BINOMIAL THEOREM
EXERCISE # 1

Question based on General Term

Q.1 If the 4^{th} term in the expansion of $\left(ax+\frac{1}{x}\right)^n$ is $\frac{5}{2}$ then the values of a and n respectively are-

(A) 2, 6 (B) $\frac{1}{2}$, 6

(C) $\frac{1}{2}$, 5 (D) None of these

Sol. [B] $T_4 = {}^nC_3(ax)^{n-3}\left(\frac{1}{x}\right)^3$

$= {}^nC_3\, a^{n-3}\, x^{n-6} = \frac{5}{2}$ given

$\therefore\ n - 6 = 0 \Rightarrow n = 6$

$T_4 = {}^6C_3 a^{6-3} = \frac{5}{2}$

$= a^3 = \frac{1}{8} = \left(\frac{1}{2}\right)^3 \Rightarrow a = \frac{1}{2}$

$\Rightarrow a = \frac{1}{2},\ n = 6$

Q.2 The 4^{th} term from the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^7$ is-

(A) 70x (B) $70x^2$

(C) $-35x^6$ (D) None of these

Sol. [A] 4^{th} term from the end

$= (7 - 4 + 2)^{th}$ term from beginning

$\Rightarrow 5^{th}$ term from beginning

$T_5 = {}^7C_4\left(\frac{x^3}{2}\right)^3\left(-\frac{2}{x^2}\right)^4 = \frac{35}{8}.\ 16x = 70x$

Q.3 The coefficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$ is-

(A) 30 (B) 60

(C) 40 (D) None of these

Sol. [B] General term is

$= ({}^5C_r x^{2r})\ ({}^4C_p x^p) = {}^5C_r\ {}^4C_p\ x^{2r+p}$

coefficient of x^5 is

(i) when r = 1, p = 3 $\Rightarrow {}^5C_1\ {}^4C_3 = 20$

(ii) when r = 2, p = 1 $\Rightarrow {}^5C_2\ {}^4C_1 = 40$

coefficient of x^5 is = 20 + 40 = 60

Question based on Middle Term / Numerically greatest term and Algebrically greatest & least term

Q.4 The middle term in the expansion of $(1-3x + 3x^2 - x^3)^6$ is -

(A) ${}^{18}C_{10}\ x^{10}$ (B) ${}^{18}C_9\ (-x)^9$

(C) ${}^{18}C_9\ x^9$ (D) None of these

Sol. [B] $(1 - 3x + 3x^2 - x^3)^6 = (1 - x)^{18}$

Middle term is $\frac{18+2}{2} = 10^{th}$ term

$T_{10} = {}^{18}C_9(-x)^9$

Q.5 The numerically greatest terms in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$ is-

(A) T_4 (B) T_5

(C) T_6 (D) None of these

Sol. [A,B] Numerically greatest term = $\frac{x+1}{|x/a|+1}$

$= \frac{15+1}{\left|\frac{3}{-5.\frac{1}{5}}\right|+1}$ when $x = \frac{1}{5} = \frac{16}{4} = 4$

greatest term are T_4 and T_5

Question based on Properties of Binomial Coefficient

Q.6 The sum of the last eight coefficients in the expansion of $(1 + x)^{16}$ is equal to-

(A) 2^{15} (B) 2^{14}

(C) $2^{15} - \frac{1}{2}\frac{(16)!}{(8!)^2}$ (D) None of these

Sol. [C] $(1 + x)^{16} = {}^{16}C_0 + {}^{16}C_1x + {}^{16}C_2x^2 + \ldots + {}^{16}C_{16}x^{16}$

$2^{16} = {}^{16}C_0 + {}^{16}C_1 + {}^{16}C_2 + \ldots + {}^{16}C_{16}$

$\Rightarrow 2^{16} = 2({}^{16}C_9 + {}^{16}C_{10} + {}^{16}C_{11} + \ldots + {}^{16}C_{16}) + {}^{16}C_8$

$\Rightarrow$ Sum of last eight coffi.

$= \frac{2^{16} - {}^{16}C_8}{2}$

$= 2^{15} - \dfrac{\underline{|16}}{2(\underline{|8})^2}$

Q.7 $\dfrac{1}{1!\cdot(n-1)!} + \dfrac{1}{3!\cdot(n-3)!} + \dfrac{1}{5!\cdot(n-5)!} + ...$ is equal to -

(A) $\dfrac{2^{n-1}}{n!}$ for even values of n only

(B) $\dfrac{2^{n-1}+1}{n!} - 1$ for odd values of n only

(C) $\dfrac{2^{n-1}}{n!}$ for all $n \in N$

(D) None of these

Sol. **[C]**

$\dfrac{1}{1!\cdot(n-1)!} + \dfrac{1}{3!\cdot(n-3)!} + \dfrac{1}{5!\cdot(n-5)!} + \ldots$

$= \dfrac{1}{n!}\left[\dfrac{n!}{1!(n-1)!} + \dfrac{n!}{3!(n-3)!} + \dfrac{n!}{5!(n-5)!} + ...\right]$

$= \dfrac{1}{n!}\,[{}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots]$

Since,

${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_4 + {}^nC_5 + \ldots + {}^nC_n = 2^n$

${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + \ldots + (-1)^n\,{}^nC_n = 0$

Subtracting, we get

$2[{}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots] = 2^n$

${}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = \dfrac{2^n}{2} = 2^{n-1}$

$\dfrac{n!}{1!(n-1)!} + \dfrac{n!}{3!(n-3)!} + \dfrac{n!}{5!(n-5)!} + \ldots = 2^{n-1}$

$\dfrac{1}{1!(n-1)!} + \dfrac{1}{3!(n-3)!} + \dfrac{1}{5!(n-5)!} + \ldots = \dfrac{2^{n-1}}{n!}$

$\therefore$ Option (C) is correct Answer.

Question based on Application of Binomial theorem

Q.8 The coefficient of y^{49} in

$(y-1)(y-3)\ldots\ldots(y-99)$ is-

(A) 2500 (B) – 2500

(C) – 99 × 50 (D) 99 × 50

Sol. $(y-1)(y-3)\ldots\ldots(y-99)$

Total term is 50

coffi of y^{49} is $-(1 + 3 + 5 + \ldots\ldots + 99)$

$= -\dfrac{50}{2}[2 + 49 \times 2]$

$= -2500$

Q.9 The coefficient of x^m in

$(1 + x)^k + (1 + x)^{k+1} + \ldots.. + (1 + x)^n$, $(k \le m \le n)$, is given by

(A) ${}^{n+1}C_m$ (B) ${}^nC_m$

(C) ${}^{n-1}C_{m-1}$ (D) ${}^{n+1}C_{m+1}$

Sol. $(1 + x)^k + (1 + x)^{k+1} + \ldots\ldots + (1 + x)^n$

Here n – k + 1 term then

coffi of x^m in $\left[\dfrac{(1+x)^k\left((1+x)^{n-k+1}-1\right)}{(1+x-1)}\right]$

coffi of x^m in $\dfrac{(1+x)^{n+1}-(1+x)^k}{x}$

coffi of x^m in $\dfrac{(1+x)^{n+1}}{x}$ $\quad \Theta\ k \le m$

coffi of x^{m+1} in $(1 + x)^{n+1}$

$= {}^{n+1}C_{m+1}$

Q.10 If ${}^mC_3 + {}^mC_4 > {}^{m+1}C_3$, then least value of m is -

(A) 6 (B) 7

(C) 5 (D) None of these

Sol. ${}^mC_3 + {}^mC_4 > {}^{m+1}C_3$

$\Rightarrow \dfrac{{}^{m+1}C_4}{{}^{m+1}C_3} > 1$

$\Rightarrow \dfrac{\underline{|m+1}}{\underline{|4}\ \underline{|m-3}} \cdot \dfrac{\underline{|3}\ \underline{|m-2}}{\underline{|m+1}} > 1$

$\Rightarrow \dfrac{m-2}{4} > 1 \Rightarrow m > 6$

Least value of m is 7

Q.11 If the coefficients of $(2r + 4)^{th}$, $(r - 2)^{th}$ terms in the expansion of $(1 + x)^{18}$ are equal, then r is-

(A) 4 (B) 6

(C) 8 (D) None of these

Sol. Given that

${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow 2r + 3 + r - 3 = 18 \Rightarrow r = 6$

Q.12 Number of terms free from radical sign in the expansion of $(1 + 3^{1/3} + 7^{1/7})^{10}$ is-

(A) 4 (B) 5

(C) 6 (D) 8

Sol. Here $\dfrac{\underline{|10}}{\underline{|p}\ \underline{|q}\ \underline{|r}}\ 1^p . 3^{q/3} . 7^{r/7}$ where p + q + r = 10

case are possible which

p q r

10	0	0
7	3	0
4	6	0
1	9	0
3	0	7
0	3	7

Total cases are 6

Q.13 Let $(5 + 2\sqrt{6})^n = p + f$ where $n \in N$ and $p \in N$ and $0 < f < 1$ then the value of $f^2 - f + pf - p$ is

(A) a natural number (B) a negative integer

(C) a prime number (D) an irrational number

Sol. $(5 + 2\sqrt{6})^n = p + f$, $n, p \in N$, $0 < f < 1$

Let $f' = (5 - 2\sqrt{6})^n$, $0 < f' < 1$

$\Rightarrow p + f + f' = 2(5^n + {}^nC_2 5^{n-2}(2\sqrt{6})^2 + \ldots)$

$= 2k$, $k \in N$

$\Rightarrow p + f + f' =$ Integer

$\Rightarrow f + f' =$ Integer $- p =$ Integer

but $0 < f + f' < 2$

$\Rightarrow f + f' = 1$

$\Rightarrow f' = 1 - f$

$\because f^2 - f + pf - p = (f - 1)(p + f)$

$= -f'(p + f)$

$= -(5 - 2\sqrt{6})^n (5 + 2\sqrt{6})^n$

$= -(25 - 24)^n = -1$

$\Rightarrow$ a negative integer.

Q.14 The term independent of x in the expansion of $(1 - x + 2x^3)\left(x^2 - \frac{1}{x}\right)^8$ is given by -

(A) –56 (B) 56

(C) 0 (D) None

Sol. **[C]**

$(1 - x + 2x^3)\left(x^2 - \frac{1}{x}\right)^8$

Let (r + 1)th term of $\left(x^2 - \frac{1}{x}\right)^8$ be independent of x

$(1 - x + 2x^3).\,{}^nC_r.\left(-\frac{1}{x}\right)^r.(x^2)^{8-r}$

$= (1 - x + 2x^3).\,{}^nC_r.(-1)^r.\,x^{16-2r-r}$

$= (1 - x + 2x^3).\,{}^nC_r.(-1)^r.\,x^{16-3r}$

$= {}^nC_r(-1)^r[x^{16-3r} - x^{17-3r} + 2.x^{19-3r}]$

Then $16 - 3r = 0 \Rightarrow r = 16/3$ Not possible due to fraction value

$17 - 3r = 0 \Rightarrow r = 17/3$ Not possible due to fraction value

$19 - 3r = 0 \Rightarrow r = 19/3$ Not possible due to fraction value

Q.15 The number of irrational terms in the expansion of $(\sqrt[8]{5} + \sqrt[6]{2})^{100}$ is-

(A) 97 (B) 98

(C) 96 (D) 99

Sol. **[A]**

$\left[(5)^{\frac{1}{8}} + (2)^{\frac{1}{6}}\right]^{100}$

Let us make a combination of digits which are divisible separately by 8 and 6 whose sum will be 100.

	✕	✓	✕	✕	✓	✕	✕	✓	✕	✕	✓	✕
	8	16	24	32	40	48	56	64	72	80	88	96
	92	84	76	68	60	52	44	36	28	20	12	4
Sum =	100	100	100	100	100	100	100	100	100	100	100	100

There will be 4-Rational terms.

Hence, No of irrational terms = 101 – 4

= 97

Q.16 If $1 \le r \le n-1$, then ${}^{n-1}C_r + {}^{n-2}C_r + \ldots\ldots + {}^rC_r$ equals-

(A) ${}^nC_r$ (B) ${}^nC_{r+1}$

(C) ${}^{n+1}C_r$ (D) None of these

Sol. **[B]**

$1 \le r \le n - 1$

${}^{n-1}C_r + {}^{n+2}C_r + \ldots + {}^rC_r = {}^nC_{n+1}$

$\therefore$ Option (B) is correct Answer.

Q.17 The expression $\left(x + (x^3 - 1)^{\frac{1}{2}}\right)^5 + \left(x - (x^3 - 1)^{\frac{1}{2}}\right)^5$ is a polynomial of degree-

(A) 5 (B) 6

(C) 7 (D) 8

Sol. **[C]**

Q.18 If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals-

(A) $(n - 1)\,a_n$ (B) na_n

(C) $\frac{1}{2}na_n$ (D) None of these

Sol. **[C]**

If $a_n = \sum_{r=0}^{n} \frac{1}{^nC_r} = \frac{1}{^nC_0} + \frac{1}{^nC_1} + \frac{1}{^nC_2} + \frac{1}{^nC_3} + \frac{1}{^nC_4} + \dots + \frac{1}{^nC_n}$

$\sum_{r=0}^{n} \frac{r}{^nC_r} = 0 + \frac{1}{^nC_1} + \frac{2}{^nC_2} + \frac{3}{^nC_3} + \frac{4}{^nC_4} + \dots\dots + \frac{(n-3)}{^nC_{n-3}} + \frac{(n-2)}{^nC_{n-2}} + \frac{(n-1)}{^nC_{n-1}} + \frac{n}{^nC_n}$

$= \frac{1}{2}\left[\frac{n}{^nC_1} + \frac{n}{^nC_2} + \frac{n}{^nC_3} + \frac{n}{^nC_4} + \dots\right]$

$= \frac{n}{2} \times \left[\frac{1}{^nC_1} + \frac{1}{^nC_2} + \frac{1}{^nC_3} + \dots\right]$

$= \frac{n}{2} \times a_n$

$\therefore$ Option (C) is correct Answer.

Question based on **Multinomial theorem**

Q.19 The coefficient of x^3y^4z in the expansion of $(1 + x + y - z)^9$ is-

(A) $2 \cdot {^9C_7} \cdot {^7C_4}$ (B) $-2 \cdot {^9C_2} \cdot {^7C_3}$

(C) ${^9C_7} \cdot {^7C_4}$ (D) None of these

Sol. **[D]**

$(1 + x + y - z)^9$

$= {^9C_0} + {^9C_1}(y + x - z) + {^9C_2}(y + x - z)^2 + {^9C_3}(y + x - z)^3 + {^9C_4}(y + x - z)^4 + {^9C_5}(y + x - z)^5 + {^9C_6}(y + x - z)^6 + {^9C_7}(y + x - z)^7 + {^9C_8}(y + x - z)^8 + {^9C_9}(y + x - z)^9$.

We have to find out coefficient of x^3y^4z.

${^9C_4}(y + x - z)^4 = {^9C_4}[{^4C_0}y^4 + {^4C_1}y^3(x - z) + {^4C_2}y^2(x - z)^2 + {^4C_3}y(x - z)^3 + {^4C_4}(x - z)^4]$.

= No terms give coefficient of x^3y^4z.

${^9C_5}(y + x - z)^5 = {^9C_5}[{^5C_0}y^5 + {^5C_1}y^4(x - z) + {^5C_2}y^3(x - z)^2 + {^5C_3}y^2(x - z)^3 + {^5C_4}y(x - z)^4 + {^5C_5}(x - z)^5]$

= No terms give coefficient of x^3y^4z.

${^9C_6}(y + x - z)^6 = {^9C_6}[{^6C_0}y^6 + {^6C_1}y^5(x - z) + {^6C_2}y^4(x - z)^2 + {^6C_3}y^3(x - z)^3 + {^6C_4}y^2(x - z)^4 + {^6C_5}y(x - z)^5 + {^6C_6}(x - z)^6]$

= No terms give coefficient of x^3y^4z.

${^9C_7}(y + x - z)^7 = {^9C_7}[{^7C_0}y^7 + {^7C_1}y^6(x - z) + {^7C_2}y^5(x - z)^2 + {^7C_3}y^4(x - z)^3 + {^7C_4}y^3(x - z)^4 + {^7C_5}y^2(x - z)^5 + {^7C_6}y(x - z)^6 + {^7C_7}(x - z)^7]$

Only term ${^9C_7} \times {^7C_4}y^4[{^4C_0}x^4 - {^4C_1}x^3z + {^4C_2}x^2z^2 - {^4C_3}xz^3 + {^4C_4}z^4]$

give coefficient of x^3y^4z

$= -{^9C_7} \times {^7C_4} \times {^4C_1}$

$= \frac{-9!}{7! \times 2!} \times \frac{7!}{4! \times 3!} \times 4!$

${^9C_8}(y + x - z)^8 = {^9C_8}[{^8C_0}y^8 + \dots + {^8C_4}y^4(x - z)^4 + \dots + {^8C_8}y^0(x - z)^8]$

Term ${^9C_8} \times {^8C_4}y^4(x - z)^4$ give coefficient of x^3y^4z.

${^9C_8} \times {^8C_4}y^4(x - z)^4 = {^9C_8} \times {^8C_4} \times y^4[{^4C_0}x^4 + {^4C_1}x^3(-z) + {^4C_2}x^2z^2 - {^4C_3}xz^3 + {^4C_4}z^4]$

Coefficient of $x^3y^4z = {^9C_8} \times {^8C_4} \times (-{^4C_1})$

Hence, coefficient x^3y^4z

$= -{^9C_7} \times {^7C_4} \times {^4C_1} - {^9C_8} \times {^8C_4} \times {^4C_1}$

$= -{^4C_1}[{^9C_7} \times {^7C_4} + {^9C_8} \times {^8C_4}]$

$= -{^4C_1} \times \left[{^9C_7} \times {^7C_4} + \frac{9!}{8!} \times \frac{8!}{4! \times 4!}\right]$

$= -{^4C_1} \times \left[{^9C_7} \times {^7C_4} + \frac{9! \times 2!}{8 \times 7! \times 2!} \times \frac{8 \times 7!}{4! \times 3! \times 4}\right]$

$= -{^4C_1}\left[{^9C_7} \times {^7C_4} + \frac{9!}{7! \times 2!} \times \frac{7!}{4! \times 3!} \times \frac{1}{2}\right]$

$= -{^4C_1}\left[{^9C_7} \times {^7C_4} + {^9C_7} \times {^7C_4} \times \frac{1}{2}\right]$

$= -{^4C_1} \times {^9C_7} \times {^7C_4}\left[1 + \frac{1}{2}\right]$

$= -{^4C_1} \times {^9C_7} \times {^7C_4} \times 3/2$

$= -4 \times \frac{3}{2} \times {^9C_4} \times {^7C_4} = -6 \times {^9C_7} \times {^7C_4}$.

= Option (D) is correct Answer.

Q.20 The coefficient of $a^8b^6c^4$ in the expansion of $(a + b + c)^{18}$ is-

(A) ${^{18}C_{14}} \cdot {^{14}C_8}$ (B) ${^{18}C_{10}} \cdot {^{10}C_6}$

(C) ${^{18}C_6} \cdot {^{12}C_8}$ (D) ${^{18}C_4} \cdot {^{14}C_6}$

Sol. **[A,B,C,D]**

$(a + b + c)^{18}$

$= {^{18}C_0}a^{18} + {^{18}C_1}a^{17}(b + c) + {^{18}C_2}a^{16}(b + c)^2 + \dots + {^{18}C_{10}}a^{18-10}(b + c)^{10} + \dots + {^{18}C_{18}}(b + c)^{18}$.

We have to find coefficient of $a^8b^6c^4$.

Only term ${^{18}C_{10}}a^8(b + c)^{10}$ gives required value.

$= {^{18}C_{10}}a^8[{^{10}C_0}b^{10} + {^{10}C_1}b^9.c + {^{10}C_2}b^8c^2 + {^{10}C_3}b^7c^3 + {^{10}C_4}b^6c^4 + \dots]$

Hence, coefficient of $a^8b^6c^4$ is

${^{18}C_{10}} \times {^{10}C_4} = {^{18}C_{10}} \times {^{10}C_6}$

$\therefore$ (B) is correct answer.

$(a + b + c)^{18}$ can be expanded as also.

$= {^{18}C_0}b^{18} + {^{18}C_1}b^{17}(a + c) + {^{18}C_2}b^{16}(a + c)^2 + \dots + {^{18}C_{12}}b^6(a + c)^{12} + \dots$

$^{18}C_{12}\ b^6\ (a + c)^{12}$ would give coefficient of $a^8b^6c^4$ as :

$^{18}C_{12}\ b^6\ (a + c)^{12} = {}^{18}C_{12}\ b^6\ [{}^{12}C_0a^{12} + {}^{12}C_1\ a^{11}\cdot c + \ldots + {}^{12}C_4\ a^8c^4 + \ldots]$

$\therefore$ coefficient of $a^8b^6c^4 = {}^{18}C_{12} \times {}^{12}C_4$

$= {}^{18}C_6 \times {}^{12}C_8$

$\therefore$ Option (C) is correct Answer.

$(a + b + c)^{18}$ can also expanded in another way as.

$(c + a + b)^{18} = {}^{18}C_0\ c^{18} + {}^{18}C_1\ c^{17}\ (a + b) + \ldots + {}^{18}C_4\ c^4\ (a + b)^{14} + \ldots$

only $^{18}C_{14}\ c^4\ (a + b)^{14}$ will give coefficient of $a^8b^6c^4$ as

$^{18}C_{14}c^4 \times (a + b)^{14} = {}^{18}C_{14} \times c^4 \times [{}^{14}C_0\ a^{14} + {}^{14}C_1\ a^{13}b + \ldots + {}^{14}C_6\ a^8b^6 + \ldots]$

$\therefore$ Coefficient $a^8b^6c^4 = {}^{18}C_{14} \times {}^{14}C_6$

$= {}^{18}C_{14} \times {}^{14}C_8$

$= {}^{18}C_4 \times {}^{14}C_6$

$\therefore$ Option (A), (D) are correct answer

$\therefore$ Option (A), (B), (C) and (D) are correct Answers.

EXERCISE # 2

Part-A **Only single correct answer type questions**

Q.1 If the r^{th} term is the middle term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{20}$ then the $(r + 3)^{th}$ term is-

(A) ${}^{20}C_{14}.\frac{1}{2^{14}}.x$ (B) ${}^{20}C_{12}.\frac{1}{2^{12}}.x^2$

(C) $-\frac{1}{2^{13}}.{}^{20}C_7 \cdot x$ (D) None of these

Sol. **[C]**

$\left(x^2 - \frac{1}{2x}\right)^{20}$

middle term will be $\left(\frac{20}{2}+1\right)$th term = 11^{th} term

Hence (r + 3)th term

$= {}^{20}C_{13}\left(-\frac{1}{2x}\right)^{13}(x^2)^{20-13}$

$= {}^{20}C_{13}\left(-\frac{1}{2x}\right)^{13}x^{14}$

$= -{}^{20}C_7\frac{1}{2^{13}}x$

Q.2 The greatest value of the term independent of x in the expansion of $(x \sin \alpha + x^{-1} \cos \alpha)^{10}$, $\alpha \in R$, is-

(A) 2^5 (B) $\frac{10\,!}{(5\,!)^2}$

(C) $\frac{1}{2^5}.\frac{10\,!}{(5\,!)^2}$ (D) None of these

Sol. **[C]**

$\left(x\sin\alpha + \frac{\cos\alpha}{x}\right)^{10}$, $\alpha \in R$

General term, $T_{r+1} = {}^{10}C_r\left(\frac{\cos\alpha}{x}\right)^r.(x \sin \alpha)^{10-r}$

$= {}^{10}C_r\ (\cos\alpha)^r.(\sin\alpha)^{10-r}(x)^{10-2r}$

Term independent of x $\Rightarrow 10 - 2r = 0 \Rightarrow r = 5$

Term independent of x= ${}^{10}C_5(\cos\alpha)^5(\sin\alpha)^5.x^0$

$= {}^{10}C_5\frac{(\sin 2\alpha)^5}{2^5}$

$= \frac{10!}{(5!)^2}\times\frac{1}{2^5}(\sin 2\alpha)^5$

Hence, term greatest of independent of x.

$\frac{10!}{(5!)^2}\times\frac{1}{(2)^5}$

Q.3 The coefficient of x^k $(0 \le k \le n)$ in the expansion of $E = 1+ (1 + x) + (1 + x)^2 + ... + (1+x)^n$ is-

(A) ${}^{n+1}C_{k+1}$ (B) ${}^{n+1}C_{n-k}$

(C) ${}^{n}C_{n-k-1}$ (D) both (A) and (B)

Sol. **[D]**

$E = 1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^n$

$= 1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^k + (1 + x)^{k+1} + (1 + x)^{k+2} + (1 + x)^{k+3} + (1 + x)^{k+4} + \ldots$
$\ldots + (1 + x)^{n-3} + (1 + x)^{n-2} + (1 + x)^{n-1} + (1 + x)^n$

$(0 \le k \le n)$

Hence, coefficient of x^k in the expansion of

$(1 + x)^k + (1 + x)^{k+1} + (1 + x)^{k+2} + (1 + x)^{k+3} + .. + (1 + x)^{n-3} + (1+ x)^{n-2} + (1 + x)^{n-1} + (1 + x)^n$

$= {}^kC_k + {}^{(k+1)}C_k + {}^{(k+2)}C_k + {}^{(k+3)}C_k + \ldots + {}^{(n-3)}C_k + {}^{(n-2)}C_k + {}^{(n-1)}C_k + {}^nC_k = {}^{n+1}C_{k+1}$

$= {}^{n+1}C_{k+1} = {}^{n+1}C_{n-k}$

$\therefore$ Option (D) is correct answer.

Q.4 If the second, third and fourth terms in the expansion of $(a + b)^n$ are 135, 30 and 10/3 respectively, then

(A) a = 3 (B) b = 1/3

(C) n = 5 (D) all are correct

Sol. **[D]**

$T_2 = {}^nC_1.(b)^1.(a)^{n-1} = 135$

$T_3 = {}^nC_2.(b)^2.(a)^{n-2} = 30$

$T_4 = {}^nC_3.(b)^3.(a)^{n-3} = 10/3$

$\frac{n!}{(n-1)!}\left(\frac{b}{a}\right).a^n = 135 \Rightarrow na^n\ (b/a) = 135 \ldots(1)$

$\frac{n!}{2!(n-2)!}b^2\,a^n\,a^{-2} = 30 \Rightarrow \frac{n(n-1)}{2}(b/a)^2\,a^n = 30 \ldots(2)$

$\frac{n!}{3!(n-3)!}(b/a)^3\,a^n = \frac{10}{3}$

$\Rightarrow \frac{n(n-1)(n-2)}{6}$ $(b/a)^3 a^n = 10/3$ … (3)

from (1) and (2). we get

$\left(\frac{n-1}{2}\right) \times 135 \times \frac{a}{b}$ $(b/a)^2 = 30$

$\Rightarrow \left(\frac{n-1}{2}\right) \times \frac{b}{a} \times 135 = 30 \Rightarrow \frac{n-1}{2} \times \frac{b}{a}$

$= \frac{30}{135} = \frac{6}{27} = \frac{2}{9}$

$\Rightarrow \frac{n-1}{2} \times (b/a) = \frac{2}{9}$

from (2) and (3) we get

$\frac{(n-2)}{3} \times \left(\frac{b}{a}\right)^3 \times \frac{30}{(b/a)^2} = \frac{10}{3} \Rightarrow \frac{n-2}{3} \times \left(\frac{b}{a}\right)$

$= \frac{1}{9}$

$\Rightarrow \frac{n-1}{2} \times \frac{3}{n-2} \times \frac{1}{9} = \frac{2}{9}$

$\Rightarrow (n-1) \times 3 = 4(n-2)$

$\Rightarrow 3n - 3 = 4n - 8 \Rightarrow n = 5$

$\frac{n-2}{3} \times (b/a) = \frac{1}{9}$

$\Rightarrow \frac{3}{3} \times (b/a) = \frac{1}{9} \Rightarrow a = 9b$

Hence, $a = 3 \Rightarrow b = 1/9$

$\therefore$ Option (D) is correct Answer

Q.5 The middle term of $(1 + x)^{2n}$ is-

(A) $\frac{1.3.5...(2n+1)}{n!} 2^n x^n$

(B) $\frac{1.3.5...(2n-1)}{n!} 2^n x^n$

(C) $\frac{1.3.5...(2n+1)}{n!} 2^{n+1} x^n$

(D) $\frac{1.3.5...(2n-1)}{n!} 2^{n-1} x^n$

Sol. **[B]**

Middle term will be $\left(\frac{2n}{2}+1\right)$ th

Hence, middle term = ${}^{2n}C_n\ (1)^n.x^n$

$= {}^{2n}C_n x^n$

$= \frac{2n!}{n! \times n!}\ x^n$

Since, $\frac{2n!}{n! \times n!}$

$= \frac{(2n)(2n-1)(2n-2)(2n-3)...6.5.4.3.2.1}{n! \times n!}$

$= \frac{\{(2n-1)(2n-3)...5.3.1\}\{2n(2n-2)...6.4.2\}}{n! \times n!}$

$= \frac{1.3.5...(2n-1).2^n.(1.2.3...n)}{n! \times n!}$

$= \frac{1.3.5...(2n-1)}{n!} \times 2^n$

Hence ${}^{2n}C_n\ x^n = \frac{2n!}{n! \times n!}\ x^n$

$= \frac{1.3.5...(2n-1)}{n!}\ x^n \times 2^n$

$\therefore$ Option (B) is correct Answer.

Q.6 If n is an integer between 0 and 21, then the minimum value of $\lfloor n \times \lfloor 21-n$ is attained for n =

(A) 1 (B) 10

(C) 9 (D) 20

Sol. **[B]**

$n! \times (21 - n)!\ ;\ n \in [0, 21]$

$n = 1\ ;\ 1! \times 20! = 20! \longrightarrow$ maximum

$n = 10;\ 10! \times 11! = 11\,(10!)^2 \to$ minimum

$n = 9\ ;\ 9! \times 12! = 12 \times 11 \times 10(9!)^2 \to$ second maximum

$n = 20\ ;\ 20! \times 1! = 20! \to$ maximum

Hence, option (B) is correct Answer.

Q.7 If the coefficient of the 5^{th} term be the numerically greatest coefficient in the expansion of $(1 - x)^n$, then the positive integral value of n is-

(A) 9 (B) 8

(C) 7 (D) 10

Sol. **[B]**

$(1 - x)^n$

$r \le \frac{n+1}{\left|\frac{x}{a}\right|+1}$

$r \le \frac{n+1}{\left|\frac{1}{-1}\right|+1}$

$r \le \frac{n+1}{2}$

$4 \le \frac{n+1}{2}$

$\Rightarrow n + 1 \ge 8$

$\Rightarrow n \ge 7 \Rightarrow n = 8$

$\therefore$ Option (B) is correct Answer.

Q.8 The interval in which x must lie so that the greatest term in the expansion of $(1 + x)^{2n}$ has the greatest coefficient is -

(A) $\left(\frac{n-1}{n}, \frac{n}{n-1}\right)$

(B) $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$

(C) $\left(\frac{n}{n+2}, \frac{n+2}{n}\right)$

(D) None of these

Sol. **[B]**

For terms to be greatest

$\left|\frac{T_{n+1}}{T_n}\right| \ge 1$ and $\left|\frac{T_{n+2}}{T_{n+1}}\right| \le 1$

$\Rightarrow T_{n+1} \ge T_n$ and $T_{n+2} \le T_{n+1}$.

$\Rightarrow T_n \le T_{n+1} \ge T_{n+2}$

$\Rightarrow T_{n+1} \ge T_n$

$\Rightarrow {}^{2n}C_n x^n \ge {}^{2n}C_{n-1} x^{n-1}$

$\Rightarrow \frac{2n!}{n!n!} x^n \ge \frac{2n!}{(n-1)!(n+1)!} x^{n-1}$

$\Rightarrow \frac{2n!}{n(n-1)!\,n!} x^n \ge \frac{2n!}{(n-1)!(n+1)n!} x^{n-1}$

$\Rightarrow \frac{2n!}{n(n-1)!n!} x^n \ge \frac{2n!}{(n+1)(n-1)n!} x^{n-1}$

$\Rightarrow \frac{1}{n} \ge \frac{1}{(n+1)} \times \frac{1}{x} \Rightarrow x \ge \frac{n}{n+1}$

$\Rightarrow T_{n+1} \ge T_{n+2}$

$\Rightarrow {}^{2n}C_n x^n \ge {}^{2n}C_{n+1}. x^{n+1}$

$\Rightarrow \frac{2n!}{n! \times n!} x^n \ge \frac{2n!}{(n+1)!\,(n-1)!} \times x^{n+1}$

$\Rightarrow \frac{2n!}{n(n-1)! \times n!} x^n \ge \frac{2n!}{(n+1)n! \times (n-1)!} \times x^n.x$

$\Rightarrow \frac{n+1}{n} \ge x \Rightarrow \frac{n}{n+1} \le x \le \frac{n+1}{n}$

Q.9 The sum of the coefficients of all the integral powers of x in the expansion of $(1 + 2\sqrt{x})^{40}$ is-

(A) $3^{40} + 1$

(B) $3^{40} - 1$

(C) $\frac{1}{2}(3^{40} - 1)$

(D) $\frac{1}{2}(3^{40} + 1)$

Sol. **[D]**

$(1 + 2\sqrt{x})^{40} = {}^{40}C_0 (2\sqrt{x})^0 + {}^{40}C_1 (2\sqrt{x})^1 + {}^{40}C_2 (2\sqrt{x})^2 + {}^{40}C_3 (2\sqrt{x})^3 + {}^{40}C_4 (2\sqrt{x})^4 + {}^{40}C_5 (2\sqrt{x})^5 + {}^{40}C_6 (2\sqrt{x})^6 + \ldots$

Put x = 1 and x = – 1 in above expansion simultaneously we get

$(1 + 2)^{40} = {}^{40}C_0 + {}^{40}C_1.2 + {}^{40}C_2 (2)^2 + {}^{40}C_3 (2)^3 + {}^{40}C_4(2)^4 + {}^{40}C_5 (2)^5 + {}^{40}C_6 (2)^6 + {}^{40}C_7 (2)^7 + {}^{40}C_8 (2)^8 + \ldots$

$(1 - 2)^{40} = {}^{40}C_0 + {}^{40}C_1.2 + {}^{40}C_2(2)^2 + {}^{40}C_3(2)^3 + {}^{40}C_4(2)^4 - {}^{40}C_5 (2)^5 + {}^{40}C_6 (2)^6 - {}^{40}C_7 (2)^7 + {}^{40}C_8 (2)^8 + \ldots$

Adding we get

$3^{40} + 1 = 2[{}^{40}C_0 + {}^{40}C_2(2)^2 + {}^{40}C_4(2)^4 + {}^{40}C_6 (2)^6 + {}^{40}C_8 (2)^8 + \ldots]$

$\therefore {}^{40}C_0 + {}^{40}C_2(2)^2 + {}^{40}C_4(2)^4 + {}^{40}C_6(2)^6 + {}^{40}C_8 (2)^8 + \ldots$

$= \frac{3^{40}+1}{2}$

Q.10 If C_0, C_1, C_2, ... C_n are the Binomial coefficients in the expansion of $(1 + x)^n$, n being even, then $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \ldots + (C_0 + C_1 + C_2 + \ldots + C_{n-1})$ is equal to

(A) $n . 2^n$

(B) $n.2^{n-1}$

(C) $n.2^{n+3}$

(D) $n.2^{n-3}$

Sol. **[B]**

$(1 + x)^n = {}^nC_0 + {}^nC_1.x + {}^nC_2 x^2 + {}^nC_3 x^3 + {}^nC_4 x^4 + {}^nC_5 x^5 + \ldots + {}^nC_n x^n$

Differentiating w.r.t. x we get

$n(1 + x)^{n-1} = 0 + {}^nC_1 + 2x\,{}^nC_2 + 3x^2\,{}^nC_3 + 4x^3\,{}^nC_4 + 5x^4\,{}^nC_5 + \ldots + nx^{n-1}\,{}^nC_n$

Put x = 1, we get

$n.(2)^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + 4\,{}^nC_4 + 5\,{}^nC_5 + \ldots + n\,{}^nC_n$

Since ${}^nC_1 = {}^nC_{n-1}$ & ${}^nC_0 = {}^nC_n$

${}^nC_2 = {}^nC_{n-2}$

$n.2^{n-1} = C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + (C_0 + C_1 + C_2 + C_3) + ... + (C_0 + C_1 + C_2 + C_3 + \ldots + C_{n-1})$

Hence, option (B) is correct Answer.

Q.11 If n > 2 than $\sum_{r=0}^{n} (-1)^r \; (n-r)\,(n-r+1)\,C_r =$

(A) 0 (B) n

(C) 2^n (D) $(n-1)\,2^{n-1}$

Sol. **[A]**

If n > 2,

We have to find value of $\sum_{r=0}^{n} (-1)^r \,(n-r)(n-r+1)\, C_r$

Let $x\,(x+1)^n =$

$x[{}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + {}^nC_3x^{n-3} + \ldots + {}^nC_nx^0]$

$= {}^nC_0x^{n+1} + {}^nC_1x^n + {}^nC_2\,x^{n-1} + {}^nC_3\,x^{n-2} + \ldots + {}^nC_nx$

Differentiating two times w.r.t. x we get

$\frac{d}{dx}\,[x.n(x+1)^{n-1} + (x+1)^n] = n(x+1)^{n-1} + x.n$

$(n-1)\,(x+1)^{n-2} + n\,x^{n-1}]$

$= 2n\,(x+1)^{n-1} + n(n-1).x.(x+1)^{n-2}$

$= {}^nC_0.(n+1)\,n.x^{n-1} + n(n-1)x^{n-2}\,{}^nC_1$

$+ {}^nC_2\,(n-1)\,(n-2)x^{n-3} + (n-2)(n-3)x^{n-4}\,{}^nC_3$

$+ \ldots\ldots\ldots\ldots$

Now put x = – 1, we get

(Let n is odd)

$\Rightarrow {}^nC_0n(n+1)(1) - n(n-1)^nC_1 + {}^nC_2\,(n-1)$

$(n-2) - (n-2)(n-3)\,{}^nC_3 + \ldots = 0 + 0$

$\Rightarrow n(n+1)\;{}^nC_0 - n(n-1)\;{}^nC_1 + (n-1)(n-2)$

${}^nC_2 - (n-2)(n-3)\;{}^nC_3 + \ldots = 0$

$\therefore$ Option (A) is correct Answer.

Q.12 The coefficient of x^{n-2} in the polynomial $(x-1)\,(x-2)\,(x-3)\,...(x-n)$ is-

(A) $\frac{n\,(n^2+2)\,(3n+1)}{24}$

(B) $\frac{n\,(n^2-1)\,(3n+2)}{24}$

(C) $\frac{n\,(n^2+1)\,(3n+4)}{24}$

(D) None of these

Sol. **[B]**

Q.13 If n is a positive integer and $(3\sqrt{3} + 5)^{2n+1} = \alpha + \beta$ where α is an integer and $0 < \beta < 1$ then consider the following statements -

(i) α is an even integer

(ii) $(\alpha + \beta)^2$ is divisible by 2^{2n+1}

(iii) the integer just below $(3\sqrt{3} + 5)^{2n+1}$ divisible by 3

(iv) α is divisible by 10

In the above correct statements are-

(A) (i), (ii) & (iii)

(B) (ii), (iii) and (iv)

(C) (i), (ii) and (iv)

(D) only (ii) and (iii)

Sol. **[C]**

$\alpha + \beta = (3\sqrt{3} + 5)^{2n+1}$

$\alpha \in I$; $0 < \beta < 1$

Let $G = (3\sqrt{3} - 5)^{2n+1}$; $0 < G < 1$

Then $\alpha + \beta - G = (3\sqrt{3} + 5)^{2n+1} - (3\sqrt{3} - 5)^{2n+1}$

$= 2[{}^{2n+1}C_1(3\sqrt{3})^{2n}\,5 + {}^{2n+1}C_3\,(3\sqrt{3})^{2n-2}$

$(5)^3 + \ldots]$

Since, $0 < \beta < 1$ and $0 < G < 1 \Rightarrow 0 < \beta - G < 0$

$\Rightarrow \beta = G$

Hence, α = Even integer.

Also, α is divisible 10.

Hence, option (C) is correct Answer.

Q.14 Let n be an odd natural number greater than 1. Then the number of zeros at the end of the sum $99^n + 1$ is-

(A) 3 (B) 4

(C) 2 (D) None of these

Sol. **[C]**

$99^n + 1 = (100 - 1)^n + 1 = {}^nC_0\ (100)^n\ (-1)^0 + {}^nC_1\ (100)^{n-1}\ (-1)^1 + \ldots + {}^nC_{n-1}\ (100)\ (-1)^{n-1} + {}^nC_n\ (100)^0\ (-1)^n + 1$

$= 100\ m + n \times 100 - 1 + 1$

$= 100\ m + n \times 100$; $m \in I^+$

n odd integer greater than 1.

Hence, at the end of sum, at least two zeros will be.

$\therefore$ Option (C) is correct Answer.

Q.15 Let $n \in N$ and $n < (\sqrt{2} + 1)^6$. Then the greatest value of n is -

(A) 199 (B) 198

(C) 197 (D) 196

Sol. **[C]**

$n \in N, n < (\sqrt{2} + 1)^6$

$(\sqrt{2} + 1)^6 = {}^6C_0(\sqrt{2})^6 + {}^6C_1(\sqrt{2})^5 + {}^6C_2(\sqrt{2})^4 + {}^6C_3(\sqrt{2})^3 + {}^6C_4(\sqrt{2})^4 + {}^6C_5(\sqrt{2}) + {}^6C_6(\sqrt{2})^0$

$= 8 + 6 \times 4\sqrt{2} + 15 \times 4 + 20 \times 2\sqrt{2} + 15 \times 2 + 6 \times \sqrt{2} + 1$

$= 99 + 70\sqrt{2}$

$= 197.70$

$n < 197.70$

$\Rightarrow$ Greatest value of n is 197

$\therefore$ Option (C) is correct statement.

Q.16 If $(1 + x - 2x^2)^8 = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{16}x^{16}$, then the sum $a_1 + a_3 + a_5 + \ldots + a_{15}$ is equal to

(A) -2^7 (B) 2^7

(C) 2^8 (D) None of these

Sol. **[A]**

Q.17 The coefficient of x^{10} in the expansion of $(1 + x^2 - x^3)^8$ is -

(A) 476 (B) 496

(C) 506 (D) 528

Sol. **[A]**

$(1 + x^2 - x^3)^8$

$= {}^8C_0 + {}^8C_1(x^2 - x^3) + {}^8C_2(x^2 - x^3)^2 + {}^8C_3(x^2 - x^3)^3 + {}^8C_4(x^2 - x^3)^4 + {}^8C_5(x^2 - x^3)^5 + {}^8C_6(x^2 - x^3)^6 + {}^8C_7(x^2 - x^3)^7 + {}^8C_8(x^2 - x^3)^8$.

Coefficient of x^{10} given by ${}^8C_4\ (x^2 - x^3)^4$ and ${}^8C_5\ (x^2 - x^3)^5$

${}^8C_4x^8(1-x)^4 = {}^8C_4x^8\ [{}^4C_0 + {}^4C_1(-x) + {}^4C_2(-x)^2 + \ldots]$

${}^8C_5x^{10}(1-x)^5 = {}^8C_4x^{10}[{}^5C_0 + {}^5C_1(-x) + {}^5C_2(-x)^2 + \ldots]$

Hence, coefficient of x^{10}

$= {}^8C_4 \times {}^4C_2 + {}^8C_5 \times {}^5C_0$

$= \frac{8!}{4! \times 4!} \times \frac{4!}{2! \times 2!} + \frac{8!}{5! \times 3!} \times 1$

$= \frac{8\times7\times6\times5\times4!}{4!\times4\times3\times2\times1} \times \frac{4!}{2!\times2!} + \frac{8\times7\times6\times5!}{5!\times3\times2}$

$= 70 \times 6 + 56$

$= 420 + 56$

$= 476$

$\therefore$ Option (A) is correct Answer.

Q.18 If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{40}x^{40}$, then $a_1 + a_3 + a_5 + \ldots\ldots + a_{37}$ equals -

(A) $2^{19}(2^{20} - 21)$ (B) $2^{20}(2^{19} - 19)$

(C) $2^{19}(2^{20} + 21)$ (D) None of these

Sol. **[A]**

Q.19 The number of terms in the expansion of $\left(x^2 + 1 + \frac{1}{x^2}\right)^n$, $n \in N$, is-

(A) 2n (B) 3n

(C) 2n+1 (D) 3n+1

Sol. **[C]**

$\left(x^2 + 1 + \frac{1}{x^2}\right)^n = \left(x^2 + \frac{1}{x^2} + 2 - 1\right)^n$

$= \left[\left(x + \frac{1}{x}\right)^2 - 1\right]^n$

$= {}^nC_0\left(x + \frac{1}{x}\right)^{2n} + {}^nC_1\left(x + \frac{1}{x}\right)^{2(n-1)}(-1) + {}^nC_2\left(x + \frac{1}{x}\right)^{2(n-2)}(-1)^2 + \ldots + C_n\left(x + \frac{1}{x}\right)^{2(n-n)}(-1)^n$

Hence, Total number of terms = 2n + 1

Part-B One or more than one correct answer type questions

Q.20 If 'a' be the sum of the odd terms and 'b' the sum of the even terms of the expansion of $(1 + x)^n$, then $(1 - x^2)^n =$

(A) $a^2 - b^2$ (B) $a^2 + b^2$

(C) $b^2 - a^2$ (D) None of these

Sol.[A] $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n = a + b$

$(1 - x)^n = C_0 - C_1x + C_2x^2 \ldots\ldots (-1)^n C_nx^n = a - b$

$(1 - x^2)^n = (a + b)(a - b) = a^2 - b^2$

Q.21 The term independent of x in the expansion of $(1 + x)^n \cdot \left(1 - \frac{1}{x}\right)^n$ is –

(A) 0, if n is odd

(B) $(-1)^{\frac{n-1}{2}} \cdot {}^nC_{\frac{n-1}{2}}$, if n is odd

(C) $(-1)^{n/2}.\ {}^nC_{n/2}$, if n is even

(D) None of these

Sol. **[A, C]**

$(1 + x)^n.\left(1 - \frac{1}{x}\right)^n$

$= (1 + x)^n.\ \frac{(x-1)^n}{x^n}$

$= \frac{(x^2-1)^n}{x^n}$

Let T_{r+1} term to be independent of x, then.

$T_{r+1} = {}^nC_r\ \frac{(x^2)^{n-r}}{x^n}\ (-1)^r = {}^nC_r\ x^{2n-2r-n}\ (-1)^r$.

$T_{r+1} = {}^nC_r\ x^{n-2r}\ (-1)^r$

Put $n - 2r = 0 \Rightarrow r = n/2$

$T_{\left(\frac{n}{2}+1\right)} = {}^nC_{n/2}\ (-1)^{n/2}$

If n is odd, $T_{\left(\frac{n}{2}+1\right)}$ will be imaginary. Hence, term independent of x would be zero.

$\therefore$ (A) is correct Answer.

If n is even, $T_{\frac{n}{2}+1} = (-1)^{n/2}\ {}^nC_{n/2}$.

$\therefore$ (C) is correct Answer.

$\therefore$ (A) and (C) are correct options.

Q.22 In the expansion of $(x + y + z)^{25}$-

(A) every term is of the form ${}^{25}C_r \cdot {}^rC_k \cdot x^{25-r} \cdot y^{r-k} \cdot z^k$

(B) the coefficient of $x^8\ y^9\ z^9$ is 0

(C) the number of terms is 325

(D) None of these

Sol. **[A, B]**

$(x + y + z)^{25}$

$= {}^{25}C_r.\ x^{25-r}\ (y + z)^r$

$= {}^{25}C_r.\ x^{25-r}\ {}^rC_k\ y^{r-k}\ z^k$

$\therefore$ (A) is correct option.

$(x + y + z)^{25} = {}^{25}C_0\ x^{25} + {}^{25}C_1\ x^{24}\ (y + z) + {}^{25}C_2\ x^{23}\ (y + z)^2 + \ldots + {}^{25}C_{17}\ x^8\ (y + z)^{17}$

We have to find out coefficient of $x^8y^9z^9$ only term ${}^{25}C_{17}\ x^8\ (y + z)^{17}$ give coefficient of $x^8y^9z^9$ ${}^{25}C_{17}\ x^8\ (y + z)^{17} = {}^{25}C_{17}\ x^8\ [{}^{17}C_0y^{17} + {}^{17}C_1y^{16}\ z + {}^{17}C_2y^{15}z^2 + {}^{17}C_3y^{14}z^3 + {}^{17}C_4y^{13}z^4 + \ldots + {}^{17}C_8y^9z^8 + {}^{17}C_9y^8z^9 + \ldots]$

Hence, coefficient of $x^8y^9z^9 = 0$

Number of terms in the expansion of $(x + y + z)^{25}$

$= {}^{25+3-1}C_{3-1}$

$= {}^{27}C_2$

$= \frac{27!}{2! \times 25!} = \frac{27 \times 26 \times 25!}{2! \times 25!}$

$= \frac{27 \times 26}{2}$

$= 27 \times 13$

$= 351$

$\therefore$ Options (A) and (B) are correct Answers.

Q.23 The sum of the coefficients in the expansion of $(1 - 2x + 5x^2)^n$ is a and the sum of the coefficients in the expansion of $(1 + x)^{2n}$ is b. Then

(A) a = b

(B) $(x - a)^2 - (x - b)^2 = 0$

(C) $\sin^2 a + \cos^2 b = 1$

(D) ab = 1

Sol.[A,B,C]

Sum of coefficient of $(1 - 2x + 5x^2)^n$

$a = 4^n = 2^{2n}$

Sum of coefficient of $(1 + x)^{2n}$

$b = 2^{2n}$

clearly a = b

$\sin^2 a + \cos^2 b = \sin^2 2^{2n} + \cos^2 2^{2n} = 1$

Q.24 In the expansion of $\left(\sqrt[3]{4} + \frac{1}{\sqrt[4]{6}}\right)^{20}$

(A) the number of rational terms = 4

(B) the number of irrational terms = 19

(C) the middle term is irrational

(D) the number of irrational terms = 17

Sol. **[B, C]**

$\left(4^{\frac{1}{3}} + \frac{1}{6^{1/4}}\right)^{20} = \left(4^{\frac{1}{3}} + 6^{-1/4}\right)^{20}$

Let (r + 1)th term will be rational.

$T_{r+1} = {}^{20}C_r (4)^{\frac{20-r}{3}} \; 6^{\frac{-r}{4}}$

We have to find values of r for which T_{r+1} would be Rational term

For r = 20, 8, T_{r+1} would be rational

Hence, No. of rational terms 2

No. of irrational terms = 21 – 2 = 19

∴ Option (B) is correct Answer.

$\left(\frac{20}{2}+1\right)$th i.e. 11th form will be middle term

∴ Middle term,

$T_{11} = {}^{20}C_{10} \left(4^{\frac{1}{3}}\right)^{(20-10)} \times \left(6^{\frac{-1}{4}}\right)^{10}$

$= {}^{20}C_{10} \times 4^{\frac{10}{3}} \times (6)^{\frac{-10}{4}}$

= which is irrational.

∴ Option (C) is also correct

∴ Option (B) and (C) are correct Answers.

Q.25 $7^9 + 9^7$ is divisible by

(A) 16 (B) 24

(C) 64 (D) 72

Sol.[A,C]

Θ $7^9 + 9^7 = (8-1)^9 + (8+1)^7$

$= ({}^9C_0\, 8^9 - {}^9C_1\, 8^8 + \ldots .. {}^9C_8\, 8 - 1) + ({}^7C_0\, 8^7 + {}^7C_1\, 8^6 + \ldots\ldots + 1)$

$= 8^9 - 9.8^8 + \ldots .. + 9.8 + 8^7 + 7.8^6 + \ldots .. + 7.8$

$= 128 + (8^9 - 9.8^8 + \ldots\ldots + {}^9C_7\, 8^2) + (8^7 + 7.8^6 + \ldots\ldots + {}^7C_6\, 8^2)$

= It is divisible by 16 and 64

Q.26 If $(1 + x)^{2n} = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$ then -

(A) $a_0+a_2+a_4+\ldots = \frac{1}{2}(a_0+a_1+a_2+a_3+\ldots)$

(B) $a_{n+1} < a_n$

(C) $a_{n-3} = a_{n+3}$

(D) None of these

Sol. **[A,B,C]**

$(1 + x)^{2n} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \ldots + a_{2n}x^{2n}$

$= {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + {}^{2n}C_3x^3 + {}^{2n}C_4x^4 + {}^{2n}C_5x^5 + {}^{2n}C_6x^6 + {}^{2n}C_7x^7 + {}^{2n}C_8x^8 + \ldots + {}^{2n}C_{2n}x^{2n}$

$a_0 + a_1 + a_2 + a_3 + a_4 + \ldots + a_{2n} = {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_4 + {}^{2n}C_5 + {}^{2n}C_6 + {}^{2n}C_7 + {}^{2n}C_8 + \ldots + {}^{2n}C_{2n}$

$\therefore {}^{2n}C_0 + {}^{2n}C_2 + {}^{2n}C_4 + \ldots + {}^{2n}C_{2n} = \frac{1}{2}({}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + {}^{2n}C_3 + \ldots + {}^{2n}C_{2n})$

i.e. $a_0 + a_2 + a_4 + \ldots + a_{2n} = \frac{1}{2}(a_0 + a_1 + a_2 + a_3 + a_4 + \ldots + a_{2n})$

∴ (A) is correct Answer.

$a_n > a_{n+1} \Rightarrow {}^{2n}C_n > {}^{2n}C_{n+1}$

$\Rightarrow \frac{2n!}{n!\times n!} > \frac{2n!}{(n+1)!(n-1)!}$

$\Rightarrow \frac{1}{n(n-1)! \times n!} > \frac{1}{(n+1)n! \times (n-1)!}$

$\Rightarrow (n + 1) > n$

$\Rightarrow 1 > 0$

∴ (B) is also correct answer.

$a_{n-3} = {}^{2n}C_{n-3} = \frac{2n!}{(n-3)!\,(n+3)!}$

$a_{n+3} = {}^{2n}C_{n+3} = \frac{2n!}{(n+3)!\,(n-3)!}$

$\therefore a_{n-3} = a_{n+3}$

∴ (C) is also correct Answer

∴ Option (A), (B) and (C) are correct Answer.

Part-C Assertion-Reason type questions

The following questions 27 to 30 consists of two statements each, printed as Statement (1) and Statement (2). While answering these questions you are to choose any one of the following four responses.

(A) If both Statement (1) and Statement (2) are true & the Statement (2) is correct explanation of the Statement (1).

(B) If both Statement (1) and Statement (2) are true but Statement (2) is not correct explanation of the Statement (1).

(C) If Statement (1) is true but the Statement (2) is false.

(D) If Statement (1) is false but Statement (2) is true

Q.27 **Statement (1) :** If n is even then

${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \ldots\ldots\ldots + {}^{2n}C_{n-1} = 2^{2n-1}$

Statement (2) : ${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \ldots + {}^{2n}C_{2n-1} = 2^{2n-1}$

Sol. **[D]**

$(1 + x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2x^2 + {}^{2n}C_3 x^3 + {}^{2n}C_4x^4 + {}^{2n}C_5x^5 + {}^{2n}C_6x^6 + \ldots + {}^{2n}C_{n-3} (x)^{n-3} + {}^{2n}C_{n-2} (x)^{n-2} + {}^{2n}C_{n-1} (x)^{n-1} + {}^{2n}C_n (x)^n + {}^{2n}C_{n+1} (x)^{n+1} + \ldots + {}^{2n}C_{2n-3} (x)^{2n-3} + {}^{2n}C_{2n-2} (x)^{2n-2} + {}^{2n}C_{2n-1} (x)^{2n-1} + {}^{2n}C_{2n} (x)^{2n}$

Put x = 1 in above expansion.

$(2)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + {}^{2n}C_3 + {}^{2n}C_4 + {}^{2n}C_5 + {}^{2n}C_6 + \ldots + {}^{2n}C_{n-3} + {}^{2n}C_{n-2} + {}^{2n}C_{n-1} + {}^{2n}C_n + {}^{2n}C_{n+1} + \ldots + {}^{2n}C_{2n-3} + {}^{2n}C_{2n-2} + {}^{2n}C_{2n-1} + {}^{2n}C_{2n}$ … (1)

Now Put x = – 1

$0 = {}^{2n}C_0 - {}^{2n}C_1 + {}^{2n}C_2 - {}^{2n}C_3 + {}^{2n}C_4 - {}^{2n}C_5 + {}^{2n}C_6 - {}^{2n}C_7 + \ldots - {}^{2n}C_{n-3} + {}^{2n}C_{n-2} - {}^{2n}C_{n-1} + {}^{2n}C_n - {}^{2n}C_{n+1} + \ldots - {}^{2n}C_{2n-3} + {}^{2n}C_{2n-2} - {}^{2n}C_{2n-1} + {}^{2n}C_{2n}$ …(2)

subtracting (2) from (1), we get

$2^{2n} = 2\,[{}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \ldots + {}^{2n}C_{n-3} + {}^{2n}C_{n-1} + {}^{2n}C_{n+1} + {}^{2n}C_{n+3} + \ldots + {}^{2n}C_{2n-3} + {}^{2n}C_{2n-1}]$

$2^{2n} = 2 \times 2\,[{}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \ldots + {}^{2n}C_{n-1}]$

$\Rightarrow {}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \ldots + {}^{2n}C_{n-1} = 2^{2n-2}$

$\Rightarrow {}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \ldots + {}^{2n}C_{2n-1} = 2^{2n-1}$

Assertion false but Reason is true.

$\therefore$ Option (D) is correct Answer.

Q.28 **Statement (1) :** The term independent of x in the expansion of $\left(x + \frac{1}{x} + 2\right)^m$ is $\frac{(2m)!}{(m!)^2}$

Statement (2) : The coefficient of x^b in the expansion of $(1 + x)^n$ is ${}^nC_b$.

Sol. **[A]**

$(x + 1/x + 2)^m = \left[(\sqrt{x})^2 + \frac{1}{(\sqrt{x})^2} + 2\right]^m$

$= \left[\sqrt{x} + \frac{1}{\sqrt{x}}\right]^{2m}$

$T_{r+1} = {}^{2m}C_r\,(\sqrt{x})^{2m-r}.\left(\frac{1}{\sqrt{x}}\right)^r$

$= {}^{2m}C_r\,(\sqrt{x})^{2m-2r}$

Put $2m - 2r = 0 \Rightarrow m = r$

$\therefore$ Term independent of x is ${}^{2m}C_m$

$= \frac{2m!}{(m!)^2}$

$\therefore$ Assertion is correct

Reason : coefficient x^b in $(1 + x)^n$ is ${}^nC_b$.

Reason is correct also and explanation of Assertion.

$\therefore$ Option (A) is correct.

Q.29 **Statement (1) :** Any positive integral power of $(\sqrt{2} - 1)$ can be expressed as $\sqrt{N} - \sqrt{N-1}$ for some natural number N > 1.

Statement (2) : Any positive integral power of $\sqrt{2} - 1$ can be expressed as $A + B\sqrt{2}$ where A and B are integers.

Sol. **[B]**

Assertion: $(\sqrt{2} - 1)^1 = (\sqrt{N} - \sqrt{N-1})^1$ for N = 2

It holds good only for integral power of 1.

Reason : $(\sqrt{2} - 1)^2 = (2 + 1 - 2\sqrt{2})$

$= (3 - 2\sqrt{2})$

$= A + B\sqrt{2}$

Reason is correct.

Both Assertion and Reason are correct but Reason is not correct explanation of Assertion.

$\therefore$ Option (B) is correct Answer.

Q.30 **Statement (1) :** If n is an odd prime then $[(\sqrt{5} + 2)^n] - 2^{n+1}$ is divisible by 20 n., where [.] denotes greatest integer function.

Statement (2) : If n is prime than ${}^nC_1$, ${}^nC_2$,……, ${}^nC_{n-1}$ must be divisible by n.

Sol. **[D]**

Assertion : Let x = [x] + f

$= (\sqrt{5} + 2)^n - 2^{n+1}$

where, 0 < f < 1.

Also, let $f' = (\sqrt{5} - 2)^n$; $0 < f' < 1$

$\therefore [x] + f - f' = (\sqrt{5} + 2)^n - 2^{n+1} - (\sqrt{5} - 2)^n$

$= 2[{}^nC_1\,(\sqrt{5})^{n-1}\,2 + {}^nC_3(\sqrt{5})^{n-3}.\,2^3 + {}^nC_5(\sqrt{5})^{n-5}\,2^5 + \ldots] - 2^{n+1}$

Since, $0 < f < 1 \quad \Rightarrow 0 < f - f' < 0$

$0 < f' < 1$

$\Rightarrow f = f'$

$\therefore$ [x] = Integral part of x

$= 2\,[{}^nC_1(\sqrt{5})^{n-1}\,2 + {}^nC_3(\sqrt{5})^{n-3}\,2^3 + \ldots] - 2^{n+1}$

= which is not necessarily divisible by 20 n

$\therefore$ Assertion is false.

Reason : ${}^{n}C_1 = \dfrac{n!}{(n-1)!} = n$

$${}^{n}C_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$
$$\vdots$$
$${}^{n}C_{n-1} = \frac{n!}{(n-1)!} = n$$

All divisible by n.

$\therefore$ Reason is True

$\therefore$ Option (D) is correct Answer.

Part-D Column Matching type questions

Q.31 Match the Column:

Column-I	**Column-II**
(A) If the second term in the expansion of $\left(x^{\frac{1}{13}} + \dfrac{x}{\sqrt{\frac{1}{x}}}\right)^n$ is $14\,x^{5/2}$ then $\left(\dfrac{{}^{n}C_3}{{}^{n}C_2}\right)$ may be	(P) less than 5
(B) If in the expansion of $\left(2^x + \dfrac{1}{4^x}\right)^n$, $\dfrac{T_3}{T_2}$ is equal to 7 and the sum of the binomial coefficient of 2nd and 3rd term is 36. Then x may be	(Q) $-\dfrac{1}{3}$
(C) 5^{99} is divided by 13 then remainder is	(R) 8
(D) Digit at unit place of 17^{256} is	(S) 4

Sol. **A → P,S, B → P,Q, C → R, D → P**

(A) $T_2 = {}^{n}C_1\ x^{\frac{n-1}{13}}\,(x^{3/2}) = 14\,x^{5/2}$

$\Rightarrow\ \because\ \dfrac{n-1}{13} = 1 \Rightarrow n = 14$

$$\Rightarrow \frac{{}^{14}C_3}{{}^{14}C_2} = \frac{14-3+1}{3} = \frac{12}{3} = 4$$

(B) $\dfrac{T_3}{T_2} = 7$

$$\Rightarrow \frac{{}^{n}C_2(2^x)^{n-2}.\left(\frac{1}{2^{2x}}\right)^2}{{}^{n}C_1(2^x)^{n-1}\left(\frac{1}{2^{2x}}\right)^1} = 5$$

$\Rightarrow 2^{3x} = \dfrac{n-1}{14}$ ……(1)

and ${}^{n}C_1 + {}^{n}C_2 = 36 \Rightarrow {}^{n+1}C_2 = 36$

$\Rightarrow n(n+1) = 72 \Rightarrow n = 8$

from (1) $2^{3x} = \dfrac{1}{2}$

$\Rightarrow 3x = -1$

$\Rightarrow x = -\dfrac{1}{3}$

(C) $5.(5)^{98} = 5(25)^{49} = 5(13+12)^{49}$

$= 5((13)^{49} + {}^{49}C_1\,(13)^{48}.12 + {}^{49}C_2\,(13)^{47}.(12)^2 + \ldots\ldots + 12)$

$= 13(5.13^{48} + {}^{49}C_1.60.(13)^{48} + \ldots.. + 4) + 8$

Remainder is 8

(D) $(17)^{256} = (289)^{128} = (290-1)^{128}$

$= (290)^{128} - {}^{128}C_1(290)^{127} + \ldots\ldots + 1$

Clearly digit at unit place is 1

Q.32 Match the column:

Column-I	**Column-II**
(A) The sum $\displaystyle\sum_{0\le i<j\le n}\sum\left({}^{n}C_i + {}^{n}C_j\right)^2$ may be	(P) $\dfrac{1}{2}[2^{2n} - {}^{2n}C_n]$
(B) The sum $\displaystyle\sum_{0\le i<j\le n}\sum\left({}^{n}C_i.{}^{n}C_j\right)$ may be	(Q) $n^2.\,2^{n-1}$
(C) The sum $\displaystyle\frac{1}{n}\sum_{0\le i<j\le n}\sum(i+j)\left({}^{n}C_i.{}^{n}C_j\right)$ may be	(R) $\displaystyle\sum_{r=0}^{n-1}{}^{2n}C_r$
(D) The sum	(S) $(n-1)\,{}^{2n}C_n + 2^{2n}$

$$\sum_{0 \le i < j \le n} \sum \left(i.{}^nC_i + j.{}^nC_j\right)$$

may be

Sol. A → S; B → P, R; C → P, R ; D → Q

Q.33 Match the column:

	Column-I	**Column-II**
(A)	If $(r + 1)^{th}$ term is the first negative term in the expansion of $(1 + x)^{7/2}$, then the value of r (where $\|x\| < 1$) is	(P) divisible by 2
(B)	The coefficient of y in the expansion of $(y^2 + 1/y)^5$ is	(Q) divisible by 5
(C)	${}^nC_r$ is divisible by n. $(1 \le r < n)$ if n is always	(R) divisible by 10
(D)	The coefficient of x^4 in the expression $(1+2x+3x^2+4x^3+ \ldots \text{ upto } \infty)^{1/2}$ is c, $(c \in N)$, then c + 1 (where $\|x\| < 1$) is	(S) a prime number

Sol. A → Q, S; B → P, Q, R; C → P ; D → P, S

Q.34 Match the column :

Column-I	**Column-II**
(A) $C_0 + 3C_1 + 5C_2 + \ldots\ldots\ldots =$	(P) $2^n - (n + 2)$
(B) ${}^nC_2 + {}^nC_3 + {}^nC_4 + \ldots + {}^nC_{n-1} =$	(Q) $\dfrac{1}{(n+1)(n+2)}$
(C) $\dfrac{C_0}{2} - \dfrac{C_1}{3} + \dfrac{C_2}{4} - \dfrac{C_3}{5} + \ldots. =$	(R) $\dfrac{\lfloor 2n-1}{\lfloor n-1 \lfloor n-1}$
(D) $C_1^2 + 2C_2^2 + 3C_3^2 + \ldots. + nC_n^2 =$	(S) $(n + 1)2^n$

Sol. A → S, B → P, C → Q, D → R

(A) $C_0 + {}^3C_1 + {}^5C_2 + \ldots\ldots..$

$$= \sum_{r=0}^{n} (2r+1)\,{}^nC_r = \sum_{r=0}^{n} (2r\,{}^nC_r + {}^nC_r)$$

$$= 2\sum_{r=0}^{n} r\,{}^nC_r + \sum_{r=0}^{n} {}^nC_r$$

$$= 2\sum_{r=1}^{n} r.\frac{n}{r}\,{}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r$$

$= 2n\,2^{n-1} + 2^n$

$= (n + 1)2^n$

(B) ${}^nC_2 + {}^nC_3 + {}^nC_4 + \ldots..+ {}^nC_{n-1}$

$$= \sum_{r=0}^{n} {}^nC_r - ({}^nC_0 + {}^nC_1 + {}^nC_n)$$

$= 2^n - (1 + n + 1) = 2^n - (n + 2)$

(C) $\dfrac{C_0}{2} - \dfrac{C_1}{3} + \dfrac{C_2}{4} - \dfrac{C_3}{5} + \ldots\ldots.$

$$= \sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{r+2}$$

$$= \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n} (-1)^r \frac{(n+1)(n+2)(r+1)}{(r+1)(r+2)}\,{}^nC_r$$

$$= \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n} (-1)^r\,(r + 1)\,{}^{n+2}C_{r+2}$$

$$= \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n} (-1)^r\,[(r + 2) - 1]\,{}^{n+2}C_{r+2}$$

$$= \frac{1}{(n+1)(n+2)} \left[\sum_{r=0}^{n} (-1)^r (r+2)\,{}^{n+2}C_{r+2} - \sum_{r=0}^{n} (-1)^r\,{}^{n+2}C_{r+2}\right]$$

Solving we get

$$= \frac{1}{(n+1)(n+2)}$$

(D) $C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots. + n\,C_n^2$

$$= \sum_{r=1}^{n} r.({}^nC_r)^2$$

$$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1}\;{}^nC_r = n.\,{}^{2n-1}C_{n-1}$$

$$= n.\frac{\lfloor 2n-1}{\lfloor n-1 \lfloor n} = \frac{\lfloor 2n-1}{\lfloor n-1 \lfloor n-1}$$

Q.35 For some positive integer n

$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots.+ a_{2n}\,x^{2n}$

then match the following column.

Column-I	Column-II
(A) $\sum_{r=0}^{2n} a_r$ is equal to	(P) $(n - r)\ a_r + (2n-r+1)a_{r-1}$
(B) a_r is equal to $(0 \le r \le 2n)$	(Q) 3^n
(C) The value of $(r + 1)a_{r+1}$ =	(R) $\frac{3^n - a_n}{2}$
(D) The value of $a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1}$ is	(S) a_{2n-r}

Sol. A → Q, B → S, C → P, D → R

(A) $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n}$

$\Rightarrow a_0 + a_1 + a_2 + \ldots\ldots + a_{2n} = 3^n$

(B) $\because (1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$

$\therefore \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} \frac{a_r}{x^r}$

$\Rightarrow (1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^{2n-r}$

$\Rightarrow \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r}$

$\Rightarrow a_{2n-r} = a_r$ for $0 \le r \le 2n$

(C) $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$

differentiate both the side we get

$n(1 + 2x)\ (1 + x + x^2)^n = (1 + x + x^2) \sum_{r=0}^{2n} r\ a_r x^{r-1}$

equating the coefficient of x^r we get

$n\ a_r + 2n\ a_{r-1} = (r + 1)\ a_{r+1} + ra_r + (r - 1)\ a_{r-1}$

$\Rightarrow (r + 1)a_{r+1} = (n - r)\ a_r + (2n - r + 1)\ a_{r-1}$

(D) we have

$a_r = a_{2n-r}$ for $0 \le r \le 2n$

$\Rightarrow \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_{2n-r}$

$\Rightarrow a_0 + a_1 + \ldots\ldots + a_{n-1} = a_{2n} + a_{2n-1} + \ldots\ldots + a_{n+1}$

$\Rightarrow 2(a_0 + a_1 + \ldots\ldots + a_{n-1}) + a_n = a_0 + a_1 + \ldots\ldots + a_{2n}$

$\Rightarrow 2(a_0 + a_1 + \ldots\ldots + a_{n-1}) + a_n = 3^n$

$\Rightarrow a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1} = \frac{3^n - a_n}{2}$

Q.36 If $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$ and

$(1 + x)^n = \sum_{r=0}^{n} C_r x^r$ then match the following -

Column-I	Column-II
(A) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots\ldots + a_{2n}^2$ =	(P) 0
(B) $a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + \ldots$ ….. $-a_{2n-1}a_2$ =	(Q) a_{n+r}
(C) $a_0a_{2r} - a_1a_{2r+1} + a_2a_{2r+2} + \ldots$ …..$+a_{2n-2r}a_{2n}$ =	(R) ${}^nC_{n/3}$
(D) $\sum_{r=0}^{n} (-1)^r a_r\ {}^nC_r$ is equal to if n = 3x	(S) a_n

Sol. A → S, B → P, C → Q, D → R

(A) $\because (1 + x + x^2)^n = a_0 + a_1x + \ldots\ldots + a_{2n}x^{2n}$ …..(1)

$\therefore \left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots\ldots + \frac{a_{2n}}{x^{2n}}$

$\Rightarrow (x^2 - x + 1)^n = a_0x^{2n} - a_1x^{2n-1} + \ldots\ldots + a_{2n}$ …..(2)

multiplying (1) and (2) we have

$\sum_{r=0}^{2n} a_r x^{2r} = (a_0 + a_1x + \ldots\ldots + a_{2n}x^{2n})$

$(a_0x^{2n} - a_1x^{2n-1} + \ldots\ldots + a_{2n})$ ……(3)

Equating the coefficient of x^{2n} in (3) we get

$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots\ldots + a_{2n}^2 = a_n$

(B) Equating the coefficient of x^{2n+1} in (3) we get

$0 = a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + \ldots\ldots - a_{2n-1}a_{2n}$

(C) Equating the coefficient of x^{2n+2r} in (3) we get

$a_{n+r} = a_0a_{2r} - a_1a_{2r+1} + a_2a_{2r+2} + \ldots\ldots + a_{2n-2r}.a_{2n}$

(D) $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n}$

and $(x - 1)^n = {}^nC_0\, x^n - {}^nC_1x^{n-1} + \ldots\ldots (-1)^n\ {}^nC_n\, x^n$

multiplying both and equating the coefficient of x^n then

$(-1)^n\ (-1)^m\ {}^{3m}C_m = {}^{3m}C_m$

$= {}^nC_{n/3}$ $\because n = 3m$

EXERCISE # 3

Part-A Subjective Type Questions

In all the questions given below you have to consider $C_r = {}^nC_r$ & $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$.

Q.1 Find the value of :

$C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2$

Sol. $\because (x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n$

and $(1 - x)^n = C_0 - C_1x + C_2x^2 - .... + (-1)^n C_nx^n$

multiplying and equating the coefficient of x^n we have

$C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2$ = coff. of x^n in $(1 - x^2)^n$

$= (-1)^n\, {}^nC_{n/2}$ if n is even

= 0 if n is odd

Q.2 Find the co-efficient of x^9 in the polynomial given by, $(x + 1)(x + 2)......(x + 10) + (x + 2)(x + 3).......(x + 11) + ... + (x + 11)(x + 12)........(x + 20)$ is-

Sol. 1155

Q.3 Show that

$C_1^2 + \frac{1+2}{2}C_2^2 + \frac{1+2+3}{3}C_3^2 + ...$

upto n terms $= \frac{1}{2}[n.{}^{2n-1}C_{n-1} + {}^{2n}C_n - 1]$

Q.4 In the binomial expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the 7^{th} term from the beginning to the 7^{th} term from the end is 1 : 6 ; find n.

Sol. $\left(2^{\frac{1}{3}} + \frac{1}{3^{1/3}}\right)^n$

7^{th} term from the beginning =

${}^nC_6 \frac{1}{\left(3^{\frac{1}{3}}\right)^6}\left(2^{\frac{1}{3}}\right)^{n-6}$

7^{th} term from the end = (n – 7 + 2)th term from the beginning i.e. (n – 5)th term.

Hence, $\dfrac{{}^nC_6 \frac{1}{\left(3^{\frac{1}{3}}\right)^6}\left(2^{\frac{1}{3}}\right)^{n-6}}{{}^nC_{n-6} \frac{1}{\left(3^{\frac{1}{3}}\right)^{n-6}} \times \left(2^{\frac{1}{3}}\right)^6} = \frac{1}{6}$

$\Rightarrow \left(3^{\frac{1}{3}}\right)^{n-12} \times \left(2^{\frac{1}{3}}\right)^{n-12} = 6^{-1}$

$\Rightarrow \left(6^{\frac{1}{3}}\right)^{n-12} = 6^{-1}$

$\Rightarrow (6)^{\frac{n-12}{3}} = 6^{-1}$

$\Rightarrow \frac{n-12}{3} = -1$

$\Rightarrow n - 12 = -3$

$\Rightarrow n = 9$ Ans.

Q.5 Prove that $\frac{(72)!}{(36!)^2} - 1$ is divisible by 73.

Q.6 Prove that

$(1.2)\,C_2 + (2.3)C_3 + ...((n-1).n)C_n = n(n-1)\,2^{n-2}$

Q.7 Prove that,

$C_1/2 + C_3/4 + C_5/6 + ... = (2^n - 1)/(n + 1)$

Sol. Let $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + {}^nC_nx^n$

Integrating both sides w.r.t. x, we get

$\int (1 + x)^n dx = \int ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + {}^nC_nx^n)dx$

$\frac{(1+x)^{n+1}}{n+1} = {}^nC_0.x + {}^nC_1x^2/2 + {}^nC_2x^3/3 + {}^nC_3x^4/4$

${}^nC_4x^5/5 + ...\ {}^nC_n \frac{x^{n+1}}{n+1} + C$

Where C is constant of integration.

Put x = 0,

$\frac{1}{n+1} = 0 + 0 + 0 + ... + 0 + C \Rightarrow C = \frac{1}{n+1}$

$$\frac{(1+x)^{n+1}}{n+1} = {}^nC_0.x + {}^nC_1x^2/2 + {}^nC_2x^3/3 + {}^nC_3x^4/4$$

$${}^nC_4x^5/5 + \ldots + {}^nC_n \frac{x^{n+1}}{n+1} + \frac{1}{n+1}$$

Put x = 1

$$\frac{2^{n+1}}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} + \ldots + \frac{C_n}{n+1} + \frac{1}{(n+1)} \quad \ldots (1)$$

Put x = – 1

$$0 = - {}^nC_0 + \frac{C_1}{2} - \frac{C_2}{3} + \frac{C_3}{4} - \frac{C_4}{5} + \ldots + C_n \frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} \quad \ldots (2)$$

Adding (1) and (2), we get

$$\frac{2^{n+1}}{n+1} = 2\left[\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots\right] + \frac{2}{(n+1)}$$

$$\frac{2^{n+1}}{n+1} - \frac{2}{n+1} = 2\left[\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots\right]$$

$$\Rightarrow \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots = \frac{2^{n-1}}{n+1} \text{ Proved.}$$

Q.8 Prove that $\sum_{r=0}^{n} \frac{C_r}{(r+1)2^{r+1}} = \frac{3^{n+1} - 2^{n+1}}{(n+1)2^{n+1}}$.

Sol. $(1 + x)^n = \sum_{r=0}^{n} C_r.x^r$

$$\sum_{r=0}^{n} \frac{C_r}{(r+1)2^{r+1}} = \frac{3^{n+1} - 2^{n+1}}{(n+1)2^{n+1}}.$$

$$= \frac{C_0}{2} + \frac{C_1}{2.2^2} + \frac{C_2}{3.2^3} + \frac{C_3}{4.2^4} + \frac{C_4}{4.2^4} + \ldots + \frac{C_n}{(n+1)2^{n+1}}$$

using, $(1 + x)^n = C_0x + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$

Integrating both sides, we get

$$\frac{(1+x)^{n+1}}{(n+1)} = C_0.x + C_1.x^2/2 + C_2.x^3/3 + C_3.x^4/4 + \ldots + C_n.\frac{x^{n+1}}{n+1} + A$$

Where, A is constant of integration

Put $x = 0 \Rightarrow A = \frac{1}{(n+1)}$

$$\frac{(1+x)^{n+1}}{(n+1)} - \frac{1}{(n+1)} = C_0.x + C_1.x^2/2 + C_2.x^3/3 + C_3.x^4/4 + \ldots + C_n.\frac{x^{n+1}}{(n+1)}$$

Replace x by 1/x, we get

$$\frac{\left(1+\frac{1}{x}\right)^{n+1}}{(n+1)} - \frac{1}{(n+1)} = C_0/x + C_1/2x^2 + C_2/3x^3 + C_3/4x^4 + \ldots + C_n/(n+1).x^{(n+1)}$$

$$\frac{(1+x)^{n+1} - x^{n+1}}{(n+1).x^{(n+1)}} = C_0/x + C_1/2x^2 + C_2/3x^3 + C_3/4x^4 + \ldots + C_n/(n+1).x^{n+1}$$

Put, x = 2

$$\frac{3^{n+1} - 2^{n+1}}{(n+1).2^{(n+1)}} = C_0 + C_1/2.2^2 + C_2/3.2^3 + C_3/4.2^4 + \ldots + C_n/(n+1)2^{n+1}$$

$$= \sum_{r=0}^{n} \frac{C_r}{(r+1)2^{r+1}}. \text{ Hence, Proved}$$

Q.9 Prove that $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \ldots\ldots\ldots (C_{n-1} + C_n) = \frac{C_0 \cdot C_1 \cdot C_2 \ldots . C_{n-1}(n+1)^n}{n!}$

Sol. $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \ldots (C_{n-1} + C_n)$

using, ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

$\therefore {}^{n+1}C_1 . {}^{n+1}C_2 . {}^{n+1}C_3 \ldots {}^{n+1}C_n$

$$= \frac{(n+1)!}{n!} . \frac{(n+1)!}{2!(n-1)!} . \frac{(n+1)!}{3!(n-2)!} \ldots \frac{(n+1)!}{n!}$$

$$= \frac{(n+1)n!}{n!} . \frac{(n+1)n!}{(n-1)(n-2)!2!} . \frac{(n+1)n!}{(n-2)3!(n-3)!} \ldots \frac{(n+1)n!}{n!}$$

$$= (n+1)^n \frac{C_1}{n} \frac{C_2}{(n-1)} \frac{C_3}{(n-2)} \ldots \frac{C_n}{1}$$

$$= \frac{C_0.C_1.C_2 \ldots C_n.(n+1)^n}{n!}. \text{ Hence, Proved}$$

Q.10 If P = a + (a + d) + (a + 2d) + ...+ (a + nd) and S = a + (a + d). ${}^nC_1$ + (a + 2d).${}^nC_2$ +..... (a + nd).${}^nC_n$ then prove that (n + 1) S = 2^n.P

Sol. S = a + (a + d) + (a + 2d) + ...+ (a + nd)

$$= (n+1)\,a + d\,\frac{n(n+1)}{2}$$

$= (n + 1)\left[a + \frac{nd}{2}\right]$

$= \frac{(n+1)}{2}[2a + nd]$

Also, $S = a + (a + d)\ {}^nC_1 + (a + 2d)\ {}^nC_2 + \ldots + (a + nd){}^nC_n$

Let $x^a (1 + x^d)^n = x^a[{}^nC_0 + {}^nC_1x^d + {}^nC_2x^{2d} + {}^nC_3x^{3d} + \ldots + {}^nC_nx^{nd}]$

$= x^a\ {}^nC_0 + {}^nC_1x^{a+d} + {}^nC_2x^{a+2d} + \ldots + {}^nC_nx^{a+nd}$

Differentiating both sides w.r.t. = x, we get

$x^a.\ n(1 + x^d)^{n-1}.dx^{d-1} + ax^{a-1}(1 + x^d)^n$

$= ax^{a-1}\ {}^nC_0 + (a + d)\ x^{(a+d)-1}\ {}^nC_1 + (a + 2d)x^{a+2d-1}\ {}^nC_2 + (a + 3d)x^{a+3d-1}\ {}^nC_3 + \ldots + (a + nd)x^{a+nd-1}\ {}^nC_n$

Put x = 1

$n(2)^{n-1}\ d + a\ (2)^n = {}^aC_0 + (a + d)\ C_1 + (a + 2d)\ C_2 + (a + 3d)\ C_3 + \ldots + (a + nd)\ C_n$

$\Rightarrow S = a.2^n + nd\ 2^{n-1}$
$= 2a.2^{n-1} + nd.2^{n-1}$
$= 2^{n-1}\ [2a + nd]$

Since, $S = a + (a + d) + (a + 2d) + \ldots + (a + nd)$

$= (n + 1)\ a + \frac{n(n+1)}{2} d$

$= (n + 1)\ [a + \frac{nd}{2}]$

$= \frac{(n+1)}{2}\ [2a + nd]$

$\Rightarrow S = 2^{n-1}\ [2a + nd] = \frac{(n+1)}{2}\ [2a + nd]$

$\Rightarrow (n + 1) = 2^n$. Hence, Proved

Q.11 Prove that

$$({}^{2n}C_1)^2 + 2.({}^{2n}C_2)^2 + 3.({}^{2n}C_3)^2 + \ldots\ldots + 2n.({}^{2n}C_{2n})^2 = \frac{(4n-1)!}{(2n-1)!(2n-1)!}.$$

Sol. We have to prove that

$({}^{2n}C_1)^2 + 2.({}^{2n}C_2)^2 + 3.({}^{2n}C_3)^2 + \ldots + 2n.({}^{2n}C_{2n})^2$

$= \frac{(4n-1)!}{(2n-1)!(2n-1)!}$

Let $(1 + x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + {}^{2n}C_3x^3 + {}^{2n}C_4x^4 + {}^{2n}C_5x^5 + \ldots + {}^{2n}C_{2n}x^{2n}$

Differentiating w.r.t. x, we get

$2n(1 + x)^{2n-1} = 0 + {}^{2n}C_1 + 2x\,{}^{2n}C_2 + 3x^2\ {}^{2n}C_3 + 4x^3\ {}^{2n}C_4 + 5x^4\ {}^{2n}C_5 + \ldots + 2n\ x^{2n-1}\ {}^{2n}C_{2n}.$ …(1)

Now, $(x + 1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + {}^{2n}C_3x^{2n-3} + \ldots + {}^{2n}C_{2n}.1$ …(2)

Now, multiplying (1) and (2), we get

$2n(1 + x)^{2n-1}\ (x + 1)^{2n} = [{}^{2n}C_1 + 2x\,{}^{2n}C_2 + 3x^2\ {}^{2n}C_3 + 4x^{3\,2n}C_4 + 5x^4\ {}^{2n}C_5 + \ldots + 2nx^{2n-1}\ {}^{2n}C_{2n}]$
$\times [{}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + {}^{2n}C_3x^{2n-3} + {}^{2n}C_4x^{2n-4} + \ldots + {}^{2n}C_{2n}]$

Now, comparing coefficients of x^{2n-1}, we get

$({}^{2n}C_1)^2 + 2({}^{2n}C_2)^2 + 3({}^{2n}C_3)^2 + 4({}^{2n}C_4)^2 + \ldots + 2n({}^{2n}C_{2n})^2] = 2n \times {}^{4n-1}C_{2n-1}$

$= 2n \times \frac{(4n-1)!}{(2n-1)!(4n-1-2n+1)!}$

$= 2n \times \frac{(4n-1)!}{(2n-1)! \times (2n)!}$

$= \frac{(4n-1)! \times 2n}{(2n-1)! \times 2n\ (2n-1)!}$

$= \frac{(4n-1)!}{(2n-1)! \times (2n-1)!}$ Proved.

Q.12 Prove that

$$C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \ldots\ldots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{\{(n+1)!\}^2}$$

Sol. We have to prove that

$C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \ldots. + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{\{(n+1)!\}^2}$

using, $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + {}^nC_4x^4 + \ldots + {}^nC_nx^n$

Integrating both sides w.r.t. x, we get

$\frac{(1+x)^{n+1}}{(n+1)} = {}^nC_0x + {}^nC_1x^2/2 + {}^nC_2x^3/3 + {}^nC_3x^4/4 + \ldots + C_n \frac{x^{n+1}}{n+1} + A$

Where A is constant of integration.

Put x = 0, we get

$A = \frac{1}{n+1}$

$$\therefore \frac{x^{n+1}}{n+1} - \frac{1}{n+1} = C_0\, x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \ldots + C_n \frac{x^{n+1}}{n+1} \quad \ldots(1)$$

$(x + 1)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + {}^nC_3x^{n-3} + \ldots + {}^nC_n \quad \ldots (2)$

Now, multiplying (1) and (2), we get

$$\frac{1}{(1+n)}[(1 + x)^{n+1} - 1] \times (1 + x)^n$$

$$= \frac{1}{1+n}[(1+x)^{2n+1} - (1+x)^n]$$

$$= (C_0x + C_1x^2/2 + C_2x^3/3 + C_3x^4/4 + \ldots + C_n \frac{x^{n+1}}{n+1}) \times$$
$$(C_0\, x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots + C_n)$$

Now, comparing coefficients of x^{n+1} both sides, we get

$$\frac{1}{(n+1)}[{}^{2n+1}C_{n+1} - 0]$$

$$= \left[C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \frac{C_3^2}{4} + \ldots + \frac{C_n^2}{(n+1)}\right]$$

$$\Rightarrow C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \frac{C_3^2}{4} + \ldots + \frac{C_n^2}{(n+1)}$$

$$= \frac{1}{(n+1)} \times \frac{(2n+1)!}{(n+1)!\,(2n+1-n-1)!}$$

$$= \frac{(2n+1)!}{(n+1)(n+1)!\,n!} = \frac{(2n+1)!}{(n+1)!\,(n+1)!} \text{ Proved.}$$

Q.13 Sum the series $\sum_{r=0}^{n} (r + 1).\ {}^{2n}C_r$.

Sol. $(n+1)2^{2n-1} + \frac{2n!}{2.(n)!(n)!}$

Q.14 Evaluate

${}^{2n+1}C_0^2 + {}^{2n+1}C_1^2 + {}^{2n+1}C_2^2 + \ldots.. + {}^{2n+1}C_n^2$.

Sol. $(1 + x)^{2n+1} = [{}^{2n+1}C_0 + {}^{2n+1}C_1x + {}^{2n+1}C_2x^2 + {}^{2n+1}C_3\, x^3 + {}^{2n+1}C_4\, x^4 + \ldots + {}^{2n+1}C_{n-1}x^{n-1} + {}^{2n+1}C_n\, x^n + \ldots] \quad \ldots(1)$

Also,

$(1 + x)^{2n+1} = [{}^{2n+1}C_0 + {}^{2n+1}C_1x + {}^{2n+1}C_2x^2 + {}^{2n+1}C_3x^3 + {}^{2n+1}C_4\, x^4 + \ldots + {}^{2n+1}C_nx^n + {}^{2n+1}C_{n+1}x^{n+1} + + {}^{2n+1}C_{n+2}x^{n+2} + \ldots + {}^{2n+1}C_{2n+1}\, x^{2n+1}] \quad \ldots(2)$

Multiplying (1) and (2), and comparing coefficients of x^{2n+1}

$\frac{1}{2}\ {}^{(4n+2)}C_{2n+1} = [{}^{2n+1}C_0.\ {}^{2n+1}C_{2n+1} + {}^{2n+1}C_{2n}.\ {}^{2n+1}C_1 + {}^{2n+1}C_{2n+1}.\ {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n.\ {}^{2n+1}C_{n+1}]$

Since, ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1}$

${}^{2n+1}C_{2n} = {}^{2n+1}C_1$

${}^{2n+1}C_{2n-1} = {}^{2n+1}C_2$

${}^{2n+1}C_n = {}^{2n+1}C_{n+1}$

$({}^{2n+1}C_0)^2 + ({}^{2n+1}C_1)^2 + ({}^{2n+1}C_2)^2 + \ldots + ({}^{2n+1}C_n)^2$

$$= \frac{1}{2}\ {}^{4n+2}C_{2n+1}$$

$$= \frac{(4n+2)!}{(2n+1)!(4n+2-2n-1)!} \times \frac{1}{2}$$

$$= \frac{(4n+2)!}{(2n+1)!(2n+1)!} \times \frac{1}{2} \text{ Ans.}$$

Q.15 Show that

$$\sum_{r=0}^{n} \frac{C_r\, .3^{r+4}}{(r+1)(r+2)(r+3)(r+4)}$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left[4^{n+4} - \sum_{s=0}^{3} {}^{n+4}C_s\, .3^s\right]$$

Q.16 Prove that $C_0 - 2^2.\ C_1 + 3^2\ .\ C_2 - \ldots\ldots\ldots\ldots. + (-1)^n\ (n + 1)^2\ C_n = 0,\ n > 2$

Sol. We have to prove that

$C_0 - 2^2.C_1 + 3^2.C_2 \ldots + (-1)^n\ (n + 1)^2\ C_n = 0,\ n > 2$

using, $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_n\, x^n$

Multiplying both sides by x, we get

$x\ (1 + x)^n = C_0.x + C_1x^2 + C_2x^3 + C_3x^4 + \ldots + C_nx^{n+1}$

Differentiating w.r.t. x, we get

$1.\ (1 + x)^n + x.n(1 + x)^{n-1} = C_0 + 2C_1\, x + 3x^2\, C_2 + 4x^3C_3 + 5x^4\, C_4 + \ldots + (n + 1)x^nC_n$

Again multiplying by x on both sides.

$x(1 + x)^n + x^2.n.\ (1 + x)^{n-1} = C_0x + 2C_1x^2 + {}^3C_2x^3 + 4x^4C_3 + 5x^5C_4 + 6x^6\, C_5 + \ldots + (n + 1)\ x^{(n+1)}\ C_n.$

Again differentiating w.r.t. x, we get

1. $(1 + x)^n + x\, n (1 + x)^{n-1} + 2x.n (1 + x)^{n-1} + x^2.n (n - 1) (1 + x)^{n-2}$
$= C_0 + 2^2.x\, C_1 + 3^2 x^2 C_2 + 4^2 x^3 C_3 + 5^2 x^4 C_4 + 6^2 x^5 C_5 + \ldots + (n + 1)^2 x^n C_n$
Put x = – 1, we get
$0 + 0 + 0 + 0 = C_0 - 2^2C_1 + 3^2C_2 - 4^2C_3 + 5^2C_4 - 6^2 C_5 + \ldots + (-1)^n (n + 1)^2. C_n$
$\therefore C_0 - 2^2 C_1 + 3^2 C_2 - 4^2C_3 + 5^2C_4 - 6^2C_5 + \ldots + (-1)^n (n + 1)^2 C_n = 0$
Hence, Proved.

Q.17 Given p + q = 1,

show that $\sum_{r=0}^{n} r^2 \;.{}^nC_r.\, p^r.q^{n-r} = np\, [(n - 1)p + 1]$.

Sol. Given p + q = 1
we have to prove that

$$\sum_{r=0}^{n} r^2 \; {}^nC_r\, p^r.q^{n-r} = np\, [(n - 1)p + 1]$$

Let $(q + px)^n = {}^nC_0\, q^n (px)^0 + {}^nC_1\, q^{n-1} (px) + {}^nC_2 (px)^2 q^{n-2} + {}^nC_3 (px)^3 q^{n-3} + {}^nC_4 (px)^4 q^{n-4} + {}^nC_5 (px)^5 q^{n-5} + \ldots + {}^nC_n (px)^n q^0$

Differentiating w.r.t. x, we get

$n (q + px)^{n-1} p = {}^nC_1\, q^{n-1} p + 2x\, p^2\, {}^nC_2\, q^{n-2} + {}^nC_3 \times 3x^2 p^3 q^{n-3} + {}^nC_4 \times 4x^3 p^4 q^{n-4} + \ldots + {}^nC_n\, p^n \times nx^{n-1}$

Multiplying both sides by x, we get

$np.\, [x (q + px)^{n-1}] = {}^nC_1\, q^{n-1}(px) + 2x^2 p^2\, {}^nC_2\, q^{n-2} + {}^nC_3 \times 3x^3 p^3 q^{n-3} + {}^nC_4 \times 4x^4 p^4 q^{n-4} + \ldots + {}^nC_n \times p^n \times n\, x^n$.

Differentiating w.r.t. = x, we get

$np\, [1. (q + px)^{n-1} + x.(n - 1)(q + px)^{n-2}.p]$
$= {}^nC_1\, pq^{n-1} + 2^2\, x\, p^2\, {}^nC_2\, q^{n-2} + {}^nC_3 \times 3^2 x^2 p^3 q^{n-3} + {}^nC_4 \times 4^2 x^3 p^4 q^{n-4} + \ldots + {}^nC_n \times n^2 x^{n-1} \times p^n$

Put x = 1, we get

$np\, [(p + q)^{n-1} + 1\, (n - 1)(p + q)^{n-2}\, p]$

$= \sum_{r=0}^{n} r^2 \;.p^r.{}^nC_r.q^{n-r}$

$= np\, [1 + (n - 1)\, p]\ (\because p + q = 1)$

Hence, Proved

Q.18 Find the coefficients of x^{50} in the polynomials obtained after parentheses have been removed and like terms have been collected in the expansions

(i) $(1 + x)^{1000} + x (1 + x)^{999} + x^2 (1 + x)^{998} + \ldots\ldots\ldots + x^{1000}$

(ii) $(1 + x) + 2 (1 + x)^2 + 3 (1 + x)^3 + \ldots\ldots\ldots + 1000 (1 + x)^{1000}$

Sol. (i) $(1 + x)^{1000} + x (1 + x)^{999} + x^2 (1 + x)^{998} + \ldots + (1 + x)x^{999} + x^{1000}$

It forms G.P. with common ratio $\left(\frac{x}{1+x}\right)$

$$= \frac{(1+x)^{1000}\left[1-\left(\frac{x}{1+x}\right)^{1001}\right]}{\left(1-\frac{x}{1+x}\right)}$$

$$= \frac{(1+x)^{1000}\left[(1+x)^{1001}-x^{1001}\right]}{(1+x)^{1001}\left(\frac{1+x-x}{1+x}\right)}$$

$$= \frac{(1+x)^{1001}}{(1+x)^{1001}} \times [(1 + x)^{1001} - x^{1001}]$$

$= (1 + x)^{1001} - x^{1001}$
= Coefficient of x^{50}
$= {}^{1001}C_{50}$

(ii) Let $P(x) = (1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + \ldots + 1000 (1 + x)^{1000}$ …(1)

Multiplying by (1 + x) in (1)

$P(x) \times (1 + x) = (1 + x)^2 + 2(1 + x)^3 + 3(1 + x)^4 + \ldots + 999 (1 + x)^{1000} + 1000 (1 + x)^{1001}$ …(2)

Subtract (2) from (1), we get

$P(x) - P(x) (1 + x) = - x. P(x)$.

$= (1 + x) + (1 + x)^2 + (1 + x)^3 + (1 + x)^4 + \ldots + (1 + x)^{1000} - 1000 (1 + x)^{1001}$

$$-x.P(x) = \frac{(1+x)\left[(1+x)^{1000}-1\right]}{x} - 1000\,(1+x)^{1001}$$

$$= \frac{(1+x)\left[(1+x)^{1000}-1\right]}{1+x-1} - 1000\,(1+x)^{1001}$$

$$x.P(x) = 1000\,(1+x)^{1001} - \frac{(1+x)}{x}\,[(1+x)^{1000} - 1]$$

$$P(x) = 1000\,\frac{(1+x)^{1001}}{x} - \frac{(1+x)^{1001}}{x^2} + \frac{1+x}{x}$$

= Coefficient of x^{50}

$P(x) = 1000 \times {}^{1001}C_{51} - {}^{1001}C_{52}$ Ans.

Q.19 Find the coefficient of x^{49} in the polynomial

$$\left(x - \frac{C_1}{C_0}\right)\left(x - 2^2\frac{C_2}{C_1}\right)\left(x - 3^2\frac{C_3}{C_2}\right)$$
$$\ldots\left(x - 50^2\frac{C_{50}}{C_{49}}\right)$$

Where $C_r = {}^{50}C_r$.

Sol. –22100

Part-B Passage based objective questions

Passage- 1 (Q. No. 20 to Q. 22)

Let P be a product given by $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$ and let $S_1 = a_1 + a_2 + \ldots\ldots + a_n$

$= \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum\sum_{i<j<k} a_i.a_j.a_k$

and so on, then it can be shown that

$P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots. + S_n$

Q.20 The coefficient of x^8 in the expression $(2 + x)^2 (3 + x)^3 (4 + x)^4$ must be

(A) 26 (B) 27

(C) 28 (D) 29

Sol. **[D]**

Coefficient of x^8 in the expansion

$(2+ x)^2 (3 + x)^3 (4 + x)^4$

$= {}^2C_1.2.{}^3C_3.{}^4C_4 + {}^2C_2.{}^3C_2.3 \cdot {}^4C_4 + {}^2C_2.{}^3C_3.{}^4C_3.4$

$= 4 + 9 + 16$

$= 29$

∴ Option (D) is correct Answer.

Q.21 The coefficient of x^{203} in the expression $(x – 1)(x^2 – 2)(x^2 – 3) \ldots\ldots(x^{20} – 20)$ must be

(A) 11 (B) 12

(C) 13 (D) 15

Sol. **[C]**

Q.22 The coefficient of x^{98} in the expression of $(x – 1)(x – 2) \ldots\ldots (x – 100)$ must be

(A) $1^2 + 2^2 + 3^2 + \ldots\ldots.+ 100^2$

(B) $(1 + 2 + 3 + \ldots\ldots.+ 100)^2 – (1^2 + 2^2 + 3^2 + \ldots..+ 100^2)$

(C) $\frac{1}{2}[(1 + 2 + 3 + \ldots..+ 100)^2 – (1^2 + 2^2 + 3^2 + \ldots..+ 100^2)]$

(D) None of these

Sol. **[B]**

$(x – 1)(x – 2)\ldots\ldots(x – 100)$

Coefficient of x^{98} in above expansion

$= \sum\sum_{i<j} a_i\, a_j$

from, $(1 + 2 + 3 + 4 + \ldots + 100)^2$

$= (1^2 + 2^2 + 3^2 + 4^2 + \ldots + 100^2) + 2\sum\sum_{i<j} a_i\, a_j$

$= \sum\sum_{i<j} a_i\, a_j$

$= \frac{(1+2+3+\ldots+100)^2 - [1^2+2^2+3^2+\ldots+100^2]}{2}$

∴ Option (B) is correct Answer.

Passage-2 (Q. No. 23 to Q. 25)

If n and x are positive integers then by binomial theorem $(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots\ldots.$ The RHS can be written in several ways. For instance $(1 + x)^n = 1 +$ (A multiple of x) $= 1 + \lambda x$ where λ is an integer.

OR

$(1+ x)^n = 1 + nx + \lambda x^2 = 1 + nx +$ a multiple of x^2 etc. From the last representation we can conclude that $(1+ x)^n –1 – nx$ is divisible by x^2.

Q.23 If $n \ge 5$ the expression $2^{2n+1} – 9n^2 + 3n – 2$ must be divisible by

(A) 36 (B) 48

(C) 54 (D) None of these

Sol. [C] $\because (1 + x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \lambda x^3$

put x = 3 we have

$2^{2n} = \frac{2+6n+9n^2-9n}{2} + \lambda(3)^3$

$\Rightarrow 2^{2n+2} – 9n^2 + 3n – 2 = 54\lambda$

$\Rightarrow$ divisible by 54

Q.24 If $f(n) = 5^{2n+2} – 24n – 25$ then select the best choice from the following choices

(A) f(n) is divisible by 12 for all n

(B) f(n) is divisible by 24 for all n

(C) f(n) is divisible by 576 for all n

(D) f(n) is divisible by 24^3 for all n

Sol. **[C]**

$\because (1 + x)^n = 1 + n x + \lambda x^2$

Put x = 24 we have

$5^{2n} = 1 + 24n + \lambda(24)^2$

$\Rightarrow 5^{2n+2} = 25 + 600n + \lambda(24)^2$

$\Rightarrow 5^{2n+2} - 24\lambda - 25 = 576n + \lambda(24)^2 = 576(n + \lambda)$

$\Rightarrow$ it is divisible by 12, 24 and 576 for all n

$\Rightarrow$ option C is best choice

Q.25 The last three digits of the number 3^{1000} must be

(A) 249 (B) 751

(C) 001 (D) 003

Sol. **[C]**

$3^{1000} = 9^{500} = (10 - 1)^{500}$

$= {}^{500}C_0\, 10^{500} - {}^{500}C_1\, 10^{499} + \ldots\ldots - {}^{500}C_{499}\, 10 + 1$

$= 10^3(10^{497} - {}^{500}C_1\, 10^{496} + \ldots\ldots - 5) + 1$

Last three digit = 001

EXERCISE # 4

Old IIT-JEE Questions

Q.1 For any positive integers m, n (with n ≥ m), let $\binom{n}{m} = {}^nC_m$. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \ldots + \binom{m}{m} = \binom{n+1}{m+1}$$

Hence or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \ldots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$ **[IIT- 2000]**

Sol. First, we have to prove that

${}^nC_m + {}^{(n-1)}C_m + {}^{(n-2)}C_m + \ldots + {}^mC_m = {}^{(n+1)}C_{m+1}$

By observation,

${}^nC_m$ is the coefficient of x^m expansion of $(1 + x)^n$

${}^{(n-1)}C_m$ is the coefficient of x^m in the expansion of $(1 + x)^{(n-1)}$

${}^{(n-2)}C_m$ is the coefficient of x^m in the expansion of $(1 + x)^{n-2}$

……………………………………………………

……………………………………………………

${}^mC_m$ is the coefficient of x^m in the expansion of $(1 + x)^m$

$\therefore S = (1+x)^n + (1+x)^{n-1} + (1+x)^{n-2} + (1+x)^{n-3} + \ldots + (1+x)^m$

It forms a G.P. with common ratio $\frac{1}{(1+x)}$.

$$S = \frac{(1+x)^n\left[1 - \frac{1}{(1+x)^{n-m+1}}\right]}{1 - \frac{1}{1+x}}$$

$$S = \frac{(1+x)^n\left[(1+x)^{n-m+1} - 1\right] \times (1+x)}{(1+x)^{n-m+1}.x}$$

$$S = \frac{(1+x)^{(n+1)}\left[(1+x)^{n-m+1} - 1\right]}{x.(1+x)^{(n-m+1)}}$$

$$S = \frac{(1+x)^{(n+1)}\left[(1+x)^{n-m+1} - 1\right]}{x.(1+x)^{n-m+1}}$$

$$S = \frac{(1+x)^{(n+1)}\left[(1+x)^{n-m+1} - 1\right]}{x.(1+x)^{(n+1)}.x^{-m}}$$

$$S = \frac{x^m}{x}\left[(1+x)^{n-m+1} - 1\right]$$

$$S = \frac{(1+x)^{n+1} - x^m}{x}$$

= Coefficient of x^m

$= {}^{(n+1)}C_{m+1} - 0$

Hence, ${}^nC_m + {}^{(n-1)}C_m + {}^{(n-2)}C_m + {}^{(n-3)}C_m + \ldots + {}^mC_m = {}^{(n+1)}C_{(m+1)}$

Secondly, we have to prove that

${}^nC_m + 2{}^{(n-1)}C_m + 3{}^{(n-2)}C_m + \ldots + (n - m + 1)\, {}^mC_m = {}^{(n+2)}C_{m+2}$

By above same observations, we can write

$S = (1 + x)^n + 2\,(1 + x)^{n-1} + 3\,(1 + x)^{n-2} + \ldots + (n - m + 1)\, {}^mC_m$

It also forms A.P. – G.P. with common ratio $\frac{1}{(1+x)}$

$S = (1 + x)^n + 2(1 + x)^{n-1} + 3\,(1 + x)^{n-2} + \ldots + (n - m + 1)\,(1 + x)^{m+1}$

$S \times \frac{1}{(1+x)} = (1 + x)^{n-1} + 2(1 + x)^{n-2} + \ldots + (n - m + 1)\,(1 + x)^m$

Subtracting we have

$S\left(1 - \frac{1}{1+x}\right) = (1 + x)^n + (1 + x)^{(n-1)} + (1 + x)^{(n-2)} + \ldots - (n - m + 1)\, x^m$

$$S.\left(\frac{x}{1+x}\right) = \frac{(1+x)^n\left[1 - \frac{1}{(1+x)^{n-m+2}}\right]}{\left(1 - \frac{1}{1+x}\right)}$$

$$S.\left(\frac{x}{1+x}\right) = \frac{(1+x)^n[(1+x)^{n-m+2} - 1]}{\frac{(1+x-1)}{(1+x)} \times (1+x)^{n-m+2}}$$

$$S.\left(\frac{x}{1+x}\right) = \frac{(1+x)^{n+1} \times \left[(1+x)^{n-m+2} - 1\right]}{x \times (1+x)^{(n-m+2)}}$$

$$S = \frac{(1+x)^{(n+2)}\left[(1+x)^{n-m+2} - 1\right]}{x^2.(1+x)^{n-m+2}}$$

$$S = \frac{(1+x)^{(n+2)} \times \left[(1+x)^{(n-m+2)} - 1\right]}{x^2 \times (1+x)^{(n+2)} \times (1+x)^{-m}}$$

$$S = \frac{x^m}{x^2} \times [(1 + x)^{n-m+2} - 1]$$

$$S = \frac{(1+x)^m}{x^2} \times [(1 + x)^{n-m+2} - 1]$$

$$S = \frac{(1+x)^{n+2} - (1+x)^m}{x^2}$$

= Coefficients of x^m

$= {}^{(n+2)}C_{(m+2)}$

Hence, ${}^nC_m + 2\,{}^{(n-1)}C_m + \ldots + (n-m+1)\,{}^mC_m$

$= {}^{(n+2)}C_{(m+2)}$

Proved

Q.2 In the binomial expansion of $(a - b)^n$, $n \ge 5$, the sum of the 5^{th} and 6^{th} terms is zero. Then $\frac{a}{b}$ equals- **[IIT-Scr.-2001]**

(A) $\frac{n-5}{6}$ (B) $\frac{n-4}{5}$

(C) $\frac{5}{n-4}$ (D) $\frac{6}{n-5}$

Sol. **[B]**

$(a - b)^n = {}^nC_0\,a^n + {}^nC_1\,a^{n-1}(-b) + {}^nC_2\,a^{n-2}(-b)^2 + {}^nC_3\,a^{n-3}(-b)^3 + {}^nC_4\,a^{n-4}(-b)^4 + {}^nC_5\,a^{n-5}(-b)^5$

Sum of fifth & sixth terms = 0

$\Rightarrow {}^nC_4\,a^{n-4}(-b)^4 + {}^nC_5\,a^{n-5}(-b)^5 = 0$

$\Rightarrow {}^nC_4\,a^{n-4}b^4 = {}^nC_5\,a^{n-5}b^5$

$$\Rightarrow \frac{n!}{4!(n-4)!}a^{n-4}b^4 = \frac{n!}{5!(n-5)!} \times \frac{a^{n-4}}{a} \times b^4 \times b$$

$$\Rightarrow \frac{n!}{4!(n-4)(n-5)!}a^{n-4}b^4 = \frac{n!}{5\times 4! \times (n-5)!} \times \frac{a^{n-4}}{a} \times b^4 \times b$$

$$\Rightarrow \frac{1}{(n-4)} = \frac{1}{5} \times (b/a) \Rightarrow \frac{a}{b} = \frac{n-4}{5}$$

$\therefore$ Option (B) is correct Answer.

Q.3 The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q} = 0$ if $p < q$) is maximum when m is- **[IIT-Scr. 2002]**

(A) 5 (B) 10

(C) 15 (D) 20

Sol. **[B]**

$\sum_{i=0}^{m} {}^{10}C_i\,{}^{20}C_{m-i} = {}^{10}C_0\,{}^{20}C_m + {}^{10}C_1\,{}^{20}C_{m-1} + {}^{10}C_2\,{}^{20}C_{m-2} + {}^{10}C_3\,{}^{20}C_{m-3} + {}^{10}C_4\,{}^{20}C_{m-4} + {}^{10}C_5\,{}^{20}C_{m-5} + {}^{10}C_6\,{}^{20}C_{m-6} + {}^{10}C_7\,{}^{20}C_{m-7} + {}^{10}C_8\,{}^{20}C_{m-8} + {}^{10}C_9\,{}^{20}C_{m-9} + {}^{10}C_{10}\,{}^{20}C_{m-10} + \{{}^{10}C_{11}\,{}^{20}C_{m-11} + {}^{10}C_{12}\,{}^{20}C_{m-12} + \ldots + {}^{10}C_{20}\,{}^{20}C_0\}$

Hence, sum will be maximum when m = 10

$\therefore$ Option (B) is correct Answer.

Q.4 Find the coefficient of t^{24} in the expansion of $(1 + t^2)^{12}(1 + t^{12})(1 + t^{24})$ is-**[IIT-Scr. 2003]**

(A) ${}^{12}C_6 + 2$ (B) ${}^{12}C_6 + 1$

(C) ${}^{12}C_6 + 3$ (D) ${}^{12}C_6$

Sol. **[A]**

$(1 + t^{12})(1 + t^{24})$

$= (1 + t^{12} + t^{24} + t^{36})$

$\therefore (1 + t^{12} + t^{24} + t^{36})(1 + t^2)^{12}$

$= (1 + t^{12} + t^{24} + t^{36})\,[{}^{12}C_0 + \ldots + {}^{12}C_6\,t^{12} + \ldots + {}^{12}C_{12}\,t^{24}]$

Coefficient of $t^{24} = {}^{12}C_0 + {}^{12}C_6 + {}^{12}C_{12}$

$= 2 + {}^{12}C_6$

$\therefore$ Option (A) is correct Answer.

Q.5 Prove that :

$$2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} \ldots\ldots + (-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$$

[IIT-2003]

Sol. We have to prove that

$2^k\,{}^nC_0\,{}^nC_k - 2^{(k-1)}\,{}^nC_1\,{}^{(n-1)}C_{(k-1)} + 2^{(k-2)}\,{}^nC_2\,{}^{(n-2)}C_{(k-2)} + \ldots + (-1)^k\,{}^nC_k\,{}^{(n-k)}C_0 = {}^nC_k$

$$= \sum_{r=0}^{k} 2^{(k-r)}\,{}^nC_r\,{}^{(n-r)}C_{(k-r)}$$

$$= 2^k \sum_{r=0}^{k} 2^{-r} \times \frac{n!}{r! \times (n-r)!} \times \frac{(n-r)!}{(k-r)! \times (n-k)!}$$

$$= 2^k \sum_{r=0}^{k} 2^{-r} \times \frac{n!}{r! \times (k-r)! \times (n-k)!}$$

$$= 2^k \sum_{r=0}^{k} 2^{-r} \times \frac{n!}{k!(n-k)!} \times \frac{k!}{r!(k-r)!}$$

$$= 2^k \sum_{r=0}^{k} 2^{-r} \times {}^nC_k \times {}^kC_r$$

$$= 2^k \times {}^nC_k \times \sum_{r=0}^{k} 2^{-r} \times \times {}^kC_r$$

$$= 2^k \times {}^nC_k \times \left(1 - \frac{1}{2}\right)^k = 2^k \times {}^nC_k \times \frac{1}{2^k}$$

$= {}^nC_k$, R.H.S.

Proved

Q.6 If ${}^{n-1}C_r = (k^2 - 3)\,{}^nC_{r+1}$, then k lies between- **[IIT Scr. 2004]**

(A) $(-\infty, -2)$ (B) $(2, \infty)$

(C) $[-\sqrt{3}, \sqrt{3}]$ (D) $(\sqrt{3}, 2]$

Sol. **[D]**

${}^{n-1}C_r = (k^2 - 3)\,{}^nC_{r+1}$ … (1)

using, ${}^nC_r = \frac{r+1}{n+1}\,{}^{n+1}C_{r+1}$ … (2)

Comparing (1) and (2), we get

$k^2 - 3 = \frac{r+1}{n+1}$

$\Rightarrow k^2 = \frac{r+1}{n+1} + 3$

Since $n \ge r$

$\Rightarrow n + 1 \ge r + 1$

$\Rightarrow 1 \ge \frac{r+1}{n+1}$

$\Rightarrow 1 + 3 \ge \frac{r+1}{n+1} + 3$

$\Rightarrow k^2 \le 4$

$\Rightarrow -2 \le k \le 2$

But at $k = \pm\sqrt{3}$

${}^{n-1}C_r = 0$ which is not true.

Hence, $k \in (\sqrt{3}, 2]$

$\therefore$ Option (D) is correct Answer.

Q.7 $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \ldots\ldots + \binom{30}{20}\binom{30}{30} =$

[IIT Scr.2005]

(A) $\binom{30}{10}$ (B) $\binom{60}{20}$ (C) $\binom{31}{10}$ (D) $\binom{31}{11}$

Sol. **[A]**

${}^{30}C_0\,{}^{30}C_{10} - {}^{30}C_1\,{}^{30}C_{11} + {}^{30}C_2\,{}^{30}C_{12} - {}^{30}C_3\,{}^{30}C_{13} + {}^{30}C_4\,{}^{30}C_{14} - {}^{30}C_5\,{}^{30}C_{15} + \ldots + {}^{30}C_{20}\,{}^{30}C_{30}$

$(1 + x)^{30} = {}^{30}C_0 + {}^{30}C_1 x + {}^{30}C_2 x^2 + {}^{30}C_3 x^3 + {}^{30}C_4 x^4 + {}^{30}C_5 x^5 + {}^{30}C_6 x^6 + \ldots + {}^{30}C_{30} x^{30}$ … (1)

$\left(1-\frac{1}{x}\right)^{30} = {}^{30}C_0 - {}^{30}C_1 \frac{1}{x} + {}^{30}C_2 \frac{1}{x^2} - {}^{30}C_3 \frac{1}{x^3} + {}^{30}C_4 \frac{1}{x^4} - {}^{30}C_5 \frac{1}{x^5} + {}^{30}C_6 \frac{1}{x^6} - \ldots + {}^{30}C_{10} \frac{1}{x^{10}} - {}^{30}C_{11} \frac{1}{x^{11}} + {}^{30}C_{12} \frac{1}{x^{12}} \ldots + {}^{30}C_{30} \frac{1}{x^{30}}$ … (2)

Multiplying (1) and (2), we get

$(1 + x)^{30} \times \left(1-\frac{1}{x}\right)^{30} = \frac{(x^2-1)^{30}}{x^{30}}$

$= ({}^{30}C_0 + {}^{30}C_1.\ x + {}^{30}C_2.x^2 + {}^{30}C_3.x^3 + {}^{30}C_4x^4 + {}^{30}C_5x^5 + {}^{30}C_6\,x^6 + \ldots + {}^{30}C_{30}\,x^{30}) \times ({}^{30}C_0 - {}^{30}C_1.\ \frac{1}{x} + {}^{30}C_2 \frac{1}{x^2} - {}^{30}C_3 \frac{1}{x^3} + \ldots + {}^{30}C_{10} \frac{1}{x^{10}} - {}^{30}C_{11} \frac{1}{x^{11}} + {}^{30}C_{12} \frac{1}{x^{12}} + \ldots + {}^{30}C_{30} \frac{1}{x^{30}})$

Comparing coefficient of $\frac{1}{x^{10}}$ both sides, we get

$\Rightarrow {}^{30}C_0\,{}^{30}C_{10} - {}^{30}C_1\,{}^{30}C_{11} + {}^{30}C_2\,{}^{30}C_{12} - \ldots + {}^{30}C_{20}\,{}^{30}C_{30} = {}^{30}C_{20} = {}^{30}C_{10}$

$\therefore$ Option (A) is correct answer.

Q.8 For r = 0, 1, …,10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1+ x)^{30}$.

Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

(A) $B_{10} - C_{10}$ (B) $A_{10}\left(B_{10}^2 - C_{10}A_{10}\right)$

(C) 0 (D) $C_{10} - B_{10}$

[IIT - 2010]

Sol. **[D]**

$A_r = {}^{10}C_r$, $B_r = {}^{20}C_r$, $C_r = {}^{30}C_r$

$\sum_{r=1}^{10} {}^{10}C_r\,({}^{20}C_{10}\,{}^{20}C_r - {}^{30}C_{10}\,{}^{10}C_r)$

$= {}^{20}C_{10} \sum_{r=1}^{10} {}^{10}C_r\,{}^{20}C_{20-r} - {}^{30}C_{10} \sum_{r=1}^{10} {}^{10}C_r.{}^{10}C_r$

$= {}^{20}C_{10}\,[{}^{30}C_{20} - {}^{10}C_0\,{}^{20}C_{20}] - {}^{30}C_{10}\,[{}^{20}C_{10} - ({}^{10}C_0)^2]$

$= {}^{20}C_{10}{}^{30}C_{20} - {}^{20}C_{10} - {}^{30}C_{10}\,{}^{20}C_{10} + {}^{30}C_{10}$

$= {}^{30}C_{10} - {}^{20}C_{10}$

$= C_{10} - B_{10}$

EXERCISE # 5

Q.1 Find the sum of the series **[IIT 1985]**

$$\sum_{r=0}^{n}(-1)^r\, {}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+.....\text{upto m terms}\right]$$

Sol. $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$

Q.2 Using mathematical induction or otherwise prove that $\sum_{k=0}^{n} k^2\,{}^nC_k = n(n+1)2^{n-2}$ for $n > 1$

[IIT 1986]

Q.3 Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+4}$. **[IIT 1988]**

Q.4 Using mathematical induction, prove that ${}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + ... + {}^mC_k\,{}^nC_0 = {}^{(m+n)}C_k$, where m, n, k are positive integers, and ${}^pC_q = 0$ for $p < q$. **[IIT 1989]**

Q.5 Prove that $C_0 - 2^2 C_1 + 3^2 C_2 - ...... + (-1)^n (n+1)^2 C_n = 0$, $n > 2$, where $C_r = {}^nC_r$ **[IIT 1989]**

Q.6 Using induction or otherwise, prove that for any non- negative integers m, n, r and k,

$$\sum_{m=0}^{k}(n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1}-\frac{k}{r+2}\right]$$

[IIT 1991]

Q.7 If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \geq n$, then show that $b_n = {}^{2n+1}C_{n+1}$.

[IIT 1992]

Q.8 Prove that $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and n is an even positive integer. **[IIT 1993]**

Q.9 Let n be a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + ... + a_{2n}x^{2n}$

Show that $a_0^2 - a_1^2 .... + a_{2n}^2 = a_n$

[IIT 1994]

Q.10 Prove that $\dfrac{3!}{2(n+3)} = \sum_{r=0}^{n}(-1)^r\left(\dfrac{{}^nC_r}{{}^{r+3}C_r}\right)$

[IIT-1997]

Q.11 If n is an odd natural number then show that $\sum_{r=0}^{n}\dfrac{(-1)^r}{{}^nC_r} = 0$. **[IIT-1998]**

Sol. From L.H.S.

$$\sum_{r=0}^{n}\frac{(-1)^r}{{}^nC_r} = \frac{1}{{}^nC_0}-\frac{1}{{}^nC_1}+\frac{1}{{}^nC_2}-\frac{1}{{}^nC_3}+\frac{1}{{}^nC_4}-\ldots-\frac{1}{{}^nC_{n-4}}+\frac{1}{{}^nC_{n-3}}-\frac{1}{{}^nC_{n-2}}+\frac{1}{{}^nC_{n-1}}-\frac{1}{{}^nC_n}$$

Since, n is odd natural number. Then there will be (n +1) terms in the expansion. All will be vanished separately as follows

${}^nC_1 = {}^nC_{n-1}$

${}^nC_2 = {}^nC_{n-2}$

${}^nC_3 = {}^nC_{n-3}$

M M

${}^nC_n = {}^nC_0$

$\Rightarrow \sum_{r=0}^{n}\dfrac{(-1)^r}{{}^nC_r} = 0$ Hence, Proved.

Q.12 Prove that

$$\sum_{r=1}^{n}(-1)^{r-1}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{r}\right){}^nC_r = \frac{1}{n}.$$

Q.13 If $(1 - x^3)^n = \sum_{r=0}^{n} a_r\, x^r\,(1-x)^{3n-2r}$ then find a_r, where $n \in N$.

Sol. $(1 - x^3)^n = \sum_{r=0}^{n} a_r x^r (1 - x)^{3n - 2r}$

$(1 - x)^n (x^2 + x + 1)^n = \sum_{r=0}^{n} a_r . x^r \frac{(1-x)^{3n}}{(1-x)^{2r}}$

$\frac{(1-x)^n (x^2+x+1)^n}{(1-x)^{3n}} = \sum_{r=0}^{n} a_r \left(\frac{x}{(1-x)^2}\right)^r$

$\Rightarrow \frac{(x^2+x+1)^n}{(x^2-2x+1)^n} = \sum_{r=0}^{n} a_r . \left(\frac{x}{(1-x)^2}\right)^r$

$\frac{(x^2-2x+1+3x)^n}{(1-x)^{2n}} = \sum_{r=0}^{n} a_r . \left(\frac{x}{(1-x)^2}\right)^r$

$\left[\frac{(1-x)^2+3x}{(1-x)^2}\right]^n = \sum_{r=0}^{n} a_r . \left(\frac{x}{(1-x)^2}\right)^r$

$\left[1+\frac{3x}{(1-x)^2}\right]^n = \sum_{r=0}^{n} a_r . \left(\frac{x}{(1-x)^2}\right)^r$

Let $\frac{x}{(1-x)^2} = A$

Then $[1 + 3A]^n = \sum_{r=0}^{n} a_r .(A)^r$

$\Rightarrow$ (r + 1)th term is

${}^nC_r (3A)^r = a_r . (A)^r$

$\Rightarrow {}^nC_r 3^r = a_r$.

Q.14 Show that

$C_1(1 - x) - (C_2/2) (1 - x)^2 + (C_3/3) (1 - x)^3 + ... + (-1)^{n-1} (1/n)(1 - x)^n$

$= (1 - x) + (1/2) (1 - x^2) + (1/3) (1 - x^3) + ... + (1/n) (1 - x^n)$

Q.15 If C_r stands for ${}^nC_r$, then the sum of the series

$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - .... + (-1)^n (n + 1) C_n^2],$

where n is an even positive integer, is equal to.

Sol. $(-1)^{n/2} (n + 2)$

Q.16 If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 & –6 respectively, then value of m is?

Sol. 12

ANSWER KEY

EXERCISE # 1

1. (B) **2.** (A) **3.** (B) **4.** (B) **5.** (A, B) **6.** (C) **7.** (C) **8.** (B)
9. (D) **10.** (B) **11.** (B) **12.** (C) **13.** (B) **14.** (C) **15.** (A) **16.** (B)
17. (C) **18.** (C) **19.** (B) **20.** (A, B, C, D)

EXERCISE # 2

Qus.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Ans.	C	C	D	D	B	B	B	B	D	B	A	B	C	C	C	A	A	A	C	A
Qus.	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**										
Ans.	A,C	A,B	A,B,C	B, C	A,C	A,B,C	D	A	B	A										

31. $A \to P, S$; $B \to P, Q$; $C \to R$; $D \to P$

32. $A \to S$; $B \to P, R$; $C \to P, R$; $D \to Q$

33. $A \to Q, S$; $B \to P, Q, R$; $C \to P$; $D \to P, S$

34. $A \to S$; $B \to P$; $C \to Q$; $D \to R$

35. $A \to Q$; $B \to S$; $C \to P$; $D \to R$

36. $A \to S$; $B \to P$; $C \to Q$; $D \to R$

EXERCISE # 3

1. zero when n is odd; $(-i)^{n/2}\ {}^{n}C_{n/2}$ when n is even. **2.** 1155 **4.** n = 9

13. $(n+1)2^{2n-1} + \dfrac{2n!}{2.(n)!(n)!}$ **18.** (i) ${}^{1001}C_{50}$; (ii) $\dfrac{51050\ (1001)!}{(52)!\ (950)!}$ **19.** –22100 **20.** (D)

21. (C) **22.** (C) **23.** (C) **24.** (C) **25.** (C)

EXERCISE # 4

2. (B) **3.** (C) **4.** (A) **6.** (D) **7.** (A) **8.** (D)

EXERCISE # 5

1. $\dfrac{2^{mn}-1}{2^{mn}(2^{n}-1)}$ **13.** $a_r = {}^{n}C_r . 3^r$ **15.** $(-1)^{n/2}(n+2)$ **16.** 12