 0$

(D) $5(x^2 + y^2) - 28x - 14y = 0$

Sol. **[B]**

We have

$S + \lambda L = 0$

$(x^2 + y^2 + 4x - 6y) + \lambda(x + 2y) = 0$

$\Rightarrow x^2 + y^2 + x(4 + \lambda) + y(2\lambda - 6) = 0$

its centre is $\left(-\frac{4+\lambda}{2}, -(\lambda-3)\right)$

$\equiv \left(-\frac{\lambda+4}{2}, 3-\lambda\right)$

This centre must lie on $x + 2y = 0$

$\Rightarrow -\frac{\lambda+4}{2} + 2(3 - \lambda) = 0$

$\Rightarrow -(\lambda + 4) + 4(3 - \lambda) = 0$

$\Rightarrow -\lambda - 4 + 12 - 4\lambda = 0$

$\Rightarrow -5\lambda = -8$

$\Rightarrow \lambda = 8/5$

$\therefore$ Reqd. circle is

$(x^2 + y^2 + 4x - 6y) + \frac{8}{5}(x + 2y) = 0$

$5x^2 + 5y^2 + 20x - 30y + 8x + 16y = 0$

$5x^2 + 5y^2 + 28x - 14y = 0$

$5(x^2 + y^2) + 28x - 14y = 0$

Q.21 The common chord of the circle $x^2 + y^2 + 6x + 8y - 7 = 0$ and a circle passing through the origin, and touching the line y = x, always passes through the point

(A) (– 1/2, 1/2) (B) (1, 1)

(C) (1/2, 1/2) (D) None of these

Sol. **[C]**

Let the second circle be

$x^2 + y^2 + 2gx + 2fy = 0$

The common chord has the equation

$2(g - 3)x + 2(f - 4)y + 7 = 0$

But y =x touches the circle

$\therefore x^2 + x^2 + 2gx + 2fx = 0$ has equal roots

i.e. $f + g = 0$

$\therefore$ the equation of the common chord is

$2(g - 3)x + 2(f - 4)y + 7 = 0$

or $(-6x - 8y + 7) + g(2x - 2y) = 0$

which passes through the point of intersection of $-6x - 8y + 7 = 0$ and $2x - 2y = 0$

$\therefore 6x + 8y - 7 = 0$ & $x = y$

$6x + 8y - 7 = 0$

$\Rightarrow 14x = 7$

$$x = \frac{7}{14} = \frac{1}{2}$$

$$\therefore x = y = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Q.22 The members of a family of circles are given by the equation $2(x^2 + y^2) + \lambda x - (1 + \lambda^2)y - 10 = 0$. The number of circles belonging to the family that are cut orthogonally by the fixed circle $x^2 + y^2 + 4x + 6y + 3 = 0$ is

(A) 2 (B) 1

(C) 0 (D) None of these

Sol. **[A]**

$$x^2 + y^2 + \frac{\lambda}{2}x - \frac{1+\lambda^2}{2}y - 5 = 0$$

and $x^2 + y^2 + 4x + 6y + 3 = 0$

cut orthogonally; if

$$2 . \frac{\lambda}{4} . 2 + 2 . \left(-\frac{1+\lambda^2}{4}\right) . 3 = -5 + 3$$

$$\Rightarrow \lambda - \frac{3}{2}(1 + \lambda^2) = -2$$

$\Rightarrow 2\lambda - 3 - 3\lambda^2 = -4$

$\Rightarrow 3\lambda^2 - 2\lambda - 1 = 0$

$\Rightarrow 3\lambda^2 - 3\lambda + \lambda - 1 = 0$

$\Rightarrow 3\lambda(\lambda - 1) + 1(\lambda - 1) = 0$

$\Rightarrow (\lambda - 1)(3\lambda + 1) = 0$

$$\lambda = 1, -\frac{1}{3}$$

$\therefore$ Two real values of λ

Hence There are two circles belonging to this family.

Q.23 The circle passing through three distinct points (1, t), (t, 1) and (t, t) passes through the point

(A) (1, 1) (B) (–1, –1)

(C) (–1, 1) (D) (1, –1)

Sol. **[A]**

Let the equation of circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

It is passing through (1, t), (t, 1) and (t,t)

than $1 + t^2 + 2g + 2ft + c = 0$ .....(i)

$t^2 + 1 + 2gt + 2f + c = 0$ ......(ii)

$2t^2 + 2gt + 2ft + c = 0$ ....(iii)

on solving (i), (ii) & (iii), we get

$2g(t - 1) + 2f(1 - t) = 0$

or $g - f = 0$ and $t^2 - 1 + 2f(t - 1) = 0$

$$\therefore f = -\frac{(t+1)}{2} = g$$

from (iii), $2t^2 - t(t+1) - t(t+1) + c = 0$ [$\therefore c = 2t$]

$\therefore x^2 + y^2 - (t + 1)x - (t + 1)y + 2t = 0$

$\Rightarrow (x^2 + y^2 - x - y) - t(x + y - 2) = 0$

$\therefore P + \lambda Q = 0$ $\therefore P = 0$ and $Q = 0$

Then $x^2 + y^2 - x - y = 0$ ....(iv)

$x + y - 2 = 0$ ......(v)

form (iv) & (v)

$x = 1$ and $y = 1$

$\therefore$ (1, 1)

Q.24 If the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x + 2y + 1 = 0$, then

(A) $g = 3/4$ and $f \neq 2$ (B) $g \neq 3/4$ and $f = 2$

(C) $g = 3/4$ or $f = 2$ (D) None of these

Sol. **[C]**

The given circles are

$S_1 : x^2 + y^2 + 2gx + 2fy + c = 0$

$$S_2 : x^2 + y^2 + \frac{3}{2}x + 4y + c = 0$$

The equation of the radical axis of the two circles is

$S_1 - S_2 = 0$

$$\text{i.e. } \left(2g - \frac{3}{2}\right)x + (2f - 4)y = 0$$

i.e. $(4g - 3)x + (4f - 8)y = 0$

since it touches the circle $x^2+y^2+2x + 2y + 1 = 0$

$S_1 - S_2 = 0$

$$\therefore \left|\frac{(4g-3)(-1)+(4f-8)(-1)}{\sqrt{(4g-3)^2+(4f-8)^2}}\right| = 1$$

$\Rightarrow (4g-3)^2 + (4f-8)^2 + 2(4g-3)(4f-8)$

$= (4g-3)^2 + (4f-8)^2$

$\Rightarrow (4g-3)(4f-8) = 0 \Rightarrow g = 3/4$ or $f = 2$

Q.25 The coordinates of the radical centre of the three circles $x^2 + y^2 = 9$, $x^2 + y^2 - 2x - 2y = 5$ and $x^2 + y^2 + 4x + 6y = 19$ are

(A) (– 1, 1) (B) (1, – 1)

(C) (1, 1) (D) (0, 0)

Sol. **[C]**

$S_1 = x^2 + y^2 - 9 = 0$

$S_2 = x^2 + y^2 - 2x - 2y - 5 = 0$

$S_3 = x^2 + y^2 + 4x + 6y - 19 = 0$

$S_1 - S_2 = 0$ & $S_2 - S_3 = 0 \Rightarrow 2x + 2y - 4 = 0$

$\Rightarrow x + y - 2 = 0$ ....(1)

from $S_2 - S_3$,

$-6x - 8y + 14 = 0$

$\Rightarrow 3x + 4y - 7 = 0$ ....(ii)

from (i) & (ii) we get

$3x + 3y - 6 = 0$

$3x + 4y - 7 = 0$

$-\quad -\quad +$

$-y + 1 = 0$

$y = 1$

$\therefore x = 1$

$\therefore$ Radical centre is (1, 1)

Q.26 The radical centre of three circles described on the three sides of a triangle as diameter is the

(A) orthocentre (B) circumcentre

(C) incentre (D) centroid

Sol. **[A]**

The radical centre of three circles described on the three sides of a triangle as diameter is the orthocentre of the triangle.

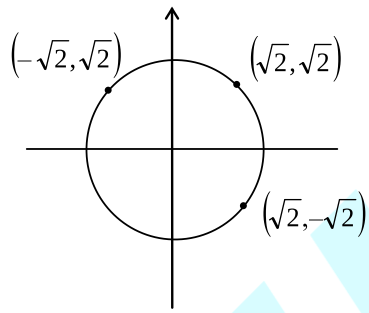
Part-B One or more than one correct answer type questions

Q.27 P($\sqrt{2}$, $\sqrt{2}$) is a point on the circle $x^2 + y^2 = 4$ and Q is another point on the circle such that arc PQ = $\frac{1}{4}$ × circumference. The coordinates of Q are

(A) ($-\sqrt{2}$, $-\sqrt{2}$) (B) ($\sqrt{2}$, $-\sqrt{2}$)

(C) ($-\sqrt{2}$, $\sqrt{2}$) (D) None of these

Sol. **[B,C]**



Clearly from above diagram

Required co-ordinates of Q are given by

$(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$

Q.28 If A and B are two points on the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ which are farthest and nearest respectively from the point (7, 2) then

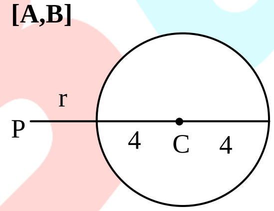
(A) A = $(2 - 2\sqrt{2}, -3 - 2\sqrt{2})$

(B) B = $(2 + 2\sqrt{2}, -3 + 2\sqrt{2})$

(C) A = $(2 + 2\sqrt{2}, -3 + 2\sqrt{2})$

(D) B = $(2 - 2\sqrt{2}, -3 + 2\sqrt{2})$

Sol. **[A,B]**



Slope of PC = $\frac{-3-2}{2-7} = \frac{-5}{-5} = 1$

If $\tan\theta = 1$, Θ $\theta = 45^o$

Equation of PA is

$$\frac{x-7}{1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = r$$

$\therefore \left(7+\frac{r}{\sqrt{2}}, 2+\frac{r}{\sqrt{2}}\right)$ lie on circle then

$$\left(7+\frac{r}{\sqrt{2}}\right)^2 + \left(2+\frac{r}{\sqrt{2}}\right)^2 - 4\left(7+\frac{r}{\sqrt{2}}\right) + 6\left(2+\frac{r}{\sqrt{2}}\right) - 3 = 0$$

$\Rightarrow r^2 + 10\sqrt{2}\, r + 34 = 0$

$\therefore r = -5\sqrt{2} \pm 4$

$\therefore$ points are $\left(7+\frac{-5\sqrt{2}\pm 4}{\sqrt{2}}, 2+\frac{-5\sqrt{2}\pm 4}{\sqrt{2}}\right)$

$\Rightarrow (2 \pm 2\sqrt{2}, -3 + 2\sqrt{2})$

$\therefore$ B $(2 + 2\sqrt{2}, -3 + 3\sqrt{2})$

A $(2 - 2\sqrt{2}, -3 - 2\sqrt{2})$

Q.29 If (a cos θ_i, a sin θ_i) i = 1, 2, 3 represent the vertices of an equilateral triangle inscribed in a circle, then

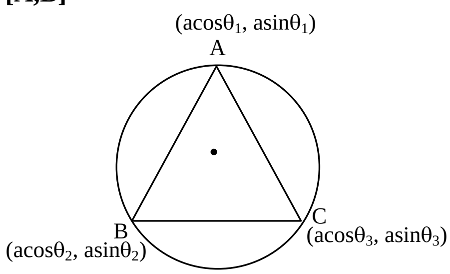
(A) $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$

(B) $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0$

(C) $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = 0$

(D) $\cot\theta_1 + \cot\theta_2 + \cot\theta_3 = 0$

Sol. **[A,B]**



ABC is an equilateral triangel

Since ($a\cos\theta_i$; $a\sin\theta_i$) lie on the circle whose equation is $x^2 + y^2 = r^2$ whose centre is (0, 0) and radius = r

since centre of circle and centroid of ΔABC are the same point

$\therefore$ centroid of ΔABC is given by

$$\frac{a\cos\theta_1 + a\cos\theta_2 + a\cos\theta_3}{3} = 0$$

and $\dfrac{a\sin\theta_1 + a\sin\theta_2 + a\sin\theta_3}{3} = 0$

$\therefore \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$

and $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0$

Q.30 If the circle $x^2 + y^2 + ax + by + c = 0$ meets the axis of x at (α, 0) and (β, 0) such that $\alpha + \beta = \alpha\beta$ then (α, β) are the roots of the equation.

(A) $x^2 + ax - a = 0$ (B) $x^2 - cx + c = 0$

(C) $x^2 + bx - b = 0$ (D) $x^2 - bx + b = 0$

Sol. **[A,B]**

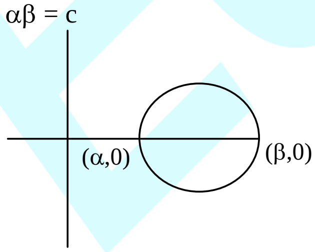
$x^2 + y^2 + ax + by + c = 0$ meets the x-axis

$\therefore x^2 + ax + c = 0$

its roots are α & β

$\alpha + \beta = a$

$\alpha\beta = c$



$\therefore$ equation of whose roots are α, & β are

$x^2 - x(\alpha + \beta) + \alpha\beta = 0 \Rightarrow x^2 - x(-a) + c = 0$

$\Rightarrow x^2 + ax + c = 0 \;\; \Rightarrow x^2 + ax - a = 0$

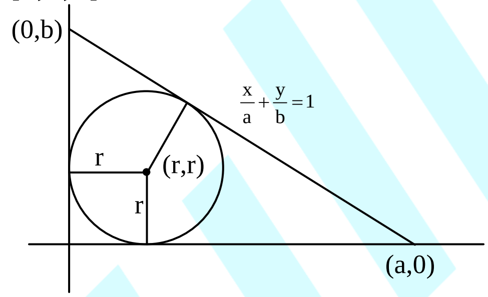
[$\therefore \alpha + \beta = \alpha\beta \;\therefore\; c = -a$] and $x^2 - cx + c = 0$

Q.31 Equations of a circle which touches the axes and x/a + y/b = 1, centre being in positive quadrant is $x^2 + y^2 - 2rx - 2ry + r^2 = 0$, where r =

(A) $\dfrac{a+b+\sqrt{a^2+b^2}}{2}$ (B) $\dfrac{a+b-\sqrt{a^2+b^2}}{2}$

(C) $\dfrac{ab}{a+b-\sqrt{a^2+b^2}}$ (D) $\dfrac{ab}{a+b+\sqrt{a^2+b^2}}$

Sol. **[B,C,D]**



Θ circle touches $\dfrac{x}{a} + \dfrac{y}{b} = 1$

$\therefore$ p = r

$$\therefore \frac{\left|\frac{r}{a} + \frac{r}{b} - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = r$$

$$\Rightarrow \left|\frac{r}{a} + \frac{r}{b} - 1\right| = r\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

$$\Rightarrow r\left(\frac{a+b}{ab}\right) \pm r\sqrt{\frac{a^2+b^2}{a^2b^2}} = 1$$

$$r\left(\frac{a+b}{ab}\right) \pm r\frac{\sqrt{a^2+b^2}}{ab} = 1$$

$$\Rightarrow r\,[(a+b) \pm \sqrt{a^2+b^2}\,] = ab$$

$$\Rightarrow r = \frac{ab}{a+b \pm \sqrt{a^2+b^2}}$$

$$\Rightarrow r = \frac{ab}{a+b+\sqrt{a^2+b^2}} \;\;\&\;\; r = \frac{ab}{a+b-\sqrt{a^2+b^2}}$$

$$\Rightarrow r = \frac{a+b \mp \sqrt{a^2+b^2}}{(a+b)^2 - (a^2+b^2)} = \frac{a+b \mp \sqrt{a^2+b^2}}{2ab}$$

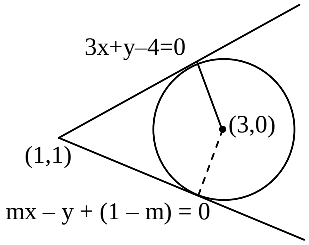
$$\Rightarrow r = \frac{a+b+\sqrt{a^2+b^2}}{2} \text{ or } \frac{a+b-\sqrt{a^2+b^2}}{2}$$

Q.32 The equation of one of the tangents from (1, 1) to a circle with its centre at (3, 0) is $3x + y - 4 = 0$. The equation of the other tangent is-

(A) $5x - y - 4 = 0$ (B) $3y - x - 2 = 0$

(C) $3y + x - 4 = 0$ (D) $3x - y - 2 = 0$

Sol. **[A,B,D]**



Let equation of tangent is

$y - 1 = m(n - 1)$

$y - 1 = mx - m$

$mx - y + (1 - m) = 0$

$\therefore \dfrac{3(3)+0-4}{\sqrt{9+1}} = \dfrac{3m-0+1(1-m)}{\sqrt{m^2+1}}$

$\Rightarrow \dfrac{5}{\sqrt{10}} = \dfrac{2m+1}{\sqrt{m^2+1}}$

$\Rightarrow 25(m^2 + 1) = (2m + 1)^2 .10$

$\Rightarrow 5m^2 + 5 = 8m^2 + 8m + 2$

$\Rightarrow 3m^2 + 9m - m - 3 = 0$

$\Rightarrow 3m(m + 3) - 1(m + 3) = 0$

$\Rightarrow (m + 3)(3m - 1) = 0$

$\Rightarrow m = -3, 1/3$

but $m \neq -3$

$\therefore m = \dfrac{1}{3}$

$\Rightarrow \dfrac{x}{3} - y + \left(1 - \dfrac{1}{3}\right) = 0$

$\Rightarrow \dfrac{x}{3} - y + \dfrac{2}{3} = 0$

$\Rightarrow x - 3y + 2 = 0$

$\Rightarrow 3y - x - 2 = 0$

Q.33 Equation of circles which pass through the point (1, –2) and (3, –4) and touch the x-axis is-

(A) $x^2 + y^2 + 6x + 2y + 9 = 0$

(B) $x^2 + y^2 + 10x + 20y + 25 = 0$

(C) $x^2 + y^2 - 6x + 4y + 9 = 0$

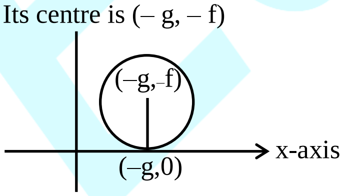
(D) None of these

Sol. Since required circles passes through (1, –2) and (3, – 4) and also touches x-axis

Let equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{.........(1)}$$

Its centre is (– g, – f)



$\because$ it touches x-axis at (–g, 0) so it is also passes (– g, 0)

Now we have to find equation of circle which passes the three point (1, –2), (3, –4) and (–g, 0)

for $(1, -2) \Rightarrow 1 + 4 + 2g - 4f + c = 0$

$\Rightarrow 2g - 4f + c + 5 = 0$ ....(i)

for $(3, -4) \Rightarrow 9 + 16 + 6g - 8f + c = 0$

$\Rightarrow 6g - 8f + c + 25 = 0$ ....(ii)

for $(-g, 0) \Rightarrow g^2 - 2g^2 + c = 0$

$\Rightarrow g^2 = c$ ....(iii)

from (i) & (ii) $\Rightarrow$

$$\begin{array}{r} 4g - 8f + 2c + 10 = 0 \\ 6g - 8f + c + 25 = 0 \\ - \quad + \quad - \quad - \\ \hline -2g + c - 15 = 0 \end{array}$$

$\Rightarrow 2g = c - 15$

(i) × 2

from (iii) $c = g^2$ putting in this equation

$g^2 - 2g - 15 = 0$

$\Rightarrow g^2 - 5g + 3g - 15 = 0$

$\Rightarrow g(g - 5) + 3(g - 5) = 0$

$\Rightarrow (g - 5)(g + 3) = 0$

$\Rightarrow g = -3, 5$

$\therefore$ C = 9, 25 when g = –3, 5 responsing

when g = –3, c = 9, we get

$\therefore 2(-3) - 4f + 9 + 5 = 0$

$\Rightarrow f = 2$

when g = 5, c = 25 we get

$10 - 4f + 30 = 0$

$f = 10$

$\therefore$ Reqd. equation of circle is given by

when g = – 3, f = 2, c = 9

$x^2 + y^2 - 6x + 4y + 9 = 0$ Ans.

find when g = 5, f = 10, c = 25

$x^2 + y^2 + 10x + 20y + 25 = 0$

Part-C Assertion-Reason type questions

The following questions 34 to 37 consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

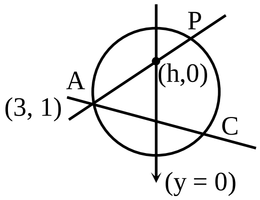
(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true

Q.34 **Assertion (A) :** Two distinct chords drawn from the point (3, 1) on the circle $x^2 + y^2 - 3x - y = 0$ are bisected by the x-axis.

Reason (R) : If point of bisection is (h, 0) then equation of chord given by $T = S_1$ passing through (3, 1) will be quadratic in h giving two distinct values of h.

Sol. **[A]**



$S \equiv x^2 + y^2 - 3x - y = 0$

Let the mid point on the chord be (h, 0), $h \in R$

then, equation of chord is given by

$$T = S_1$$

i.e., $hx + 0.y - \frac{3}{2}(x + h) - \frac{1}{2}(y + 0)$

$$= h^2 - 3h$$

i.e., $hx - \frac{3}{2}x + \frac{3}{2}h - \frac{y}{2} - h^2 = 0$

$\because$ it satisfies (3, 1),

$$3h - \frac{3}{2} + \frac{3}{2}h - \frac{1}{2} - h^2 = 0$$

$\therefore \quad h^2 + \frac{9}{2}h - 2 = 0$

$\therefore \quad 2h^2 + 9h - 4 = 0$

$D = 9^2 - 4.2.(-4) = 81 + 32 = 113 > 0$

it shows that two such chords are possible

Q.35 **Assertion (A) :** Angle between line x + y = 3 and circle $x^2 + y^2 - 2x - 4y - c^2 = 0$ will not depend on c.

Reason (R) : As line passes through centre of circle so angle is 90º.

Sol. **[A]**

$\because$ the centre (1, 2) lies on the line x + y = 3

this is a diameter of the circle. Therefore, it subtends angle $\frac{\pi}{2}$ at any point on the circle.

Q.36 **Assertion (A) :** If three circles which are such that their centres are non-colliner, then exactly one circle exists which cuts the three circles orthogonally.

Reason (R) : Radical axis for two intersecting circles is the common chord.

Sol.[B] Assertion (A)

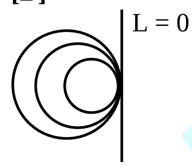
If 3 circles which are such that their centers are non-collinear, then exactly one circle exists which cuts the 3 circles orthogonally & is called as radical or orthogonal circle.

Reason is correct but it is not correct explanation.

Q.37 **Assertion (A) :** If a line L = 0 is tangent to the circle S = 0, then it will also be a tangent to the circle $S + \lambda L = 0$.

Reason (R) : If a line touches a circle, then perpendicular distance of the line from the centre of the circle is equal to the radius of the circle.

Sol. **[B]**



as shown in figure 1

Part-D Column Matching type questions

Q.38

Column 1	**Column 2**
(A) Circles $x^2 + y^2 - 5x - 3 = 0$ and $3x^2 + 3y^2 + 2x + 4y - 6 = 0$	(P) Line joining centres of two circles is x + y = 3
(B) $x^2 + y^2 - 2x - 4y - 5 = 0$, $x^2 + y^2 - 4x - 2y - 3 = 0$	(Q) Radical axis of this two circles is 17x + 4y + 3 = 0
(C) $x^2 + y^2 = 9$, $x^2 + y^2 - 2x - 2y - 5 = 0$	(R) Radical axis of two circles x + y = 2
(D) Equation of common chord of circles $x^2 + y^2 + 2x + 2y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$ is	(S) 2x + y + 1 = 0

Sol. **A→ Q, B→ P, C → R, D → S**

(A) $S_1 : x^2 + y^2 - 5x - 3 = 0$ ....(i)

$S_2 : 3x^2 + 3y^2 + 2x + 4y - 6 = 0$

$\Rightarrow x^2 + y^2 + \frac{2}{3}x + \frac{4}{3}y - 2 = 0$ ....(ii)

$\therefore S_1 - S_2 = 0$

$\Rightarrow \left(-5x - \frac{2}{3}x\right) - \frac{4}{3}y - 3 + 2 = 0$

$\Rightarrow (-15x - 2x) - 4y - 3 = 0$

$\Rightarrow 17x + 4y + 3 = 0$

(B) $S_1 : x^2 + y^2 - 2x - 4y - 5 = 0$

$S_2 : x^2 + y^2 - 4x - 2y - 3 = 0$

$S_1 - S_2 = 0$

$\Rightarrow -2x - 2y - 2 = 0$

$\Rightarrow x + y + 1 = 0$

$C_1 : (1, 2)$ $C_2 (2, 1)$

$y - 2 = -1 (x - 1)$

$y - 2 = -x + 1$

$x + y = 3$

$\therefore$ line joining centre of two circle is $x + y = 3$

(C) $S_1 : x^2 + y^2 = 9$ ....(i)

$S_2 : x^2 + y^2 - 2x - 2y - 5 = 0$ ....(ii)

$S_1 - S_2 = 0$

$\Rightarrow 2x + 2y - 4 = 0$

$x + y = 2$

(D) $S_1 - S_2 = 0 \Rightarrow 2x + y + 1 = 0$

Q.39 **Column-1** **Column-2**

(A) Number of values of a for which the common chord of the circles $x^2 + y^2 = 8$ and $(x - a)^2 + y^2 = 8$ subtends a right angle at the origin is — (P) 4

(B) A chord of the circle $(x-1)^2 + y^2 = 4$ lies along the line $y = 22\sqrt{3}(x-1)$. The length of the chord is equal to — (Q) 2

(C) The number of circles touching all the three lines $3x + 7y = 2$, $21x + 49y = 5$ and $9x + 21y = 1$ are — (R) 0

(D) If radii of the smallest and largest circle passing through the point $(\sqrt{3}, \sqrt{2})$ and touching the circle $x^2 + y^2 - 2\sqrt{2}\,y - 2 = 0$ are r_1 and r_2 respectively, then the mean of r_1 and r_2 is — (S) 1

Sol. **A→ Q, B→ P, C → R, D→ S**

(A) equation of common chord is

$$x = \frac{a}{2}$$

Since it subtend $\frac{\pi}{2}$ at the origin,

$$x^2 + y^2 = \left(\frac{2x}{a}\right)^2$$

i.e., $\left(1 - \frac{4}{a^2}\right) x^2 + y^2 = 0$

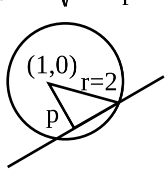
$\therefore$ co-eff of x^2 + co-eff of $y^2 = 0$

$\therefore \quad 1 - \frac{4}{a^2} + 1 = 0$

$$a = \pm\sqrt{2}$$

(B) length of chord $= 2\sqrt{r^2 - p^2}$

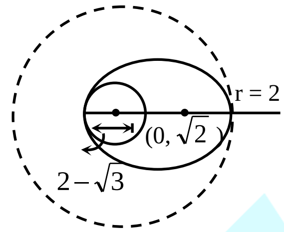
$p = 0$



$\therefore$ length of the chord $= 2 \times 2 = 4$

(C) Since all three lines are parallel, a circle touching all the lines.

(D)



$(\sqrt{3}, \sqrt{2})$ lies inside the circle

$$S = x^2 + y^2 - 2\sqrt{2}\,y - 2 = 0$$

$r_1 = 2 - \sqrt{3}$ and $r_2 = 2 + \sqrt{3}$

$\therefore$ mean $= \frac{r_1 + r_2}{2} = 2$

Q.40 **Column-1** **Column-2**

(A) Number of common tangents of the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ is — (P) 1

(B) Number of indirect common tangents of the circles $x^2 + y^2 - 10y - 4x + 4 = 0$ & $x^2 + y^2 - 6x - 12y - 55 = 0$ is — (Q) 2

(C) Number of common tangents of the circles $x^2 + y^2 - 2x - 4y = 0$ & $x^2 + y^2 - 8y - 4 = 0$ is — (R) 3

(D) Number of direct common tangents of the circles $x^2+y^2+2x-8y + 13 = 0$ & $x^2 + y^2 - 6x - 2y + 6 = 0$ is — (S) 0

Sol. **A→ R, B→ S, C → P, D→ Q**

(A) $C_1 \equiv (1, 0)$, $r_1 = 1$

$C_2 = (-3, 3)$, $r_2 = 4$

$C_1C_2 = r_1 + r_2 = 5$

\# tangents = 3

(B) $C_1 = (5, 2)$, $r_1 = 2$

$C_2 = (3, 4)$, $r_2 = 10$

$C_1C_2 < r_2 - r_1$

\# indirect tangents = 0

(C) $C_1 \equiv (1, 2)$, $r_1 = \sqrt{5}$

$C_2 \equiv (0, 4)$, $r_2 = 2\sqrt{5}$

$C_1C_2 = \sqrt{5} = r_2 - r_1$

$\therefore$ C_1 touches C_2 internally

$\therefore$ # tangents = 1

(D) $C_1 \equiv (-1, 4)$, $r_1 = 2$

$C_2 \equiv (3, 1)$, $r_2 = 2$

$C_1C_2 > r_1 + r_2$

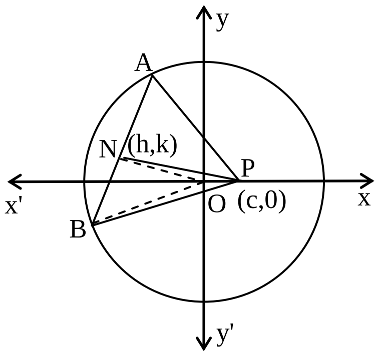
$\therefore$ # direct tangents = 2

EXERCISE # 3

Part-A Subjective Type Questions

Q.1 Find the locus of the middle points of chords of circle $x^2 + y^2 = a^2$ which subtend a right angle at the point (c, 0).

Sol.



Let N (h, k) be the middle point of any chord AB, which subtends a right angle at P(c,0)

since $\angle APB = 90^\circ$

$\therefore$ NA = NB = NP

(since distance of the vertices from middle point of the hypotenuse are equal)

or $(NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2$ ....(i)

But also $\angle BNO = 90^\circ$

$\therefore (OB)^2 = (ON)^2 + (NB)^2$

$- (NB)^2 = (ON)^2 - (OB)^2$

$\Rightarrow - [(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

or $2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

$\therefore$ Locus of N(h, k) is

$2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$

Q.2 A variable circle passes through the point A (a, b) & touches the x-axis ; show that the locus of the other end of the diameter through A is $(x - a)^2 = 4by$.

Sol. Let the other end of the diameter through A(a,b) be B(p,q) so that centre is $\left(\frac{p+a}{2}, \frac{q+b}{2}\right)$

and $(\text{diameter})^2 = 4r^2 = (p-a)^2 + (q - b)^2$ ....(i)

Since the circle touches x-axis, its radius is ordinate of centre i.e. $\frac{q+b}{2} = r$

or $(q + b)^2 = 4r^2$ ....(ii)

$\therefore (q + b)^2 -(q - b)^2 = (p - \alpha)^2$ [from (i) & (ii)]

or $(p - q)^2 = 4bq$

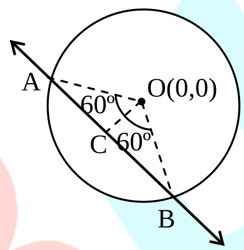
Hence locus of (p, q) is $(x - a)^2 = 4by$

Q.3 Find the equations of straight lines which pass through the intersection of the lines $x - 2y - 5 = 0$, $7x + y = 50$ & divide the circumference of the circle $x^2 + y^2 = 100$ into two arcs whose lengths are in the ratio 2 : 1.

Sol. The equation of line passing through the point of intersection of the lines $x - 2y - 5 = 0$ and $7x + y - 50 = 0$ is given by

$(x - 2y - 5) + \lambda(7x + y - 50) = 0$

$\Rightarrow x(1 + 7\lambda) + y (\lambda - 2) - 50\lambda - 5 = 0$ ....(i)



Suppose this line cuts the given circle at A and B such that arc APB : arc AQB = 2 : 1

$\therefore \angle AOB = 120^\circ$

$\Rightarrow \angle AOC = \angle BOC = 60^\circ$

Now, $OC = \left|\frac{-50\lambda - 5}{\sqrt{(1+7\lambda)^2 + (\lambda - 2)^2}}\right| = \frac{50\lambda + 5}{\sqrt{50\lambda^2 + 10\lambda + 5}}$

and, OA = (radius of the circle $x^2 + y^2 = 100$)

$\therefore$ OA = 10

In ΔAOC, we have $\cos 60^\circ = \frac{OC}{OA}$

$\Rightarrow OC = OA \cos 60^\circ$

$\Rightarrow OC = 10. \frac{1}{2} = 5$

$\Rightarrow \frac{50\lambda + 5}{\sqrt{50\lambda^2 + 10\lambda + 5}} = 5$

$\Rightarrow (10\lambda + 1)^2 = 50\lambda^2 + 10\lambda + 5$

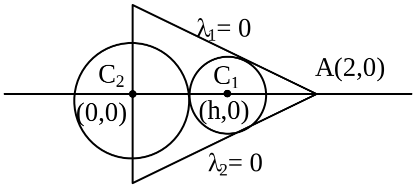
$\Rightarrow 25\lambda^2 + 5\lambda - 2 = 0$

$\Rightarrow (5\lambda + 2)(5\lambda - 1) = 0 \Rightarrow \lambda = \frac{1}{5}, \frac{-2}{5}$

Putting the values of λ in (i), we get the reqd. lines $4x - 3y = 25$ and $3x + 4y = 25$

Q.4 Find an equation of the circle which touches the straight - lines $x + y = 2$, $x - y = 2$ and also touches the circle $x^2 + y^2 = 1$ externally.

Sol.



$\lambda_1 = 0$ & $\lambda_2 = 0$ are tangents to required circle which intersect at A(2, 0). The centre of the required circle will be lie on bisector of acute angle between these tangnets

i.e. on $\dfrac{x+y-2}{\sqrt{2}} = \dfrac{x-y-2}{\sqrt{2}}$ i.e. y = 0

let it be (h, 0) where h is +ve and if r be the radius then it toueches $x^2 + y^2 = 1$ externally.

$C_1C_2 = r_1 + r_2$ or h = 1 + r form (i)

P = r with any tangent gives $\dfrac{h-2}{\sqrt{2}} = r$

$\therefore$ $h = 2 \pm \sqrt{2}$ r = 1 + r form (i)

$1 = (1 + \sqrt{2})r \;\therefore\; r = \dfrac{1}{1+\sqrt{2}} = \sqrt{2} - 1$

Only as r = + ve

$\therefore$ $h = 2 \pm \sqrt{2}\;(\sqrt{2} - 1)$

$= 4 - \sqrt{2}\,,\ \sqrt{2}$

$\therefore$ centres of circles are $(4 - \sqrt{2}\,, 0), (\sqrt{2}\,, 0)$

and radius $(\sqrt{2} - 1)$

$\Rightarrow (x - 4 + \sqrt{2}\,)^2 + y^2 = (\sqrt{2} - 1)^2$

and $(x - \sqrt{2}\,)^2 + y^2 = (\sqrt{2} - 1)^2$

Q.5 A tangent is drawn to each of the circle $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$. Show that if the two tangents are mutually perpendicular, the locus of their points of intersection is a circle concentric with the given circles.

Sol. Any tangent to first circle $(x^2 + y^2 = a^2)$ is given by x cosα + y sin α = a

As its distance from centre (0, 0) is equal to radius a

any tangent to $x^2 + y^2 = b^2$ but perpendicular to above is obtained by replacing α by α – 90º and its equation is

$x\cos(\alpha - 90^\circ) + y\sin(\alpha - 90^\circ) = b$

or $x\cos(90^\circ - \alpha) - y\sin(90^\circ - \alpha) = b$

or $x\sin\alpha - y\cos\alpha = b$

locus of the point of intersection of these tangents is $x^2 + y^2 = a^2 + b^2$

which is a circle concentric with the given circles.

Q.6 Find the equation of the smallest circle passing through the intersection of $x^2 + y^2 - 2x - 4y - 4 = 0$ and the line x + y – 4 = 0.

Sol. Any circle passing through the points of intersection of the given circle and the line has the equation as

$(x^2 + y^2 - 2x - 4y - 4) + \lambda(x + y - 4) = 0$

its centre is $\left(-\dfrac{\lambda-2}{2}, -\dfrac{\lambda-4}{2}\right)$ ....(i)

The circle is the smallest if its certre lie on the given line (chord)

$\therefore -\dfrac{\lambda-2}{2} - \dfrac{\lambda-4}{2} - 4 = 0$

$\Rightarrow \dfrac{2-\lambda}{2} + \dfrac{4-\lambda}{2} - 4 = 0$

$\Rightarrow 2 - \lambda + 4 - \lambda - 8 = 0$

$\Rightarrow -2\lambda - 2 = 0$

$\Rightarrow 2\lambda = -2$

$\Rightarrow \lambda = -1$

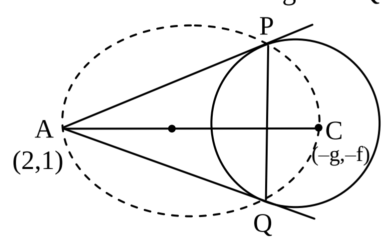
Putting λ = –1 in equation (i), weget

$x^2 + y^2 - 3x - 5y = 0$

Hence $x^2 + y^2 - 3x - 5y = 0$

Q.7 A point A (2, 1) is outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ & AP, AQ are tangents to the circle. Find the equation of the circumcircle of the triangle APQ.

Sol.



centre of the required circle is the mid-point of A(2, 1) and centre C of the given circle

$\therefore \left(\dfrac{2-g}{2}, \dfrac{1-f}{2}\right)$ is the centre of required circle

and radius $= \sqrt{\left(\dfrac{2-g}{2} - 2\right)^2 + \left(\dfrac{1-f}{2} - 1\right)^2}$

$= \dfrac{1}{2}\sqrt{(g+2)^2 + (f+1)^2}$

$= \dfrac{1}{2}\sqrt{g^2 + 4 + 4g + f^2 + 2f + 1}$

$\therefore$ Reqd. circle is

$$\left(x-\frac{2-g}{2}\right)^2+\left(y-\frac{1-f}{2}\right)^2$$

$$=\frac{(g^2+f^2+2f+4g+5)}{4}$$

$[2x–(2–g)]^2 + [2y–(1–f)]^2 = g^2 + f^2 + 2f + 4g + 5$

$4x^2+(2-g)^2-4x(2-g)+4y^2-4y(1-f)+(1-f)^2$

$= g^2 + f^2 + 2f + 4g + 5$

$4x^2 + 4y^2 - 8x - 4y + 4xg + 4fy - 2f - 4g = 0$

$\Rightarrow x^2 + y^2 - 2x - y - xg - fy = 0$

$\Rightarrow x^2 + y^2 + x(g-2) + y(f-1) - 2g - f = 0$

Q.8 If $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ are two circles with radii r_1 and r_2 respectively, show that the points at which the circles subtend equal angles lie on the circle $\frac{S_1}{r_1^2} = \frac{S_2}{r_2^2}$

Sol. $\tan\theta = \frac{r_1}{t_1} = \frac{r_2}{t_2}$

$\therefore t_1r_1 = t_2r_1$

on squaring

$t_1^2\, r_2^2 = t_2^2\, r_1^2$

or $S'_2 r_1^2 = S'_1 r_2^2 = 0$

on generalisation

$\Rightarrow r_2^2\, S_1 - r_1^2\, S_2 = 0$

$\Rightarrow \frac{S_1}{r_1^2} = \frac{S_2}{r_2^2}$

Q.9 Find the locus of the mid point of all chords of the circle $x^2 + y^2 - 2x - 2y = 0$ such that the pair of lines joining (0, 0) & the point of intersection of the chords with the circles make equal angle with axis of x.

Sol. The given circle is

$x^2 + y^2 - 2x - 2y = 0$

$\Rightarrow (x-1)^2 + (y-1)^2 = 2$

Whose centre is (1, 1) and radius equal to $\sqrt{2}$ it passes through origin (0, 0) i.e. O(0, 0)

Let OP be any chord through origin whose equation is y = mx where m = tanθ

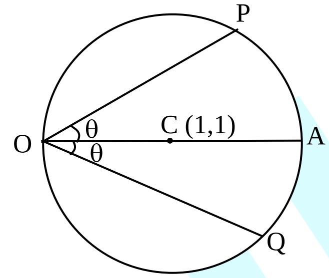
solving with circle $x^2 + y^2 - 2x - 2y = 0$

we get $x^2 + m^2x^2 - 2x - 2y = 0$

$\Rightarrow x(x + m^2x - 2 - 2m) = 0$

$\Rightarrow x = 0$ and $x = \frac{2+2m}{1+m^2}$

$\therefore$ P is $\left(\frac{2+2m}{1+m^2}, \frac{m(2+2m)}{1+m^2}\right)$



Since QQ is equally inclined to x-axis

$\therefore \tan(-\theta) = -\tan\theta = -m$

$\therefore$ Point Q is $\left(\frac{2-2m}{1+m^2}, \frac{-m(2-2m)}{1+m^2}\right)$

If (h, k) be the mid point of PQ then

$h = \frac{2}{1+m^2}$ and $k = \frac{2m^2}{1+m^2}$

$\Rightarrow h + k = 2$

$\therefore$ locus is x + y = 2

Q.10 The centre of the circle S = 0 lie on the line $2x - 2y + 9 = 0$ & S = 0 cuts orthogonally the circle $x^2 + y^2 = 4$. Show that circle S = 0 passes through two fixed points & find their coordinates.

Sol. Let circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ..(i)

since centre of this circle (–g,–f) lie on $2x - 2y + 9 = 0$

$\therefore -2g + 2f + 9 = 0$

and the circle S = 0 and $x^2 + y^2 - 4 = 0$ cuts orthogonally

$\therefore 2g \times 0 + 2f(0) = c - 4$

$\therefore c = 4$ ....(iii)

Substituting the values of g and c from (ii) and (iii) in (i) then

$\therefore x^2 + y^2 + (2f + 9)x + 2fy + 4 = 0$

or $(x^2 + y^2 + 9x + 4) + 2f(x + y) = 0$

Hence the circle S = 0 passes through fixed point

$\therefore x^2 + y^2 + 9x + 4 = 0$ and $x + y = 0$

After solving, we get

$(-4, 4)$ or $\left(-\frac{1}{2}, \frac{1}{2}\right)$

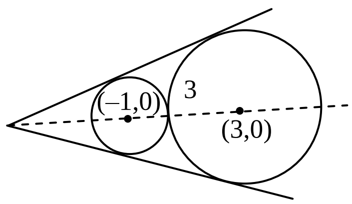
Q.11 Show that the common tangents to the circles $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 + 2x = 0$ form an equilateral triangle.

Sol. Here the circle $x^2 + y^2 - 6x = 0$ has the centre = (3, 0) and radius=3 and the circle $x^2+y^2 + 2x = 0$ has the centre = (–1, 0) &

radius = $\sqrt{(3+1)^2 + 0^2} = 4$

and sum of radii = 3 + 1 = 4

$\therefore$ the circles touch externally



obviously, the point of contect = (0,0)

let y = mx + c be a common tangent to the circle except at the point of contact. Then the length of the perpendicular from (–1, 0) to y = mx + c is equal to the radius of second circle

$$\therefore \left|\frac{-m+c}{\sqrt{1+m^2}}\right| = 1$$

or $(-m + c)^2 = 1 + m^2$ ....(i)

and the lenght of the perpendicular from (3, 0) to y = mx + x is equal to the radius of the first circle

$$\therefore \left|\frac{3m+c}{\sqrt{1+m^2}}\right| = 3$$

or $(3m + c)^2 = 9\ (1+m^2)$ ......(ii)

from (i) $\Rightarrow c^2 - 2mc = 1$ ......(iii)

form (ii) $\Rightarrow 6mc + c^2 = 9$ ....(iv)

$\therefore (6mc + c^2) - (c^2 - 2mc) = 9 - 1 = 8$

or 8 mc = 8 $\therefore$ mc = 1

$\therefore$ (iii) given $c^2 = 3$, $\therefore c = \pm\sqrt{3}$

when $c = \sqrt{3}$ from (iii), weget

$3 - 2m\sqrt{3} = 1 \ \therefore\ m = \frac{1}{\sqrt{3}}$

when $c = -\sqrt{3}$ form (iii), we get

$3 + 2m\sqrt{3} = 1 \ \therefore\ m = -\frac{1}{\sqrt{3}}$

$\therefore$ two common tangents are

$y = \frac{1}{\sqrt{3}}x + \sqrt{3}$ and $y = -\frac{1}{\sqrt{3}}x - \sqrt{3}$

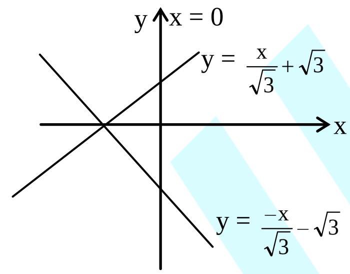
The tangent at the point of contanct (0, 0) is

$x.0 + y.0 + (x + 0) = (x + 0) = 0 \Rightarrow x = 0$

Now we have to prove that lines

$x = 0$, $y = \frac{-x}{\sqrt{3}} - \sqrt{3}$ and $y = \frac{x}{\sqrt{3}} + \sqrt{3}$

Forms an equilatarl triangle



Clearly, the indination of $y = \frac{-x}{\sqrt{3}} - \sqrt{3}$ with the x-axis is $\frac{5\pi}{6}$ and that of $y = \frac{x}{\sqrt{3}} + \sqrt{3}$ is $\frac{\pi}{6}$ as shown in

Also their point of intersection is (–3, 0)

$\therefore$ we get from the figure that the triangle is equilateral.

Q.12 Find the equation to the four common tangents to the circles $x^2 + y^2 = 25$ and $(x - 12)^2 + y^2 = 9$.

Sol. The centres of similitud from where the common tangents pass are easily found to be the points (15/2, 0) and (30, 0)

$\therefore$ Required four common tangents are given by

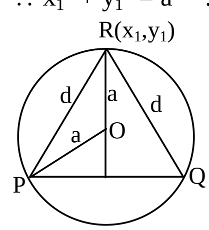
$\sqrt{5}\,y = 2x - 15,\ \sqrt{5}\,y = -2x + 15$

$\sqrt{35}\,y = x - 30,\ \sqrt{35}\,y = -x + 30$

Q.13 Show that the equation of a straight line meeting the circle $x^2 + y^2 = a^2$ in two points at equal distances 'd' from a point (x_1, y_1) on its circumference is $xx_1 + yy_1 - a^2 + \frac{d^2}{2} = 0$.

Sol. Let R (x_1,y_1) be a point on the circle $x^2 + y^2 = a^2$

$\therefore x_1^2 + y_1^2 = a^2$ ....(i)



Let P and Q be two points on the given circle such that PR = QR = d

Now line joining R and O (the centre of the circle) is perpendicular to PQ and meets PQ at L (say).

$\therefore$ let equation of PQ be

$y = \frac{-x_1}{y_1} x + c$

or $xx_1 + yy_1 = cy_1$ ....(ii)

$\Rightarrow OL = \left|\frac{-cy_1}{\sqrt{x_1^2 + y_1^2}}\right| = \left|\frac{-cy_1}{a}\right|$

and $RL = \left|\frac{x_1^2 + y_1^2 - cy_1}{\sqrt{x_1^2 + y_1^2}}\right| = \left|a - \frac{cy_1}{a}\right|$

from ΔRPL, $PL^2 = d^2 - \left(a - \frac{cy_1}{a}\right)^2$

from ΔOPL, $PL^2 = a^2 - \left(\frac{cy_1}{a}\right)^2$

Equating the two values, we get

$d^2 - \left(a - \frac{cy_1}{a}\right)^2 = a^2 - \left(\frac{cy_1}{a}\right)^2$

$= d^2 - a^2 + 2cy_1 - \left(\frac{cy_1}{a}\right)^2 = a^2 - \left(\frac{cy_1}{a}\right)^2$

$\Rightarrow d^2 - a^2 + 2cy_1 = a^2 \Rightarrow 2cy_1 = 2a^2 - d^2$

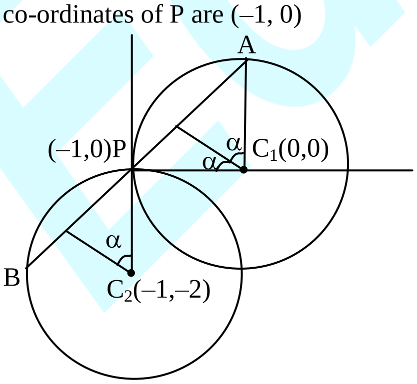
$\Rightarrow cy_1 = a^2 - \frac{d^2}{2}$

∴ from (ii), we get the requird equation of the line as

$xx_1 + yy_1 = a^2 - \frac{d^2}{2} \Rightarrow xx_1 + yy_1 - a^2 + \frac{d^2}{2} = 0$

Q.14 Through the point of intersection P which has integral coordinates of the circles $x^2 + y^2 = 1$ and $x^2 + y^2 + 2x + 4y + 1 = 0$, a common chord APB is drawn terminating on the two circles such that the chords AP and BP of the given circles subtend equal angles at the centre. Find the equation of this chord.

Sol. The given circle intersect at the point (–1, 0) and (3/5, –4/5).since P has integral co-ordinates, the co-ordinates of P are (–1, 0)



let an equation of the chord AB through P be

$y = m(x + 1)$

let AP and BP make the same angle 2a at the centres $C_1(0, 0)$ and $C_2(-1,-2)$ respectively of the gieven circles.

$\angle APC_1 = \angle BPC_2 = \frac{\pi}{2} - \alpha$ ....(i)

Now slope of $C_1P = 0$ and slope of $C_2P = 2/m'$ where m' = 0

from (i), we have

$\tan\left(\frac{\pi}{2} - \alpha\right) = \left|\frac{m-0}{1+m.0}\right| = \left|\frac{m - \frac{2}{m}}{1 + m.\frac{2}{m}}\right|$

$\Rightarrow |m| = \left|\frac{mm'-2}{m'+2m}\right| = \left|\frac{1}{m}\right|$

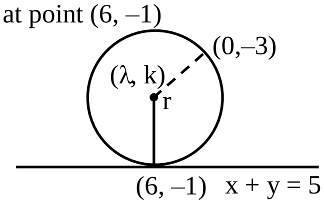
$\Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

so that the required equation of the chord is $y = x + 1$ corresponding to the value m = 1 (for m = –1, the equation will represent the chord PAB, A and B lying on the same side of P, Hence we donote consider it).

Q.15 From a point P tangent drawn to the circles $x^2 + y^2 + x - 3 = 0$, $3x^2 + 3y^2 - 5x + 3y = 0$ & $4x^2 + 4y^2 + 8x + 7y + 9 = 0$ are of equal lengths. Find the equation of the circle through P which touches the line x + y = 5 at the point (6, –1).

Sol. Here point P is the radical centre of these circles. On solving like $S_1 - S_2 = 0$ & $S_2 - S_3 = 0$ we get point P(0, –3).

Given reqd. circle passes through P and touches x + y = 5 at point (6, –1)



Let centre of reqd. circle is (h, k)

$\therefore \sqrt{h^2 + (k+3)^2} = \sqrt{(h-6)^2 + (k+1)^2}$

$\Rightarrow h^2 + k^2 + 6k + 9 = h^2 - 12h + 36 + k^2 + 2k + 1$

$\Rightarrow 3h + k = 7$ ……..(1)

Now equation of line perpendicular to x + y = 5 is given by

$x - y = \lambda$

since it passes (6, –1) then

$6 + 1 = \lambda \Rightarrow \lambda = 7$

$\therefore x - y = 7$

this also passes (h, k)

$\therefore h - k = 7$ ……..(2)

from (1) & (2), we get 4h = 14

$h = 7/2 \quad \therefore k = \frac{7}{2} - 7 = -\frac{7}{2}$

$\therefore$ (h, k) $\equiv \left(\frac{7}{2}, -\frac{7}{2}\right)$

$\therefore$ radius r = $\sqrt{\left(\frac{7}{2}\right)^2 + \left(3 - \frac{7}{2}\right)^2}$

$= \sqrt{\frac{49}{4} + \frac{1}{4}} = \sqrt{\frac{50}{4}} =$

since centre (h, k) = $\left(\frac{7}{2}, -\frac{7}{2}\right)$

$\therefore$ radius r = $\sqrt{\frac{50}{4}}$

$\therefore$ Equation of required circle is given by

$$\left(x - \frac{7}{2}\right)^2 + \left(y + \frac{7}{2}\right)^2 = \left(\sqrt{\frac{50}{4}}\right)^2$$

$\Rightarrow \left(x - \frac{7}{2}\right)^2 + \left(y + \frac{7}{2}\right)^2 = \frac{50}{4}$

$\Rightarrow x^2 + y^2 - 7x + 7y + \left(\frac{98}{4} - \frac{50}{4}\right) = 0$

$\Rightarrow x^2 + y^2 - 7x + 7y + 12 = 0$

$\therefore x^2 + y^2 - 7x + 7y + 12 = 0$

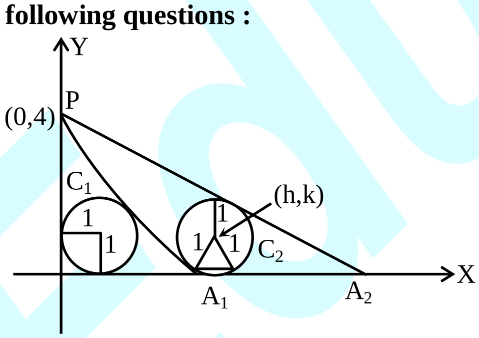
Part-B Passage based objective questions

Passage I (Question 16 to 18)

Let C_1, C_2 are two circles each of radius 1 touching internally the sides of triangles POA_1, PA_1A_2 respectively where P $\equiv$ (0, 4), O is origin, A_1, A_2 are the points on positive x-axis.

On the basis of above passage, answer the following questions :

Sol.



$C_1 \equiv (x - 1)^2 + (y - 1)^2 - 1 = 0$

$x^2 + y^2 - 2x - 2y + 1 = 0$

let equation of PA, is y – 4 = mx

Θ PA_1 is tangent to C_1

$\therefore \frac{|m+3|}{\sqrt{1+m^2}} = 1$

$\therefore m^2 + 6m + 9 = 1 + m^2$

$\therefore$ m = ∞ and $-\frac{4}{3}$

$\therefore$ equation of PA_1 is

4x + 3y = 12

equation of C_2 is

$(x - h)^2 + (y - k)^2 = 1^2$

$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - 1 = 0$

Θ PA_1 is tangent to C_2

$\therefore \frac{|4h - 9|}{5} = 1$

$h = \frac{9 \pm 5}{4} = 1$ or $\frac{7}{2}$

$\therefore h = \frac{7}{2}$ (as (1, 1) is centre of another circle)

Q.16 Angle subtended by circle C_1 at P is-

(A) $\tan^{-1}\frac{2}{3}$ (B) $2\tan^{-1}\frac{2}{3}$

(C) $\tan^{-1}\frac{3}{4}$ (D) $2\tan^{-1}\frac{3}{4}$

Sol.[C] Angle subtended by circle C_1 at P is

$2\tan^{-1}\frac{r}{\sqrt{S_1}}$

$= 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2/3}{1-(1/3)^2} = \tan^{-1}\left(\frac{3}{4}\right)$

Q.17 Centre of circle C_2 is-

(A) (3, 1) (B) $(3\frac{1}{2}, 1)$

(C) $(3\frac{3}{4}, 1)$ (D) None of these

Sol.[B] Centre of circle C_2 is $\left(3\frac{1}{2}, 1\right)$

Q.18 Slope of line PA_2 is-

(A) $\frac{-4}{3}$ (B) $\frac{-3}{4}$ (C) $\frac{-8}{15}$ (D) $\frac{-2}{3}$

Sol.[C] Slope of PA_2 = ?

equation of tangent from (0, 4) to the circle C_2 is

$y - 1 = m\left(x - \frac{7}{2}\right) + \sqrt{1+m^2}$

Θ (0, 4) satisfies this equation

$3 = m\left(-\frac{7}{2}\right) + \sqrt{1+m^2}$

$\Rightarrow m = -\frac{4}{3}$ or $-\frac{8}{15}$

Passage II (Question 19 to 21)

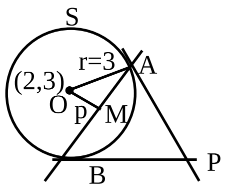
A line 2x + y = 3 intersects circle $x^2 + y^2 - 4x - 6y + 4 = 0$ at A and B. Tangents at A and B are drawn which meets at P. M is the mid point of AB

On the basis of above passage, answer the following questions :

Q.19 Length of chord AB is equal to-

(A) $\sqrt{\frac{29}{5}}$ (B) $2\sqrt{\frac{29}{5}}$ (C) $\frac{2\sqrt{29}}{5}$ (D) $\frac{\sqrt{29}}{5}$

Sol.[B]



length of chord(λ) = $2\sqrt{r^2 - p^2}$

$p = \frac{4}{\sqrt{5}}$

$\therefore\ \lambda = 2\sqrt{3^2 - \frac{16}{5}} = 2\sqrt{\frac{29}{5}}$

Q.20 Co-ordinate of M is-

(A) $\left(\frac{2}{5}, \frac{11}{5}\right)$ (B) $\left(\frac{1}{5}, \frac{13}{5}\right)$

(C) $\left(\frac{3}{5}, \frac{9}{5}\right)$ (D) None of these

Sol.[A] Equation of AB is L = $2x + y = 3$

equation of OP is $x - 2y = -4$

intersection point of AB and OP is

$M \equiv \left(\frac{2}{5}, \frac{11}{5}\right)$

Q.21 Equation of circum circle of triangle PAB is-

(A) $4x^2 + 4y^2 + 4x - 15y - 11 = 0$

(B) $4x^2 + 4y^2 + 2x - 15y - 11 = 0$

(C) $4x^2 + 4y^2 - 2x - 15y - 11 = 0$

(D) $4x^2 + 4y^2 - 4x - 15y - 11 = 0$

Sol.[B] Equation of circle passing through intersection point of the circle S ≡ 0 and L ≡ 0 is S + λL = 0

Θ P satisfies it $\Rightarrow \lambda = \frac{7}{4}$

$\therefore$ equation is $4x^2 + 4y^2 + 2x - 15y - 11 = 0$

Passage III (Question 22 to 24)

Two circles $S_1 : x^2 + y^2 - 2x - 2y - 7 = 0$ and $S_2 : x^2 + y^2 - 4x - 4y - 1 = 0$ intersects each other at A and B. Tangents at A & B to circle S_2 meet at Q.

On the basis of above passage, answer the following questions :

Sol Equation of common chord is

$x + y = 3$

Q.22 Equation of circle passing through A and B whose AB is diameter-

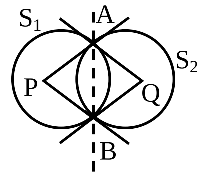
(A) $x^2 + y^2 - 3x - 3y - 5 = 0$

(B) $x^2 + y^2 - 3x - 3y - 4 = 0$

(C) $x^2 + y^2 + 3x + 3y - 4 = 0$

(D) $x^2 + y^2 + 3x + 3y - 5 = 0$

Sol.[B]



Equation of circle passing through AB is given by

$S_1 + \lambda L = 0$ (or $S_2 + \lambda L = 0$) $\lambda \in R$

i.e., $x^2 + y^2 - 2x - 2y - 7 + \lambda(x + y - 3) = 0$

i.e., $x^2 + y^2 + (\lambda - 2)x + (\lambda - 2)y - (7 + 3\lambda) = 0$

$C \equiv \left(\frac{-(\lambda - 2)}{2}, \frac{-(\lambda - 2)}{2}\right)$

Θ AB is a diameter of this circle

C lies on AB

$\therefore -(\lambda - 2) = 3 \Rightarrow \lambda = -1$

$\therefore$ equation of the circle is

$x^2 + y^2 - 3x - 3y - 4 = 0$

Q.23 Co-ordinates of point Q are-

(A) (–6, –6) (B) (–5, –5)

(C) (–7, –7) (D) None of these

Sol.[C] Let Q ≡ (h, k) then equation of AB is

$hx + ky - 2(x + h) - 2(y + k) - 1 = 0$

i.e., $(h - 2)x + (k - 2)y - (2h + 2k + 1) = 0$

comparing this equation with $x + y - 3 = 0$ gives

Q ≡ (h, k) = (–7, –7).

Q.24 Mid point of AB is-

(A) $\left(\frac{5}{2}, \frac{1}{2}\right)$ (B) $\left(\frac{3}{2}, \frac{3}{2}\right)$

(C) (2, 1) (D) (1, 2)

Sol.[B] Let (a, b) be mid point of AB then equation of AB (w.r.t. $S_1 = 0$) is

$T = S_1$

i.e., $ax + by - (x + a) - (y + b) - 7 = a^2 + b^2 - 2a - 2b - 7$

Comparing this equation with $x + y = 3$ gives

$M \equiv (a, b) \equiv \left(\frac{3}{2}, \frac{3}{2}\right)$

EXERCISE # 4

Old IIT-JEE questions

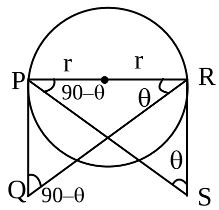
Q.1 Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then 2r equals **[IIT SCR.-2001]**

(A) $\sqrt{PQ.RS}$ (B) $\frac{PQ+RS}{2}$

(C) $\frac{2PQ.RS}{PQ.RS}$ (C) $\sqrt{\frac{PQ^2+RS^2}{2}}$

Sol. **[A]**

From following figure, it is clear that ΔPRQ & ΔRSP are similar.



$\therefore \frac{PR}{RS} = \frac{PQ}{RP}$

$\Rightarrow PR^2 = PQ.RS$

$\Rightarrow PR = \sqrt{PQ.RS}$

$\Rightarrow 2r = \sqrt{PQ.RS}$

Q.2 Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then the locus of the centroid of the triangle PAB as P moves on the circle is **[IIT SCR.-2001]**

(A) a parabola

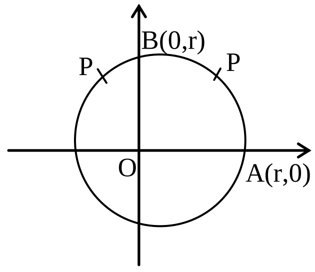
(B) a circle

(C) an ellipse

(D) a pair of straight lines

Sol. **[B]**

Choosing OA as x-axis, A = (r, 0), B = (0, r) and any point P on the circle is (rcosθ, rsinθ). if (x, y) is the centroid of ΔPAB.



$3x = r\cos\theta + r + 0$

$3y = r\sin\theta + 0 + r$

$\therefore (3x - r)^2 + (3y - r)^2 = r^2$

There fore, locus of P is a circle.

Q.3 Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA. **[IIT - 2001]**

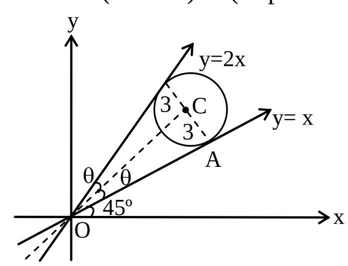
Sol. $2x^2 + y^2 - 3xy = 0$ (given) $\Rightarrow 2x^2 - 2xy - xy + y^2 = 0$

$\Rightarrow 2x(x - y) - y(x - y) = 0 \Rightarrow (2x - y)(x - y) = 0$

$\Rightarrow$ y = 2x, y = x are the equations of straight lines passing through origin.

Now, Let the angle between the lines be 2θ and the line y = x makes angel of 45º with x-axis.

therefore $\tan(45^\circ + 2\theta) = 2$(slope of the line y = 2x)



$\Rightarrow \frac{\tan 45^\circ + \tan 2\theta}{1 - \tan 45^\circ \tan 2\theta} = 2 \Rightarrow \frac{1+\tan 2\theta}{1-\tan 2\theta} = 2$

$\Rightarrow \frac{(1+\tan 2\theta)-(1-\tan 2\theta)}{(1+\tan 2\theta)+(1-\tan 2\theta)} = \frac{2-1}{2+1} = \frac{1}{3}$

$\Rightarrow \tan 2\theta = \frac{1}{3} \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{1}{3}$

$\Rightarrow \tan^2\theta + 6\tan\theta - 1 = 0$

$\Rightarrow \tan\theta = \frac{-6 \pm \sqrt{36+4\times1\times1}}{2} \Rightarrow \tan\theta = -3 \pm \sqrt{10}$

$\Rightarrow \tan\theta = -3 + \sqrt{10}$ $(\therefore 0 < \theta < \pi/4)$

Againg in ΔOCA $\tan\theta = \frac{3}{OA}$

$OA = \frac{3}{\tan\theta} = \frac{3}{-3+\sqrt{10}}$

$= \frac{3(3+\sqrt{10})}{(-3+\sqrt{10})(3+\sqrt{10})} = 3(3 + \sqrt{10})$

Q.4 Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C. **[IIT - 2001]**

Sol. Let equtION of C_1 be $x^2 + y^2 = r_1^2$
and of C_2 be $(x – a)^2 + (y – b)^2 = r_2^2$
Let centre C be (h, k) and radiucs r, then by the given condifion

$\sqrt{(h-a)^2+(k-b)^2} = r_1 + r_2$ & $\sqrt{h^2+k^2} = r_1 - r$

$\Rightarrow \sqrt{(h-a)^2+(k-b)^2} + \sqrt{h^2+k^2} = r_1 + r_2$

Required locus is

$\sqrt{(x-a)^2+(y-b)^2} + \sqrt{x^2+y^2} = r_1 + r_2$

Which represents and ellipse whose focii are at (a, b) and (0, 0)

Q.5 If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x – 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is – **[IIT SCR.-2002]**

(A) 4 (B) $2\sqrt{5}$ (C) 5 (D) 3

Sol. **[C]**

The line $5x – 2y + 6 = 0$ meets the y-axis at the point (0, 3) and therefore the tangent has to pass through the point (0, 3) and required length is therefore

$= \sqrt{x_1^2+y_1^2+6x_1+6y_1-2}$

$\Rightarrow \sqrt{0^2+3^2+6(0)+6(3)-2}$

$= \sqrt{25} = 5$

Q.6 If a > 2b > 0 then the positive value of m for which $y = mx – b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x – a)^2 + y^2 = b^2$ is - **[IIT SCR.-2002]**

(A) $\frac{2b}{\sqrt{a^2-4b^2}}$ (B) $\frac{\sqrt{a^2-4b^2}}{2b}$

(C) $\frac{2b}{a-2b}$ (D) $\frac{b}{a-2b}$

Sol. **[A]**

The given line is a tangent to first circle i.e. $y = mx \pm a\sqrt{1+m^2}$ is a tangent.

If it is to be a tangent to the second circle whose centre (a, 0), radius = b

$\therefore$ p = r apply

$\therefore \frac{ma-b\sqrt{1+m^2}}{\pm\sqrt{1+m^2}} = b$ or ma = 0 or $2b\sqrt{1+m^2}$

$\therefore$ m = 0 (rejected)

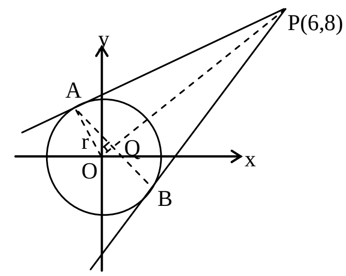
or $m^2a^2 = 4b^2 + 4m^2b^2$ or $m^2(a^2 – 4b^2) = 4b^2$

$\therefore m^2 = \frac{4b^2}{a^2-4b^2}$

$\therefore m = \frac{2b}{\sqrt{a^2-4b^2}}$

Q.7 Two tangents are drawn from the point P(6, 8) to the circle $x^2 + y^2 = r^2$. Find r such that the area of the triangle formed by the tangents and the chord of contact is maximum. **[IIT - 2003]**

Sol. To maximise area of ΔAPB ; we know OP = 10 and sinq = $\frac{r}{10}$, where $\theta \in (0, \pi/2)$ ....(i)



$\therefore$ Area = $\frac{1}{2}$ (2AQ) (PQ) = AQ. PQ

= (r cosθ) (10 – rOQ) = (r cosθ) (10 – rsinθ)

= 10 sinθ cosθ (10 – 10sin^{2}θ) [from i]

$\Rightarrow f(\theta) = 100 \cos^3\theta \sin\theta$

$\Rightarrow f'(\theta) = 100 \cos^4\theta – 300\cos^2\theta. \sin^2\theta$

Put f '(θ) = 0

$\Rightarrow \cos^2\theta = 3 \sin^2\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

at which f '(θ) < 0, Thus when $\theta = \frac{\pi}{6}$, area is max. and r = 10 sinθ form (i)

r = 10sinθ form (i)

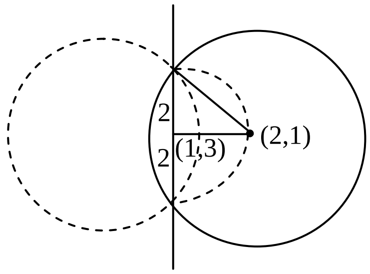
$r = 10 \sin\frac{\pi}{6} = 10. \frac{1}{2} = 5$ units.

Q.8 Diameter of the given circle $x^2 + y^2 –2x–6y + 6 = 0$ is the chord of another circle C having centre (2, 1), the radius of the circle C is - **[IIT SCR.-2004]**

(A) $\sqrt{3}$ (B) 2 (C) 3 (D) 1

Sol. **[C]**

Clearly from the figure the radius of bigger circle



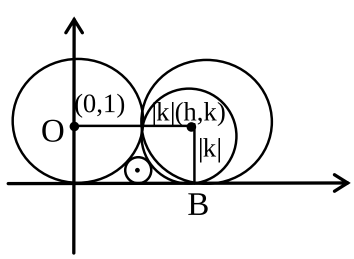
$r^2 + 2^2 + \{(2-1)^2 + (1-3)^2\}$
$r^2 = 4 + 1 + 4$
$r^2 = 9$
$r = 3$

Q.9 Find a circle orthogonal to the circle having ends of diameter (0, –1) and (–2, 3) and touching the line $2x + 3y + 1 = 0$ at the point (1, –1). **[IIT - 2004]**

Sol. The equation of circle having tanget
$2x + 3y + 1 = 0$ at (1,–1)
$\Rightarrow (x-1)^2 + (y+1)^2 + \lambda(2x + 3y + 1) = 0$
$\Rightarrow x^2 + y^2 + 2x(\lambda - 1) + y(3\lambda + 2)\ (\lambda + 2) = 0$..(i)
Which is orthogonal to the circle having end pointsof diameter (0, –1) and (–2, 3)
$\Rightarrow x(x+2) + (y+1)(y-3) = 0$
$\Rightarrow x^2 + y^2 + 2x - 2y - 3 = 0$ ....(ii)
$\therefore \frac{2(2\lambda - 2)}{2}.1 + \frac{2(3\lambda - 2)}{2}(-1) = \lambda + 2 - 3$
$\Rightarrow 2\lambda - 2 - 3\lambda - 2 = \lambda - 1 \Rightarrow 2\lambda = -3/2$
$\therefore$ from equation (i) equation of circle is
$2x^2 + 2y^2 - 10x - 5y + 1 = 0$

Q.10 Locus of the centre of circle touching to the x-axis & the circle $x^2 + (y-1)^2 = 1$ externally is- **[IIT SCR.-2005]**
(A) $\{(0, y)\ ;\ y \geq 0\}$ U $(x^2 = -4y)$
(B) $\{(0, y)\ ;\ y \leq 0\}$ U $(x^2 = y)$
(C) $\{(x, y)\ ;\ y \leq y\}$ U $(x^2 = 4y)$
(D) $\{(0, y)\ ;\ y \geq 0\}$ U $\{x^2 + (y-1)^2 = 4\}$

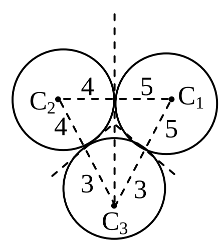
Sol. **[A]**
Let the locus of centre of circle be (h, k) touching $x^2 + (y-1)^2 = 1$ and x-axis shown as:



Clearly form figure
distance between O and A is always 1 + |k|
i.e. $\sqrt{(h-0)^2 + (k-1)^2} = 1 + |k|$
squaring both sides, we get
$h^2 + k^2 - 2k + 1 = 1 + k^2 + 2|k|$
$\Rightarrow h^2 = 2|k| + 2k$
or $x^2 = 2|y| + 2y$
where $|y| = \begin{cases} y, & y \geq 0 \\ -y, & y < 0 \end{cases}$
$\therefore x^2 = -2y + 2y,\ y \geq 0$
and $x^2 = -2y + 2y,\ y < 0$
$x^2 = 4y$ when $y \geq 0$
and $x^2 = 0$, when $y > 0$
$\therefore \{(x, y): x^2 = 4y$
when $y \geq 0\} \cup \{(0,y): (y<0)\}$

Q.11 Circle with radii 3, 4 and 5 touch each other externally if P is the point of intersection of tangents to these circles at their point of contact. Find the distance of P from the point of contact. **[IIT 2005]**

Sol. As the circle with radii 3,4 and 5 touch each other externally and P is the point of intersection of tangents



$\Rightarrow$ P is incentre of $\Delta C_1C_2C_3$
Thus distance of point P from the points of contanct is equal to In-rdius (r) of $\Delta C_1C_2C_3$
i.e. $r = \frac{\Delta}{S} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$
where 2s = 7 + 8 + 9
$\therefore$ s = 12
Hence, r $= \frac{\sqrt{(12-7)(12-8)(12-9)}}{12}$
$= \sqrt{\frac{5.4.3}{12}}$
$= \sqrt{5}$

Passage (Q. 12 to 14)

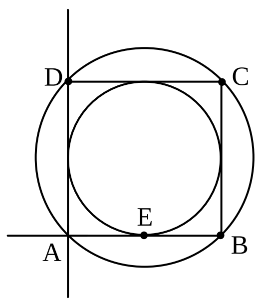
Let C_1 is a circle touching to all the sides of square ABCD of side length 2 units internally and C_2 circle is passing through the vertices of square. A line L is drawn through A.**[IIT 2006]**

Q.12 Let P is a point on C_1 and Q is on C_2, then $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ =

(A) 0.75 (B) 0.5

(C) 1.25 (D) 1

Sol. **[A]**



Take A as origin, AB as x-axis, AD as y-axis. since all the choices are numercally quautities, The ratio should be independent of choice of P and Q Taking P as E and Q as A

Required Ratio = $\dfrac{1^2 + 1^2 + 5 + 5}{4 + 4 + 8}$

$= \dfrac{12}{16} = \dfrac{3}{4} = 0.75$

Q.13 A variable circle touches to the line L and circle C_1 externally such that both circles are on the same side of the line, then the locus of center of variable circle is –

(A) ellipse (B) circle

(C) hyperbola (D) parabola

Sol. **[D]**

The variable circle's centre is equidistant from a point (centre of C_1) and from a line parallel to the gieven line L on the other side.

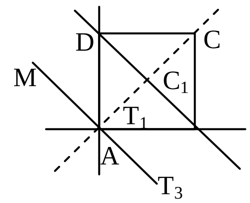
Thus locus of P is a parabola whose focus is centers of C_1 and whose directrix is line drawn parallel to L.

Q.14 A line M through A is drawn parallel to BD. Locus of point R, which moves such that its distances from the line BD and the vertex A are equal, cuts to line M at T_2 and T_3 and AC at T_1, then area of triangle $T_1T_2T_3$ is-

(A) $\dfrac{1}{2}$ (sq.units) (B) 2(sq.units)

(C) 1 (sq.units) (D) $\dfrac{4}{3}$ (sq.units)

Sol. **[C]**



Since C_1 is centre of C_1

$\therefore AC_1 = \sqrt{2}$

Also, $AT_1 = T_1C_1 = \dfrac{1}{\sqrt{2}}$

($\therefore$ T_1 is vertex, A is focus and BD is diretrix)

T_2T_3 = latus reetun of the parabola

$= 4 \times \dfrac{1}{\sqrt{2}}$

$\Rightarrow$ Area of $\Delta T_1T_2T_3$

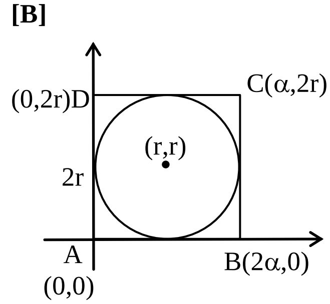
$= \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}} \times \dfrac{4}{\sqrt{2}}$

= 1

Q.15 Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and AB = 2 CD. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral ABCD touching all the sides, then its radius is **[IIT 2007]**

(A) 3 (B) 2 (C) 3/2 (D) 1

Sol. **[B]**



Let CD = α so that AB = 2α be two parallel lines. Taking A as origin the co-ordinates are A(0, 0), B(2α, 0), D(0, 2α) and C(α, 2r). since the circle is touching the axes of coordianates of it is form

$(x - r)^2 + (y - r)^2 = r^2$ ....(i)

centre (r, r) radius = r

Equation of BC is $y - 0 = -\frac{2r}{\alpha}(x - 2\alpha)$

or $\alpha y + 2rx = 4r\alpha$ ....(ii)

The above line (ii) is a tangent to circle (i) apply the condition of tangency i.e. p = r

$= \frac{\alpha r + 2r^2 - 4r\alpha}{\sqrt{\alpha^2 + 4r^2}} = r$

or $(2r - 3\alpha)^2 = \alpha^2 + 4r^2$

$8\alpha^2 - 12r\alpha = 0 \quad \therefore 2a = 3r$ ....(iii)

Area of quadilatel i.e. trapiziun ABCD is

$\frac{1}{2}(\alpha + 2\alpha).\ 2r = 18 \therefore \alpha r = 6$ ....(iv)

or $\frac{3}{2}r^2 = 6 \quad \therefore r^2 = 4$

r = 2

Q.16 Tangents are drawn from the point (17,7) to the circle $x^2 + y^2 = 169$. **[IIT 2007]**

Statement-1 : The tangents are mutually perpendicular.

Because

Statement -2 : The locus of the points from which mutually perpendicular tangents can be drawn to given circle is $x^2 + y^2 = 338$.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 False, Statement-2 is True

Sol. **[A]**

The locus of the point of intersection of two perpendicular tangents to a circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$ and 169 × 2 = 338

$\therefore$ (A) is correct option.

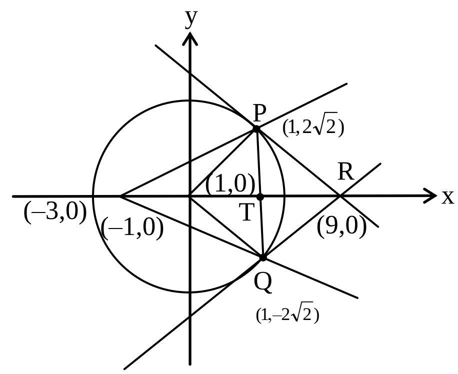
Passage (Q. 17 to 19)

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. **[IIT 2007]**

Q.17 The ratio of the areas of the triangles PQS and PQR is -

(A) $1 : \sqrt{2}$ (B) 1 : 2 (C) 1 : 4 (D) 1 : 8

Sol. **[C]**



The points P,Q are $(1,2\sqrt{2})$ and $(1,-2\sqrt{2})$ respectinely. such that PQ = $4\sqrt{2}$

For circle :

TP is $x.1 + y.\ 2\sqrt{2} = 9$

TQ is $x.1 - y.\ 2\sqrt{2} = 9$

Solving, we get the point as R as (9, 0)

For Parabola :

TP is $y.2\sqrt{2} = 4(x + 1)$

TQ is $y.(-2\sqrt{2}) = 4(x + 1)$

Solving we get the point S as (–1, 0)

Also PQ = $4\sqrt{2}$

$\therefore \Delta PQR = \frac{1}{2}.\ PQ.PT = \frac{1}{2}.4\sqrt{2}.8 = 16\sqrt{2}$

$\Delta PQS = \frac{1}{2}\ PQ.ST = \frac{1}{2}.\ 4\sqrt{2}.2 = 4\sqrt{2}$

$\therefore \frac{\Delta PQS}{\Delta PQR} = \frac{4\sqrt{2}}{16\sqrt{2}} = 1 : 4$

Q.18 The radius of the circumcircle of the triangle PRS is -

(A) 5 (B) $3\sqrt{3}$ (C) $3\sqrt{2}$ (D) $2\sqrt{3}$

Sol. **[B]**

We know that circumradius $R = \frac{abc}{4\Delta}$

For trinangle PRS, PA = $\sqrt{4+8} = 2\sqrt{3}$

PR = $\sqrt{64+8} = 6\sqrt{2}$, RS = 10

and Δ = area = $\frac{1}{2}.10.\ 2\sqrt{2} = 10\sqrt{2}$

$\therefore R = \frac{2\sqrt{3}.6\sqrt{2}.10}{4.10\sqrt{2}} = 3\sqrt{3}$

Q.19 The radius of the incircle of the triangle PQR is

(A) 4 (B) 3 (C) 8/3 (D) 2

Sol. **[D]**

Radius of incircle of ΔPQR is $r = \frac{\Delta}{s}$

$2s = PQ + QR + RP$

$= 4\sqrt{2} + \sqrt{72} + \sqrt{72}$

$= 16\sqrt{2}$

$\therefore s = 8\sqrt{2}$

Also, $D = \frac{1}{2}$ PQ.TR

$= \frac{1}{2} . 4\sqrt{2} . 8$

$= 16\sqrt{2}$

$\therefore r = \frac{16\sqrt{2}}{8\sqrt{2}}$

$\therefore r = 2$

Q.20 Consider

$L_1 : 2x + 3y + p - 3 = 0$

$L_2 : 2x + 3y + p + 3 = 0$

where p is a real number, and

$C : x^2 + y^2 + 6x - 10y + 30 = 0$.

Statement- 1 : If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.

and

Statement-2 : If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

[IIT-2008]

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is not a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True.

Sol. Given that L_1 and L_2 are two parallel lines and distance between these two lines is less than radius.

So option (B) correct Ans.

Passage (Q. 21 to 23)

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ. **[IIT 2008]**

Q.21 The equation of circle C is

(A) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$

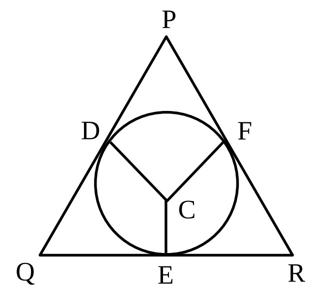
(B) $(x - 2\sqrt{3})^2 + (y + \frac{1}{2})^2 = 1$

(C) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$

(D) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Sol. [D] Let C be the centre of circle and PQ line is given by $\sqrt{3}x + y - 6 = 0$

By using parametric form we can find centre $(\sqrt{2}, 1)$



circle C : $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Q.22 Points E and F are given by

(A) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$

(B) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

(C) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(D) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Sol. [A] We have to find the equation of sides making 60° with PQ, these are $y = \sqrt{3}x$ and $y = 0$ and Now we can find the mid points E and F.

Q. 23 Equations of the sides QR, RP are

(A) $y = \frac{2}{\sqrt{3}}x + 1, y = -\frac{2}{\sqrt{3}}x - 1$

(B) $y = \frac{1}{\sqrt{3}}x, y = 0$

(C) $y = \frac{\sqrt{3}}{2}x + 1,\ y = \frac{\sqrt{3}}{2}x - 1$

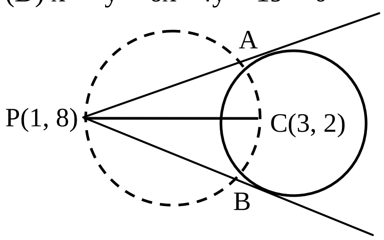
(D) $y = \sqrt{3}\,x,\ y = 0$

Sol. [D] We have to find the equation of sides making 60° with PQ, these are $y = \sqrt{3}x$ and $y = 0$.

Q.24 Tangents drawn from the point P(1, 8) to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is - **[IIT- 2009]**

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$
(B) $x^2 + y^2 - 4x - 10y + 19 = 0$
(C) $x^2 + y^2 - 2x + 6y - 29 = 0$
(D) $x^2 + y^2 - 6x - 4y + 19 = 0$

Sol. [B]



clearly PC is diameter of the circle

$\therefore$ from diameter form

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

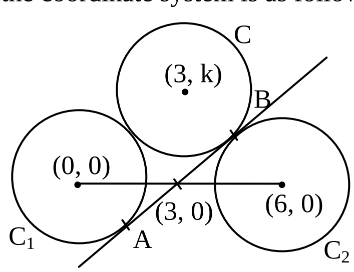
$\Rightarrow (x - 1)(x - 3) + (y - 8)(y - 2) = 0$

$\Rightarrow x^2 - 3x - x + 3 + y^2 - 2y - 8y + 16 = 0$

$\Rightarrow x^2 + y^2 - 4x - 10y + 19 = 0$

Q.25 The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is : **[IIT 2009]**

Sol. Let the coordinate system is as follows



equation of AB is $y = m(x - 3)$

Θ AB is tangent to C_1 so $m = \pm\frac{1}{2\sqrt{2}}$

But m should be positive $m = \frac{1}{2\sqrt{2}}$

So equation of AB = $2\sqrt{2} = x - 3$

Θ C_1 & C are touching each other externally.

So $CC_1 = r_1 + r \Rightarrow 9 + k^2 = (r + 1)^2$ … (1)

Θ AB is tangent to circle C so

$r = \frac{|2\sqrt{2}k - 3 + 3|}{\sqrt{8+1}} \Rightarrow k = \frac{3r}{2\sqrt{2}}$ … (2)

So solving (1) and (2) r = 8

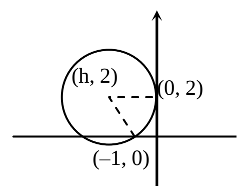
Q.26 The circle passing through the point (–1, 0) and touching the y-axis at (0, 2) also passes through the point - **[IIT 2011]**

(A) $\left(-\frac{3}{2}, 0\right)$ (B) $\left(-\frac{5}{2}, 2\right)$

(C) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (D) (– 4, 0)

Sol. [D] $\therefore (h - 0)^2 + (2 - 2)^2 = (h + 1)^2 + (2 - 0)^2$

$h^2 = h^2 + 1 + 2h + 4$



$h = -\frac{5}{2}$

Equation of circle is

$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = \left(-\frac{5}{2} - 0\right)^2$

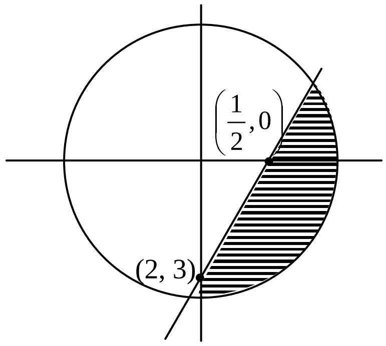
$x^2 + \frac{25}{4} + 5x + y^2 + 4 - 4y = \frac{25}{4}$

$x^2 + y^2 + 5x - 4y + 4 = 0$

from given points only point (–4, 0) satisfies this equation.

Q.27 The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts. If $S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\}$, then the number of point(s) in S lying inside the smaller part is **[IIT 2011]**

Sol.



Pont (x_1, y_1) lies inside the region if $x_1^2 + y_1^2 - 6 \le 0$ & $2x_1 - 3y_1 - 1 \le 0$.

$P_1 \equiv \left(2, \frac{3}{4}\right)$ $\quad 4 + \frac{9}{16} - 6 \le 0 \quad$ True

$\quad 4 - \frac{9}{4} - 1 > 0 \quad$ True

$P_2 \equiv \left(\frac{5}{2}, \frac{3}{4}\right)$ $\quad \frac{25}{4} + \frac{9}{16} - 6 \le 0 \quad$ False

$P_3 \equiv \left(\frac{1}{4}, \frac{-1}{4}\right)$ $\quad \frac{1}{16} + \frac{1}{16} - 6 \le 0 \quad$ True

$\quad \frac{2}{4} + \frac{3}{4} - 1 > 0 \quad$ True

$P_4 \equiv \left(\frac{1}{8}, \frac{1}{4}\right)$ $\quad \frac{1}{64} + \frac{1}{16} - 6 \le 0$ True

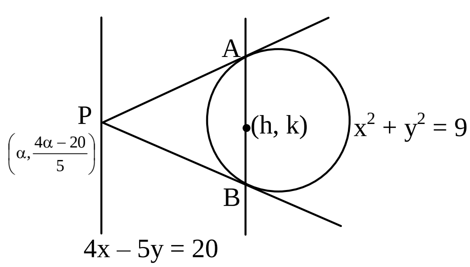
$\quad \frac{2}{8} - \frac{3}{4} - 1 > 0$ False

So P_1 & P_3 lies in the interval

Q.28 The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is **[IIT 2012]**

(A) $20(x^2 + y^2) - 36x + 45y = 0$
(B) $20(x^2 + y^2) + 36x - 45y = 0$
(C) $36(x^2 + y^2) - 20x + 45y = 0$
(D) $36(x^2 + y^2) + 20x - 45y = 0$

Sol. **[A]**



Equation of chord AB is T = 0

$$\alpha x + \left(\frac{4\alpha - 20}{5}\right) y = 9 \quad \ldots(i)$$

& $hx + ky - 9 = h^2 + k^2 - 9$ …(ii)

Θ Equation (i) & (ii) both represent the same line

So $\frac{\alpha}{h} = \frac{\frac{4\alpha - 20}{5}}{k} = \frac{9}{h^2 + k^2}$

$\alpha = \frac{9h}{h^2 + k^2} = \frac{45k + 20(h^2 + k^2)}{4(h^2 + k^2)}$

$36h = 45k + 20(h^2 + k^2)$

$20(x^2 + y^2) - 36x + 45y = 0$

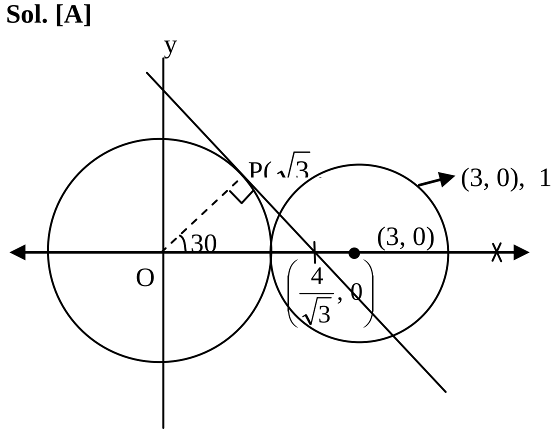
Passage (Q. 29 to Q. 30)

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P\left(\sqrt{3}, 1\right)$. A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$. **[IIT 2012]**

Q.29 A possible equation of L is

(A) $x - \sqrt{3}\, y = 1$ (B) $x + \sqrt{3}\, y = 1$
(C) $x - \sqrt{3}\, y = -1$ (D) $x + \sqrt{3}\, y = 5$

Sol. [A]



Slope of PT = tan (120º) = $-\sqrt{3}$

Slope of line L = $\frac{1}{\sqrt{3}}$

Line L ≡ $x - \sqrt{3}\, y + \lambda = 0$

tangent to $(x - 3)^2 + y^2 = 1$

$\frac{|3 + \lambda|}{2} = 1$

$\lambda + 3 = 2, -2$

$\lambda = -1, -5$

$x - \sqrt{3}\, y - 1 = 0$

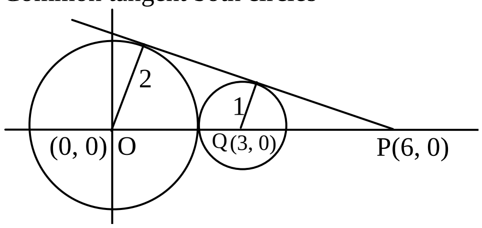
or $\quad x - \sqrt{3}\, y - 5 = 0$

Q.30 A common tangent of the two circles is

(A) $x = 4$ (B) $y = 2$

(C) $x + \sqrt{3}\, y = 4$ (D) $x + 2\sqrt{2}\, y = 6$

Sol. [D] Common tangent both circles



So $P \equiv (6, 0)$

line through P

$\lambda x - y - 6\lambda = 0$

tangent to circle $\dfrac{|3\lambda|}{\sqrt{1+\lambda^2}} = 1$

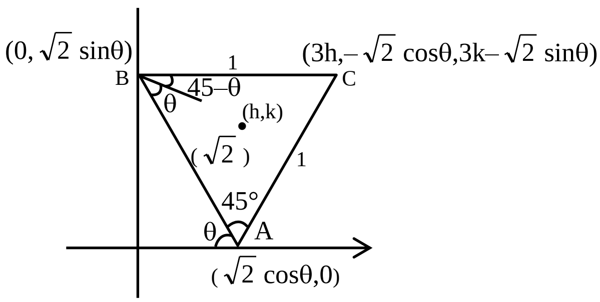
$9\lambda^2 = 1 + \lambda^2 \Rightarrow \lambda^2 = \dfrac{1}{8}$

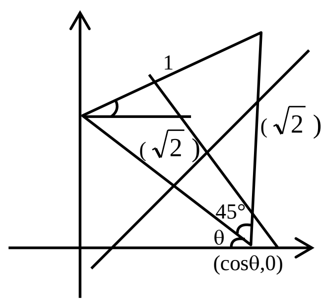
$\lambda = \dfrac{1}{2\sqrt{2}}\ ,\ \dfrac{-1}{2\sqrt{2}}$

Equation of tangent $x + 2\sqrt{2}\, y = 6$

Q.1 An isosceles right angled triangle whose sides are 1, 1, $\sqrt{2}$ lies entirely in the first quadrant with the ends of the hypotenuse on the coordinate axes. If it slides prove that the locus of its centroid is $(3x - y)^2 + (x - 3y)^2 = \frac{32}{9}$.

Sol.





Slope of AC = tan(180 – (θ + 45))

= tan(135 – θ) =

$\therefore C \equiv (\sqrt{2}\cos\theta + \cos(135 - \theta), 0 + \sin(135 - \theta))$

$= (3h - \sqrt{2}\cos\theta, 3k - \sqrt{2}\sin\theta)$

comparing y – co-ordinat gives

$\sin(135 - \theta) = 3k - \sqrt{2}\sin\theta$

$\therefore \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) = 3k - \sqrt{2}\sin\theta$

$\therefore \cos\theta + 3\sin\theta = 3\sqrt{2}k$ ……(1)

slope of BC = tan (45 – θ)

$\therefore C \equiv (0 + 1.\cos(45 - \theta), \sqrt{2}\sin\theta + 1.\sin(45 - \theta))$

$= (3h - \sqrt{2}\cos\theta, 3k - \sqrt{2}\sin\theta)$

comparing x-co-ordinate gives

$\cos(45 - \theta) = 3h - \sqrt{2}\cos\theta$

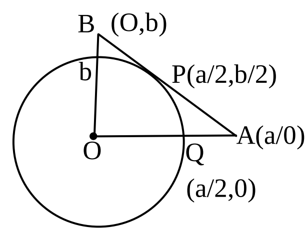
$\therefore \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) = 3h - \sqrt{2}\cos\theta$

Equating cos θ and sin θ from (1) & (2) and using cos2θ + sin2θ = 1 gives

$(3x - y)^2 + (x - 3y)^2 = \frac{32}{9}$

Q.2 In a right-angled triangle, the length of the sides are a and b(0 < a < b). Show that the radius of the circle passing through the mid-point of the smaller side and touching the hypotenuse at its mid-point is $b\sqrt{a^2 + b^2}/4a$.

Sol. We have to choose the perpendicular sides along axes of co-ordinates so that the hypotenuse is $\frac{x}{a} + \frac{y}{b} = 1$, which is a tangent at mid-point $\left(\frac{a}{2}, \frac{b}{2}\right)$



The euation of the cricle is given by

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 + \lambda\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0$$

it passes through mid-point Q of OA i.e. $\left(\frac{a}{2}, 0\right)$

$$\therefore \frac{b^2}{4} - \frac{\lambda}{2} = 0 \text{ or } \lambda = \frac{b^2}{2}$$

Reuquird circle is

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 + \frac{b^2}{2}\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0$$

or $x^2 + y^2 + x\left(-a + \frac{b^2}{2a}\right) + y\left(-b + \frac{b}{2}\right) + \frac{a^2 + b^2}{4} - \frac{b^2}{4} = 0$

or $x^2 + y^2 + \frac{b^2 - 2a^2}{2a}x - \frac{b}{2}y + \frac{a^2 - b^2}{4} = 0$

$\therefore r^2 = g^2 + f^2 - c$

$$\therefore r^2 = \left(\frac{b^2 - 2a^2}{4a}\right)^2 + \left(\frac{b}{4}\right)^2 - \frac{(a^2 - b^2)}{4}$$

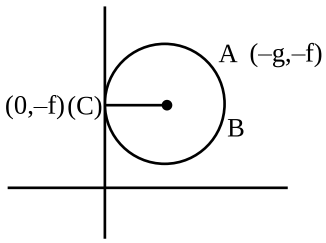
$$= \frac{b^4}{16a^2} + \frac{a^2}{4} - \frac{b^2}{4} + \frac{b^2}{16} - \frac{a^2}{4} + \frac{b^2}{4}$$

$$= \frac{b^4 + a^2b^2}{16a^2}$$

$$\therefore\ r = \sqrt{\frac{b^4 + a^2b^2}{16a^2}} = \frac{b}{4a}\sqrt{a^2 + b^2}$$

Q.3 Find the equation of the circle passing through the point A(4, 3) & B(2, 5) & touching the axis of y. Also find the point P on the y-axis such that the angle APB has largest magnitude.

Sol.



Let equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

since circle passes through A(4, 3), B(2, 5), C(0, –f)

passes (4, 3) $\Rightarrow 16 + 9 + 8g + 6f + c = 0$

$\Rightarrow 8g + 6f + c + 25 = 0$ …..(i)

passes (2, 5) $\Rightarrow 4 + 25 + 4g + 10f + c = 0$

$\Rightarrow 4g + 10f + c + 29 = 0$ ….(ii)

passes (0,–f) $\Rightarrow f^2 - 2f^2 + c = 0$

$\Rightarrow f^2 = c$ ……(iii)

from (i), (ii) we get

$$\begin{array}{r} 8g + 6f + c + 25 = 0 \\ 8g + 25f + 2c + 58 = 0 \\ - \quad - \quad - \quad - \\ \hline -14f - c - 33 = 0 \end{array}$$

$\Rightarrow 14f + c + 33 = 0$

$\Rightarrow f^2 + 14f + 33 = 0$

$\Rightarrow f^2 + 11f + 3f + 33 = 0$

$\Rightarrow f(f + 11) + 3(f + 11) = 0$

$\Rightarrow (f + 11)(f + 3) = 0$

$f = -3, -11 \ \therefore\ c = 9, 121$ respectively

from (i) when $f = -3, c = 9$ then

$8g + 6(-3) + 9 + 25 = 0$

$\Rightarrow 8g - 18 + 34 = 0$

$\Rightarrow 8g = -16$

$g = -2$

and when $f = -11, c = 121$, then

$8g + 6(-11) + 121 + 25 = 0$

$\Rightarrow 8g - 66 + 121 + 25 = 0$

$\Rightarrow 8g + 146 - 66 = 0 \Rightarrow 8g = -80$

$g = -10$

$\therefore$ Reqd. circle is

$x^2 + y^2 - 4x - 6y + 9 = 0$

or $x^2 + y^2 - 20x - 22y + 121 = 0$

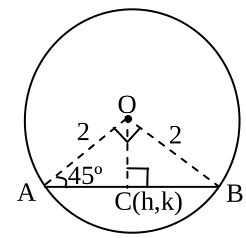
Q.4 The locus of the mid-point of a chord of the circle $x^2 + y^2 = 4$ which subtends a right angle at the origin is **[IIT 1984]**

(A) $x + y = 2$ (B) $x^2 + y^2 = 1$

(C) $x^2 + y^2 = 2$ (D) $x + y = 1$

Sol. **[C]**

As we have to find locus of mid-point of chord and we know perpendicular from centre bisects the chord



clearly $\angle OAC = 45°$

or $\frac{OC}{OA} = \sin 45°$

$\Rightarrow OC = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\therefore\ \sqrt{h^2 + k^2} = OC$

$\Rightarrow h^2 + k^2 = OC^2$

or $x^2 + y^2 = 2$

whcih is the required equation of locus of mid-point of chord subtentding right angle at centre.

Q.5 From the origin chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. The equation of the locus of the mid-points of these chords is.......... **[IIT 1985]**

Sol. For the equation of circle $x^2 + y^2 - 2x = 0$,

Let the mid-point of chords be (h, k)

$\therefore$ Equation of chord bisected at the point is $T = S_1$

$\Rightarrow h^2 + k^2 - 2h = xh + yk - (x + h)$

Which passes through (0, 0)

$\Rightarrow h^2 + k^2 - 2h = -h$

$\Rightarrow h^2 + k^2 - h = 0$

$\Rightarrow$ Locus is

$x^2 + y^2 - x = 0$

Q.6 From the point A (0, 3) on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that

AM = 2AB. The equation of the locus of M is .... **[IIT 1986]**

Sol. $(x + 2)^2 + (y – 3)^2 = 4$

let if M (h, k) where B is mid-point of A and M.

$\Rightarrow B\left(\frac{h}{2}, \frac{k+3}{2}\right)$

But AB is the chord of circle $x^2+4x+(y–3)^2 = 0$

Thus B must satisfy above equation

i.e. $\frac{h^2}{4} + \frac{4h}{2} + \left\{\frac{1}{2}(k+3)-3\right\}^2 = 0$

$\Rightarrow h^2 + k^2 + 8h – 6k + 9 = 0$

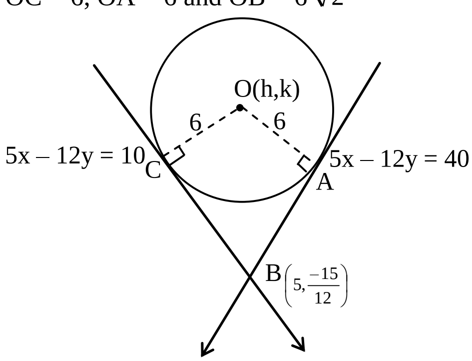
$\therefore$ locus of M is the circle

$$x^2 + y^2 + 8x – 6y + 9 = 0$$

Q.7 Lines $5x + 12y – 10 = 0$ and $5x – 12y – 40 = 0$ touch a circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines. **[IIT 1986]**

Sol. Since, $5x + 12y – 10 = 0$ and $5x – 12y – 40 = 0$ are both perpendicular tangents to the circle, C_1 $\therefore$ OABC forms a square.

Let the centre co-ordinate be (h, k) where OC = 6, OA = 6 and OB = $6\sqrt{2}$

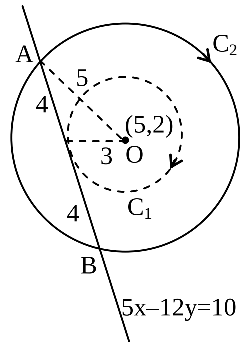


$\Rightarrow \frac{5h+12k-10}{13} = 3$

$\Rightarrow 5h+12k–10 = \pm 39$ and $5h – 12k – 40 = \pm 39$

on solving above equations. The co-ordinates which lie in Ist quadrant are (5, 2)

$\therefore$ Centre for $C_1(5, 2)$



To obtain euqtion of circle concentric with C_1 and making on intercept of lenght 8cm $5x + 12y = 10$ and $5x – 12y = 40$

$\Rightarrow$ required equation of circle C_2 has centre (5, 2) and radius 5.

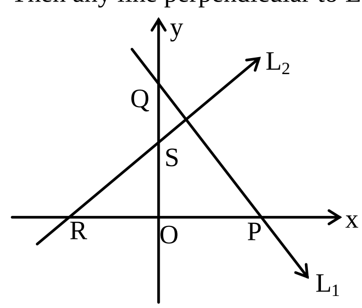
$\Rightarrow (x – 5)^2 + (y – 2)^2 = 5^2$

$\Rightarrow x^2 + y^2 –10x – 4y + 4 = 0$

Q.8 Let a given line L_1 intersect the x and y axes at P and Q respectively. Let another line L_2, perpendicular to L_1, cut the x and y axes at R and S, respectively. Show that the locus of the point of intersection of the lines PS and QR is a circle passing through the origin. **[IIT 1987]**

Sol. Let the equation of L_1 be $x \cos\alpha + y\sin\alpha = P_1$

Then any line perpendicular to L_1 is given by



$x \sin\alpha – y \sin\alpha = p_2$

Where p_2 is a variable.

Then L_1 meets x-axis at $P(p_1\sec\alpha, 0)$ and y-axis at Q (0, p, cosecα)

Similarly L_2 meets x-axis at $R(p_2 \operatorname{cosec}\alpha, 0)$ and y-axis at $S(O, – p_2 \sec\alpha)$

Now equation of PS is –

$$\frac{x}{p_1\sec\alpha} + \frac{x}{-p_2\sec\alpha} = 1$$

$$\Rightarrow \frac{x}{p_1} - \frac{x}{p_2} = \sec\alpha \quad ....(i)$$

Smilarly equation of QR is

$$\frac{x}{P_2\operatorname{cosec}\alpha} + \frac{x}{P_1\operatorname{cosec}\alpha} = 1$$

$\Rightarrow \frac{x}{P_2} + \frac{y}{P_1} = \sec\alpha$

...........(ii)

Locus of point of intersection of PS and QR can be obtained by eliminating variable P_2 from (i) and (ii)

i.e. $\left(\frac{x}{p_1} - \sec\alpha\right)\frac{x}{y} + \frac{y}{P_1} = \text{cosec } \alpha$

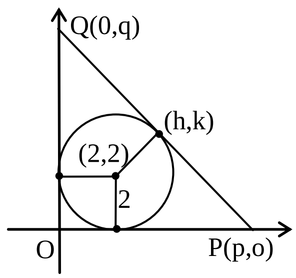
$\Rightarrow (x - p_1\sec \alpha)\ x + y_2 = p_1\ y \text{ cosece } \alpha$

$\Rightarrow x^2 + y^2 - p_1\ x \text{ seca} - p_1 y \text{ cosec}\alpha = 0$

Which is a circle through origin.

Q.9 The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the co-ordinate axes. The locus of the circumcentre of the triangle is $x + y - xy + k\ (x^2 + y^2)^{1/2} = 0$. Find k. **[IIT 1987]**

Sol.



Let the equation of third side be $\frac{x}{p} + \frac{y}{q} = 1$

Since OPQ is an Right angled triangle, whose circumcentre will bethe mid-point of PQ. Let (h, k) is the circum centre of the triangle.

Since line PQ touches the given circle

$\therefore$ Perpendicular distance from centre of circle = radius of circle

Centre is (2, 2) and given line is

$\frac{x}{p} + \frac{y}{q} = 1$ i.e $qx + py - pq = 0$

$\frac{2q + 2p - pq}{\sqrt{q^2 + p^2}} = 2$

since (h, k) is mid-point of PQ

$\therefore P = 2h,\ q = 2k$

$2(2k) + 2(2h) - 4hk = \pm 2\sqrt{4h^2 + 4k^2}$

$\Rightarrow h + k - hk \pm \sqrt{h^2 + k^2} = 0$

$\therefore$ locus is

$x + y - xy \pm \sqrt{x^2 + y^2} = 0$

$\therefore k = \pm 1$

Q.10 The equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, are **[IIT 1988]**

(A) $x = 0$

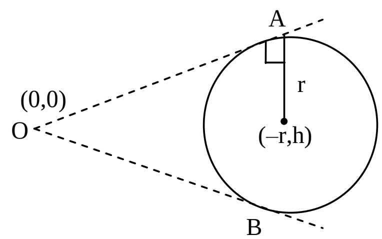
(B) $y = 0$

(C) $(h^2 - r^2)x - 2rhy = 0$

(D) $(h^2 - r^2)x + 2rhy = 0$

Sol. **[C]**

Since tangents are drawn from origin. so let equation of tangent be y = mx where m is slope of tangent.



Length of perpendicular from origin = radius

$\Rightarrow \frac{mr - h}{\sqrt{m^2 + 1}} = r$

$\Rightarrow m^2r^2 + h^2 - 2mrh = r^2\ (m^2 + 1)$

$\Rightarrow m = \frac{-r^2 + h^2}{2rh} = \frac{h^2 - r^2}{2hr}$

or $h^2 + m^2r^2 - 2mhr = r^2 + m^2r^2$

or $0.m^2 + 2mhr + (r^2 - h^2) = 0$

$\therefore m = \infty,\ m = \frac{h^2 - r^2}{2hr}$

Tangents are $x = 0$ for $m = \infty$

and $(h^2 - r^2)x - 2rhy = 0$

Q.11 If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = k^2$ orthogonally, then the equation of the locus of its centre is **[IIT 1988]**

(A) $2ax + 2by - (a^2 + b^2 + k^2) = 0$

(B) $2ax + 2by - (a^2 - b^2 + k^2) = 0$

(C) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - k^2) = 0$

(D) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - k^2) = 0$

Sol. **[A]**

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts $x^2 + y^2 = k^2$ orthogonally, if

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$\Rightarrow -2g.0 - 2f.0 = c - k^2$

$\Rightarrow c = k^2$ ....(i)

Also, $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through (a, b)

$\therefore a^2 + b^2 + 2ga + 2fb + c = 0$ ......(ii)

$\Rightarrow$ Required equation of locus of centre is

$-2ax - 2by + a^2 + b^2 + k^2 = 0$

$\Rightarrow 2ax + 2by - (a^2 + b^2 + k^2) = 0$

Q.12 If the circle $C_1 : x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that common chord is of maximum length and has a slope equal to 3/4, then the coordinates of the centre of C_2 are......... **[IIT 1988]**

Sol. $C_1 : x^2 + y^2 = 16$

$C_2 : (x - h)^2 + (y - k)^2 = 25$

common chords by $S_1 - S_2 = 0$ is

$2hx + 2ky = h^2 + k^2 - 9$

$\therefore$ its slope $= -\frac{h}{k} = \frac{3}{4}$ (given)

If p be the length of perpendicular on it from the centre (0, 0) of C_1 of radius 4, then

$$p = \frac{h^2 + k^2 - 9}{\sqrt{4h^2 + 4k^2}}$$

Also, the lenght of the chord is

$2\sqrt{r^2 - p^2} = 2\sqrt{4^2 - p^2}$

The chord will be of maximum length, if $\phi = 0$

or $h^2 + k^2 - 9 = 0$

or $h^2 + \frac{16}{9}h^2 = 9$

or $h = \pm 9/5$

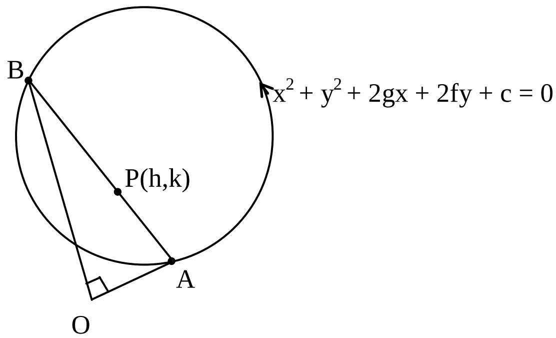
$\therefore k = \mu \frac{12}{5}$

$\therefore$ centres are $\left(\frac{9}{5}, \frac{-12}{5}\right)$

and $\left(-\frac{9}{5}, \frac{12}{5}\right)$

Q.13 Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends a right angle at the origin. **[IIT 1988]**

Sol. Let P(h, k) be the foot of perpendicular drawn from origin, 0(0, 0) on the chord AB of the given circle such that the chord AB subtends a right angle at the origin.



The equation of chord AB is

$$y - k = -\frac{h}{k}(x - h)$$

$\Rightarrow hx + ky = h^2 + k^2$

the combined equation of OA and OB is homogeneous equation of second degree obtained by the help of the given circle and the chord AB and is given by

$$x^2+y^2+(2gx + 2fy)\left(\frac{hx+ky}{h^2+k^2}\right)+c\left(\frac{hx+ky}{h^2+k^2}\right) = 0$$

$\Rightarrow 2(h^2 + k^2) + 2(gh + fk) + c = 0$

$\Rightarrow h^2 + k^2 + gh + fk + \frac{c}{2} = 0$

$\therefore$ Required eqution of locus is

$x^2 + y^2 + gx + fy + \frac{c}{2} = 0$

Q.14 If the two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then **[IIT 1989]**

(A) $2 < r < 8$ (B) $r < 2$

(C) $r = 2$ (D) $r > 2$

Sol. **[A]**

As, if the two circles intersect in two distinct point

$\Rightarrow$ distance between centres lies between $|r_1 - r_2|$ and $|r_1 + r_2|$

i.e. $|r - 3| < \sqrt{(4-1)^2 + (1+3)^2} < |r + 3|$

$\Rightarrow |r - 3| < 5 < |r + 3|$

$\therefore r < 8$ or $r > 2$

$\therefore 2 < r < 8$

Q.15 The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. Then the equation of this circle is - **[IIT 1989]**

(A) $x^2 + y^2 + 2x - 2y = 62$

(B) $x^2 + y^2 + 2x - 2y = 47$

(C) $x^2 + y^2 - 2x + 2y = 47$

(D) $x^2 + y^2 - 2x + 2y = 62$

Sol. **[C]**

Since, $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle, this implies that their point of intersection is centre i.e. (1, –1) and area is $\pi r^2 = 154$ given

$\Rightarrow \pi r^2 = 154$

$\Rightarrow r^2 = 154 \times \frac{7}{22}$

$\Rightarrow r = 7$

$\therefore$ Required equation of circle is

$$(x - 1)^2 + (y+1)^2 = 7^2$$

or $x^2 + y^2 - 2x + 2y = 47$

Q.16 A circle touches the line y = x at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point (–10, 2) in its interior and the length of its chord on the line x + y = 0 is $6\sqrt{2}$. Determine the equation of the circle. **[IIT 1990]**

Sol. The parametric form of OP is

$$\frac{x-0}{\cos 45^\circ} = \frac{y-0}{\sin 45^\circ}$$

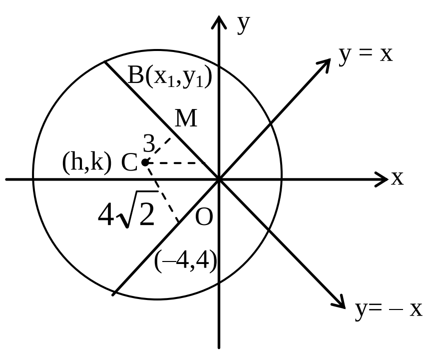
Since, $OP = 4\sqrt{2}$

So, the co-ordinate of P are given by

$$\frac{x-0}{\cos 45^\circ} = \frac{y-0}{\sin 45^\circ} = -4\sqrt{2}$$

So, P (– 4,– 4)

Let C(h, k) be the centre of circle and r be its radius.



Now, CP $\perp$ OP

$\Rightarrow \frac{k+4}{h+4}(1) = -1$

$\Rightarrow k + 4 = -h - 4$

$\Rightarrow h + k = -8$ ....(i)

Also, $CP^2 = (h + 4)^2 + (k + 4)^2$

$\Rightarrow (h + 4)^2 \neq (k + 4)^2 = r^2$ ....(ii)

In ΔACM, we have

$$AC2 = \left(3\sqrt{2}\right)^2 + \left(\frac{h+k}{\sqrt{2}}\right)^2$$

$\Rightarrow r^2 = 18 + 32 = 50$

$\Rightarrow r = 5\sqrt{2}$

............(iii)

Also, CP = r

$\Rightarrow \left|\frac{h-k}{\sqrt{2}}\right| = r$

$\Rightarrow h - k = \pm 10$ ....(iv)

from (i) & (iv), we get

(h = – 9, k = 1) or (h =1, k = –9)

Thus the equation of the circles are

$(x + 9)^2 + (y - 1)^2 = \left(5\sqrt{2}\right)^2$

and $(x - 1)^2 + (y + 9)^2 = \left(5\sqrt{2}\right)^2$

or $x^2 + y^2 + 18x - 2y + 18y + 32 = 0$

and $x^2 + y^2 - 2x + 18y + 32 = 0$

Clearly, (–10, 2) lies interior of

$x^2 + y^2 + 18x - 2y + 32 = 0$

Hence, the required equation of circle is $x^2 + y^2 + 18x - 2y + 32 = 0$

Q.17 If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ = ... **[IIT 1991]**

Sol. As, the point of intersection of the co-ordinate axes with the line $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$ forms the circle.

$$\therefore (\lambda x - y + 1)(x - 2y + 3) = 0$$

represents a circle if,

cofficient of x^2 = coefficient of y^2

and cofficient of xy = 0

$\Rightarrow \lambda = 2$ and $-2\lambda - 1 = 0$

$\Rightarrow \lambda = 2$ and $l = -\frac{1}{2}$

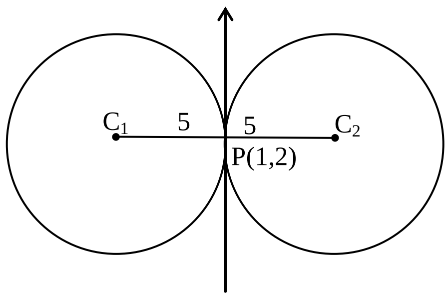
$\Rightarrow \lambda = 2$ and –1/2

Q.18 Two circles, each of radius 5 units, touch each other at (1, 2). If the equation of their common tangent is 4x + 3y = 10, find the equation of the circles. **[IIT 1991]**

Sol. We have,

Slope of the common tangent = $\frac{-4}{3}$

$\therefore$ slope of $C_1C_2 = \frac{3}{4}$



if C_1C_2 makes an angle θ with x-axis, then cosθ = 4/5 and sinθ = 3/5

So, the equation of C_1C_2 in parametric form is

$$\frac{x-1}{4/5} = \frac{y-2}{3/5} \quad \text{............(i)}$$

Since, C_1 and C_2 are points on (i) at a distance of 5 units from P. so, the co-ordinates of C_1 & C_2 are given by

$$\frac{x-1}{4/5} = \frac{y-2}{3/5} = \pm 5$$

⇒ x = 1 ± 4 and y = 2 ± 3

Thus, the co-ordinates of C_1 and C_2 are (5, 5) and (–3, 1) respectively.

Hence the rqutions of the two circles are

$(x-5)^2 + (y-5)^2 = 5^2$

and $(x+3)^2 + (y+1)^2 = 5^2$

Q.19 The centre of a circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is - **[IIT 1992]**

(A) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (B) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(C) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, -2^{\frac{1}{2}}\right)$

Sol. **[D]**

Let (h, k) be the centre of the requird circle, then

$$\sqrt{(h-0)^2 + (k-0)^2} = \sqrt{(h-1)^2 + (k-0)^2}$$

$$\Rightarrow \sqrt{h^2 + k^2} = \sqrt{h^2 - 2h + 1 + k^2}$$

$\Rightarrow h^2 + k^2 = h^2 - 2h + 1 + k^2$

$\Rightarrow -2h + 1 = 0$

$\Rightarrow h = 1/2$

Now (0, 0) and (1, 0) lie inside the circle $x^2 + y^2 = 9$ Therefore, the required circle can touch the given circle internally.

i.e. $C_1.C_2 = r_1 - r_2$

$$\Rightarrow \sqrt{h^2 + k^2} = 3 - \sqrt{h^2 + k^2}$$

$$\Rightarrow 2\sqrt{h^2 + k^2} = 3$$

$$\Rightarrow 2\sqrt{\frac{1}{4} + k^2} = \frac{3}{2}$$

$$\Rightarrow \frac{1}{4} + k^2 = 9/4$$

$\Rightarrow k^2 = 2$

$\Rightarrow k = \pm \sqrt{2}$

$\therefore$ Requried centre = $\left(\frac{1}{2}, \pm\sqrt{2}\right)$

$= \left(\frac{1}{2}, \pm 2^{1/2}\right)$

from option $\left(\frac{1}{2}, -2^{1/2}\right)$ is correct answer.

Q.20 Let a circle be given by 2x(x – a) + y(2y – b) = 0, (a ≠ 0, b ≠ 0). Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from $\left(a, \frac{b}{2}\right)$ **[IIT 1992]**

Sol. The given circle is

2x(x – a) + y (2y – b) = 0

or x(x – a) + y (y – b/2) = 0

or $x^2 + y^2 - ax - \frac{by}{2} = 0$ ....(i)

Let one of the chord through $\left(a, \frac{b}{2}\right)$ be bisected at the point (h, 0). then the equation of the chord having (h, 0) as mid-point is $T = S_1$

$\Rightarrow h.x + 0.y - \frac{a}{2}(x + h) - \frac{b}{4}(y + 0)$

$\Rightarrow \left(h - \frac{a}{2}\right)x - \frac{by}{4} - \frac{ah}{2} = h^2 - ah$...........(ii)

Now, (ii) will pass through $\left(a, \frac{b}{2}\right)$ if

$\left(h - \frac{a}{2}\right)a - \frac{b}{4}.\frac{b}{2} - \frac{a}{2}.h = h^2 - ah$

$\Rightarrow h^2 - \frac{3}{2}ah + \frac{a^2}{2} + \frac{b^2}{8} = 0$ ....(iii)

According to the given condiation, (iii) must have two distinct real roots. this is possible if the discriminant of (iii) is greater then O.

i.e. if $\frac{9}{4}a^2 - 4\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$

$\Rightarrow \frac{a^2}{4} - \frac{b^2}{2} > 0$

$\Rightarrow a^2 > 2b^2$

Q.21 The locus of the centre of a circle, which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis, is given by the equation: **[IIT 1993]**

(A) $x^2 - 6x - 10y + 14 = 0$
(B) $x^2 - 10x - 6y + 14 = 0$
(C) $y^2 - 6x - 10y + 14 = 0$
(D) $y^2 - 10x - 6y + 14 = 0$

Sol. **[D]**

Let (h, k) be the centre of the circle which touches the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and y-axis. Now, $x^2 + y^2 + 2(-3)x + 2(-3)y + 14 = 0$, the centre of this circle is (3, 3)

and radius is $\sqrt{3^2 + 3^2 - 14}$

$= \sqrt{18 - 14} = 2$

since the circle touches y-axis, the distance from its centre to y-axis must be equal to its radius, therefore its radius is h. Again the circles meets externally therefore the distance between two centres = sum of the radii of the two circles.

Hence $(h - 3)^2 + (k - 3)^2 = (2 + h)^2$

i.e. $h^2 + 9 - 6h + k^2 + 9 - 6k = 4 + h^2 + 4h$

i.e. $k^2 - 10h - 6k + 14 = 0$

Thus the locus of (h, k) is

$y^2 - 10x - 6y + 14 = 0$

Q.22 The equation of the locus of the mid-points of the chord of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $2\pi/3$ at its centre is..................... **[IIT 1993]**

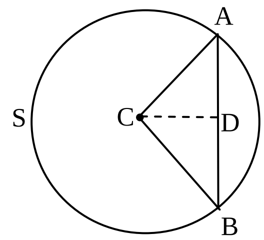
Sol. Given circle is $4x^2 + 4y^2 - 12x + 4y + 1 = 0$

i.e. $x^2 + y^2 - 3x + y + \frac{1}{4} = 0$

its centre is $\left(\frac{3}{2}, \frac{-1}{2}\right)$ and radius

$\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 - \frac{1}{4}}$

$= \sqrt{\frac{9}{4} + \frac{1}{4} - \frac{1}{4}} = \frac{3}{2}$



Again, Let S is a circle with centre at C and AB is given chord and AD subtent angle $2\pi/3$ at the centre and D be the mid-point of AB and let its coordinates are (h, k)

Now $\angle DCA = \frac{1}{2}\ (\angle BCA) = \frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$

Again $\frac{DA}{\sin\frac{\pi}{3}} = \frac{CA}{\sin\frac{\pi}{2}}$

$\Rightarrow DA = CA \sin\pi/3$

$= \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$

Now, in ΔACD,

$CD^2 = CA^2 - AD^2$

$CD^2 = \frac{9}{4} - \frac{27}{16} = \frac{9}{16}$

But $CD^2 = (h-3/2)^2 + \left(k + \frac{1}{2}\right)^2 = \frac{9}{16}$

on genralising, we get

$$\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{9}{16}$$

$\Rightarrow 16x^2 + 16y^2 - 48x + 16y + 31 = 0$

Q.23 Consider a family of circles passing through two fixed points A (3, 7) and B (6, 5). Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinate of this point. **[IIT 1993]**

Sol. The equation of the circle on the line joining the points A(3, 7) and B(6, 5) as diameter is

$(x - 3)(x - 6) + (y - 7)(y - 5) = 0$ ....(i)

and the equation of the line joining the points A(3, 7) and B(6, 5) is

$y - 7 = \frac{7-5}{3-6}(x - 3)$

$\Rightarrow 2x + 3y - 27 = 0$ ....(ii)

Now the equation of family of circles passing through the point of intersection of (i) and (ii) is

$S + \lambda P = 0$

$\Rightarrow (x-3)(x-6)+(y-7)(y-5)+\lambda(2x+3y-27)=0$

$\Rightarrow x^2 - 6x - 3x + 18 + y^2 - 5y - 7y + 35 + 2\lambda x + 3\lambda y - 27\lambda = 0$

$\Rightarrow S_1 \equiv x^2 + y^2 - x(2\lambda - 9) + y(3\lambda - 12) + (53 - 27\lambda) = 0$ ....(iii)

Again the circle, which cuts the memners of family of circles, is

$S_2 \equiv x^2 + y^2 - 4x - 6y - 3 = 0$ ....(iv)

and the equation of common chord to circles S_1 and S_2 is –

$S_1 - S_2 = 0$

$\Rightarrow \{x(2\lambda - 9) + y(3\lambda - 12) + (53 - 27\lambda)\} - \{-4x - 6y - 3\} = 0$

$\Rightarrow x\{2\lambda - 9 + 4\} + y(3\lambda - 12 + 6) + (53 - 27\lambda + 3) = 0$

$\Rightarrow 2\lambda x - 5x + 3\lambda y - 6y + 56 - 27\lambda = 0$

$\Rightarrow (5x - 6y + 56) + \lambda(2x + 3y - 27) = 0$

Which represents equatious of two straight lines passing through the fixed point whose co-ordinates are obtaind by solving the two equations.

$5x + 6y - 56 = 0$ and $2x + 3y - 27 = 0$

on solving, weget

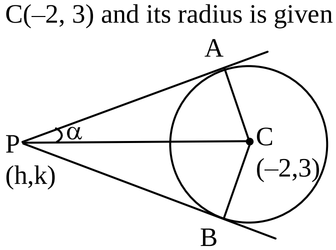
$x = 2$ and $y = 23/3$

Q.24 The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. The equation of the locus of the point P is- **[IIT 1996]**

(A) $x^2 + y^2 + 4x - 6y + 4 = 0$
(B) $x^2 + y^2 + 4x - 6y - 9 = 0$
(C) $x^2 + y^2 + 4x - 6y - 4 = 0$
(D) $x^2 + y^2 + 4x - 6y + 9 = 0$

Sol. **[B,D]**

Centre of the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is $C(-2, 3)$ and its radius is given by



$\sqrt{(-2)^2 + (3)^2 - 9\sin^2\alpha - 13\cos^2\alpha}$

$= \sqrt{4 + 9 - 9\sin^2\alpha - 13\cos^2\alpha}$

$= \sqrt{13 - 13\cos^2\alpha - 9\sin^2\alpha}$

$= \sqrt{13(1 - \cos^2\alpha) - 9\sin^2\alpha}$

$= \sqrt{13\sin^2\alpha - 9\sin^2\alpha}$

$= \sqrt{4\sin^2\alpha} = 2\sin\alpha$

Let (h, k) be any point P and $\angle APC = \alpha$, $\angle PAC = \pi/2$ i.e. triangle APC is a right angle triangle.

Thus, $\sin\alpha = \dfrac{AC}{PC} = \dfrac{2\sin\alpha}{\sqrt{(h+2)^2 + (k-3)^2}}$

$\Rightarrow \sqrt{(h+2)^2 + (k-3)^2} = 2$

$\Rightarrow (h + 2)^2 + (k - 3)^2 = 4$

$\Rightarrow h^2 + 4 + 4h + k^2 + 9 - 6k = 4$

$\Rightarrow h^2 + k^2 + 4h - 6k + 9 = 0$

Thus, required equation of the locus is

$x^2 + y^2 + 4x - 6y + 9 = 0$

Q.25 (a) The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Find the equation of the circle with AB as a diametre.

(b) Find the the intervals of values of a for which the line $y + x = 0$ bisects two chords drawn from a point to $\left(\dfrac{1+\sqrt{2}a}{2}, \dfrac{1-\sqrt{2}a}{2}\right)$ the circle

$2x^2 + 2y^2 - (1 + \sqrt{2}\,a)x - (1 - \sqrt{2}\,a)y = 0.$

[IIT 1996]

Sol. Equation of any circle passing through the point of intersection of $x^2 + y^2 - 2x = 0$ and $y = x$ is

$(x^2 + y^2 - 2x) + \lambda(y - x) = 0$

$\Rightarrow x^2 + y^2 - (2 + \lambda)x + \lambda y = 0$

its centre is $\left(\dfrac{2+\lambda}{2}, \dfrac{-\lambda}{2}\right)$

$\Rightarrow$ For AB to be the diameter of the requried circle the must be on AB.

i.e. $2 + \lambda = -\lambda$

$\Rightarrow 2\lambda = -2 \Rightarrow \lambda = -1$

Therefore, equation of the required circle is

$x^2 + y^2 - (2 - 1)x - 1.y = 0$

$\Rightarrow x^2 + y^2 - x - y = 0$

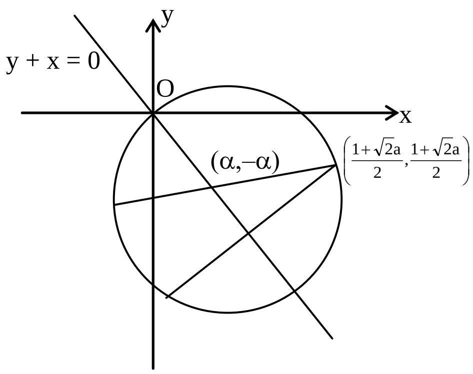
$\Rightarrow x^2 - x + y^2 - y = 0$

$\Rightarrow \left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{1}{2}$

(b) $2x^2 + 2y^2 - (1 + \sqrt{2}\,a)x - (1 - \sqrt{2}\,a)y = 0$

$\Rightarrow x^2 + y^2 - \left(\frac{1+\sqrt{2}a}{2}\right)x - \left(\frac{1-\sqrt{2}a}{2}\right)y = 0$

since, y + x = 0 bisects two chords of this circle, mid-point of the chords must be of the form (α, –α).



Equation of the chord having (α,–α) as mid-point is $T = S_1$

$\Rightarrow x\alpha + y(-\alpha) - \left(\frac{1+\sqrt{2}a}{4}\right)(x+\alpha) - \left(\frac{1-\sqrt{2}a}{4}\right)(y-\alpha)$

$= \alpha^2 + (-\alpha)^2 - \left(\frac{1+\sqrt{2}a}{4}\right)\alpha - \left(\frac{1-\sqrt{2}a}{4}\right)(-\alpha)$

$\Rightarrow 4x\alpha - 4y\alpha - (1+\sqrt{2}a)x - (1-\sqrt{2}a)y - (1-\sqrt{2}a)y + (1+\sqrt{2}a)\alpha$

$= 4\alpha^2 + 4\alpha^2 - (1+\sqrt{2}a) . 2\alpha + (1-\sqrt{2}a) . 2\alpha$

$\Rightarrow 4\alpha x + 4\alpha y - (1+\sqrt{2}a) x. - (1-\sqrt{2}a)y$

$= 8a^2 - (1+\sqrt{2}a)\alpha + (1-\sqrt{2}a)\alpha$

But this chord will pass through the point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$

$4\alpha\left(\frac{1+\sqrt{2}a}{2}\right) - 4\alpha\left(\frac{1-\sqrt{2}a}{2}\right) - \frac{(1+\sqrt{2}a)(1+\sqrt{2}a)}{2}$

$- \frac{(1-\sqrt{2}a)(1-\sqrt{2}a)}{2}$

$= 8a^2 - 2\sqrt{2}a\alpha$

$\Rightarrow 2\alpha[(1+\sqrt{2}a - 1 + \sqrt{2}a)] = 8\alpha^2 - 2\sqrt{2}a\alpha$

$\Rightarrow 4\sqrt{2}a\alpha - \frac{1}{2}[2 + 2(\sqrt{2}a)^2] = 8\alpha^2 - 2\sqrt{2}a\alpha$

$[\Theta\ (a + b^2) + (a-b)^2 = 2(a^2 + b^2)]$

$\Rightarrow 8\alpha^2 - 6\sqrt{2}a\alpha + 1 + 2a^2 = 0$

But this quadratic equation will have two distinct roots if

$(6\sqrt{2}a)^2 - 4(8)(1+2a^2) > 0$

$\Rightarrow 72a^2 - 32(1 + 2a^2) > 0$

$\Rightarrow 72a^2 - 32 - 64a^2 > 0$

$\Rightarrow 8a^2 - 327 > 0$

$\Rightarrow a^2 - 47 > 0$

$\Rightarrow a^2 > 4$

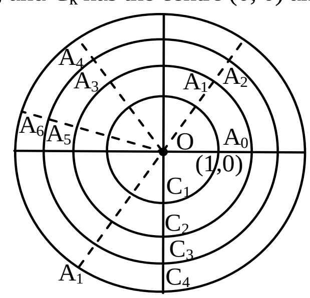
$\Rightarrow a < -2 \cup a > 2$

$\therefore a \in (-\infty, -2) \cup (2, \infty)$

Q.26 For each natural number k, let C_k denotes the circle with radius k centimetres and centre at the origin. On the circle C_k, α-particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1}in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n then n = ...

[IIT 1997]

Sol. It is given that C_1 has centre (0, 0) and radius equal to 1 similarly; C_2 has centre (0, 0) and radius 2, and C_k has the centre (0, 0) and radius k.



Now partical starts it motion from (1, 0) and moves 1 radian on first circle then particle shifts form C_1 to C_2

After that particle moves 1 radian on C_2 and than particle shifts from C_2 to C_3 similarly, particle move on n circles

Now $n \geq 2\pi$ because particle crosses the x-axis for the first time on C_n, then n is least positive integer.

Therefore. n = 7 is the answer.

Q.27 (a) Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line drawn from the point P intersects the curve at points Q and R. If the products PQ. PR is independent of the slope of the line, then show that the curve is a circle

(b) Two vertices of an equilateral triangle are (–1,0) and (1,0) and its third vertex lies above the x-axis. Find the equation of its circumcircle. **[IIT - 97]**

Sol. (a) The given circle is

$ax^2 + 2hxy + by^2 = 1$

Let the point P not lying on (i) be (x_1,y_1), Let θ be the indintion of the line through P whin intersects the given curve at Q and R. Then equation of line through Pis

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$\Rightarrow x = x_1 + r\cos\theta,\ y = y_1 + r\ \text{sinq}$

for point Q and R, above point must lie on (i)

$\Rightarrow a(x_1 + r\cos\theta)^2 + 2h(x_1 + r\cos\theta)(y_1 + r\sin\theta) + b(y_1 + r\sin\theta)^2 = 1$

$\Rightarrow (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)r^2 + 2(ax_1\cos\theta + hx_1\sin\theta + hy_1\cos\theta + by_1\sin\theta)r^2 + 2(ax_1\cos\theta + hx_1\sin\theta + hy_1\cos\theta + by_1\sin\theta)r + (ax_1^2 + 2x_1y_1 + by_1^2 - 1) = 0$

It is quadratic in r, giving two values fo r as PQ and PR.

$$\text{Q } PQ.PR = \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}$$

Here, $ax_1^2 + 2hx_1y_1 + by_1^2 - 1 \neq 0$ cos (x_1,y_1) does not lie on (i),Also,

$a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$

$= a + 2h\sin\theta\cos\theta + (b - a)\sin^2\theta$

$= a + \sin\theta(2h\cos\theta + (b - a)\sin\theta)$

$= a + \sin\theta\sqrt{4h^2 + (b-a)^2}\ .\ [\cos\theta\sin\phi + \sin\theta\cos\phi]$

where $\tan\theta = \dfrac{b-a}{2h}$

$= a + \sqrt{4h^2 + (b-a)^2}\ \sin\theta\sin(\theta + \phi)$

which will be independent of θ, if

$4h^2 + (b - a)^2 = 0$

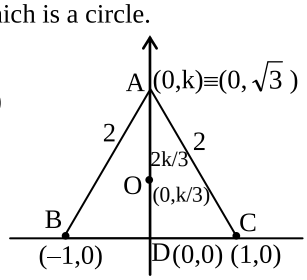
$\Rightarrow h = 0$ and $b = a$

$\therefore$ equation (i) reduees to

$$x^2 + y^2 = \frac{1}{a}$$

which is a circle.

(b)



here BC = 2 = AB = AC as the traingle is euilateral. its vertex will be lie on the right bisector of BC. i.e. in y-axis. let it be (0, k) where k is +ve and equal to $2\sin 60^o = \sqrt{3}$

its centre O will coincide with centroed $G\left(0, \frac{k}{3}\right)$

i.e. $\left(0, \frac{1}{\sqrt{3}}\right)$ and radius $r = OA = \frac{2k}{3} = \frac{2}{\sqrt{3}}$

circle is $x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$

$\Rightarrow x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$

Q.28 The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through the point........... **[IIT-97]**

Sol. A point on the line $2x + y = 4$ is of the form $(h, 4 - 2h)$ Equation of the chord of contact is T = 0

i.e. $hx + (4 - 2h)y = 1$

or $(4y - 1) + h(x - 2y) = 0$

This line passes through the point of intersection of $4y - 1 = 0$ and $x - 2y = 0$

i.e. through the point $\left(\frac{1}{2}, \frac{1}{4}\right)$

Q.29 Let C be any circle with centre $(0, \sqrt{2})$. Prove that at most two rational points can be there on C. (A rational point is a point both of whose coordinates are rational numbers) **[IIT 1997]**

Sol. Equations of any circle C with centre at $(0, \sqrt{2})$ is given by

$(x - 0)^2 + (y - \sqrt{2})2 = r^2$

or $x^2 + y^2 - 2\sqrt{2}y + 2 = r^2$ ....(i)

where $r > 0$.

Let (x_1,y_1), (x_2, y_2), (x_3, y_3) be three distinct rational point on(i). since a straight line parallel to x-axis meets a circle in atmost two points, either, y_1,y_2 or y_1,y_3 putting these in (i), we get

$x_1^2 + y_1^2 - 2\sqrt{2}y_1 = r^2 - 2$ ....(ii)

$x_2^2 + y_2^2 - 2\sqrt{2}y_2 = r^2 - 2$ ....(iii)

$x_3^2 + y_3^2 - 2\sqrt{2}y_3 = r^2 - 2$ ....(iv)

subtracting (ii) from (iii), we obtain

$p_1 - \sqrt{2}\, q_1 = 0$

where $p_1 = x_2^2 + y_2^2 - x_1^2 - y_1^2$

$q_1 = y_2 - y_1$

Subtreting (ii) form (iv), we obtain

$p_2 - \sqrt{2}\, q_2 = 0$

where $p_2 = y_3 - y_2$

$x_3^2 + y_3^2 - x_1^2 - y_1^2$

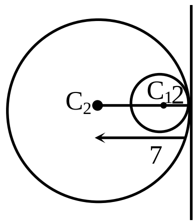
Now p_1, p_2, q_1, q_2 are rational numbers. Also either $q_1 \neq 0$ or $q_2 \neq 0$. If $q_1 \neq 0$. then

$\sqrt{2}$ =p_2/p_2 In aby case $\sqrt{2}$ is a rational number. This is a contradiction.

Q.30 The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is **[IIT-98]**

(A) 0 (B) 1 (C) 3 (D) 4

Sol. **[B]**



$S_1 \equiv x^2 + y^2 - 4 = 0$

$C_1 \equiv (0, 0)$; $r_1 = 2$

$S_2 \equiv x^2 + y^2 - 6x - 8y - 24 = 0$

$C_2 \equiv (3, 4)$; $r_2 = 7$

Θ $C_1C_2 = 5$

$\therefore$ $C_1C_2 = r_2 - r_1$

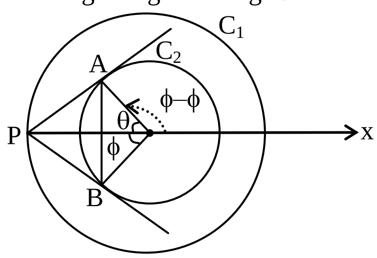
So, only one tangent possible

Q.31 C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. **[IIT-98]**

Sol. Let the point P be (2r cosθ, 2r sinθ)

we have OA = r, OP = 2r

As ΔOAP is a right angled triangle.



$\cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$

$\therefore$ co-ordinates of A are

$$\left[r\cos\left(\theta-\frac{\pi}{3}\right), r\sin\left(\theta-\frac{\pi}{3}\right)\right]$$

and that of B are $\left[r\cos\left(\theta+\frac{\pi}{3}\right), r\sin\left(\theta+\frac{\pi}{3}\right)\right]$

If p,q is the centroid of ΔPAB

then $p = \frac{1}{3}\left[r\left\{\cos\left(\theta-\frac{\pi}{3}\right)+r\cos\left(\theta+\frac{\pi}{3}\right)\right\}+2r\cos\theta\right]$

$= \frac{1}{3}\left[r\left(2\cos\frac{\theta-\frac{\pi}{3}+\theta+\frac{\pi}{3}}{2}.\cos\frac{\theta-\frac{\pi}{3}-\theta-\frac{\pi}{3}}{2}\right)+2r\cos\theta\right]$

$= \frac{1}{3}$ [r{2cosθ cos $\frac{\pi}{3}$ } + 2rcosθ]

$= \frac{1}{3}$ [rcosθ + 2r cosθ] = r cosθ

and $q = \frac{1}{3}\left[r\sin\left(\theta-\frac{\pi}{3}\right)+r\sin\left(\theta+\frac{\pi}{3}\right)+2r\sin\theta\right]$

$= \frac{1}{3}\left[r\left\{\sin\left(\theta-\frac{\pi}{3}\right)+\sin\left(\theta+\frac{\pi}{3}\right)\right\}+2r\sin\theta\right]$

$= \frac{1}{3}\left[r\left(2\sin\frac{\theta-\frac{\pi}{3}+\theta+\frac{\pi}{3}}{2}.\cos\frac{\theta-\frac{\pi}{3}-\theta-\frac{\pi}{3}}{2}\right)+2r\sin\theta\right]$

$= \frac{1}{3}$ [r{2sinθ cos $\frac{\pi}{3}$ } + 2rsin θ]

$= \frac{1}{3}$ [r{sinθ} + 2r sinθ] = r sinθ

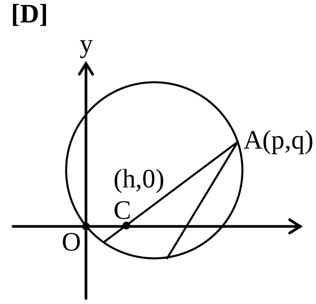
Now, (p, q)= (rcosθ, rsinθ) lies on $x^2 + y^2 = r^2$ which is called C_1.

Q.32 If two distinct chords drawn from the point (p,q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the x-axis then **[IIT-99]**

(A) $p^2 = q^2$ (B) $p^2 = 8q^2$

(C) $p^2 < 8q^2$ (D) $p^2 > 8q^2$

Sol. **[D]**



From equation of circle it is clealr that circle passes through origin. Let AB is chord of the circle.

A ≡ (p, q) and C(h, 0) is mid-point then B(– p + 2h,– q) and lies on the circle,

$x^2 + y^2 = px + qy$

we have, $(-p + 2h)^2 + (-q)^2 = p(-p + 2h) + q(-q)$

$\Rightarrow p^2 + 4h^2 - 4ph + q^2 = -p^2 + 2ph - q^2$

$\Rightarrow 2h^2 - 3ph + p^2 + q^2 = 0$ ....(i)

There are given two distinect chords which are bisceted at x-axis, then there will be two distinct values of h restisfying (i)

so, discriminant of this qadratic equation must be < 0

$\Rightarrow D > 0$

$\Rightarrow (-3p)^2 - 4.2(p^2 + q^2) > 0$
$\Rightarrow 9p^2 - 8p^2 - 8q^2 > 0$
$p^2 - 8q^2 > 0$
$\Rightarrow p^2 > 8q^2$

Q.33 Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1 **[IIT- 99]**

(A) $x + y = 0$ (B) $x - y = 0$
(C) $x + 7y = 0$ (D) $x - 7y = 0$

Sol. **[C]**

Let euation of line L_1 be $y = mx$ Intercepts made by L_1 and L_2 on the circle will be equal i.e. L_1 and L_2 are at the same distacne from the centre of the circle.

centre of the given circle is (1/2,–3/2), therefore

$$\frac{\left|\frac{1}{2}-\frac{3}{2}-1\right|}{\sqrt{1+1}} = \frac{\left|\frac{m}{2}+\frac{3}{2}\right|}{\sqrt{m^2+1}}$$

$$\Rightarrow \frac{2}{\sqrt{2}} = \frac{|m+3|}{2\sqrt{m^2+1}}$$

$\Rightarrow 8m^2 + 8 = m^2 + 6m + 9$

$\Rightarrow 7m^2 - 6m - 1 = 0$

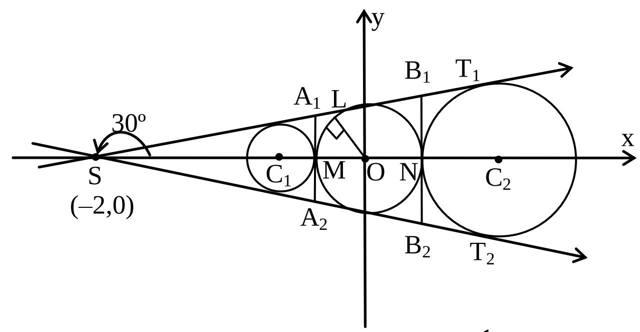
$\Rightarrow (7m + 1)(m - 1) = 0 \Rightarrow m = 1, -1/7$

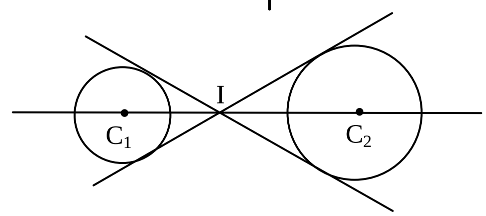
Thus two chords are $x + 7y = 0$ and $y - x = 0$

Hence $x - y = 0$ & $x + 7y = 0$ represents L_1

Q.34 Let T_1, T_2 be two tangents drawn from (–2 , 0) onto the circle C : $x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents, Further , find the equations of all possible common tangents to these circles, when taken two at a time. **[IIT-99]**

Sol.





From figure it is clear that triagle OLS is a right angle triangle with right angle at L.

Also, OL = 1 and OS = 2

$\therefore \sin(\angle LSO) = \frac{1}{2}$

$\Rightarrow \angle LSO = -30º$

since $SA_1 = SA_2$ $\therefore$ ΔSA_1A_2 is an equilateral triangle The circle with centre at C_1 is a circle Inscribed in the ΔSA_1A_2. Therefore centre C_1 is centroid of ΔSA_1A_2 This C_1 divides SM in the ratio 2 : 1 Therefore co-ordinates of C_1 are (–4/3, 0) and its radius = C_1M = 1/3

$\therefore$ its equation is $\left(x+\frac{4}{3}\right)^2 + y^2 = \left(\frac{1}{3}\right)^2$ .....(i)

The other circle touches the equilrteral triangle SB_1B_2 extreally its radius r is given by $r = \frac{\Delta}{s-a}$

where $B_1B_2 = a$

But $\Delta = \frac{1}{2}(a)(SN) = \frac{3}{2}a$

and $s - a = \frac{3}{2}a - a = \frac{a}{2}$

Thus r = 3

$\Rightarrow$ co-ordinates of C_2 are (4, 0)

$\therefore$ equation of circle with centre at C_2 is

$(x - 4)^2 + y^2 = 3^2$ ....(ii)

Equations of common tangents to circle (i) and circle are $x - 1$ and $y = \pm \frac{1}{\sqrt{3}}(x + 2)$ [T_1 & T_2]

Two tangents common to (i) and (ii) adre T_1 and To find the remaining two transverse tangent and (ii); we find a point I which divides the C_1C_2 in the ratio $r_1 : r_2 = \frac{1}{3} : 3 = 1 : 9$

Therefore, co-ordinates of I are (–4/5, 0)

Equation of any line through I is $y = m\left(x + \frac{4}{5}\right)$

it will touch (i) if $\frac{\left|m\left(-\frac{4}{5}+\frac{4}{5}\right)-0\right|}{\sqrt{1+m^2}} = \frac{1}{3}$

$\Rightarrow 39m^2 = 25 = m = \pm 5/\sqrt{39}$

Therefore, These tangents are

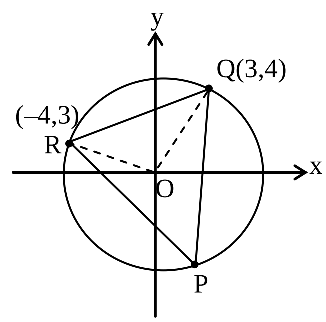
$y = \pm \frac{5}{\sqrt{39}}\left(x + \frac{4}{5}\right)$

Q.35 The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have co-ordinates (3, 4) and (–4, 3) respectively, then angle QPR is equal to **[IIT SCR.-2000]**

(A) π/2 (B) π/3 (C) π/4 (D) π/6

Sol. **[C]**

O is the point at centre and Pis the point at circumfrecens therefore, angle QOR is double the angle QPR. so it is sufficient to find the angle QOR.



Now slope of OQ = 4/3

slope of OR = –3/4

Again $m_1m_2 = -1$

Therefore, ∠QOR = 90º

which implies that ∠QPR = 45º

ANSWER KEY

EXERCISE # 1

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	B	D	A	B	C	D	B	C	B	A	A	A	A	B	D	C	C	B	C	C
Q.No.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36				
Ans.	A	A	C	C	C	B	B	A	D	A	D	B	A	A	A	B				

37. True **38.** True **39.** True **40.** True **41.** (–b,a) or (b, –a) **42.** $2\sqrt{c^2-2}$

EXERCISE # 2

(PART-A)

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13
Ans.	B	B	B	A	C	C	C	A	A	A	C	B	A
Q.No.	14	15	16	17	18	19	20	21	22	23	24	25	26
Ans.	C	A	B	A	A	B	B	C	A	A	C	C	A

(PART-B)

Qus.	27	28	29	30	31	32	33
Ans.	B,C	A,B	A,B	A, B	A,B,C,D	B	B,C

(PART-C)

Qus.	34	35	36	37
Ans.	A	A	B	B

(PART-D)

38. $A\to Q$, $B\to P$, $C\to R$, $D\to S$

39. $A\to Q$, $B\to P$, $C\to R$, $D\to S$

40. $A\to R$, $B\to S$, $C\to P$, $D\to Q$

EXERCISE # 3

1. $2(x^2+y^2)-2cx+c^2-a^2=0$

3. $4x-3y-25=0$; $3x+4y=25$

4. $(x+4+3\sqrt{2})^2+y^2=9(\sqrt{2}+1)^2$, $(x-\sqrt{2})^2+y^2=(-1+\sqrt{2})^2$

5. $x^2+y^2=a^2+b^2$

6. $x^2+y^2-3x-5y=0$

7. $x^2+y^2+(g-2)x+(f-1)y-2g-f=0$

9. $x+y=2$

10. $(-4, 4)$, $(-\frac{1}{2}, \frac{1}{2})$

12. $2x\pm y\sqrt{5}-15=0$ or $x\pm y\sqrt{35}-30=0$

14. $y=x+1$, $y+x+1=0$

15. $x^2+y^2-7x+7y+12=0$

Qus.	16	17	18	19	20	21	22	23	24
Ans.	C	B	C	B	A	B	B	C	B

EXERCISE # 4

1. (A) **2.** (B) **3.** $9 + 3\sqrt{10}$ **4.** Ellipse **5.** (C) **6.** (A) **7.** $r = 5$

8. (C) **9.** $2x^2 + 2y^2 - 10x - 5y + 1 = 0$ **10.** (A) **11.** $\sqrt{5}$ **12.** (A) **13.** (D)

14. (C) **15.** (B) **16.** (A) **17.** (C) **18.** (B) **19.** (D) **20.** (C)

21. (D) **22.** (A) **23.** (D) **24.** (B) **25.** 8 **26.** (D) **27.** 2

28. (A) **29.** (A) **30.** (D)

EXERCISE # 5

3. $x^2 + y^2 - 4x - 6y + 9 = 0$ or $x^2 + y^2 - 20x - 22y + 121 = 0$ **4.** (C) **5.** $x^2 + y^2 - x = 0$

6. $x^2 + y^2 + 8x - 6y + 9 = 0$ **7.** $x^2 + y^2 - 10x - 4y + 4 = 0$ **9.** $k = 1$ **10.** (A,C)

11. (A) **12.** $\left(-\frac{9}{5}, \frac{12}{5}\right)$ or $\left(\frac{9}{5}, \frac{-12}{5}\right)$ **13.** $x^2 + y^2 + gx + fy + c/2 = 0$

14. (A) **15.** (C) **16.** $x^2 + y^2 + 18x - 2y + 32 = 0$ **17.** 2

18. $x^2 + y^2 + 6x + 2y - 15 = 0$ and $x^2 + y^2 - 10x - 10y + 25 = 0$ **19.** (D) **20.** $a^2 > 2b^2$

21. (D) **22.** $16x^2 + 16y^2 - 48x + 16y + 31 = 0$ **23.** $\left(2, \frac{23}{3}\right)$ **24.** (D)

25. **(a)** $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$ **(b)** $(-\infty, -2) \cup (2, \infty)$ **26.** 7 **27.** **(b)** $x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$

28. $\left(\frac{1}{2}, \frac{1}{4}\right)$ **30.** (B) **32.** (D) **33.** (B, C)

34. $(x - 4)^2 + y^2 = 9$, $\left(x + \frac{4}{3}\right)^2 + y^2 = \frac{1}{9}$, $x = \pm 1$, $y = \pm \frac{(x+2)}{\sqrt{3}}$, $y = \pm \frac{5}{\sqrt{39}}\left(x + \frac{4}{5}\right)$ **35.** (C)