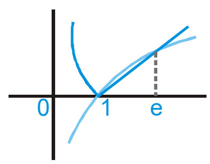
HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. $A=\int_1^e (\ln x - \ln^2 x)dx$

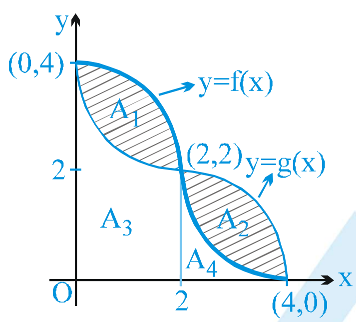
on solving it by parts we get



$A = 3x(\ln x - 1)\big|_1^e - x(\ln^2 x)\big|_1^e = 3 - e$

2. Given $\int_0^4 f(x)dx - \int_0^4 g(x)dx = 10$

$(A_1 + A_3 + A_4) - (A_2 + A_3 + A_4) = 10$



$A_1 - A_2 = 10$ ....**(ii)**

again $\int_2^4 g(x)dx - \int_2^4 f(x)dx = 5$

$(A_2 + A_4) - A_4 = 5$

$A_2 = 5$ .....**(i)**

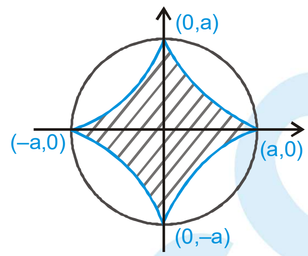
$\therefore$ $(1)+(2)$

$A_1 = 15$

3. $x = a\cos^3 t,\ y = a\sin^3 t \Rightarrow x^{2/3} + y^{2/3} = a^{2/3}$

$A = 4\int_0^{\pi/2} y\frac{dx}{dt}dt = 4\int_0^{\pi/2} 3a^2\sin^3 t\cos^2 t(-\sin t)dt$

$= \left|-12a^2\int_0^{\pi/2}\sin^4 t\cos^2 t\,dt\right| = \left|-12a^2\frac{3.1.1}{6.4.2.1}\times\frac{\pi}{2}\right|$



Figure

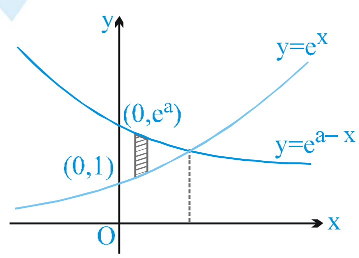
$= \frac{3}{8}\pi a^2$ sq. units

4. Solving $e^x = e^{a-x}$, we get

$e^{2x} = e^a \Rightarrow x = \frac{a}{2}$

$S = \int_0^{a/2}(e^a \cdot e^{-x} - e^x)dx$

$= \left[-(e^a \cdot e^{-x} + e^x)\right]_0^{a/2}$



$= (e^a + 1) - (e^{a/2} + e^{a/2}) = e^a - 2e^{a/2} + 1 = (e^{a/2} - 1)^2$

$\therefore \frac{S}{a^2} = \left(\frac{e^{a/2}-1}{a}\right)^2 = \frac{1}{4}\left(\frac{e^{a/2}-1}{a/2}\right)^2$

$\therefore \lim_{a\to 0}\frac{S}{a^2} = \frac{1}{4}$

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6. $m_{PQ} = \dfrac{a^2 - b^2}{a + b} = a - b$

equation of PQ

$$y - a^2 = \frac{a^2 - b^2}{a + b}(x - a)$$

or $y - a^2 = (a - b)(x - a)$

$y = a^2 + x(a - b) - a^2 + ab$

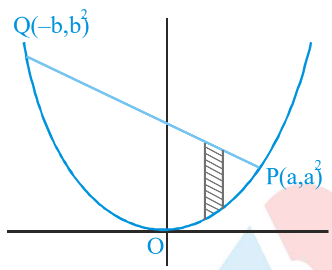
$y = (a - b)x + ab$

$\therefore \quad S_1 = \displaystyle\int_{-b}^{a} (a - b)x + ab - x^2)dx$

which simplifies to $\dfrac{(a+b)^3}{6}$ ....**(1)**

Also $S_2 = \dfrac{1}{2}\begin{vmatrix} a & a^2 & 1 \\ -b & b^2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \dfrac{1}{2}\left[ab^2 + a^2 b\right] = \dfrac{1}{2}ab(a+b)$

..........**(2)**



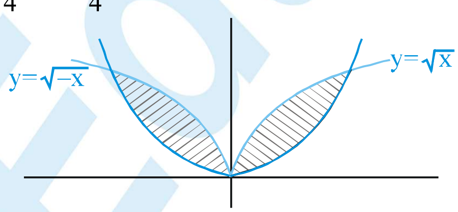
$\therefore$

$$\frac{S_1}{S_2} = \frac{(a+b)^3}{6} \cdot \frac{2}{ab(a+b)} = \frac{(a+b)^2}{3ab} = \frac{1}{3}\left[\frac{a}{b} + \frac{b}{a} + 2\right]$$

$$\therefore \quad \left.\frac{S_1}{S_2}\right|_{min.} = \frac{4}{3}$$

8. $A = \left(\dfrac{16ab}{3}\right) \cdot 2$

$a = \dfrac{1}{4};\ b = \dfrac{1}{4}$



$A = \dfrac{2}{3}$

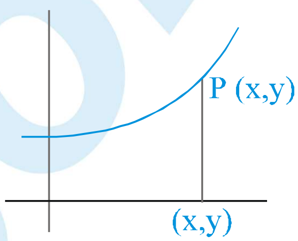
10. $\displaystyle\int_1^b f(x)\,dx = (b - 1)\sin(3b + 4)$

Area function $= \displaystyle\int_1^x f(x)\,dx = (x - 1)\sin(3x + 4)$

differentiating

$\therefore \quad f(x) = \sin(3x + 4) + 3(x - 1).\cos(3x + 4) \Rightarrow C$

12. $\displaystyle\int_0^x f(x)\,dx = y^3$



Differentiating

$f(x) = 3y^2.\dfrac{dy}{dx}$

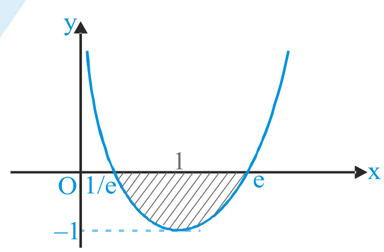
$y = 3y^2\dfrac{dy}{dx}$

$\Rightarrow \quad y = 0$ (rejected)

or $3y\,dy = dx$

$\dfrac{3y^2}{2} = x + c \Rightarrow$ parabola $\Rightarrow$ C

14. $y = \ln^2 x - 1$



$y' = \dfrac{2\ln x}{x} = 0 \qquad \Rightarrow \quad x = 1$

$x > 1, y\uparrow$ and $0 < x < 1$, y is $\downarrow$

$$A = \left|\int_{1/e}^{e} (\ln^2 x - 1)dx\right| = \left|\int_{1/e}^{e} \ln^2 x\,dx - \int_{1/e}^{e} dx\right|$$

$$= \left|\left[x\ln^2 x\right]_{1/e}^{e} - 2\int_{1/e}^{e}\left(\frac{\ln x}{x}\right)\cdot x\,dx - \left(e - \frac{1}{e}\right)\right|$$

$$= \left|\left(e - \frac{1}{e}\right) - 2\int_{1/e}^{e}\left(\frac{\ln x}{x}\right)\cdot x\,dx - \left(e - \frac{1}{e}\right)\right|$$

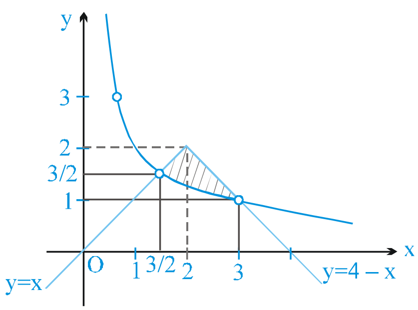
$$= \left|-2\left[\left[x\ln x\right]_{1/e}^{e} - \int_{1/e}^{e} dx\right]\right| = \left|-2\left[\left(e + \frac{1}{e}\right) - \left(e - \frac{1}{e}\right)\right]\right|$$

$$= \left|\frac{4}{e}\right| = \frac{4}{e}$$

16. $y = \begin{cases} 2-(2-x) \text{ if } x \le 2 \\ = x \\ 2-(x-2) \text{ if } x \ge 2 \\ = 4-x \end{cases}$; also $y = \begin{cases} \dfrac{3}{x} \text{ if } x > 0 \\ -\dfrac{3}{x} \text{ if } x < 0 \end{cases}$

$$A = \int_{3/2}^{2}\left(x - \frac{3}{x}\right)dx + \int_{2}^{3}\left((4-x) - \frac{3}{x}\right)dx$$

Now compute



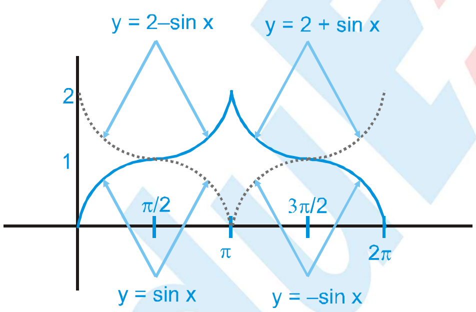
18. $\int_0^x f(x) = xe^x \Rightarrow f(x) = \frac{d}{dx}(xe^x) = xe^x + e^x$

19. $f(x) + f(\pi - x) = 2. \;\forall\; x \in \left(\frac{\pi}{2}, \pi\right]$

$f(x) = 2 - \sin(\pi - x)$

$f(x) = 2 - \sin x, \;\forall\; x \in \left(\frac{\pi}{2}, \pi\right]$

$f(x) = 2 - f(2\pi - x), \;\forall\; x \in \left(\pi, \frac{3\pi}{2}\right]$



$f(x) = 2 + \sin x, x \in \left(\pi, \frac{3\pi}{2}\right]$

$f(x) = f(2\pi - x), \;\forall\; x \in \left(\frac{3\pi}{2}, 2\pi\right]$

$f(x) = -\sin x, \;\forall\; x \in \left(\frac{3\pi}{2}, 2\pi\right]$

Clearly, from figure required area = 2π

20. Given $g(x) = 2x + 1$; $h(x) = (2x+1)^2 + 4$

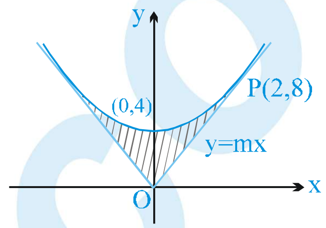
now $h(x) = f[g(x)]$

$(2x+1)^2 + 4 = f(2x+1)$

let $2x + 1 = t$

$\Rightarrow f(t) = t^2 + 4$

$\therefore f(x) = x^2 + 4$ ....**(1)**



solving $y = mx$ and $y = x^2 + 4$

$x^2 - mx + 4 = 0$

put $D = 0$

$m^2 = 16 \Rightarrow m = \pm 4$

tangents are $y = 4x$ and $y = -4x$

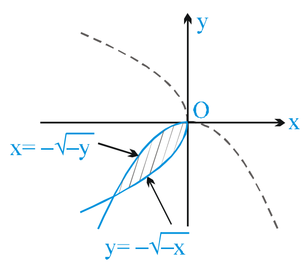
$$A = 2\int_0^2 [(x^2+4) - 4x]dx = 2\int_0^2 [(x-2)^2 dx$$

$$= \frac{2}{3}(x-2)^3\Big]_0^2 = \frac{16}{3} \text{ sq. units}$$

22. $y = -\sqrt{-x} \Rightarrow y^2 = -x$ where x & y both (–) ve

$x = -\sqrt{-y} \Rightarrow x^2 = -y$ where x & y both (–) ve

Hence $A = \frac{16ab}{3}$



where $a = b = \frac{1}{4}$

$\therefore A = \frac{1}{3} \Rightarrow$ **(B)**

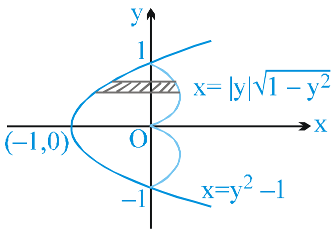
23. Required area $(b-1)\sin(3b+4)=\int_1^b f(x)\,dx$

diff. w.r.t. b

$$3(b-1)\cos(3b+4)+\sin(3b+4)=f(b)$$

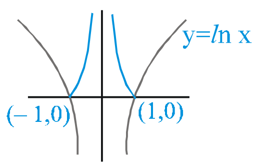
$\Rightarrow$ $f(x)=3(x-1)\cos(3x+4)+\sin(3x+4)$

24. $A=2\int_0^1\left[y\sqrt{1-y^2}-(y^2-1)\right]dy$



$=2$

26. $4\int_0^1 |\ln x|\,dx = -4\int_0^1 \ln x\,dx = 4$



28. (a, 0) lies on the given curve

$\therefore$ $0=\sin 2a-\sqrt{3}\sin a$

$\Rightarrow$ $\sin a=0$ or $\cos a=\sqrt{3}/2$

$\Rightarrow$ $a=\frac{\pi}{6}$ (as $a>0$ and the first point of intersection with positive X-axis)

and

$$A=\int_0^{\pi/6}(\sin 2x-\sqrt{3}\sin x)\,dx=\left(-\frac{\cos 2x}{2}+\sqrt{3}\cos x\right)_0^{\pi/6}$$

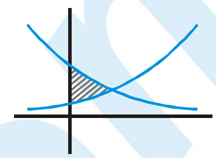
$$=\left(-\frac{1}{4}+\frac{3}{2}\right)-\left(-\frac{1}{2}+\sqrt{3}\right)=\frac{7}{4}-\sqrt{3}=\frac{7}{4}-2\cos a$$

$\Rightarrow$ $4A+8\cos a=7$

EXERCISE - 2

Part # I : Multiple Choice

1. $S=\int_0^{a/2}(e^{a-x}-e^x)\,dx$



$=-[2e^{a/2}-(e^a+1)]$

Now $\lim_{a\to 0}\frac{e^a-2e^{a/2}+1}{a^2}=\lim_{a\to 0}\left(\frac{e^{a/2}-1}{a/2}\right)^2\frac{1}{4}=\frac{1}{4}$

3. Given $f(x)=\begin{cases}\cos x & 0\le x<\frac{\pi}{2}\\ \left(\frac{\pi}{2}-x\right)^2 & \pi/2\le x<\pi\end{cases}$

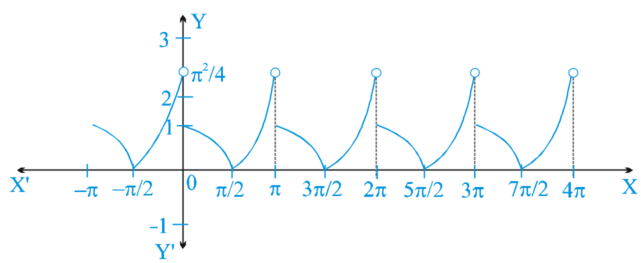
and f is periodic with period π

$\therefore$ Let us draw the graph of $y=f(x)$

From the graph, the range of the function is $\left[0,\frac{\pi^2}{4}\right)$

$\Rightarrow$ **(A)**

It is discontinuous at $x=n\pi,\ n\in I$. It is not differentiable at $x=\frac{n\pi}{2},\ n\in I$.



Area bounded by $y=f(x)$ and the X-axis from $-n\pi$ to $n\pi$ for $n\in N$

$$=2n\int_0^{\pi}f(x)\,dx=2n\left[\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}\left(\frac{\pi}{2}-x\right)^2dx\right]=2n\left(1+\frac{\pi^3}{24}\right)$$

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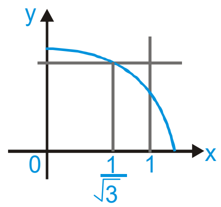
8. $A=\dfrac{1}{\sqrt{3}}+\displaystyle\int_{1/\sqrt{3}}^{1}\sqrt{\dfrac{4}{3}-x^2}\,dx$



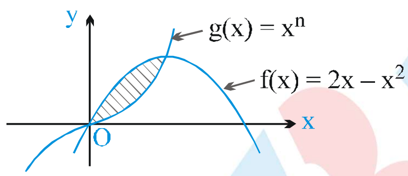
$$=\frac{1}{\sqrt{3}}+\left[\frac{x}{2}\sqrt{\frac{4}{3}-x^2}+\frac{2}{3}\sin^{-1}\left(\frac{x\sqrt{3}}{2}\right)\right]_{1/\sqrt{3}}^{1}$$

$$=\frac{1}{\sqrt{3}}+\left[\left(\frac{1}{2\sqrt{3}}-\frac{1}{2\sqrt{3}}\right)+\frac{2}{3}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right]=\frac{3\sqrt{3}+\pi}{9}$$

9. Solving $f(x)=2x-x^2$ and $g(x)=x^n$

we have $2x-x^2=x^n \Rightarrow x=0$ and $x=1$

$$A=\int_0^1(2x-x^2-x^n)\,dx=\left[x^2-\frac{x^3}{3}-\frac{x^{n+1}}{n+1}\right]_0^1$$



$$=1-\frac{1}{3}-\frac{1}{n+1}=\frac{2}{3}-\frac{1}{n+1}$$

hence, $\dfrac{2}{3}-\dfrac{1}{n+1}=\dfrac{1}{2} \Rightarrow \dfrac{2}{3}-\dfrac{1}{2}=\dfrac{1}{n+1}$

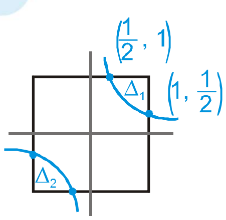
$\Rightarrow \dfrac{4-3}{6}=\dfrac{1}{n+1} \Rightarrow n+1=6$

$\Rightarrow n=5$

Hence n is a divisor of 15, 20, 30

$\Rightarrow$ **B, C, D**

10. $\Delta_2=\Delta_1=\displaystyle\int_{1/2}^{1}\left[1-\frac{1}{2x}\right]dx$



$=\dfrac{1}{2}-\dfrac{1}{2}\,\ell n\,2$

$A=4-(\Delta_1+\Delta_2)=4-(1-\ell n\,2)=3+\ell n\,2$

11. The two curves meet at

$mx=x-x^2$ or $x^2=x(1-m)$ $\therefore$ $x=0,\ 1-m$

$$\int_0^{1-m}(y_1-y_2)\,dx=\int_0^{1-m}(x-x^2-mx)\,dx$$

$$=\left[(1-m)\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{1-m}=\frac{9}{2}\ \text{if } m<1$$

or $(1-m)^3\left[\dfrac{1}{2}-\dfrac{1}{3}\right]=\dfrac{9}{2}$ or $(1-m)^3=27$

$\therefore$ $m=-2$

But if $m>1$ then $1-m$ is negative, then

$$\left[(1-m)\frac{x^2}{2}-\frac{x^3}{3}\right]_{1-m}^{0}=\frac{9}{2}$$

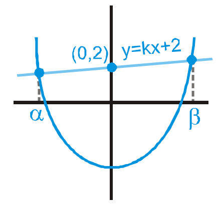
$$-(1-m)^3\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{9}{2}$$

$\therefore$ $-(1-m)^3=-27$ or $1-m=-3$ $\therefore$ $m=4$.

Part # II : Assertion & Reason

3. $A=\displaystyle\int_\alpha^\beta(kx+2-x^2+3)\,dx$

$$=\left(\frac{kx^2}{2}-\frac{x^3}{3}+5x\right)_\alpha^\beta$$



$$=\left(\frac{k(\alpha+\beta)}{2}-((\alpha+\beta)^2-\alpha\beta)\frac{1}{3}+5\right)(\beta-\alpha)$$

$$=\sqrt{k^2+20}\left[\frac{k^2}{2}-\left(\frac{k^2+5}{3}\right)+5\right]=\frac{1}{6}(k^2+20)^{3/2}$$

Hence statement I is true & II is false.

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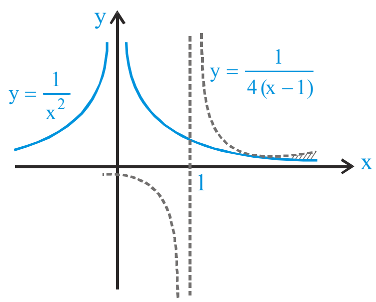
Part # II : Comprehension

Comprehension # 1

Now $\int_2^a \left[\frac{1}{4(x-1)} - \frac{1}{x^2}\right]dx = \frac{1}{a}$

$\Rightarrow \quad a = e^2 + 1$



Also $\int_b^2 \left[\frac{1}{4(x-1)} - \frac{1}{x^2}\right]dx = 1 - \frac{1}{b}$

$\Rightarrow \quad \left[\frac{1}{4}\ln(x-1) + \frac{1}{x}\right]_b^2 = 1 - \frac{1}{b}$

$\Rightarrow \quad -\ln(b-1) = 2 \quad \Rightarrow \quad b = 1 + e^{-2}$

1. $\ln\left(\frac{a}{b}\right) = \ln\left(\frac{e^2+1}{1+e^{-2}}\right) = 2$

2. $|A| = \ln(a-1)\ \ln(b-1) = -4$

$A^{-1} = \frac{-1}{4}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} = \frac{A}{4}$

3. $z = 2 - 2i$

$\arg(z) = \frac{-3\pi}{4}$

Comprehension # 2

$2f''(x)\,f'(x) = 2\,f(x)\,f'(x)$

Integrating

$(f'(x))^2 = (f(x))^2 + c$

put $x = 0 \quad \Rightarrow \quad c = 5$

$(f'(x))^2 = (f(x))^2 + 5$

put $y = f(x)$

$\frac{dy}{dx} = \pm\sqrt{y^2+5}$

$\ln\left(y + \sqrt{y^2+5}\right) = \pm x + c_1$

$x = 0, \quad y = 2 \quad \Rightarrow \quad c_1 = \ln 5$

$\frac{y + \sqrt{y^2+5}}{5} = e^{\pm x}$

$y = \frac{5e^x - e^{-x}}{2} \quad$ or $\quad y = \frac{5e^{-x} - e^x}{2}$

If $f(x) = \frac{5e^{-x} - e^x}{2}$; $f'(0) = 3$ is not satisfied

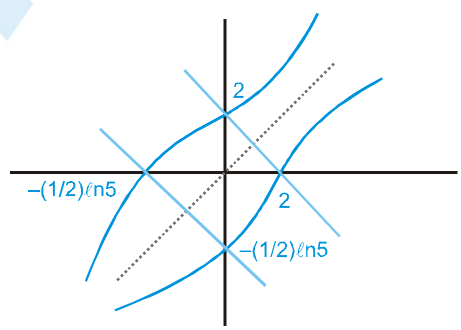
$\Rightarrow \quad f(x) = \frac{5e^x - e^{-x}}{2}$

put $f(x) = 0$

$\Rightarrow \quad 2x = \ln\left(\frac{1}{5}\right) \quad \Rightarrow \quad x = -\frac{1}{2}\ln 5$

$f'(x) = \frac{5e^x + e^{-x}}{2} > 0 \quad \Rightarrow \quad f(x)$ is increasing

Area in second quadrant $= \int_{-\frac{1}{2}\ln 5}^{0} \left(\frac{5e^x - e^{-x}}{2}\right)dx$



Figure

$= \left[\frac{5e^x + e^{-x}}{2}\right]_{-\frac{1}{2}\ln 5}^{0} = 3 - \sqrt{5}$

Area by lines $x + y = 2$, $x + y = -\frac{1}{2}\ln 5$,

$y = f(x)$ and $y = f'(x)$ is $2(3 - \sqrt{5}) + \frac{1}{2} \cdot 2.2 + \frac{1}{2}\left(\frac{1}{2}\ln 5\right)\left(\frac{1}{2}\ln 5\right)$

$= 8 - 2\sqrt{5} + \frac{1}{8}(\ln 5)^2$

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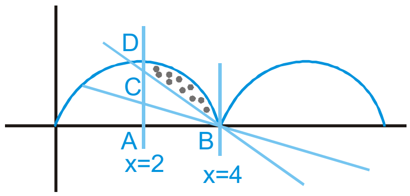
EXERCISE - 4
Subjective Type

4. Let equation of line is $y = mx - 4m$



$$A = \int_2^4 \sqrt{2}\sin\frac{\pi}{4}x\,dx = \left[-\sqrt{2}\,\frac{4}{\pi}\cos\frac{\pi x}{4}\right]_2^4 = \frac{4\sqrt{2}}{\pi} \quad \text{... (i)}$$

Also area of $\Delta ABC = \frac{1}{2}.2.(-2m_1) = -2m_1$ ... **(ii)**

from **(i)** and **(ii)**

$$-2m_1 = \frac{4\sqrt{2}}{3\pi} \Rightarrow m_1 = \frac{-2\sqrt{2}}{3\pi}$$

$$\Rightarrow \tan(\pi - \theta_1) = \frac{-2\sqrt{2}}{3\pi} \Rightarrow \pi - \theta_1 = \tan^{-1}\frac{2\sqrt{2}}{3\pi}$$

$$\Rightarrow \theta_1 = \pi - \tan^{-1}\frac{2\sqrt{2}}{3\pi} \quad \text{or} \quad \frac{1}{2}.(2)(-2m_2) = \frac{8\sqrt{2}}{3\pi}$$

$$\Rightarrow m_2 = \frac{-4\sqrt{2}}{3\pi} \Rightarrow \tan(\pi - \theta_2) = \frac{-4\sqrt{2}}{3\pi}$$

$$\Rightarrow \theta_2 = \pi - \tan^{-1}\frac{4\sqrt{2}}{3\pi}$$

5. Curve $y = a - bx^2$ passes through the point (2, 1)

$\therefore a - 4b = 1$

$$A = 2\int_0^{\sqrt{a/b}} (a - bx^2)dx = 2\left[ax - \frac{bx^3}{3}\right]_0^{\sqrt{a/b}}$$

$$= \frac{4}{3}\frac{a^{3/2}}{\sqrt{b}} = \frac{4}{3}\frac{(1+4b)^{3/2}}{\sqrt{b}}$$

$$A' = \frac{2}{3}\frac{\sqrt{1+4b}\,(8b-1)}{b^{3/2}} \Rightarrow A' = 0 \Rightarrow b = \frac{1}{8}$$

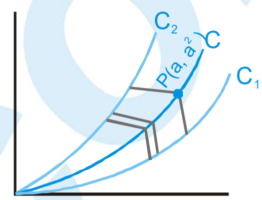
$\Rightarrow A = 4\sqrt{3}$ sq. units

8. According to question

$$\int_0^{a^2}\left(-f^{-1}(y) + \sqrt{y}\right)dy = \int_0^a\left(x^2 - \frac{x^2}{2}\right)dx$$

$$\Rightarrow [f^{-1}(a^2) - a]\,2a = -\frac{a^2}{2}$$

$$\Rightarrow f^{-1}(a^2) = \frac{3a}{4}$$



$$\Rightarrow f\left(\frac{3a}{4}\right) = a^2$$

or $f(x) = \frac{16}{9}x^2$

9. $f(x) = \text{Maximum}\,\{x^2, (1-x)^2, 2x(1-x)\}$

We draw the graph of

$y = x^2$ **(1)**

$y = 2x(1-x)$ **(2)**

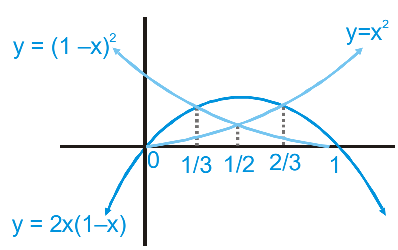
$y = 2x(1-x)$ **(3)**

Solving (1) and (3), we get $x^2 = 2x(1-x)$

$\Rightarrow 3x^2 = 2x \Rightarrow x = 0$ or $x = \frac{2}{3}$.

Solving (2) and (3) we get $(1-x)^2 = 2x(1-x)$

$\Rightarrow x = \frac{1}{3}$ and $x = 1$.



Figure

From figure it is clear that

$$f(x) = \begin{cases} (1-x)^2 \text{ for } 0 \le x \le 1/3 \\ 2x(1-x) \text{ for } 1/3 \le x \le 2/3 \\ x^2 \text{ for } 2/3 \le x \le 1 \end{cases}$$

The required area A is given by

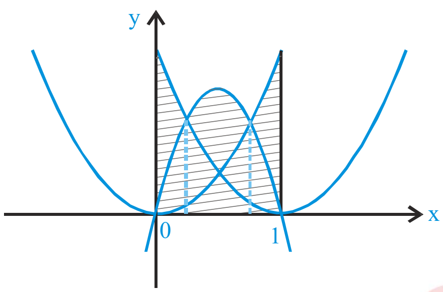
$$A=\int_0^1 f(x)\,dx=\int_0^{1/3}(1-x)^2\,dx+\int_{1/3}^{2/3}2x(1-x)\,dx+\int_{2/3}^{1}x^2dx$$

$$=\left[-\frac{1}{3}(1-x)^3\right]_0^{1/3}+\left[\left(x^2-\frac{2x^3}{3}\right)\right]_{1/3}^{2/3}+\left[\frac{x^3}{3}\right]_{2/3}^{1}$$

$$=-\frac{1}{3}\left(\frac{2}{3}\right)^3+\frac{1}{3}+\left(\frac{2}{3}\right)^2-\frac{2}{3}\left(\frac{2}{3}\right)^3-\left(\frac{1}{3}\right)^2$$

$$+\frac{2}{3}\left(\frac{1}{3}\right)^3+\frac{1}{3}-\frac{1}{3}\left(\frac{2}{3}\right)^3=\frac{17}{27}\text{ sq. units.}$$

16. $A=\int_0^{1/3}(x-1)^2\,dx+\int_{1/3}^{2/3}2x(1-x)dx+\int_{2/3}^{1}x^2dx$



17. $A_n=\int_0^{\pi/4}(\tan x)^n\,dx$

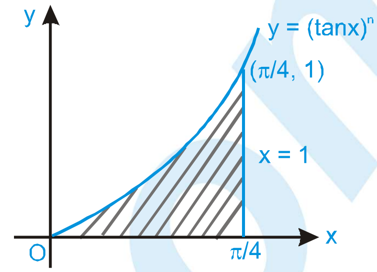
$$A_n+A_{n-2}=\int_0^{\pi/4}\left[(\tan x)^n+(\tan x)^{n-2}\right]dx$$

$$=\int_0^{\pi/4}(\tan x)^{n-2}\sec^2 x\;dx=\left[\frac{t^{n-1}}{n-1}\right]_0^1=\frac{1}{n-1}$$

Also $A_{n+2}<A_n<A_{n-2}$

$$\Rightarrow \frac{1}{n+1}<2A_n<\frac{1}{n-1}$$

18. (i) $0<\tan x<1$, when $0<x<\pi/4$, we have
$0<(\tan x)^{n+1}<(\tan x)^n$ for each $n\in N$



(ii) we have $A_n=\int_0^{\pi/4}(\tan x)^n\,dx$

$$\Rightarrow \int_0^{\pi/4}(\tan x)^{n+1}dx<\int_0^{\pi/4}(\tan x)^n\,dx \Rightarrow A_{n+1}<A_n$$

Now, for $n>0$, $A_n+A_{n+2}=\int_0^{\pi/4}[(\tan x)^n+(\tan x)^{n+2}]dx$

$$=\int_0^{\pi/4}(\tan x)^n\;(\sec^2 x)\,dx$$

$$\left[\frac{1}{(n+1)}(\tan x)^{n+1}\right]_0^{\pi/4}=\frac{1}{(n+1)}(1-0).$$

Similarly $A_n+A_{n-2}=\dfrac{1}{n-1}$

since $A_{n+2}<A_{n+1}<A_n$, we get $A_n+A_{n+2}<2A_n$

$$\Rightarrow \frac{1}{n+1}<2A_n \quad\Rightarrow\quad \frac{1}{2n+2}<A_n \quad \text{......... (1)}$$

Also for $n>2$, $A_n+A_n<A_n+A_{n-2}=\dfrac{1}{n-1}$

$$\Rightarrow 2A_n<\frac{1}{n-1} \quad \text{........ (2)}$$

$$\Rightarrow A_n<\frac{1}{2n-2}$$

Combining **(1)** and **(2)** we get $\dfrac{1}{2n+2}<A_n<\dfrac{1}{2n-2}$

Hence Proved.

20. $f(x+1) = f(x) + 2x + 1$

$\Rightarrow \; f''(x+1) = f''(x) \quad \forall\, x \in R$

Let $f''(x) = a$

$\Rightarrow \; f'(x) = ax + b$

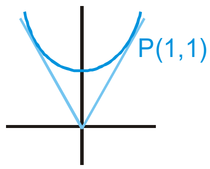
$\Rightarrow \; f(x) = \dfrac{ax^2}{2} + bx + c$

$\Rightarrow \; c = 1 \qquad [\because f(0) = 1]$

Now $f(x+1) - f(x) = 2x + 1$

$\Rightarrow \; \left[\dfrac{a}{2}(x+1)^2 + b(x+1) + c\right] - \left[\dfrac{ax^2}{2} + bx + c\right] = 2x + 1$

$\Rightarrow \; ax + \dfrac{a}{2} + b = 2x + 1$



on comparing we get $a = 2$,

or $\dfrac{a}{2} + b = 1 \quad \Rightarrow \quad b = 0$

$\therefore \; f(x) = x^2 + 1$ ... **(i)**

Now let equation of tangent be $y = mx$ ... **(ii)**

from **(i)** and **(ii)**

$x^2 - mx + 1 = 0$

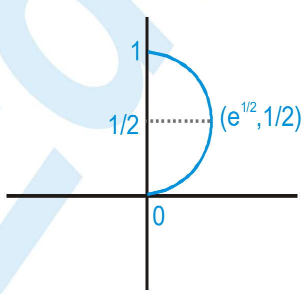
$\Rightarrow \; m = \pm 2$

$\therefore$ tangent are $y = 2x$ or $y = -2x$

$$A = 2\int_0^1 (x^2 + 1 - 2x)\,dx = \frac{2}{3}$$

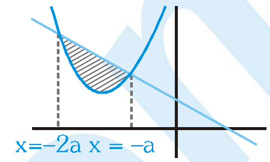
21. Area $= \displaystyle\int_0^1 e^y \sin(\pi y)\,dy$

$$= \left.\frac{e^y}{1+\pi^2}(\sin \pi y - \pi \cos \pi y)\right]_0^1 = \frac{(e+1)\pi}{1+\pi^2}$$



22. $A = \displaystyle\int_{-a}^{-2a} \frac{a^2 - ax - (x^2 + 2ax + 3a^2)}{1+a^4}\,dx$

$$= \frac{3}{2}\,\frac{a^3}{1+a^4}$$



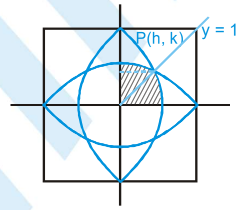
Now $f(a) = \dfrac{3}{2}\,\dfrac{a^3}{1+a^4}$

$\Rightarrow \; f'(a) = 0$

$\Rightarrow \; (1 + a^4)\, 3a^2 - a^3\, 4a^3 = 0$

$\Rightarrow \; a_{min} = 0,\ a_{max} = 3^{1/4}$

23. Distance of point P from origin is less then distance of P from $y = 1$



$\sqrt{h^2 + k^2} < k - 1$; $\sqrt{h^2 + k^2} < -k - 1$

$\Rightarrow \; x^2 + y^2 < (y-1)^2$; $x^2 + y^2 < y^2 + 2y + 1$

$\Rightarrow \; x^2 < -2\left(y - \dfrac{1}{2}\right)$; $x^2 < 2\left(y + \dfrac{1}{2}\right)$

similarly $y^2 < -2\left(x - \dfrac{1}{2}\right)$; $y^2 < 2\left(x + \dfrac{1}{2}\right)$

$\Rightarrow \; y = \dfrac{x^2 - 1}{-2}$ or $y = x = \dfrac{x^2 - 1}{-2}$

$\Rightarrow \; x^2 + 2x - 1 = 0$

$\Rightarrow \; x = -1 \pm \sqrt{2}$

$$A = 8\int_0^{\sqrt{2}-1} \left[\frac{1 - x^2}{2} - \sqrt{2} + 1\right] dx + 4(\sqrt{2} - 1)^2$$

$$= \frac{16\sqrt{2} - 20}{3}$$

24. (i) $f(x)=\min\left\{x+1,\sqrt{1-x}\right\}=\begin{cases}x+1 & -1<x<0\\ \sqrt{1-x} & 0<x<1\end{cases}$

$\therefore\ \dfrac{12}{7}\displaystyle\int_{-1}^{1} f(x)\,dx$

$$=\frac{12}{7}\left[\int_{-1}^{0}(x+1)\,dx+\int_{0}^{1}\sqrt{1-x}\,dx\right]$$

$$=\frac{12}{7}\left[\left(\frac{x^2}{2}+x\right)\Bigg|_{-1}^{0}-\frac{2}{3}(1-x)^{3/2}\Bigg|_{0}^{1}\right]$$

$$=\frac{12}{7}\left[0-\left(\frac{1}{2}-1\right)-\frac{2}{3}(0-1)\right]=\frac{12}{7}\left(\frac{1}{2}+\frac{2}{3}\right)=2$$

(ii) ✈ $0<x<\dfrac{1}{2}\ \therefore\ \{x\}=x$

$$A=\int_{0}^{1/2}x.dx=\left(\frac{x^2}{2}\right)_0^{1/2}=\frac{1}{8}$$

26. $f(x)=\begin{cases}x^2+ax+b & ; & x<-1\\ 2x & ; & -1\le x\le 1\\ x^2+ax+b & ; & x>1\end{cases}$

✈ $f(x)$ is continuous at $x=-1$ and $x=1$

$\therefore\ (-1)^2+a(-1)+b=-2$

and $2=(1)^2+a.\,1+b$

i.e., $a-b=3$

and $a+b=1$

on solving we get $a=2,\ b=-1$

$\therefore\ f(x)=\begin{cases}x^2+2x-1 & ; & x<-1\\ 2x & ; & -1\le x\le 1\\ x^2+2x-1 & ; & x>1\end{cases}$

Given curves are

$y=f(x),\ x=-2y^2$ and $8x+1=0$

solving $x=-2y^2,\ y=x^2+2x-1\ (x<-1)$ we get

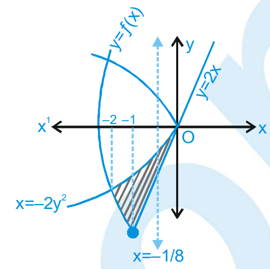
$x=-2.$

Also $y=2x,\ x=-2y^2$ meet at $(0,0)$

and $\left(-\dfrac{1}{8},-\dfrac{1}{4}\right)$

The required area is the shaded region in the figure.

$\therefore$ Required area



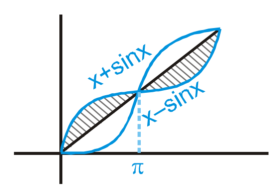
$$=\int_{-2}^{-1}\left[\sqrt{\frac{-x}{2}}-(x^2+2x-1)\right]dx+\int_{-1}^{-1/8}\left[\sqrt{\frac{-x}{2}}-2x\right]dx$$

$$=\left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3}-\frac{x^3}{3}-x^2+x\right]_{-2}^{-1}$$

$$+\left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3}-x^2\right]_{-1}^{-1/8}$$

$=\dfrac{257}{192}$ square units

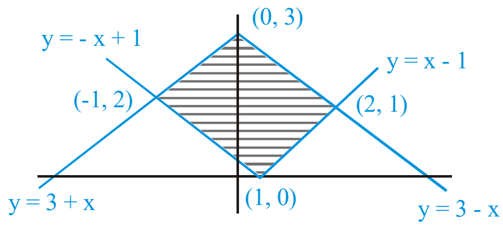
30. $A=4\displaystyle\int_{0}^{\pi}[x+\sin x-x)]dx$



EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

2.



$$A = \int_{-1}^{0} \{(3+x)-(-x+1)\}dx$$

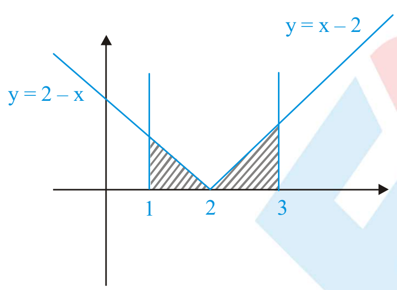
$$+\int_{0}^{1} \{(3-x)-(-x+1)\}dx + \int_{1}^{2} \{(3-x)-(-x-1)\}dx$$

$$= \int_{-1}^{0} \{(2+2x)dx + \int_{0}^{1} 2dx + \int_{1}^{2} (4-2x)dx$$

$$= \left[2x-x^2\right]_{-1}^{0} + \left[2x\right]_0^1 + \left[4x-x^2\right]_1^2$$

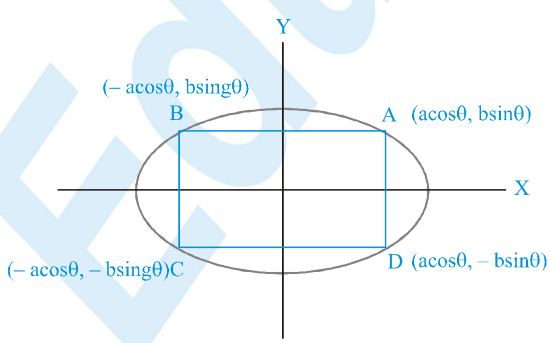
$= 0 - (-2 + 1) + (2 - 0) + (8 - 4) - (4 - 1)$
$= 1 + 2 + 4 - 3 = 4$ sq. units

3. Area $= \int_1^2 (2-x)dx + \int_2^3 (x-2)dx = 1$



4. Area of rectangle ABCD = $(2a\cos\theta)(2b\sin\theta) = 2ab\sin 2\theta$

$\Rightarrow$ Area of greatest rectangle is equal to 2ab when $\sin 2\theta = 1$.



5. Required area (OAB) $= \int_{1-e}^{0} \ln(x+e)dx$

$$= \left[x\ln(x+e) - \int \frac{1}{x+e} x dx\right]_0^1 = 1.$$

6. $y^2 = 4x$ and $x^2 = 4y$ are symmetric about line $y = x$

$\Rightarrow$ area bounded between $y^2 = 4x$

and $y = x$ is $\int_0^4 \left(2\sqrt{x} - x\right)dx = \frac{8}{3}$

$\Rightarrow A_{s_2} = \frac{16}{3}$ and $A_{s_1} = A_{s_3} = \frac{16}{3}$

$\Rightarrow A_{s_1} : A_{s_2} : A_{s_3} :: 1:1:1.$

7. Given that $\int_{\pi/4}^{\beta} f(x)dx = \beta\sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$

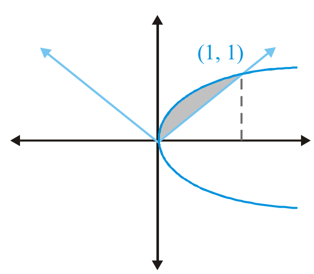
Differentiating w.r.t β

$$f(\beta)\cos\beta + \sin\beta - \frac{\pi}{4}\sin\beta + \sqrt{2}$$

$$f\left(\frac{\pi}{2}\right) = \left(1 - \frac{\pi}{4}\right)\sin\frac{\pi}{2} + \sqrt{2} = 1 - \frac{\pi}{2} + \sqrt{2}.$$

8. $A = \int_0^1 \left(\sqrt{x} - x\right)dx$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1$$



$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

9. Solving the equations we get the points of intersection $(-2, 1)$ and $(-2, -1)$

The bounded region is shown as shaded region.

The required area $= 2\int_0^1 \left(1-3y^2\right) - \left(-2y^2\right)$

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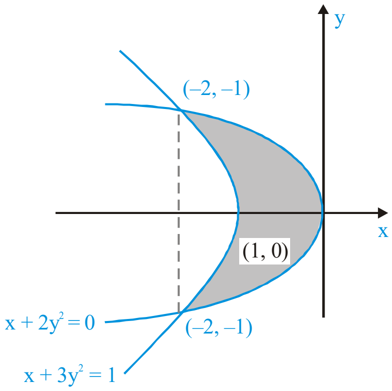


$$=2\int_0^1\left(1-y^2\right)dy=2\left[y-\frac{y^3}{3}\right]_0^1=2\times\frac{2}{3}=\frac{4}{3}.$$

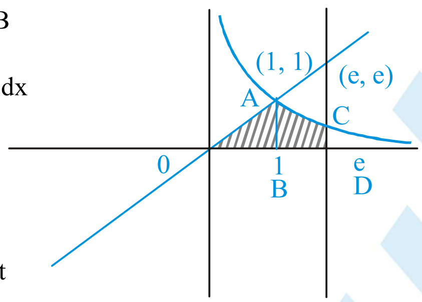
12. Required area

$= OAB + ACDB$

$$=\frac{1}{2}\times1\times1+\int_1^e\frac{1}{x}dx$$

$$=\frac{1}{2}+(\ln x)_1^e$$

$$=\frac{3}{2}\text{ square unit}$$

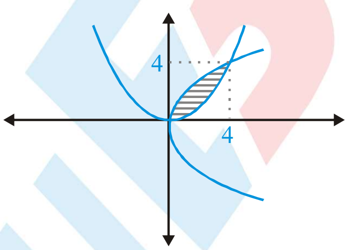


13. Area $=\int_0^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx$

$$=\left(2\left(\frac{x^{3/2}}{3/2}\right)-\frac{x^3}{12}\right)_0^4$$

$$=\frac{4}{3}\times8-\frac{64}{12}$$

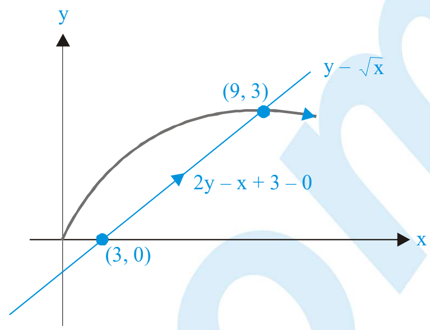
$$=\frac{32}{3}-\frac{16}{3}=\frac{16}{3}$$



14. $2.\int_0^2\left|\frac{\sqrt{y}}{2}-3\sqrt{y}\right|dy=2.\frac{5}{2}.\frac{2}{3}y^{3/2}\Big]_0^2$

$$=2.\frac{5}{3}.2\sqrt{2}=\frac{20\sqrt{2}}{3}$$

15. $y=\sqrt{x}$ and $2y-x+3=0$



$$\int_0^9\sqrt{x}\,dx-\int_3^9\left(\frac{x-3}{2}\right)dx$$

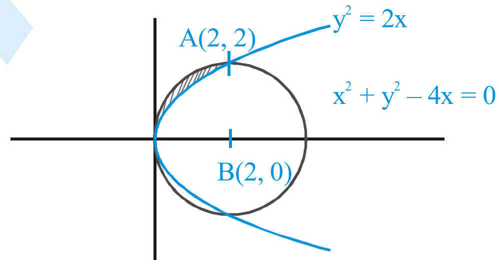
$$\left(\frac{x^{3/2}}{3/2}\right)_0^9-\left[\frac{\left(\frac{x^2}{2}-3x\right)}{2}\right]_3^9$$

$\Rightarrow$ 9 square units

18. $x^2+y^2-4x\le0$

$y^2\ge2x$

$x^2+2x-4x=0$



$\Rightarrow\ x^2-2x=0$

$\Rightarrow\ x(x-2)=0$

$\Rightarrow\ x=0, x=2$

$$\text{Area}=\int_0^2\left[\sqrt{4x-x^2}-\sqrt{2}\sqrt{x}\right]dx$$

$$=\int_0^2\left[\sqrt{2^2-(x-2)^2}-\sqrt{2}\sqrt{x}\right]dx$$

$$=\left[\left|\frac{x-2}{2}\sqrt{4x-x^2}+\frac{4}{2}\sin^{-1}\frac{x-2}{2}-\sqrt{2}\times\frac{2}{3}x^{3/2}\right|_0^2\right]$$

$$=\left[-\frac{2\sqrt{2}}{3}\times2\sqrt{2}-\left\{-2\times\frac{\pi}{2}\right\}\right]=\left[\pi-\frac{8}{3}\right]$$

Part # II : IIT-JEE ADVANCED

3. The given curves are $y = x^2$

which is an upward parabola with vertex at (0, 0)

$y = |2 - x^2|$

or $y = \begin{cases} 2 - x^2 & \text{if} \quad -\sqrt{2} < x < \sqrt{2} \\ x^2 - 2 & \text{if} \quad x < -\sqrt{2} \text{ or } x > \sqrt{2} \end{cases}$

or $x^2 = -(y - 2); -\sqrt{2} < x < \sqrt{2}$ .....(2)

a downward parabola with vertex at (0, 2)

$x^2 = y + 2 ; \quad x < -\sqrt{2}, x > \sqrt{2}$ .....(3)

On upward parabola with vertex at (0, –2)

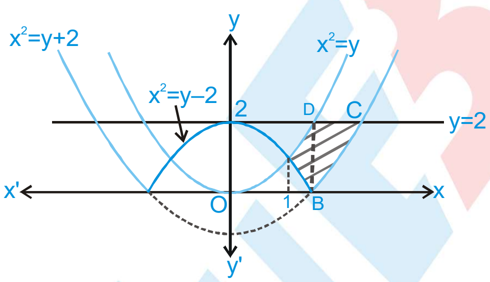
$y = 2$ .....(4)

Straight line parallel to x-axis

$x = 1$ .....(5)

Straight line parallel to y-axis

The graph of these curves is as follows.



$\therefore$ Required area = BCDEB

$$= \int_1^{\sqrt{2}} [x^2 - (2 - x^2)]dx + \int_{\sqrt{2}}^{2} [2 - (x^2 - 2)]dx$$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)dx + \int_{\sqrt{2}}^{2} (4 - x^2)dx = \left(\frac{20}{3} - 4\sqrt{2}\right) \text{ sq. units}$$

8. We have, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$

$\Rightarrow \quad 4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + 3a$

$4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + 3b$

$4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c$

Consider the equation

$4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$

or $[4f(-1) - 3]x^2 + [4f(1) - 3]\, x + f(2) = 0$

Then clearly this equation is satisfied by

$x = a, b, c$

A quadratic equation satisfied by more than two values of x means it is an identity and hence

$4f(-1) - 3 = 0 \quad \Rightarrow \quad f(-1) = 3/4$

$4f(1) - 3 = 0 \quad \Rightarrow \quad f(1) = 3/4$

$f(2) = 0 \quad \Rightarrow \quad f(2) = 0$

Let $f(x) = px^2 + qx + r$ [$f(x)$ being a quad. equation]

$f(-1) = \frac{3}{4} \quad \Rightarrow \quad p - q + r = \frac{3}{4}$

$f(1) = \frac{3}{4} \quad \Rightarrow \quad p + q + r = \frac{3}{4}$

$f(2) = 0 \quad \Rightarrow \quad 4p + 2q + r = 0$

Solving the above we get $q = 0$, $p = \frac{-1}{4}$, $r = 1$

$\therefore \quad f(x) = -\frac{1}{4} x^2 + 1$

It's maximum value occur at $f'(x) = 0$

i.e., $x = 0$ then $f(x) = 1 \qquad \therefore$ V(0, 1)

A (–2, 0) is the pt. where curve meet x-axis

Let B be the pt. $\left(h, \frac{4 - h^2}{4}\right)$

As $\quad \angle AVB = 90°$

$m_{AV} \times m_{BV} = -1$

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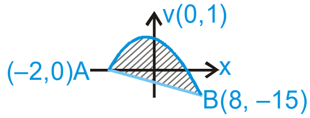
$\Rightarrow \quad \frac{1}{2} \times \left(\frac{-h}{4}\right) = -1$

$\Rightarrow \quad h = 8 \quad \therefore B(8, -15)$

Equation of chord AB is

$$y + 15 = \frac{0-(-15)}{-2-8}(x-8)$$

$\Rightarrow \quad 3x + 2y + 6 = 0$



Required area is the area of shadded region given by

$$= \int_{-2}^{8}\left[\left(-\frac{x^2}{4}+1\right)-\left(\frac{-6-3x}{2}\right)\right]dx$$

$$= \frac{125}{3} \text{ sq. units.}$$

9. **(C)** By inspection, the point of intersection of two curves $y = 3^{x-1}\log x$ and $y = x^x - 1$ is (1, 0)

For first curve $\frac{dy}{dx} = \frac{3^{x-1}}{x} + 3^{x-1}\log 3 \log x$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,0)} = 1 = m_1$

For second curve $\frac{dy}{dx} = x^x(1 + \log x)$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,0)} = 1 = m_2$

$\Rightarrow \quad m_1 = m_2 \quad \Rightarrow$ two curves touch each other

$\Rightarrow$ angle between them is 0°

$\therefore \quad \cos\theta = 1$

10. $y^3 - 3y + x = 0$

$3y^2y' - 3y' + 1 = 0 \qquad y' = \frac{-1}{3(y^2-1)}$

$f(-10\sqrt{2}) = 2\sqrt{2}$

$f'(-10\sqrt{2}) = -\frac{1}{3(7)} = -\frac{1}{21}$

$6y(y')^2 + 3y^2y'' - 3y'' = 0$

$$y'' = -\frac{2y(y')^2}{y^2-1}$$

$$f''(-10\sqrt{2}) = \frac{-2(2\sqrt{2})}{441\times 7} = \frac{-4\sqrt{2}}{7^3 3^2}$$

11. $\int_a^b f(x)dx = [xf(x)]_a^b - \int_a^b xf'(x)dx$

$$= bf(b) - af(a) + \int_a^b \frac{x}{3\left[(f(x))^2-1\right]}dx$$

$$= \int_a^b \frac{x}{3\left[(f(x))^2-1\right]}dx + bf(b) - af(a)$$

12. $\int_{-1}^{1} g'(x)dx = g(1) - g(-1)$

Now $\quad g(1) = -(g(-1))$

(as g'(x) is an even function)

so $\quad \int_{-1}^{1} g'(x)dx = 2g(1)$

13. Area $= \int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$$= \int_0^{\pi/4}\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\sqrt{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}}dx$$

$$= \int_0^{\pi/4}\frac{2\sin\frac{x}{2}}{\sqrt{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}}dx = \int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

Let $\tan\frac{x}{2} = t$

$\sec^2\frac{x}{2}dx = 2dt \quad \Rightarrow \quad dx = \frac{2dt}{(1+t^2)}$

$\therefore \quad$ Area $= \int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

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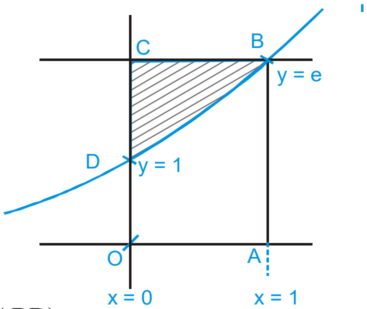
14. $A = \int_1^e \ln y \, dy$



Apply

$= \int_1^e \ln(e+1-y)dy$

$A = ar(OABC) - ar(OABD)$

$= e - \int_1^e e^x dx$.

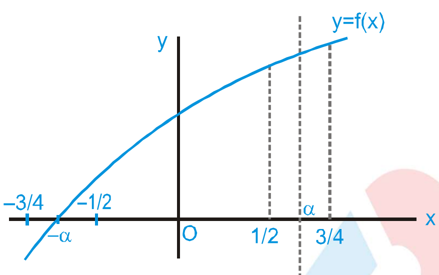
15. ✈ $f'(x) = 2 + 6x + 12x^2 > 0 \ \forall \ x \in R$

$\therefore$ f(x) is strictly increasing in R

✈ $f(0) = 1, f(-1) = -2, \ f\left(-\frac{1}{2}\right) = \frac{1}{4} \ \ \& \ \ f\left(-\frac{3}{4}\right) = -\frac{1}{2}$

$\therefore$ f(x) = 0 has only one real root lying in $\left(-\frac{3}{4}, -\frac{1}{2}\right)$

16.



Let real root is $-\alpha$

$\Rightarrow \ t = |s| = \alpha$

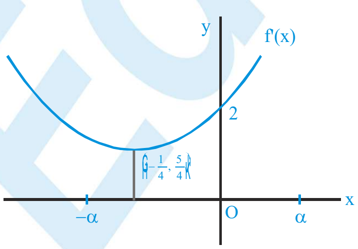
Required area

$A = \int_0^{\alpha} f(x)dx \ \ \& \ \int_0^{1/2} f(x)dx < A < \int_0^{3/4} f(x)dx$

$\Rightarrow \ \left|x + x^2 + x^3 + x^4\right|_0^{1/2} < A < \left|x + x^2 + x^3 + x^4\right|_0^{3/4} < \left|4x\right|_0^{3/4}$

$\Rightarrow \ \frac{15}{16} < A < 3$

17.



$f'(x) = 2(6x^2 + 3x + 1)$

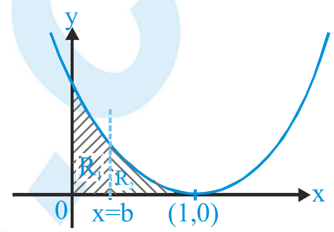
$\Rightarrow$ f'(x) is decreasing in $\left(-\alpha, -\frac{1}{4}\right)$

increasing in $\left(-\frac{1}{4}, \alpha\right)$

or f '(x) is decreasing in $\left(-t, -\frac{1}{4}\right)$

and increasing in $\left(-\frac{1}{4}, t\right)$

18. (A) ✈ $R_1 - R_2 = \frac{1}{4}$



$\Rightarrow \int_0^b (1-x)^2 dx - \int_b^1 (1-x)^2 dx = \frac{1}{4}$

$\Rightarrow -\left(\frac{(1-x)^3}{3}\right)_0^b + \left(\frac{(1-x)^3}{3}\right)_b^1 = \frac{1}{4}$

$\Rightarrow -\left\{\frac{(1-b)^3}{3} - \frac{1}{3}\right\} - \frac{(1-b)^3}{3} = \frac{1}{4}$

$\Rightarrow \frac{1}{3} - \frac{2}{3}(1-b)^3 = \frac{1}{4} \quad \Rightarrow \frac{2}{3}(1-b)^3 = \frac{1}{12}$

$\Rightarrow (1-b)^3 = \frac{1}{8} \Rightarrow \ 1 - b = \frac{1}{2}$

$\Rightarrow \ b = \frac{1}{2}$

(B) $R_2 = \int_{-1}^{2} f(x)dx, \quad R_1 = \int_{-1}^{2} x f(x)dx$

$= \int_{-1}^{2} (1-x) f(1-x)dx \quad \left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$

$= \int_{-1}^{2} (1-x) f(x)dx \qquad (\text{given } f(x) = f(1-x))$

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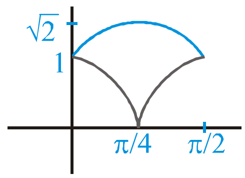
$= \int_{-1}^{2} f(x)dx - \int_{-1}^{2} x f(x)dx$

or $R_1 = R_2 - R_1 \Rightarrow 2R_1 = R_2$

19. $y = \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$



$y = |\cos x - \sin x| = \sqrt{2}\left(\cos\left(x + \frac{\pi}{4}\right)\right)$

$$\text{Area} = \int_0^{\pi/4} \left[(\sin x + \cos x) - (\cos x - \sin x)\right]dx$$

$$+ \int_{\pi/4}^{\pi/2} \left[(\sin x + \cos x) - (\sin x - \cos x)\right]dx$$

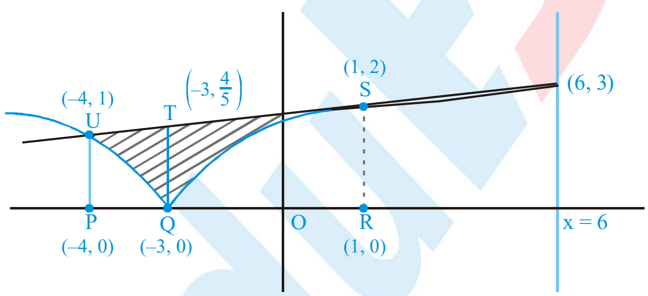
$$= \int_0^{\pi/4} 2\sin x dx + \int_{\pi/4}^{\pi/2} 2\cos x dx$$

$$= [-2\cos x]_0^{\pi/4} + [2\sin x]_{\pi/4}^{\pi/2}$$

$$= 2\sqrt{2}\left(\sqrt{2} - 1\right)$$

21. $y \geq \sqrt{|x+3|}$

$$y^2 \geq \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \end{cases}$$



$$A = \left(A(\text{trapezium PQTU}) - \int_{-4}^{-3} \sqrt{-x-3}\, dx\right)$$

$$+ \left(A(\text{trapezium QRST}) - \int_{3}^{1} \sqrt{x+3}\, dx\right)$$

$$= \left(\frac{11}{10} - \frac{2}{3}\right) + \frac{16}{15} = \frac{3}{2}$$

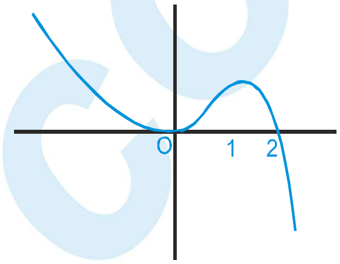
MOCK TEST

1. $y = 8x^2 - x^5 = x^2(8 - x^3)$

Case I $a < 1$

$$A = \int_a^1 (8x^2 - x^5)\, dx = \frac{16}{3}$$

or $\frac{8}{3} - \frac{1}{6} - \frac{8a^3}{3} + \frac{a^6}{6} = \frac{16}{3}$



or $(a^3 - 17)(a^3 + 1) = 0$

$\Rightarrow a = -1$, $a = (17)^{1/3}$ is not possible

Case II $a \in [1, 2]$

$$A = \int_1^a (8x^2 - x^5)\, dx = \frac{16}{3}$$

or $16a^3 - a^6 - 15 = 32$

or $a^6 - 16a^3 + 47 = 0$

This equation is not satisfied by $a = 1$, $a = 2$

Case III $a > 2$

There is no option

Hence one solution is – 1

2. **(D)**

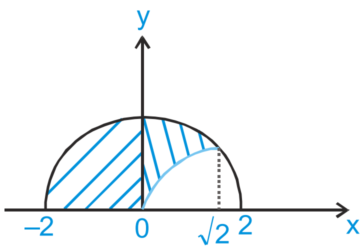
$y = \sqrt{4 - x^2}$, $y = \sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)$

intersect at $x = \sqrt{2}$

Area of the left of y-axis is π

Area to the right of y-axis =

$$\int_0^{\sqrt{2}} \left(\sqrt{4 - x^2} - \sqrt{2}\sin\frac{x\pi}{2\sqrt{2}}\right) dx$$

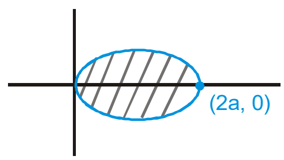


$$= \left(\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right)_0^{\sqrt{2}} + \frac{4}{\pi}\cos\left.\frac{x\pi}{2\sqrt{2}}\right|_0^{\sqrt{2}}$$

$$= \left(1 + 2\cdot\frac{\pi}{4}\right) + \frac{4}{\pi}(0-1) \Rightarrow 1 + \frac{\pi}{2} - \frac{4}{\pi} = \frac{2\pi + \pi^2 - 8}{2\pi}$$

$$\therefore \quad \text{ratio} = \frac{2\pi^2}{2\pi + \pi^2 - 8}$$

3. $a^4y^2 = (2a - x)\,x^5 = 2ax^5 - x^6$



here $(2a - x)\,.\,x^5 \geq 0$

$\Rightarrow \quad x \in [0, 2a]$

$$A = 2\int_0^{2a}\sqrt{\frac{(2a-x).x^5}{a^4}}\,dx$$

$$= \frac{2}{a^2}\int_0^{2a}\sqrt{a^2 - (x-a)^2}\;.\,x^2\,dx$$

put $\quad x - a = a\cos\theta$

then $\quad A = \dfrac{-2}{a^2}\displaystyle\int_0^{\pi} a\sin.\theta\; a^2(1+\cos\theta)^2.(-a\sin\theta)\,d\theta$

$$= 2a^2\int_0^{\pi}\left(\frac{5}{8} - \frac{\cos 2\theta}{2} - \frac{\cos 4\theta}{8} + 2\sin^2\theta\cos\theta\right)d\theta$$

$$= 2a^2\left[\frac{5\theta}{8} - \frac{1}{2}\frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{32} + \frac{2}{3}(\sin\theta)^3\right]_0^{\pi}$$

$$A = 2a^2\left[\frac{5}{8}\pi\right] = \frac{5}{4}\pi a^2$$

$$\therefore \quad \frac{\frac{5}{4}\pi a^2}{\pi a^2} = 5 : 4$$

4. **(C)**

$$y = a^2x^2 + ax + 1$$

$$\therefore \quad \text{area } A = \int_0^1 (a^2x^2 + ax + 1)\,dx$$

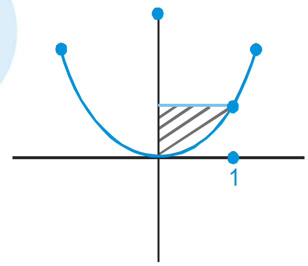
$$= \left(a^2\frac{x^3}{3} + \frac{ax^2}{2} + x\right)\Bigg|_0^1 = \frac{a^2}{3} + \frac{a}{2} + 1$$

$$= \frac{2a^2 + 3a + 6}{6} = \frac{1}{3}\left(a^2 + \frac{3}{2}a + \frac{9}{16}\right) + 1 - \frac{9}{48}$$

$$= \frac{1}{3}\left(a + \frac{3}{4}\right)^2 + \frac{39}{48} \text{ attains its least value, when } a = -\frac{3}{4}$$

5. $y = x^2$ and $y = [x] + 1$

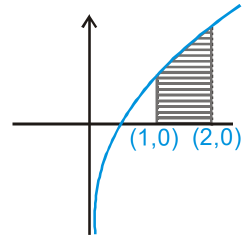
$$A = \int_0^1 (1 - x^2)\,dx = \frac{2}{3}$$



6. **(D)**

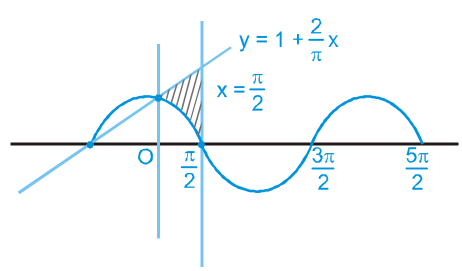
$$\text{Area} = \int_1^2 (\ell n\, x + \tan^{-1}x)\,dx$$

$$= x\ell n\, x\big|_1^2 - \int_1^2 1\,dx + x\tan^{-1}x\big|_1^2 - \int_1^2 \frac{x}{1+x^2}\,dx$$



$$= \frac{5}{2}\ell n\, 2 - \frac{1}{2}\ell n\, 5 + 2\tan^{-1}2 - \frac{\pi}{4} - 1$$

7. $A = \displaystyle\int_0^{\pi/2}\left(1 + \frac{2}{\pi}x - \cos x\right)dx$



$$= x + \frac{x^2}{\pi} - \sin x \Big|_0^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{4} - 1$$

or $\quad A = \dfrac{3\pi}{4} - 1$

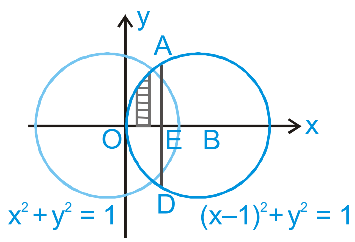
8. (A)

Solving the given equation of circle, we get

$$A \equiv \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) ; D \equiv \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Now area = 2[OBAO] = 2[area OEAO + EBAE]

$$= 2\left[\int_0^{x_E} \sqrt{[1-(x-1)^2]}\, dx + \int_{x_E}^{x_B} \sqrt{1-x^2}\, dx\right]$$



$$= 2\left[\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx + \int_{1/2}^{1} \sqrt{1-x^2}\, dx\right]$$

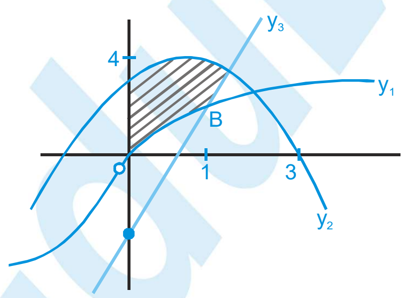
$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \text{ square units}$$

9. $y_1 = x^{1/3}$

and $\quad y_2 = -x^2 + 2x + 3 = -(x-3)(x+1)$

and $\quad y_3 = 2x - 1$

B(1, 1) and A(2, 3)



Figure

$$A = \int_0^1 (y_2 - y_1)\, dx + \int_1^2 (y_2 - y_3)\, dx$$

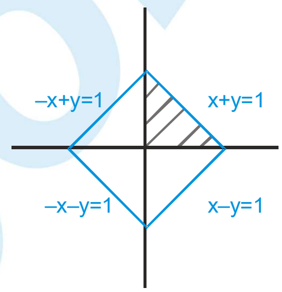
$$= \int_0^2 y_2\, dx - \int_0^1 y_1\, dx - \int_1^2 y_3\, dx$$

$$= \left[-\frac{x^3}{3} + x^2 + 3x\right]_0^2 - \left[\frac{3}{4}x^{4/3}\right]_0^1 - \left[x^2 - x\right]_1^2 = \frac{55}{12}$$

10. (A)

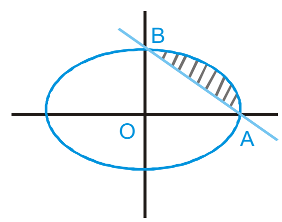
S_1 : Obvious

S_2 : Area $= 4\left(\dfrac{1}{2}.1.1\right) = 2$

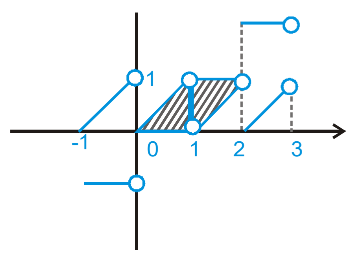


S_3 : Area $= \dfrac{1}{4}$ (Ellipse area) – ΔOAB

$$= \frac{\pi ab}{4} - \frac{ab}{2}$$



S_4 : Area $= 2.\left(\dfrac{1}{2}1.1\right) = 1$



12. $f(x) = 2^{\{x\}}$

Clearly f(x) is periodic with period 1.

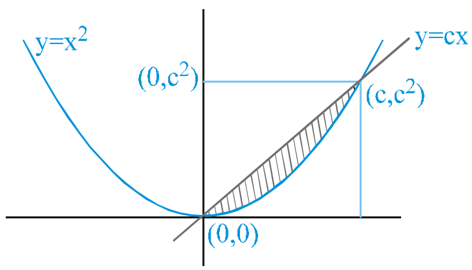
Now $\quad \displaystyle\int_0^1 2^{\{x\}} dx = \int_0^1 2^x dx = \left[\frac{2^x}{\ln 2}\right]_0^1 = \frac{1}{\ln 2} = \log_2 e$

Also $\quad \displaystyle\int_0^{100} 2^{\{x\}} dx = 100 \int_0^1 2^{\{x\}} dx = 100 \log_2 e$

$$\left[\text{Using } \int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx \text{ if a is the period of } f(x)\right]$$

13. Area (T) $= \dfrac{c \cdot c^2}{2} = \dfrac{c^3}{2}$

Area (R) $= \dfrac{c^3}{2} - \int_0^c x^2\,dx$



$= \dfrac{c^3}{2} - \dfrac{c^3}{3} = \dfrac{c^3}{6}$

$\therefore \quad \lim\limits_{c \to 0^+} \dfrac{\text{Area (T)}}{\text{Area (R)}} = \lim\limits_{c \to 0^+} \dfrac{c^3}{2} \cdot \dfrac{6}{c^3} = 3$

14. If $\quad x \le \dfrac{3}{2}$

$f(x) = \int_0^x (3-2t)\,dt = 3x - x^2$

$x > \dfrac{3}{2}$

$f(x) = \int_0^{3/2} (3-2t)\,dt + \int_{3/2}^{x} (2t-3)\,dt = \dfrac{9}{2} + x^2 - 3x$

$f(x) = \begin{cases} 3x - x^2 & , \quad x \le 3/2 \\ x^2 - 3x + 9/2 & , \quad x > 3/2 \end{cases}$

Now this is continuous at $x = \dfrac{3}{2}$

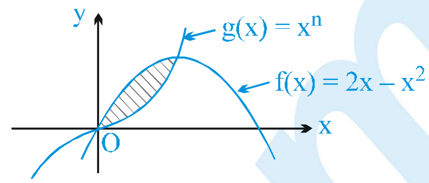
and at $x = 3$ also differentiable at $x = 0$

15. Solving $f(x) = 2x - x^2$ and $g(x) = x^n$

we have $2x - x^2 = x^n$

$\Rightarrow \quad x = 0$ and $x = 1$

$A = \int_0^1 (2x - x^2 - x^n)\,dx = \left[x^2 - \dfrac{x^3}{3} - \dfrac{x^{n+1}}{n+1} \right]_0^1$



$= 1 - \dfrac{1}{3} - \dfrac{1}{n+1} = \dfrac{2}{3} - \dfrac{1}{n+1}$

hence, $\dfrac{2}{3} - \dfrac{1}{n+1} = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{2}{3} - \dfrac{1}{2} = \dfrac{1}{n+1} \qquad \Rightarrow \quad \dfrac{4-3}{6} = \dfrac{1}{n+1}$

$\Rightarrow \quad n + 1 = 6 \qquad \Rightarrow \quad n = 5$

Hence n is a divisor of 15, 20, 30

$\Rightarrow$ **B, C, D**

16. (C)

Statement-I Let $\dfrac{p}{\sqrt{p^2+q^2}}\,x + \dfrac{q}{\sqrt{p^2+q^2}}\,y = U$

and $\dfrac{q}{\sqrt{p^2+q^2}}\,x - \dfrac{p}{\sqrt{p^2+q^2}}\,y = V$

Then the axis get rotated through an angle θ,

where $\cos\theta = \dfrac{p}{\sqrt{p^2+q^2}}$ and $\sin\theta = \dfrac{q}{\sqrt{p^2+q^2}}$

$\therefore$ the equation of the given curve becomes $|U| + |V| = a$

$\therefore$ the area bounded $= 2a^2$.

$\therefore$ statement-1 is true

Statement-II the equation of the curve is $|\alpha x + \beta y| + |\beta x - \alpha y| = a$ which is equivalent to

$$\left| \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\,x + \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\,y \right| + \left| \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\,x - \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\,y \right|$$

$= \dfrac{a}{\sqrt{\alpha^2+\beta^2}}$

$\therefore$ area bounded $= \dfrac{2a^2}{\alpha^2+\beta^2}$

$\therefore$ statement-2 is false.

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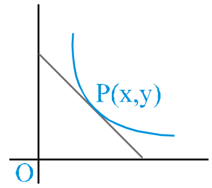
17. Equation of tangent



$$Y - y = m(X - x)$$

put $X = 0, \qquad Y = y - mx$

hence initial ordinate is

$$y - mx = x - 1 \Rightarrow mx - y = 1 - x$$

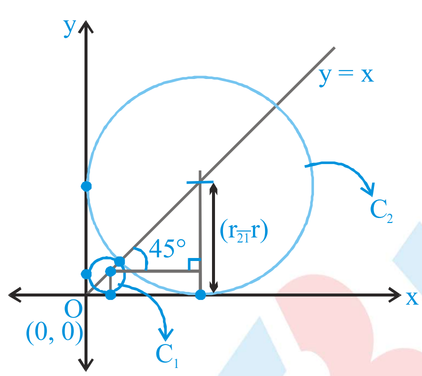
$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1-x}{x}$$ which is a linear differential equation

Hence statement-1 is correct and its degree is 1

$\Rightarrow$ statement-2 is also correct. Since every 1st degree differential equation need not be linear hence statement-2 is not the correct explanation of statement-1.

19. From the diagram,

$$\sqrt{2}(r_2 - r_1) = r_2 + r_1$$



but $r_1 = 2$

$$\sqrt{2}(r_2 - 2) = r_2 + 2$$

$$\left(\sqrt{2}-1\right)r_2 = 2 + 2\sqrt{2}$$

$$\Rightarrow r_2 = \sqrt{2}\left(\sqrt{2}+1\right)\left(\sqrt{2}+2\right)$$

Also centres of both the circles may also lie on $y = -x$.

20. **(B)**

$$\text{Area} = \int_1^3 -(x^2 - 4x + 3)\,dx = -\left(\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right)\Bigg|_1^3 = \frac{4}{3}$$

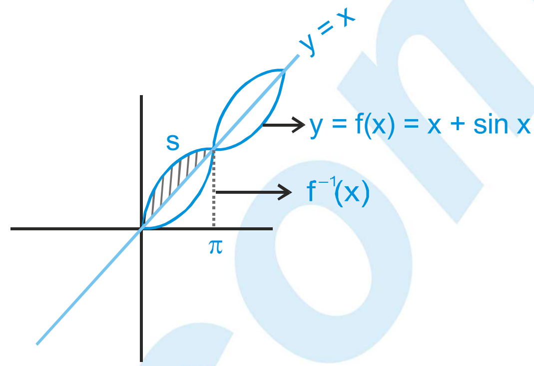
$\therefore$ **Statement-I** is true

Statement-II is true but does not explain statement-I

21. **(A) → (t), (B) → (t), (C) → (r), (D) → (s)**

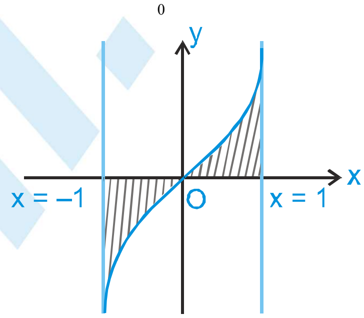
(A) Required area = 4s

$$s = \int_0^{\pi}(x + \sin x)\,dx - \int_0^{\pi} x\,dx$$



$$= \frac{\pi^2}{2} - \cos\pi + \cos 0 - \frac{\pi^2}{2} = 2 \text{ sq. units}$$

(B) Required area $= 2\int_0^1 xe^x dx = 2\left[xe^x - e^x\right]_0^1 = 2$

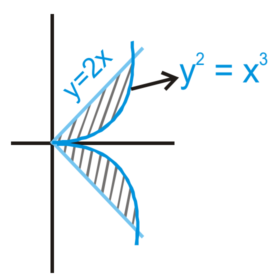


(C) $y^2 = x^3$ and $|y| = 2x$ both the curve are symmetric about y - axis

$4x^2 = x^3 \Rightarrow x = 0, 4,$

required area

$$= 2\int_0^4 (2x - x^{3/2})\,dx = \frac{16}{5}$$



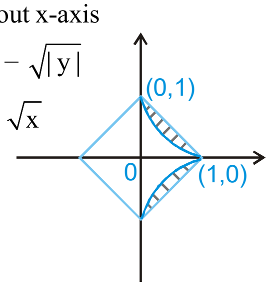
(D) $\sqrt{x} + \sqrt{|y|} = 1$

Above curve is symmetric about x-axis

$\sqrt{|y|} = 1 - \sqrt{x}$ and $\sqrt{x} = 1 - \sqrt{|y|}$

$\Rightarrow$ for $x > 0, y > 0$ $\sqrt{y} = 1 - \sqrt{x}$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$



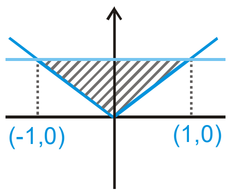
$$\frac{dy}{dx} = -\sqrt{\frac{x}{y}}$$

$\frac{dy}{dx} < 0$, function is decreasing required area

$1 - 2\int_0^1 \left(1-\sqrt{x}\right) dx = \frac{1}{3}$

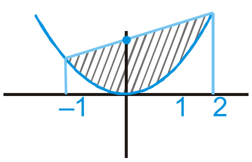
22. (A) → (t), (B) → (p), (C) → (s), (D) → (r)

(A) The area = 1 unit



(B) Area enclosed = $\int_0^{\pi} \sin x \; dx = 2$

(C) The line $y = x + 2$ intersects $y = x^2$ at



$x = -1$ and $x = 2$

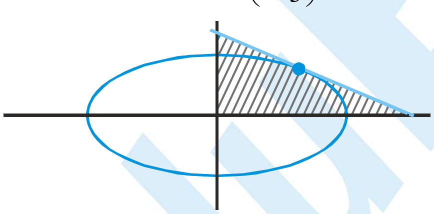
the given region is shaded region area

$$\frac{15}{2} - \int_{-1}^{2} x^2 \, dx = \frac{9}{2}$$

(D) Here $a^2 = 9, b^2 = 5, b^2 = a^2(1 - e^2)$

$\Rightarrow \; e^2 = \frac{4}{9} \qquad \Rightarrow \; e = \frac{2}{3}$

Equation of tangent at $\left(2, \frac{5}{3}\right)$ is



$$\frac{2x}{9} + \frac{y}{3} = 1$$

x intercept = $\frac{9}{2}$, y intercept = 3

Area = $4 \times \frac{9}{2} \times 3 \times \frac{1}{2} = 27$ sq. units

24.

1. (A)

$(y - 4)\, x^2 + x + 2 = 0$

the coefficient of the highest power of x

i.e. x^2 is $y - 4 = 0$

$y - 4 = 0$ is the asymptote parallel to the x-axis.

The coefficient of the highest power of y is x, so $x = 0$ is also a asymptotes.

2. (B)

$\varphi_3(m) = 1 + m^3, \varphi_2(m) = -3m$

Putting $\varphi_3(m) = 0$ or $m^3 + 1 = 0$

or $(m + 1)(m^2 - m + 1) = 0$

$$m = -1, \; m = \frac{1 \pm \sqrt{1-4}}{2}$$

Only real value of m is -1

Now we find c from the equation $c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)}$

$$c = \frac{3m}{3m^2} = \frac{1}{m} = -1$$

On putting $m = -1$ and $c = -1$ in $y = mx + c$.

The equation of asymptote is

$y = (-1)\,x + (-1)$ or $x + y + 1 = 0$

3. (B)

The coefficient of the highest power of y is $(2 - x)$,

So $x = 2$ is asymptotes.

$\therefore \; a = 1, b = 0, c = -2$

$\therefore \; |a + b + c| = 1$

26. Here $f(x + y) = f(x) + f(y) - 8xy$.

Replacing x, y by 0 we obtain $f(0) = 0$

Now, $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{y\to 0} \frac{f(x+y) - f(x)}{y}$$

$$= \lim_{y\to 0} \frac{f(x) + f(y) - 8xy - f(x)}{y}$$

$$= \lim_{y\to 0} \left\{\frac{f(y)}{y} - \frac{8xy}{y}\right\} = f'(0) - 8x = 8 - 8x \quad [\text{given } f'(0) = 8]$$

$\Rightarrow \; f'(x) = 8 - 8x$

Integrating both side,

$f(x) = 8x - 4x^2 + c$

as $f(0) = 0 \quad \Rightarrow \quad c = 0$

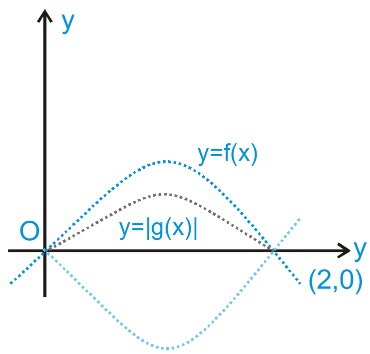
$\Rightarrow \quad f(x) = 8x - 4x^2$

also $g(x+y) = g(x) + g(y) + 3xy\,(x+y)$

Replacing x, y by 0, we obtain $g(0) = 0$

Now $g'(x) = \lim_{y \to 0} \dfrac{g(x+y) - g(x)}{y}$

$= \lim_{y \to 0} \dfrac{g(x) + g(y) + 3x^2y + 3xy^2 - g(x)}{y}$



$= \lim_{y \to 0}\left[\dfrac{g(y)}{y} + \dfrac{y(3x^2 + 3xy)}{y}\right] = g'(0) + 3x^2 = -4 + 3x^2$

$\therefore \quad g(x) = x^3 - 4x \quad (\text{as } g(0) = 0)$ ....**(ii)**

$|g(x)| = \begin{cases} x^3 - 4x, & x \in [-2, 0] \cup [2, \infty) \\ 4x - x^3, & x \in (-\infty, -2) \cup (0, 2) \end{cases}$

Points where f(x) and |g(x)| meets, we have

$f(x) = |g(x)|$

$\Rightarrow \quad x = 0\ , 2.$

Area bounded by $y = f(x)$ and $y = |g(x)|$,

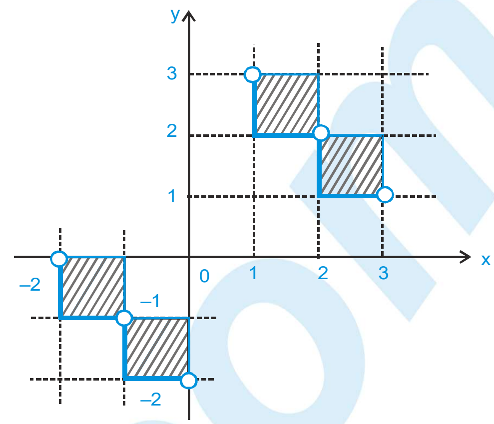
between $x = 0$ to $x = 2$ is

$\int_0^2 (x^3 - 4x^2 + 4x)\,dx = \dfrac{4}{3}.$

27. (4)

$[x] \cdot [y] = 2$

Here four cases arise



(1) $[x] = 2\ \&\ [y] = 1 \quad \Rightarrow \quad 2 \le x < 3\ \&\ 1 \le y < 2$

(2) $[x] = 1\ \&\ [y] = 2 \quad \Rightarrow \quad 1 \le x < 2\ \&\ 2 \le y < 3$

(3) $[x] = -2\ \&\ [y] = -1 \quad \Rightarrow \quad -2 \le x < -1\ \&\ -1 \le y < 0$

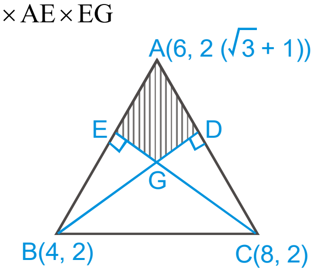
(4) $[x] = -1\ \&\ [y] = -2 \quad \Rightarrow \quad -1 \le x < 0\ \&\ -2 \le y < -1$

Area enclosed by solution set = 4

28. As the given triangle is equilateral with side lengths 4. BD and CE are angle bisectors of angle B and C resp. Any point inside the ΔAEC is nearer to AC than BC and any point inside the ΔBDA is nearer to AB than BC. So points inside the quadrilateral AEGD will satisfy the given condition

$\therefore$ Required area = 2 (ΔEAG)

$= 2 \times \dfrac{1}{2} \times AE \times EG$



$= \dfrac{4\sqrt{3}}{3}$ sq. units.

29. (8)

$ay^2 = x^2(a - x) \qquad y = \pm x\sqrt{\dfrac{a - x}{a}}$

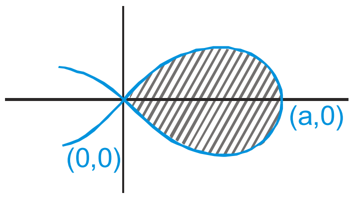
Area $= 2\int_0^a x\sqrt{\dfrac{a-x}{a}}\,dx$

put $x = a\cos\theta,\ dx = -a\sin\theta\,d\theta$

$= 2\int_0^{\pi/2} a\cos\theta\sqrt{2}\sin\dfrac{\theta}{2}\ a\sin\theta\,d\theta$

$$= 2\sqrt{2}\,a^2 \int_0^{\pi/2} \left(1 - 2\sin^2\frac{\theta}{2}\right) 2\sin^2\frac{\theta}{2} \cos\frac{\theta}{2}\, d\theta$$

put $\sin\frac{\theta}{2} = t$, $\cos\frac{\theta}{2}\, d\theta = 2dt$



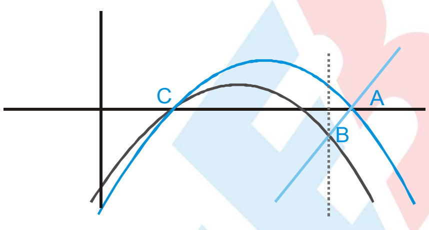
$$= 8\sqrt{2}\,a^2 \int_0^{1/\sqrt{2}} (1-2t^2)t^2\, dt = 8\sqrt{2}\,a^2 \int_0^{1/\sqrt{2}} (t^2 - 2t^4)\, dt$$

$$= 8\sqrt{2}\,a^2 \left(\frac{t^3}{3} - \frac{2t^5}{5}\right)^{1/\sqrt{2}} = 8\sqrt{2}\,a^2 \left(\frac{1}{6\sqrt{2}} - \frac{2}{20\sqrt{2}}\right)$$

$$= 8a^2 \left(\frac{1}{6} - \frac{1}{10}\right) = \frac{8a^2}{15}$$

30. $y = -(x^2 - 6x + 5) = -(x-5)(x-1)$

$y = -(x^2 - 4x + 3) = -(x-3)(x-1)$



Figure

$y = 3x - 15$

A (5, -0) B(4, – 3) C (1, 0).

$$\text{Area} = \int_1^4 \left((-x^2+6x-5) - (-x^2+4x-3)\right)dx$$

$$+ \int_4^5 \left((-x^2+6x-5) - (3x-15)\right)dx$$

$$= \int_1^5 (-x^2+6x-5)dx - \int_1^4 (-x^2+4x-3)dx - \int_4^5 (3x-15)dx$$

$$= \left(-\frac{x^3}{3} + 3x^2 - 5x\right)_1^5 - \left(-\frac{x^3}{3} + 2x^2 - 3x\right)_1^4$$

$$- \left(\frac{3x^2}{2} - 15x\right)_4^5 = \frac{32}{3} - (0) + \frac{3}{2}$$

$$\text{Area} = \frac{73}{6}$$