SOLVED EXAMPLES

Ex. 1. **(A)** Find the angle between the curve $2y^2 = x^3$ and $y^2 = 32x$

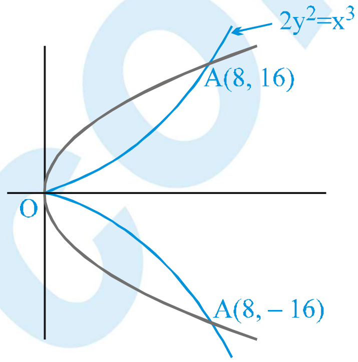
(B) Prove that the angle between the curves $y^2 = x$ and $x^3 + y^3 = 3xy$ at the point other than origin is $\tan^{-1}(16)^{1/3}$.

Sol. **(A)**

Solving $2y^2 = x^3$ and $y^2 = 32x$

we get $(0, 0)$; $(8, 16)$ and $(8, -16)$

for $\sqrt{2}\,y = x\sqrt{x}$ or $\sqrt{2}\,y = -x\sqrt{x}$



at $(0, 0)$ $\left.\dfrac{dy}{dx}\right|_{(0,0)} = 0$

for I

at $(0, 0)$ $\left.\dfrac{dy}{dx}\right|_{(0,0)} = \infty$

for II

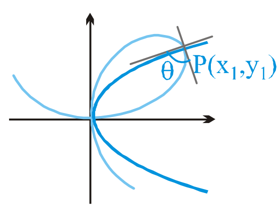
hence angle = 90°

now $\left.\dfrac{dy}{dx}\right|_{I} = \dfrac{3x^2}{4y} = \dfrac{3}{4}\cdot\dfrac{64}{16} = 3$

$\left.\dfrac{dy}{dx}\right|_{II} = \dfrac{32}{2y} = \dfrac{16}{16} = 1$ $\quad\therefore\quad \tan\theta = \dfrac{3-1}{1+3} = \dfrac{2}{4} = \dfrac{1}{2}$

(B) $y^2 = x$; $x^3 + y^3 = 3xy$

$2y\dfrac{dy}{dx} = 1 \Rightarrow \left.\dfrac{dy}{dx}\right|_{P} = \dfrac{1}{2y_1}$



again for the 2nd curve

$\left.\dfrac{dy}{dx}\right|_{P} = \dfrac{y_1 - x_1^2}{y_1^2 - x_1}$

solving $y^2 = x$ and $x^3 + y^3 = 3xy$; $y^6 + y^3 = 3y^3 \Rightarrow y^3 + 1 = 3 \Rightarrow y^3 = 2$

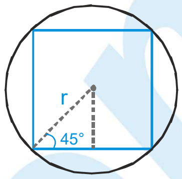
$\therefore$ $y_1 = 2^{1/3}$ and $x_1 = 2^{2/3}$

now $m_1 = \dfrac{1}{2\cdot 2^{\frac{1}{3}}} = \dfrac{1}{2^{\frac{4}{3}}}$; $\qquad m_2 = \dfrac{2^{\frac{1}{3}} - 2^{\frac{4}{3}}}{2^{\frac{2}{3}} - 2^{\frac{2}{3}}} \to \infty$

$\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\dfrac{1 - \dfrac{m_1}{m_2}}{\dfrac{1}{m_2} + m_1}\right| = \left|\dfrac{1}{m_1}\right| = 2^{\frac{4}{3}} = 16^{\frac{1}{3}}$ $\qquad\therefore\qquad \theta = \tan^{-1}(16^{\frac{1}{3}})$

Ex 2 If area of circle increases at a rate of $2cm^2/sec$, then find the rate at which area of the inscribed square increases.

Sol. Area of circle, $A_1 = \pi r^2$. Area of square, $A_2 = 2r^2$ (see figure)



$$\frac{dA_1}{dt} = 2\pi r \frac{dr}{dt}, \qquad \frac{dA_2}{dt} = 4r \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad 2 = 2\pi r \cdot \frac{dr}{dt} \qquad \Rightarrow \qquad r\frac{dr}{dt} = \frac{1}{\pi}$$

$$\therefore \quad \frac{dA_2}{dt} = 4 \cdot \frac{1}{\pi} = \frac{4}{\pi} \text{ cm}^2/\text{sec}$$

$\therefore$ Area of square increases at the rate $\frac{4}{\pi}$ cm^2/sec.

Ex. 3 The equation of the tangent to the curve $x = a\cos^3 t$, $y = a\sin^3 t$ at 't' point is

Sol.
$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = -\frac{3a\sin^2 t\ \cos t}{3a\cos^2 t\ \sin t} = -\frac{\sin t}{\cos t}$$

which is the slope of the tangent at 't' point. Hence equation of the tangent at 't' point is

$$y - a\sin^3 t = -\frac{\sin t}{\cos t}\left(x - a\cos^3 t\right) \Rightarrow \frac{y}{\sin t} - a\sin^2 t = -\frac{x}{\cos t} + a\cos^2 t$$

$$\Rightarrow \quad x \sec t + y \operatorname{cosec} t = a$$

Ex. 4 Sand is pouring from pipe at the rate of $12\ cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one - sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?

Sol. $V = \frac{1}{3}\pi r^2 h$

but $h = \frac{r}{6}$

$$\Rightarrow \quad V = \frac{1}{3}\pi (6h)^2 h$$

$$\Rightarrow \quad V = 12\pi h^3$$

$$\frac{dV}{dt} = 36\pi h^2 \cdot \frac{dh}{dt}$$

when, $\frac{dV}{dt} = 12\ cm^3/s$ and $h = 4$ cm

$$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi} \text{ cm/sec.}$$

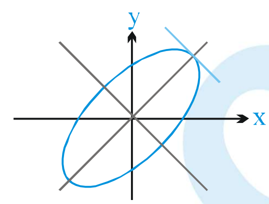
Ex. 5 Find the condition for the line $y = mx$ to cut at right angles the conic $ax^2 + 2hxy + by^2 = 1$. Hence find the direction of the axes of the conic.

Sol. $ax^2 + 2hxy + by^2 = 1$ ....**(1)**

$$2ax + 2h\left[x\frac{dy}{dx} + y\right] + 2by\frac{dy}{dx} = 0$$

$$(xh + by)\frac{dy}{dx} = -(ax + hy)$$

$$\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$



$\therefore$ $$+m\left(\frac{ax + hy}{hx + by}\right) = +1$$

$$m\left[\frac{ax + h\cdot mx}{hx + b\cdot mx}\right] = 1$$

$$m(a + hm) = h + bm$$

$$m^2h + (a - b)m - h = 0$$

Ex. 6 If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is -

Sol. $y = 1 - ax^2 \Rightarrow \frac{dy}{dx} = -2ax$ $\quad y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

Two curves intersect orthogonally if $\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2 = -1$

$\Rightarrow (-2ax)(2x) = -1 \quad \Rightarrow 4ax^2 = 1$ ..... **(i)**

Now eliminating y from the given equations we have $1 - ax^2 = x^2$

$\Rightarrow (1 + a)x^2 = 1$ ..... **(ii)**

Eliminating x^2 from **(i)** and **(ii)** we get $\frac{4a}{1 + a} = 1 \Rightarrow a = \frac{1}{3}$

Ex. 7 Find the locus of point on the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ where tangents are parallel to the axis of x.

Sol. We have $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ ....**(i)**

Differentiating w.r.t. x, we get $2y\frac{dy}{dx} = 4a\left[1 + \cos\frac{x}{a}\right]$

For points at which the tangents are parallel to x-axis,

$\frac{dy}{dx} = 0$ or $4a\left(1 + \cos\frac{x}{a}\right) = 0$

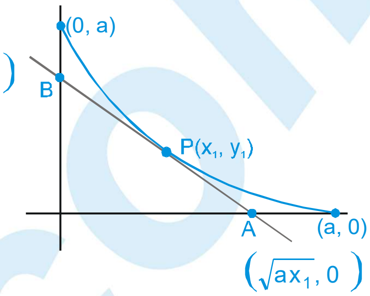
or $\cos\frac{x}{a} = -1$ or $\frac{x}{a} = (2n + 1)\pi$

For these values of x, $\sin\frac{x}{a} = 0$.

Therefore, all these points lie on the parabola $y^2 = 4ax$ [putting sinx/a = 0 in equation **(i)**.

Ex. 8 Prove that sum of intercepts of the tangent at any point to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ on the coordinate axis is constant.

Sol. Let $P(x_1, y_1)$ be a variable point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, as shown in figure.



$\Rightarrow$ equation of tangent at point P is $-\dfrac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1) = (y - y_1)$

$\Rightarrow$ $-\dfrac{x}{\sqrt{x_1}} + \sqrt{x_1} = \dfrac{y}{\sqrt{y_1}} - \sqrt{y_1}$

$\Rightarrow$ $\dfrac{x}{\sqrt{x_1}} + \dfrac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1}$

$\Rightarrow$ $\dfrac{x}{\sqrt{x_1}} + \dfrac{y}{\sqrt{y_1}} = \sqrt{a}$ $\qquad (\because \sqrt{x_1} + \sqrt{y_1} = \sqrt{a})$

Hence point A is $\left(\sqrt{ax_1}, 0\right)$ and coordinates of point B is $(0, \sqrt{ay_1})$. Sum of intercepts

$= \sqrt{a}\,(\sqrt{x_1} + \sqrt{y_1}) = \sqrt{a}\,.\sqrt{a} = a$ (which is constant)

Ex. 9 x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the first square.

Sol. Given x and y are sides of two squares. Thus the area of two squares are x^2 and y^2

We have to obtain $\dfrac{d(y^2)}{d(x^2)} = \dfrac{2y\dfrac{dy}{dx}}{2x} = \dfrac{y}{x}.\dfrac{dy}{dx}$ .... **(i)**

where the given curve is, $y = x - x^2$ $\Rightarrow$ $\dfrac{dy}{dx} = 1 - 2x$ .... **(ii)**

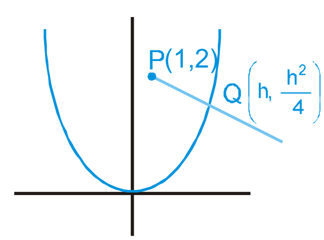
Thus, $\dfrac{d(y^2)}{d(x^2)} = \dfrac{y}{x}(1 - 2x)$ [from **(i)** and **(ii)**]

or $\dfrac{d(y^2)}{d(x^2)} = \dfrac{(x - x^2)(1 - 2x)}{x}$ $\Rightarrow$ $\dfrac{d^2(y^2)}{d(x^2)^2} = (2x^2 - 3x + 1)$

The rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$

Ex. 10 Find the equation of all possible normal/s to the parabola $x^2 = 4y$ drawn from point (1, 2).

Sol. Let point Q be $\left(h, \dfrac{h^2}{4}\right)$ on parabola $x^2 = 4y$ as shown in figure



Now, m_{PQ} = slope of normal at Q.

Slope of normal $= -\left.\dfrac{dx}{dy}\right|_{x=h} = -\dfrac{2}{h}$

$\Rightarrow$ $\dfrac{\dfrac{h^2}{4} - 2}{h - 1} = -\dfrac{2}{h}$ $\Rightarrow$ $\dfrac{h^3}{4} - 2h = -2h + 2$

$\Rightarrow$ $h^3 = 8$ $\Rightarrow$ $h = 2$

Hence coordinates of point Q is (2, 1) and so equation of required normal becomes $x + y = 3$.

Ex. 11 The equation of normal to the curve $x + y = x^y$, where it cuts x-axis is -

Sol. Given curve is $x + y = x^y$ ..... **(i)**

at x-axis y = 0,

$\therefore \quad x + 0 = x^0 \Rightarrow x = 1$

$\therefore$ Point is A(1, 0)

Now to differentiate $x + y = x^y$ take log on both sides

$$\Rightarrow \log(x+y) = y\log x \qquad \therefore \frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\} = y.\frac{1}{x} + (\log x)\frac{dy}{dx}$$

Putting $x = 1, y = 0$ $\qquad \left\{1+\frac{dy}{dx}\right\} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -1$

$\therefore$ slope of normal = 1

Equation of normal is, $\frac{y-0}{x-1} = 1 \Rightarrow y = x - 1$

Ex. 12 The length of the normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$ is -

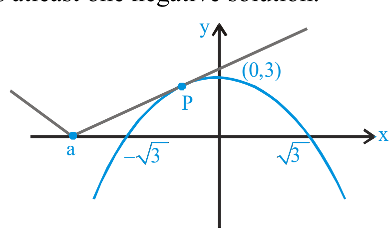
Sol.
$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2} \Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \tan\left(\frac{\pi}{4}\right) = 1$$

Also at $\theta = \frac{\pi}{2}$, $y = a\left(1 - \cos\frac{\pi}{2}\right) = a$

$\therefore$ required length of normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = a\sqrt{1+1} = \sqrt{2}a$

Ex. 13 Find the possible values of 'a' such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution.

Sol. $3 - x^2 > |x - a|$

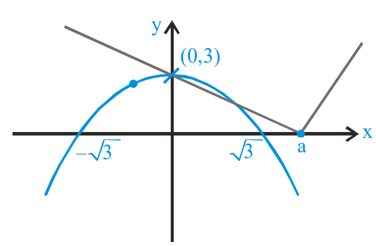


Case **(i)** if a < 0 and y = x – a is tangent of $y = 3 - x^2$

$\Rightarrow -2x = 1 \Rightarrow x = -\frac{1}{2} \Rightarrow p\left(-\frac{1}{2}, \frac{11}{4}\right)$

Since y = x – a passes the $\left(-\frac{1}{2}, \frac{11}{4}\right) \Rightarrow a = x - y$

$= -\left(\frac{11}{4} + \frac{1}{2}\right) = -\frac{13}{4}$ (minimum value of a)



Case **(ii)** a > 0 and y = – x + a passes through (0, 3),

then a = 3 (maximum value of a)

$\Rightarrow a \in \left(-\frac{13}{4}, 3\right)$

Ex. 14 For the curve $y = a\,\ell n\,(x^2 - a^2)$ show that sum of lengths of tangent & subtangent at any point is proportional to coordinates of point of tangency.

Sol. Let point of tangency be (x_1, y_1)

$$m = \left.\frac{dy}{dx}\right|_{x=x_1} = \frac{2ax_1}{x_1^2 - a^2}$$

Length of tangent + subtangent = $|y_1|\sqrt{1 + \frac{1}{m^2}} + \left|\frac{y_1}{m}\right|$

$$= |y_1|\sqrt{1 + \frac{(x_1^2 - a^2)^2}{4a^2x_1^2}} + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right| = |y_1|\frac{\sqrt{x_1^4 + a^4 + 2a^2x_1^2}}{2|ax_1|} + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right|$$

$$= \left|\frac{y_1(x_1^2 + a^2)}{2ax_1}\right| + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right| = \frac{|y_1|(2x_1^2)}{2|ax_1|} = \left|\frac{x_1\,y_1}{a}\right|$$

Ex. 15 If the relation between subnormal SN and subtangent ST at any point S on the curve $by^2 = (x + a)^3$ is $p(SN) = q(ST)^2$, then find value of $\frac{p}{q}$ in terms of b and a.

Sol. $by^2 = (x + a)^3$

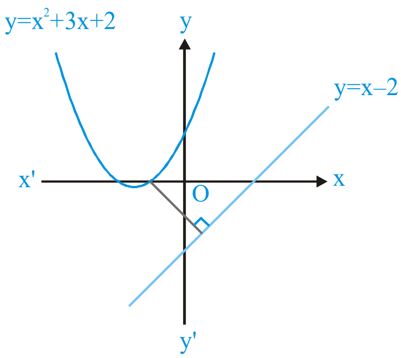
$$b\,.\,2y\frac{dy}{dx} = 3(x+a)^2 \Rightarrow \frac{dy}{dx} = \frac{3(x+a)^2}{2by}$$

Given that $p(SN) = q(ST)^2$

$$\Rightarrow \quad py\frac{dy}{dx} = q\frac{y^2}{\left(\frac{dy}{dx}\right)^2} \quad \Rightarrow \frac{p}{q} = \frac{y}{\left(\frac{dy}{dx}\right)^3} = \frac{y\,8b^3y^3}{27(x+a)^6} = \frac{8}{27}\frac{b^3(x+a)^6}{b^2(x+a)^6} = \frac{8}{27}b$$

Ex. 16 Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2+3x+2$

Sol. Let $P(x_1, y_2)$ be a point closest to the line $y = x - 2$. Then



$$\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \text{slope of line}$$

or $\quad 2x_1 + 3 = 1$

or $\quad x_1 = -1$

or $\quad y_1 = 0$

Hence, point (–1, 0) is the closest and its perpendicular distance from the line $y = x - 2$ will give the shortest distance. Therefore,

$$\text{Shortest distance} = \frac{3}{\sqrt{2}}$$

Ex. 17 Find the angle of intersection of curves, $y = [|\sin x|+|\cos x|]$ and $x^2 + y^2 = 5$ where [.] denotes greatest integral function.

Sol. We know that, $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

$\therefore \quad y = [|\sin x|+|\cos x|] = 1$

Let P and Q be the points of intersection of given curves.

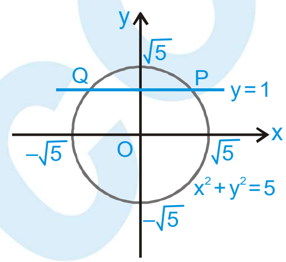
Clearly the given curves meet at points where y = 1 so, we get

$x^2 + 1 = 5$

$x = \pm 2$

Now, P(2, 1) and Q(–2, 1)

Now, $x^2 + y^2 = 5$



Differentiating the above equation w.r.t. x,

we get $2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

$$\left(\frac{dy}{dx}\right)_{(2,\,1)} = -2$$

and $$\left(\frac{dy}{dx}\right)_{(-2,\,1)} = 2$$

Clearly the slope of line y = 1 is zero and the slope of the tangents at P and Q are (–2) and (2) respectively.

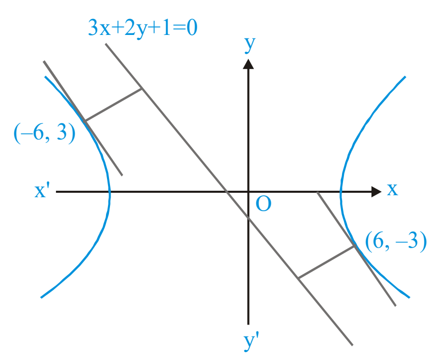
Thus, the angle of intersection is $\tan^{-1}(2)$

Ex. 18 Find the point on the curve $3x^2 - 4y^2 = 72$ which is nearest to the line $3x + 2y + 1 = 0$.

Sol. Slope of the given line $3x + 2y + 1 = 0$ is (–3/2)

Let us located the point on the curve at which the tangent is parallel to given line.

Differentiating the curve on both sides w.r.t. to x,



we get $6x - 8y\dfrac{dy}{dx} = 0$

or $$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{3x_1}{4y_1} = -\frac{3}{2} \quad \text{[since parallel to } 3x + 2y + 1 = 0\text{]}$$

or $$\frac{x_1}{y_1} = -2 \qquad \text{....(i)}$$

Also the point (x_1, y_1) lies on $3x^2 - 4y^2 = 72$. Therefore,

$3x_1^{\,2} - 4y_1^{\,2} = 72$

or $$3\frac{x_1^2}{y_1^2} - 4 = \frac{72}{y_1^2} \qquad \text{....(ii)}$$

or $\quad 3(4) - 4 = \dfrac{72}{y_1^2}$ $\qquad$ [from (i)]

or $\quad y_1^{\,2} = 9$ or $y_1 = \pm 3$

The required points are (–6, 3) and (6, –3)

Distance (–6, 3) from the given line

$$= \frac{|-18+6+1|}{\sqrt{13}} = \frac{11}{\sqrt{13}}$$

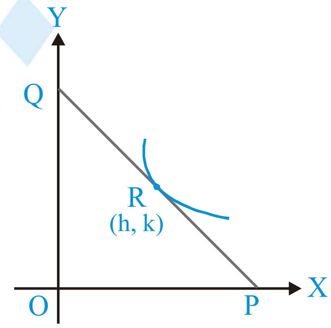
and distance of (6, –3) from the given line

$$= \frac{|18-6+1|}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13}$$

Thus, $\quad$ (–6, 3) is the required point.

Ex. 19 The tangent at any point on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the axes in P and Q. Proove that the locus of the midpoint of PQ is circle.

Sol. The given curve is $x = a\cos^3\theta$, $y = a\sin^3\theta$. Then



$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta(-\sin\theta)} = -\tan\theta$$

Therefore, equation of tange1nt at θ is

$$y - a\sin^3\theta = -\tan\theta\,(x - a\cos^3\theta)$$

or $\quad \dfrac{y}{\sin\theta} - a\sin^2\theta = -\dfrac{x}{\cos\theta} + a\cos^2\theta$

or $\quad \dfrac{x}{\cos\theta} + \dfrac{y}{\sin\theta} = a \quad$ or $\quad \dfrac{x}{(a\cos\theta)} + \dfrac{y}{(a\sin\theta)} = 1$

$\therefore \quad P \equiv (a\cos\theta, 0)$ and $Q \equiv (0, a\sin\theta)$

If midpoint of PQ is R(h, k), then

$$2h = a\cos\theta \quad \text{and} \quad 2k = a\sin\theta$$

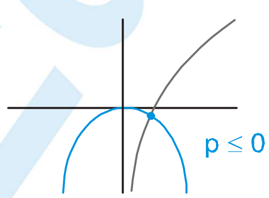
or $\quad (2h)^2 + (2k)^2 = a^2$ or $h^2 + k^2 = a^2/4$

Hence, the locus of midpoint is $x^2 + y^2 = a^2/4$, which is a circle.

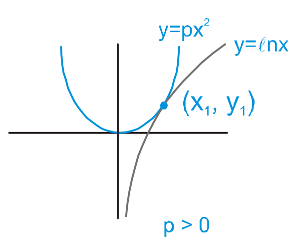
Ex. 20 Find possible values of p such that the equation $px^2 =$ ●nx has exactly one solution.

Sol. Two curves must intersect at only one point.

(i)



(ii)



I. If $p \le 0$ then there exists only one solution (see graph - **(i)**)

II. If $p > 0$

then the two curves must only touch each other

i.e. tangent at $y = px^2$ and $y =$ ●nx must have same slope at point (x_1, y_1)

$\Rightarrow \quad 2px_1 = \dfrac{1}{x_1}$

$\Rightarrow \quad x_1^{\,2} = \dfrac{1}{2p}$ ....**(i)**

also $\quad y_1 = px_1^{\,2} \quad \Rightarrow \quad y_1 = p\left(\dfrac{1}{2p}\right)$

$\Rightarrow \quad y_1 = \dfrac{1}{2}$ ....**(ii)**

and $\quad y_1 =$ ●nx_1 $\quad \Rightarrow \quad \dfrac{1}{2} =$ ●nx_1

$\Rightarrow \quad x_1 = e^{1/2}$ ....**(iii)**

✈ $\quad x_1^{\,2} = \dfrac{1}{2p} \quad \Rightarrow \quad e = \dfrac{1}{2p}$

$\Rightarrow \quad p = \dfrac{1}{2e}$

Hence possible values of p are $(-\infty, 0] \cup \left\{\dfrac{1}{2e}\right\}$

Ex. 21 Find the complete set of values of λ, for which the function $f(x) = \begin{cases} x+1, & x<1 \\ \lambda, & x=1 \\ x^2 - x + 3, & x>1 \end{cases}$ is strictly increasing at $x = 1$.

Sol. Let $g(x) = x + 1$, where $x < 1$. Then $g(x)$ is strictly increasing.

Let $h(x) = x^2 - x + 3$, where $x > 1$. $h(x)$ is also strictly increasing since $h'(x) = 2x - 1 > 0 \ \forall \ x > 1$.

Since f(x) is an increasing function,

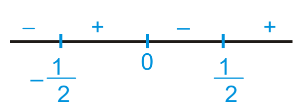
$\lim\limits_{x\to 1^-}(x+1) \le \lambda \le \lim\limits_{x\to 1^+}(x^2 - x + 3)$ or $2 \le \lambda \le 3$.

Ex. 22 Find the intervals of monotonicity of the function $y = x^2 - \log_e|x|$, $(x \ne 0)$.

Sol. Let $y = f(x) = x^2 - \log_e|x|$

$$f(x) = \begin{cases} x^2 - \log_e(-x), & x<0 \\ x^2 - \log_e(x), & x>0 \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} 2x - \dfrac{1}{(-x)}(-1), & x<0 \\ 2x - \dfrac{1}{x}, & x>0 \end{cases}$$

$\therefore f'(x) = 2x - \dfrac{1}{x}$; for all $x(x \ne 0)$



$$f'(x) = \frac{2x^2 - 1}{x} \quad \Rightarrow \quad f'(x) = \frac{(\sqrt{2}x - 1)(\sqrt{2}x + 1)}{x}$$

So $f'(x) > 0$ when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and $f'(x) < 0$ when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$

$\therefore$ f(x) is increasing when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

and decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$

Ex. 23 Find the intervals of monotonicity of the following functions.

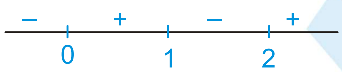
(i) $f(x) = x^2 (x-2)^2$ **(ii)** $f(x) = x \ell n x$

(iii) $f(x) = \sin x + \cos x$; $x \in [0, 2\pi]$

Sol. **(i)** $f(x) = x^2 (x-2)^2$

$f'(x) = 4x (x-1)(x-2)$

observing the sign change of $f'(x)$



Hence increasing in $[0, 1]$ and in $[2, \infty)$

and decreasing for $x \in (-\infty, 0]$ and $[1, 2]$

(ii) $f(x) = x \ell n x$

$f'(x) = 1 + \ell n x$

$f'(x) \geq 0 \quad \Rightarrow \quad \ell n x \geq -1 \quad \Rightarrow \quad x \geq \frac{1}{e}$

$\Rightarrow$ increasing for $x \in \left[\frac{1}{e}, \infty\right)$ and decreasing for $x \in \left(0, \frac{1}{e}\right]$.

(iii) $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

for increasing $f'(x) \geq 0 \quad \Rightarrow \quad \cos x \geq \sin x$

$\Rightarrow$ f is increasing in $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$

f is decreasing in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Ex. 24 $2\sin x + \tan x \geq 3x \quad (0 \leq x < \frac{\pi}{2})$

Sol. Consider , $f(x) = 2\sin x + \tan x - 3x$

$f'(x) = 2\cos x + \sec^2 x - 3$

$= \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x} = \frac{(\cos x - 1)^2 (2\cos x + 1)}{\cos^2 x}$

Hence $f'(0) > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right) \quad \Rightarrow \quad f(x)$ is $\uparrow$ in $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad f(x) > f(0) \ \forall x$ in $\left(0, \frac{\pi}{2}\right)$; but $f(0) = 0$

$\therefore \quad f(x) > 0$; as equaly holds

Hence $f(x) > 0$ in $\left[0, \frac{\pi}{2}\right) \Rightarrow$ the result

Ex. 25 Prove the following

(i) $y = e^x + \sin x$ is increasing in $x \in R^+$

(ii) $y = 2x - \sin x - \tan x$ is decreasing in $x \in (0, \pi/2)$

Sol. **(i)** $f(x) = e^x + \sin x,\ x \in R^+ \Rightarrow f'(x) = e^x + \cos x$

Clearly $f'(x) > 0 \ \forall \ x \in R^+$ (as $e^x > 1, x \in R^+$ and $-1 \leq \cos x \leq 1, x \in R^+$)

Hence f(x) is increasing.

(ii) $f(x) = 2x - \sin x - \tan x \qquad x \in (0, \pi/2)$

$\Rightarrow \quad f'(x) = 2 - \cos x - \sec^2 x$

$\Rightarrow \quad f'(x) = \cos^2 x - \cos x - (\cos^2 x + \sec^2 x - 2)$

$= \cos^2 x - \cos x - (\cos x - \sec x)^2$

$\therefore \quad f'(x) < 0, x \in (0, \pi/2) \qquad ✈ \qquad \cos^2 x < \cos x, x \in (0, \pi/2)$

Hence f(x) is decreasing in $(0, \pi/2)$

Ex. 26 Let $f : R \to R$ be a twice differentiable function such that $f(2) = 8$, $f(4) > 64$, $f(7) = 343$ then show that there exists a c $\in (2, 7)$ such that $f''(c) < 6c$.

Sol. Consider $g(x) = f(x) - x^3$

By LMVT

$\frac{g(4) - g(2)}{4 - 2} = g'(c_1),\ 2 < c_1 < 4$ and $\frac{g(7) - g(4)}{7 - 4} = g'(c_2), \quad 4 < c_2 < 7$

$g'(c_1) > 0,\ g'(c_2) < 0$

By LMVT

$\frac{g'(c_2) - g'(c_1)}{c_2 - c_1} = g''(c), \qquad c_1 < c < c_2$

$\Rightarrow \quad g''(c) < 0 \quad \Rightarrow \quad f''(c) - 6c < 0 \qquad$ for same $c \in (c_1, c_2)\ c\ (2, 7)$

Ex. 27 $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ for $0 < x < \dfrac{\pi}{2}$.

Sol. We have to show that

$$\frac{\tan x \sin x - x^2}{x \sin x} > 0 \text{ for } 0 < x < \pi/2$$

Since $x \sin x > 0$ for $0 < x < \pi/2$, it is enough to show that

$$\tan x \cdot \sin x - x^2 > 0,\ 0 < x < \pi/2$$

Let $f(x) = \tan x \sin x - x^2$ for $0 < x < \pi/2$

$$f'(x) = \sin x \sec^2 x + \tan x \cos x - 2x = \sin x \sec^2 x + \sin x - 2x$$

$$f''(x) = \cos x \sec^2 x + \sin x \,.\, 2\sec^2 x \tan x + \cos x - 2$$

$$= \sec x + \cos x - 2 + 2 \sin x \tan x \sec^2 x$$

$$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2 \sin x \tan x \sec^2 x > 0 \text{ for } 0 < x < \pi/2$$

$\therefore$ f' is strictly increasing in $[0, \pi/2)$. Also $f'(0) = 0$

$\Rightarrow$ $f'(x) > 0$ for $0 < x < \pi/2$

$\Rightarrow$ f is strictly increasing in $[0, \pi/2)$. Also $f(0) = 0$

$\Rightarrow$ $f(x) > 0$ for $0 < x < \pi/2$.

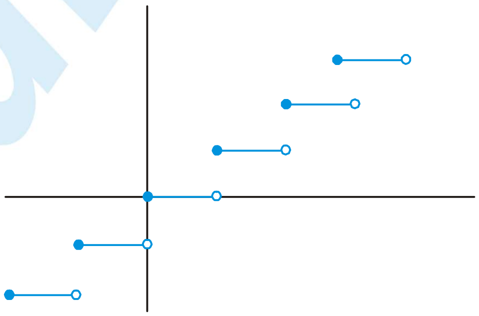
$\Rightarrow$ $\tan x \sin x - x^2 > 0$ for $0 < x < \pi/2$.

$\therefore$ $\dfrac{\tan x \sin x - x^2}{x \sin x} > 0,\ 0 < x < \pi/2$.

$\Rightarrow$ $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ for $0 < x < \dfrac{\pi}{2}$. Henced proved.

Ex. 28 f(x) = [x] is a step up function. Is it a strictly increasing function for $x \in R$.

Sol. No, f(x) = [x] is increasing (monotonically increasing) (non-decreasing), but not strictly increasing function as illustrated by its graph.

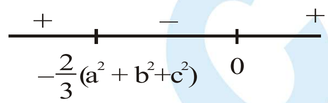


Ex. 29 If a, b, c are real, then $f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$ is decreasing in

Sol. $f'(x)=\begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$

$=\left(x+b^2\right)\left(x+c^2\right)-b^2c^2+\left(x+a^2\right)\left(x+c^2\right)-a^2c^2+\left(x+a^2\right)\left(x+b^2\right)-a^2b^2$

$=3x^2+2x\left(a^2+b^2+c^2\right)$



f(x) will be decreasing when $f'(x)<0$

$\Rightarrow \quad 3x^2+2x\left(a^2+b^2+c^2\right)<0 \quad \Rightarrow \quad x\in\left(-\frac{2}{3}\left(a^2+b^2+c^2\right),0\right)$

Ex. 30 Establish the inequality given below by examining the sign of the derivative of an appropriate function:

$\frac{1}{x+(1/2)}<\ln\left(1+\frac{1}{x}\right)<\frac{1}{x} \qquad$ for $x>0$

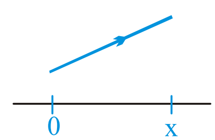
Sol. consider $f(x)=\frac{2}{2x+1}-\ln\left(1+\frac{1}{x}\right)$; $f'(x)=\frac{-4}{(2x+1)^2}-\frac{1}{1+(1/x)}\cdot\left(-\frac{1}{x^2}\right)$

$=\frac{1}{x(x+1)}-\frac{4}{(2x+1)^2}=\frac{1}{x(x+1)(2x+1)^2}$

which is always +ve for $x>0$

hence $f(x)$ is ↑ for $x>0$

i.e. $f(x)<\lim_{x\to\infty} f(x)$ **(note carefully)**



but $\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left\{\frac{2}{2x+1}-\ln\left(1+\frac{1}{x}\right)\right\}=0$

so $f(x)<0$ i.e.

$\frac{2}{2x+1}<\ln\left(1+\frac{1}{x}\right)$ Proved

Similarly consider $g(x)=\ln\left(1+\frac{1}{x}\right)-\frac{1}{x}$; $g'(x)=\frac{1}{x^2}-\frac{1}{x(x+1)}=\frac{1}{x^2(x+1)}$

$g'(x)$ is ↑ i.e.

so $g(x)<\lim_{x\to\infty} g(x)$ but $\lim_{x\to\infty} f(x)=0$

so $g(x)<0$ hence proved

Ex. 31 If $f(x) = \sin^4x + \cos^4x + bx + c$, then find possible values of b and c such that f(x) is monotonic for all $x \in R$

Sol. $f(x) = \sin^4x + \cos^4x + bx + c$

$f'(x) = 4\sin^3x\cos x - 4\cos^3x\sin x + b = -\sin 4x + b.$

Case - (i) : for M.I. $f'(x) \geq 0$ for all $x \in R$

$\Leftrightarrow$ $b \geq \sin 4x$ for all $x \in R$ $\Leftrightarrow$ $b \geq 1$

Case - (ii) : for M.D. $f'(x) \leq 0$ for all $x \in R$

$\Leftrightarrow$ $b \leq \sin 4x$ for all $x \in R$ $\Leftrightarrow$ $b \leq -1$

Hence for f(x) to be monotonic $b \in (-\infty, -1] \cup [1, \infty)$ and $c \in R$.

Ex.32 In any ΔABC prove that $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.

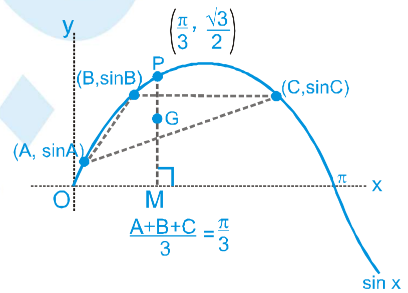
Sol. Co-ordinates of centroid G are $\left(\frac{A+B+C}{3}, \frac{\sin A + \sin B + \sin C}{3}\right)$

from figure we have

$PM \geq GM$

$$\sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A + \sin B + \sin C}{3}$$

$$\sin\left(\frac{\pi}{3}\right) \geq \frac{\sin A + \sin B + \sin C}{3}$$



$$\Rightarrow \quad \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

equality holds when triangle is equilateral.

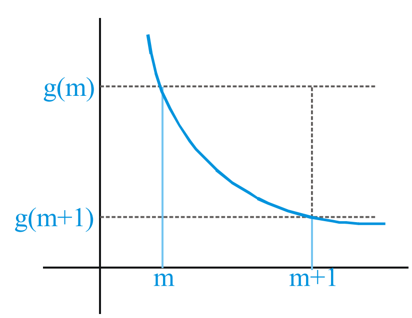
Ex. 33 If $f(x) = \ln(\ln x)$ where $x > e$

Prove that $\frac{1}{(m+1)\ln(m+1)} \leq f(m+1) - f(m) \leq \frac{1}{m \cdot \ln(m)}$ for $m > e$.

Sol. $f(x) = \ln(\ln x)$

$$g(x) = f'(x) = \frac{1}{x \ln x}$$

$$g'(x) = \frac{1}{-(x\ln x)^2}[\ln x + 1] = -\left[\frac{1}{x^2\ln^2 x} + \frac{1}{x^2 \ln x}\right]$$



which is < 0 for $x > e$

Hence g(x) is a decreasing function for $x > e$

hence $g(m) > g(m+1)$ for $m > e$

From the graph

$$g(m+1)(m+1-m) < \int_m^{m+1} g(x)\,dx < g(m)[m+1-m]$$

$$g(m+1) \leq \int_m^{m+1} f'(x)\,dx \leq g(m) \Rightarrow \frac{1}{(m+1)\ln(m+1)} \leq f(m+1) - f(m) \leq \frac{1}{m \cdot \ln(m)}$$

Ex. 34 Find possible values of 'a' such that $f(x) = e^{2x} - (a+1)\,e^x + 2x$ is monotonically increasing for $x \in R$

Sol. $f(x) = e^{2x} - (a+1)\,e^x + 2x$

$$f'(x) = 2e^{2x} - (a+1)\,e^x + 2$$

Now, $2e^{2x} - (a+1)\,e^x + 2 \ge 0 \qquad$ for all $\quad x \in R$

$\Rightarrow \quad 2\left(e^x + \frac{1}{e^x}\right) - (a+1) \ge 0 \qquad$ for all $\quad x \in R$

$(a+1) \le 2\left(e^x + \frac{1}{e^x}\right) \qquad$ for all $\quad x \in R$

$\Rightarrow \quad a + 1 \le 4 \left(\because \; e^x + \frac{1}{e^x} \text{ has minimum value } 2\right)$

$\Rightarrow \quad a \le 3$

Ex. 35 Show that between any two roots of $e^{-x} - \cos x = 0$ there exists at least one root of $\sin x - e^{-x} = 0$.

Sol. If $x = a$ and $x = b$ are two distinct roots of $e^{-x} - \cos x = 0$

then $\quad e^{-a} - \cos a = 0 \quad$ and $\quad e^{-b} - \cos b = 0 \qquad$ .... **(1)**

and $\quad$ let $f(x) = e^{-x} - \cos x$

We observe that

(i) e^{-x} and $\cos x$ are continuous as well as differentiable in [a, b] then f(x) is also continuous in [a, b] & differentiable in (a,b).

(ii) $\left.\begin{aligned} f(a) &= e^{-a} - \cos a = 0 \\ \text{and } f(b) &= e^{-b} - \cos b = 0 \end{aligned}\right\} \qquad$ {from **(1)**}

i.e. $f(a) = f(b) = 0$

Thus f satisfies all the three conditions of Rolle's theorem in [a, b]. Hence there is at least one value of x in (a, b), say c such that $f'(c) = 0$.

Now $\quad f'(c) = 0$

$\Rightarrow \quad -e^{-c} + \sin c = 0 \quad \Rightarrow \quad \sin c - e^{-c} = 0$

$\Rightarrow \quad$ c is a root of the equation $\sin x - e^{-x} = 0$.

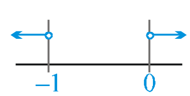
Hence between any two roots of the equation $e^{-x} - \cos x = 0$,

there is at least one root of the equation $\sin x - e^{-x} = 0$.

Ex. 36 Prove that $f(x) = \left(1 + \frac{1}{x}\right)^x$ is always an increasing function for all x in its domain. State its range and also plot the graph of the function.

Sol. $f(x) = y = \left(1 + \frac{1}{x}\right)^x = e^{x \ln\left(1 + \frac{1}{x}\right)}$

Hence for domain $\frac{x+1}{x} > 0$, $\quad \Rightarrow \quad x \in (-\infty, -1) \cup (0, \infty)$



now $\lim\limits_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e \quad ; \qquad$ Also $\quad \ln y = x \ln\left(1 + \frac{1}{x}\right)$

$\therefore \quad \lim_{x\to 0^+} \dfrac{\ln\left(1+\dfrac{1}{x}\right)}{\dfrac{1}{x}} \left(\dfrac{\infty}{\infty}\right) = \dfrac{x}{x+1}\left(\dfrac{-\dfrac{1}{x^2}}{-\dfrac{1}{x^2}}\right) = 0 \Rightarrow y = e^0 = 1 \; ; \; \therefore \lim_{x\to 0^+} f(x) = 1$

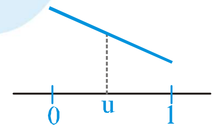
$\lim_{x\to -1^-} f(x) = \infty$ (obviously)

Now $\quad f'(x) = \left(1+\dfrac{1}{x}\right)^x \left[x\left(\dfrac{1}{1+x} - \dfrac{1}{x}\right) + \ln\dfrac{1+x}{x}\right] = \left(1+\dfrac{1}{x}\right)^x \left[\dfrac{x}{1+x} - 1 + \ln\dfrac{1+x}{x}\right]$

Let $\quad \dfrac{x}{1+x} = u \qquad$ [for $x > 0,\ u \in (0, 1)$]

consider a function

$g(u) = u - 1 - \ln u$



$g'(u) = 1 - \dfrac{1}{u} < 0$ which is negative for $u \in (0, 1)$

Hence $g(u)$ is decreasing in $(0, 1]$

$\therefore \quad g(u) > g(1) \quad \forall\ u \in (0, 1]$

But $\; g(1) = 0$

$\therefore \quad g(u) > 0\ \forall\ u \in (0, 1)$

$\therefore \quad \dfrac{x}{1+x} - 1 + \ln\dfrac{1+x}{x} > 0 \quad \forall\ x > 0$

$\therefore \quad f'(x) > 0$

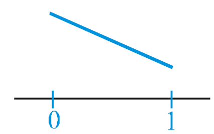
$\therefore \quad f(x)$ is increasing for $x > 0$ **....(1)**

again for $x < -1$

$f'(x) = \left(1+\dfrac{1}{x}\right)^x \left[\ln\dfrac{1+x}{x} + \dfrac{x}{1+x} - 1\right] \qquad$ put $\dfrac{1+x}{x} = u$

consider $g(u) = \ln u + \dfrac{1}{u} - 1 \qquad$ as $x \in (-\infty, -1), \quad u \in (0, 1)$

$g'(u) = \dfrac{1}{u} - \dfrac{1}{u^2} = \dfrac{u-1}{u^2} < 0 \quad$ for $u \in (0, 1)$



$g(u)$ is decreasing in $(0, 1)$

$\therefore \quad g(u) > g(1)$

$g(u) > 0$

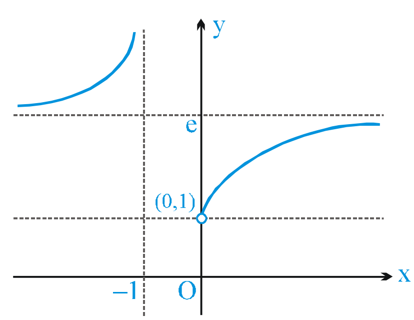
$\ln\dfrac{1+x}{x} + \dfrac{x}{1+x} - 1 > 0$

$\therefore \quad f'(x) > 0$ for $x \in (-\infty, -1)$

$\therefore \quad f(x)$ is increasing in $(-\infty, -1)$ **....(2)**

from **(1)** and **(2)**

$f(x)$ is increasing for all x in the domain of $f(x)$.



Graph of $f(x)$

Ex. 37 For $x \in (0, 1)$ prove that $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$ hence or otherwise find $\lim_{x\to 0}\left[\frac{\tan^{-1} x}{x}\right]$

Sol. Let $f(x) = x - \frac{x^3}{3} - \tan^{-1}x$

$f(x) = 1 - x^2 - \frac{1}{1+x^2}$

$f'(x) = -\frac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1)$ $\Rightarrow$ f(x) is M.D.

$\Rightarrow$ $f(x) < f(0)$ $\Rightarrow$ $x - \frac{x^3}{3} - \tan^{-1}x < 0$

$\Rightarrow$ $x - \frac{x^3}{3} < \tan^{-1}x$ ....(i)

Similarly $g(x) = x - \frac{x^3}{6} - \tan^{-1}x$

$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2}$

$g'(x) = \frac{x^2(1-x^2)}{2(1+x^2)}$

$g'(x) > 0$ for $x \in (0, 1)$ $\Rightarrow$ g(x) is M.I.

$\Rightarrow$ $g(x) > g(0)$

$x - \frac{x^3}{6} - \tan^{-1}x > 0$

$x - \frac{x^3}{6} > \tan^{-1}x$ ....(ii)

from (i) and (ii), we get

$x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6}$ Hence Proved

Also, $1 - \frac{x^2}{3} < \frac{\tan^{-1} x}{x} < 1 - \frac{x^2}{6}$, for $x > 0$

Hence by sandwich theorem we can prove that $\lim_{x\to 0}\frac{\tan^{-1} x}{x} = 1$ but it must also be noted that as $x \to 0$, value of $\frac{\tan^{-1} x}{x} \longrightarrow 1$ from left hand side i.e. $\frac{\tan^{-1} x}{x} < 1$

Ex. 38 If $g(x) = f(x) + f(1 - x)$ and $f''(x) < 0$; $0 \le x \le 1$, show that g(x) increasing in $x \in [0, 1/2]$ and decreasing in $x \in [1/2, 1]$

Sol. $f''(x) < 0$ $\Rightarrow$ f'(x) is decreasing function.

Now, $g(x) = f(x) + f(1 - x)$ $\therefore$ $g'(x) = f'(x) - f'(1 - x)$ .... (i)

Case I : If $x \ge (1 - x) \Rightarrow x \ge 1/2$

$\therefore$ $f'(x) \le f'(1 - x)$ $\Rightarrow$ $f'(x) - f'(1 - x) \le 0$

⇒ $g'(x) \le 0$

∴ g(x) decreases in $x \in \left[\frac{1}{2}, 1\right]$

Case II : If $x \le (1 - x) \Rightarrow x \le 1/2$

∴ $f'(x) \ge f'(1 - x)$

⇒ $f'(x) - f'(1 - x) \ge 0 \Rightarrow g'(x) \ge 0$

∴ g(x) increases in $x \in [0, 1/2]$

Ex. 39 Let f (x) and g (x) be differentiable functions such that $f'(x)\,g(x) \ne f(x)\,g'(x)$ for any real x. Show that between any two real solutions of f (x) = 0, there is at least one real solution of g (x) = 0

Sol. Let a, b be the solutions of f(x)=0

suppose g(x) is not equal to zero for any x belonging to [a,b] now consider h(x)= $f(x)/g(x)$ since g(x) not equal to zero

h(x) is differentiable and continuous in (a,b)

h(a) = h(b) = 0 (as f (a) = 0 and f (b) = 0 but g(a) or g (b) ≠ 0) applying rolles theorem

h ' (c)=0 for some c belonging to (a,b)

$f(x)g'(x)=f'(x)g(x)$

this gives the contadiction

hence proved

Ex. 40 Examine which is greater : sinx tanx or x^2. Hence evaluate $\lim\limits_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \frac{\pi}{2}\right)$

Sol. Let $f(x) = \sin x \tan x - x^2$

$f'(x) = \cos x \,.\, \tan x + \sin x \,.\, \sec^2 x - 2x$

⇒ $f'(x) = \sin x + \sin x \sec^2 x - 2x$

⇒ $f''(x) = \cos x + \cos x \sec^2 x + 2\sec^2 x \sin x \tan x - 2$

⇒ $f''(x) = (\cos x + \sec x - 2) + 2 \sec^2 x \sin x \tan x$

Now $\cos x + \sec x - 2 = \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2$ and $2 \sec^2 x \tan x \,.\, \sin x > 0$ becausex $\in \left(0, \frac{\pi}{2}\right)$

⇒ $f''(x) > 0 \Rightarrow f'(x)$ is M.I.

Hence $f'(x) > f'(0)$

⇒ $f'(x) > 0 \Rightarrow f(x)$ is M.I.

⇒ $f(x) > 0 \Rightarrow \sin x \tan x - x^2 > 0$

Hence $\sin x \tan x > x^2$

⇒ $\frac{\sin x \tan x}{x^2} > 1 \Rightarrow \lim\limits_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right] = 1$

⇒ $\lim\limits_{x\to 0}\left[\frac{\tan^{-1} x}{x}\right] = 0$

Ex. 41 If functions f(x) and g(x) are continuous in [a, b] and differentiable in (a, b), show that there will be at least one point c, a < c < b such that $\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$

Sol. Let $F(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} = f(a)\,g(x) - g(a)\,f(x)$ ......**(i)**

$\Rightarrow$ $F'(x) = f(a)\,g'(x) - g(a)\,f'(x)$ ......**(ii)**

Since f(x) and g(x) are continuous in [a, b] and differentiable in (a, b), therefore, from **(i)** and **(ii)** it follows that F(x) is continuous in [a, b] and differentiable in (a, b).

Also from **(i)**, $F(a) = f(a)g(a) - g(a)f(a) = 0$

And $F(b) = f(a)\,g(b) - g(a)\,f(b)$

Now by mean value theorem for F(x) in [a, b], there will be at least one point c, a < c < b

such that $F'(c) = \dfrac{F(b) - F(a)}{b-a}$

$\Rightarrow$ $f(a)\,g'(c) - g(a)\,f'(c) = \dfrac{f(a)g(b) - g(a)f(b) - 0}{b-a}$

or $f(a)g(b) - g(a)f(b) = (b-a)\{f(a)g'(c) - g(a)f'(c)\}$

or $\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$

Ex. 42 Let P (x) be a polynomial with real coefficients. Let a, b ∈ R, a < b, be two consecutive roots of P(x). Show that there exists 'c' such that $a \le c \le b$ and $P'(c) + 100\,P(c) = 0$.

Sol. Consider $f(x) = e^{100x} \cdot P(x)$.

Now $f(a) = f(b) = 0$ {as $P(a) = P(b) = 0$ }

Also as P (x) is polynomial

$\Rightarrow$ f (x) is continuous and differentiable in [a, b]

$\Rightarrow$ Rolle's theorem can be applied

$\Rightarrow$ $\exists\ c \in (a, b)$ such that $f'(c) = 0$

now $f'(x) = e^{100x}(P'(x) + 100 \cdot P(x))$

$\Rightarrow$ $e^{100c}(P'(c) + 100 \cdot P(c)) = 0$, from (1)

$\Rightarrow$ $P'(c) + 100 \cdot P(c) = 0$ (as $[e^{100c} \ne 0]$) hence proved.

Ex. 43 Prove that $f(x) = \left(1 + \dfrac{1}{x}\right)^x$ is monotonically increasing in its domain. Hence or otherwise draw graph of f(x) and find its range

Sol. $f(x) = \left(1 + \dfrac{1}{x}\right)^x$, for Domain of f(x), $1 + \dfrac{1}{x} > 0$

$\Rightarrow$ $\dfrac{x+1}{x} > 0$ $\Rightarrow$ $(-\infty, -1) \cup (0, \infty)$

Consider $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ln\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}} \frac{-1}{x^2}\right]$

$\Rightarrow \quad f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$

Now $\left(1+\frac{1}{x}\right)^x$ is always positive, hence the sign of $f'(x)$ depends on sign of $\ln\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

i.e. we have to compare $\ln\left(1+\frac{1}{x}\right)$ and $\frac{1}{1+x}$

So lets assume $g(x) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$

$$g'(x) = \frac{1}{1+\frac{1}{x}} \frac{-1}{x^2} + \frac{1}{(x+1)^2} \quad \Rightarrow \quad g'(x) = \frac{-1}{x(x+1)^2}$$

(i) for $x \in (0, \infty)$, $g'(x) < 0 \Rightarrow$ $g(x)$ is M.D. for $x \in (0, \infty)$

$g(x) > \lim\limits_{x\to\infty} g(x)$

$g(x) > 0.$

and since $g(x) > 0 \quad \Rightarrow \quad f'(x) > 0$

(ii) for $x \in (-\infty, -1)$, $g'(x) > 0 \quad \Rightarrow \quad g(x)$ is M.I. for $x \in (-\infty, -1)$

$\Rightarrow \quad g(x) > \lim\limits_{x\to-\infty} g(x)$

$\Rightarrow \quad g(x) > 0 \quad \Rightarrow \quad f'(x) > 0$

Hence from **(i)** and **(ii)** we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$

$\Rightarrow$ f(x) is M.I. in its Domain

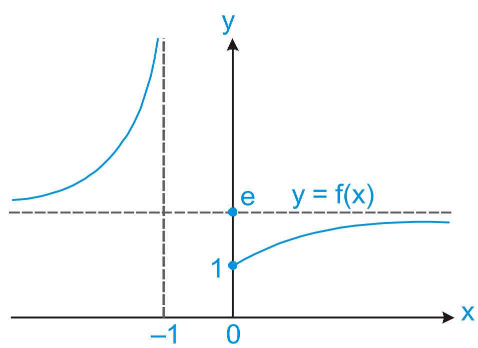
For drawing the graph of f(x), its important to find the value of f(x) at boundary points

i.e. $\pm\infty, 0, -1$

$$\lim_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to 0^+} \left(1+\frac{1}{x}\right)^x = 1 \quad \text{and} \quad \lim_{x\to -1} \left(1+\frac{1}{x}\right)^x = \infty$$

so the graph of f(x) is



Range is $y \in (1, \infty) - \{e\}$

Ex. 44 Prove that if $2a_0^2 < 15a$, all roots of $x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d = 0$ can't be real. It is given that $a_0, a, b, c, d \in R$.

Sol. Let $f(x) = x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d$

$f'(x) = 5x^4 - 4a_0x^3 + 9ax^2 + 2bx + c$

$f''(x) = 20x^3 - 12a_0x^2 + 18ax + 2b$

$f'''(x) = 60x^2 - 24a_0x + 18a = 6(10x^2 - 4a_0x + 3a)$

Now, discriminant $= 16a_0^2 - 4.\ 10.\ 3a = 8(2a_0^2 - 15a) < 0$

as $2a_0^2 - 15a < 0$ is given.

Hence the roots of $f'''(x) = 0$ can not be real.

$\therefore$ $f''(x)$ have one real root and $f'(x) = 0$ have at most two real roots so $f(x) = 0$ have at most three real roots.

Therefore all the roots of $f(x) = 0$ will not be real.

Ex. 45 Let a, b, c be three real number such that $a < b < c$, $f(x)$ is continuous in $[a, c]$ and differentiable in (a, c). Also $f'(x)$ is strictly increasing in (a, c). Prove that

$$(b - c)\, f(a) + (c - a)\, f(b) + (a - b)\, f(c) < 0$$

Sol. By LMVT in $[a, b]$ for f

$$\frac{f(b) - f(a)}{b - a} = f'(u) \quad \text{where } a < u < b$$

and applying LMVT in $[b, c]$

and $$\frac{f(c) - f(b)}{c - b} = f'(v) \quad \text{where } b < v < c$$

Since $f'(x)$ is strictly increasing

$\therefore$ $f'(u) < f'(v)$; $\therefore$ $\dfrac{f(b) - f(a)}{b - a} < \dfrac{f(c) - f(b)}{c - b}$

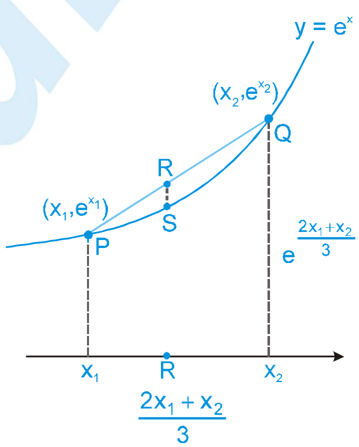
$f(b)\,(c - b + b - a) - f(a)\,(c - b) - f(c)\,(b - a) < 0$

$\therefore$ $(b - c)\, f(a) + (c - a)\, f(b) + (a - b)\, f(c) < 0$

Ex. 46 Prove that for any two numbers x_1 & x_2, $\dfrac{2e^{x_1} + e^{x_2}}{3} > e^{\frac{2x_1 + x_2}{3}}$

Sol. Assume $f(x) = e^x$ and let x_1 & x_2 be two points on the curve $y = Ex$.

Let R be another point which divides $\overline{PQ}$ in ratio 1 : 2.



y coordinate of point R is $\frac{2e^{x_1}+e^{x_2}}{3}$ and y coordinate of point S is $e^{\frac{2x_1+x_2}{3}}$. Since $f(x) = e^x$ is concave up, the point R will always be above the point S.

$$\Rightarrow \quad \frac{2e^{x_1}+e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$$

Ex. 47 Show by using mean value theorem that $\frac{\beta-\alpha}{1+\beta^2} < \tan^{-1}\beta - \tan^{-1}\alpha < \frac{\beta-\alpha}{1+\alpha^2}$ where $\beta > \alpha > 0$.

Sol. Take $f(x) = \tan^{-1}x$

$\Rightarrow \quad f'(x) = \frac{1}{1+x^2}$. By mean value theorem for f(x) in $[\alpha, \beta]$

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(c) = \frac{1}{1+c^2} \qquad \text{where } \alpha < c < \beta \qquad \text{........ (i)}$$

Now, $c > \alpha \quad \Rightarrow \quad \frac{1}{1+c^2} < \frac{1}{1+\alpha^2}$

as $\quad c < \beta \quad \Rightarrow \quad \frac{1}{1+c^2} > \frac{1}{1+\beta^2}$

$\therefore \quad$ from **(i)**, $\frac{1}{1+\beta^2} < \frac{f(\beta)-f(\alpha)}{\beta-\alpha} < \frac{1}{1+\alpha^2}$

or $\quad \frac{\beta-\alpha}{1+\beta^2} < f(\beta)-f(\alpha) < \frac{\beta-\alpha}{1+\alpha^2}$

Hence, $\frac{\beta-\alpha}{1+\beta^2} < \tan^{-1}\beta - \tan^{-1}\alpha < \frac{\beta-\alpha}{1+\alpha^2}$

Ex. 48 If the function $f : [0, 4] \rightarrow R$ is differentiable then show that

for $a, b \in (0, 4) \quad f^2(4) - f^2(0) = 8 f'(a) f(b)$

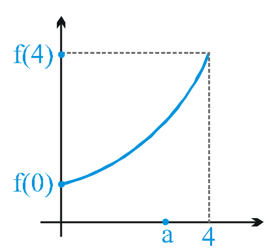
Sol. Since f is differentiable on [0, 4] so it is continuous also. Hence by LMVT,

$\exists$ some $a \in (0, 4)$ such that $f'(a) = \frac{f(4)-f(0)}{4-0}$ ....**(1)**

again, by intermediate value theorem for continuous functions,

$\exists\ b \in (0, 4)$ such that

$$f(b) = \frac{f(0)+f(4)}{2} \qquad \text{....(2)}$$



from **(1)** and **(2)**

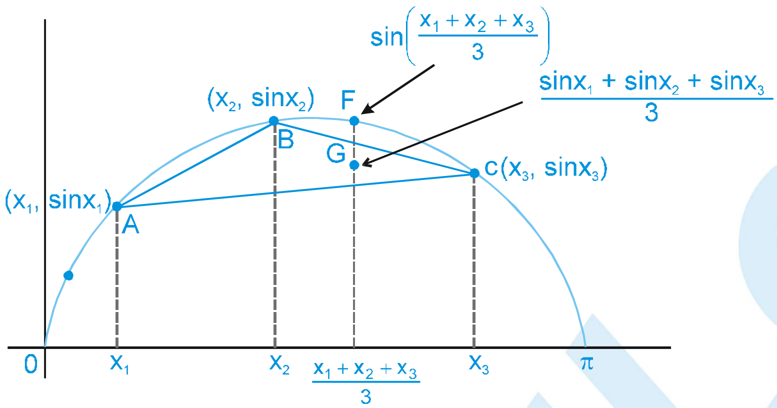
$$f'(a) \cdot f(b) = \frac{f^2(4)-f^2(0)}{8}$$

Ex.49 If $0 < x_1 < x_2 < x_3 < \pi$ then prove that $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right) > \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$.

Henceprove that : if A, B, C are angles of a triangle then maximum value of

$\sin A + \sin B + \sin C$ is $\frac{3\sqrt{3}}{2}$.

Sol.



Point A, B, C form a triangle.

y coordinate of centroid G is $\frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$ and y coordinate of point F is $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right)$.

Hence $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right) \geq \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$.

If $A + B + C = \pi$, then

$\sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A + \sin B + \sin C}{3}$ $\Rightarrow$ $\sin\frac{\pi}{3} \geq \frac{\sin A + \sin B + \sin C}{3}$

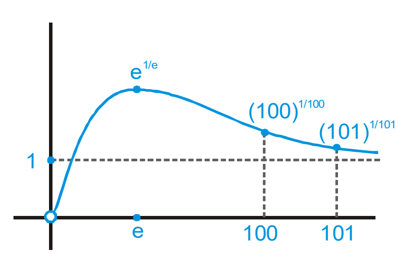
$\Rightarrow$ $\frac{3\sqrt{3}}{2} \geq \sin A + \sin B + \sin C$

$\Rightarrow$ maximum value of $(\sin A + \sin B + \sin C) = \frac{3\sqrt{3}}{2}$

Ex. 50 Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

Sol. Assume $f(x) = x^{1/x}$ and let us examine monotonic nature of f(x) , (x > 0)

$f'(x) = x^{1/x} . \left(\frac{1 - \ln x}{x^2}\right)$



$f'(x) > 0$ $\Rightarrow$ $x \in (0, e)$

and $f'(x) < 0$ $\Rightarrow$ $x \in (e, \infty)$

Hence f(x) is M.D. for $x \geq e$

and since 100 < 101

$\Rightarrow$ $f(100) > f(101)$

$\Rightarrow$ $(100)^{1/100} > (101)^{1/101}$

Ex. 51 If f and F are continuous in [a, b] and derivable in (a, b) with $F'(x) \neq 0 \ \forall \ x \in (a, b)$. Prove that

$\exists \ c \in (a, b)$ such that $\dfrac{f'(c)}{F'(c)} = \dfrac{f(b) - f(a)}{F(b) - F(a)}$.

Sol. Let $K_1 = f(b) - f(a)$ and $K_2 = F(b) - F(a)$ $\Rightarrow$ TPT $\dfrac{f'(c)}{F'(c)} = \dfrac{K_1}{K_2}$

Consider a function $\phi(x) = K_1 F(x) - K_2 f(x)$ ....(1)

$\therefore$ f(x) and F(x) are continuous in [a, b] and derivable in (a, b) hence $\phi(x)$ will also be continuous and differentiable

also $\phi(a) = K_1 F(a) - K_2 f(a)$ and $\phi(b) = K_1 F(b) - K_2 f(b)$

now $\phi(a) - \phi(b) = K_1(F(a) - F(b)) - K_2(f(a) - f(b))$

$= [f(b) - f(a)][F(a) - F(b)] - [F(b) - F(a)][f(a) - f(b)]$

$= [f(b) - f(a)]\{F(a) - F(b) + F(b) - F(a)] = 0$

$\Rightarrow$ $\phi(a) = \phi(b)$

Hence rolles theorem is applicable for $\phi(x)$

$\therefore$ $\exists$ some $c \in (a, b)$, such that, $\phi'(c) = 0$

$\varphi'(x)]_{x=c} = K_1 F'(x) - K_2 f'(x) = 0$ or $K_1 F'(c) = K_2 f'(c)$

$\therefore$ $\dfrac{f'(c)}{F'(c)} = \dfrac{K_1}{K_2} = \dfrac{f(b) - f(a)}{F(b) - F(a)}$

Ex. 52 Find the points of inflection of the function $f(x) = \sin^2 x$ $x \in [0, 2\pi]$

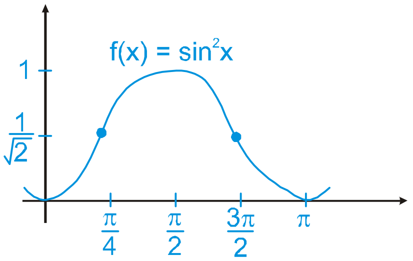
Sol. $f(x) = \sin^2 x$

$f'(x) = \sin 2x$

$f''(x) = 2\cos 2x$

$f''(0) = 0$ $\Rightarrow$ $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

both these points are inflection points as sign of $f''(x)$ change on either sides of these points.



Ex. 53 For $\forall$ x $\in$ [0, 1], let the second derivative f"(x) of a function f(x) exist and satisfy |f"(x)| ≤ 1. If f(0) = f(1) then show that |f'(x)| < 1 for all x in [0, 1].

Sol. Since f"(x) exists for all x in [0, 1]

$\therefore$ f(x) and f'(x) are differentiable as well as continuous for all x in [0, 1]

Now f(x) is continuous in [0, 1] and differentiable in (0, 1) and f(0) = f(1)

$\therefore$ By Rolle's theorem there is at least one c such that f'(c) = 0, where 0 < c < 1.

Case I : Let x = c then f'(x) = f'(c) = 0

$\therefore$ |f'(x)| = |0| = 0 < 1

Case II : Let x > c. By Largrange's mean value theorem for f'(x) in [c, x]

$$\frac{f'(x)-f'(c)}{x-c} = f''(\alpha) \text{ for at least one } \alpha,\ c < \alpha < x$$

or $f'(x) = (x-c)\, f''(\alpha)$ $\quad$ ($\because$ f'(c) = 0)

or $|f'(x)| = |x-c||f''(\alpha)|$

But $x \in [0, 1],\ c \in (0, 1)$

$\therefore$ $|x-c| < 1 - 0$ $\quad$ or $\quad$ $|x-c| < 1$

and given $|f''(x)| \le 1\ \forall\ x \in [0, 1]$

$\therefore$ $|f''(\alpha)| \le 1$

$\therefore$ $|f'(x)| < 1.1.$ $\quad$ ($\because$ $|f'(x)| = |x-c||f''(\alpha)|$

or $|f'(x)| < 1\ \forall\ x \in [0, 1]$

Case III : Let x < c then

$$\frac{f'(x)-f'(c)}{x-c} = f''(\alpha)$$

$\therefore$ $|-f'(x)| = |c-x||f''(\alpha)|$

$\Rightarrow$ $|f'(x)| < 1.1$ $\quad$ or $\quad$ $|f''(x)| < 1$

Combining all cases, we get $|f'(x)| < 1\ \forall\ x \in [0, 1]$

Ex. 54 Use the mean value theorem to prove

$e^x \ge 1 + x,\ \forall\ x \in R,$

Sol. Consider the function $f(x) = e^x - 1$ in $[0, x]$ where $x > 0$

$\therefore$ f is continuous and differentiable hence using LMVT some $c \in (0, x)$

$$f'(c) = \frac{(e^x-1)-0}{x} = \frac{e^x-1}{x} \qquad f'(c) = \frac{f(x)-f(0)}{x-0}$$

but $f'(c) = e^c$; $\quad$ hence $\dfrac{e^x-1}{x} = e^c > 1,$ for $x > 0$

$\therefore$ $e^x - 1 > x$

$\therefore$ $e^x > x + 1$ for $x > 0$ $\quad$ ....**(1)**

again consider the function

$f(x) = e^x - 1$ in $[x, 0]$ where $x < 0$

using LMVT some $c \in (x, 0)$ such that

$$f'(c) = \frac{0-(e^x-1)}{-x} = \frac{1-e^x}{-x} = \frac{e^x-1}{x}$$

but $f'(c) = e^c$ $\quad$ hence $\dfrac{e^x-1}{x} = e^c < 1$ for $c < 0$

hence $\frac{(e^x - 1)}{x} < 1$ for $x < 0$

$(e^x - 1) > x$ (as x is – ve)

$e^x - 1 - x > 0$ for $x < 0$ ....(2)

from (1) and (2)

$e^x > x + 1$ for $x \neq 0$

$\therefore$ for $x = 0$ equality holds

$\therefore$ $e^x \geq x + 1$ for $x \in R$

Ex. 55 Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to concavity and point of inflection.

Sol. $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

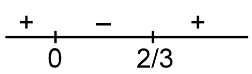
$f'(x) = 12x^2(x - 1)$

$f''(x) = 12(3x^2 - 2x)$

$f''(x) = 12x(3x - 2)$

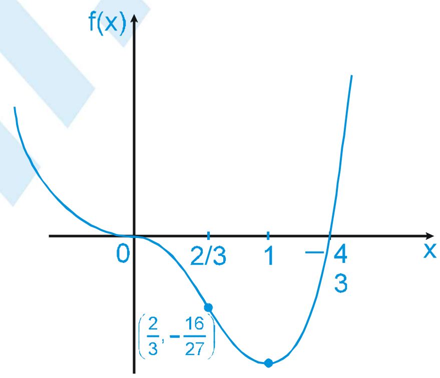
$f''(x) = 0 \Rightarrow x = 0, \frac{2}{3}$.

Again examining sign of $f''(x)$



thus $x = 0, \frac{2}{3}$ are the inflection points

Hence the graph of f(x) is



Ex. 56 Find the points of maxima, minima of $f(x) = x^3 - 12x$. Also draw the graph of this functions.

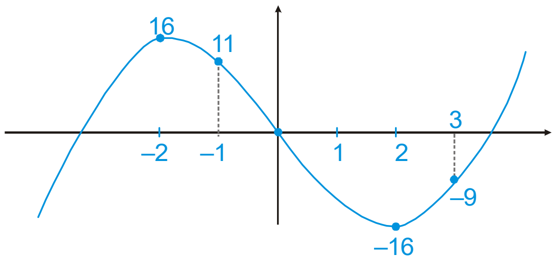
Sol. $f(x) = x^3 - 12x$

$f'(x) = 3(x^2 - 4) = 3(x - 2)(x + 2)$

$f'(x) = 0 \Rightarrow x = \pm 2$

For tracing the graph let us find maximum and minimum values of f(x).

x	f(x)
2	–16
–2	+16



Ex. 57 Find the value of a and b where a < b, for which the integral $\int_a^b (24-2x-x^2)^{1/2}dx$ has the largest value.

Sol. $I = \int_a^b [25-(x+1)^2]^{1/2}dx$

for I to be largest $(x+1)^2$ must be minimum $\Rightarrow$ x = – 1 i.e. a = – 1

$\therefore$ $I = \int_{-1}^{b} (24-2x-x^2)\,dx$

for maxima, $\frac{dI}{db} = 24 - 2b - b^2 = 0$ $\Rightarrow$ $b^2 + 2b - 24 = 0$ $\Rightarrow$ $(b+6)(b-4) = 0$

as $b \neq -6$ hence $b = 4$

hence $I = \int_{-1}^{4} (25-(x+1)^2]^{1/2}dx$ put $x + 1 = 5\sin\theta$

$= 24\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{25\pi}{4}$ Ans. Hence $a = -1;\ b = 4;\ I_{max} = \frac{25\pi}{4}$

Ex. 58 If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3, \end{cases}$, then

Sol. Given, $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$

$\Rightarrow$ $f'(x) = \begin{cases} 6x + 12, & -1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$

(A) which shows $f'(x) > 0$ for $x \in [-1, 2)$

So, f(x) is increasing on [–1, 2)

Hence, (A) is correct.

(B) for continuity of f(x). (check at x = 2)

RHL = 35, LHL = 35 and f(2) = 35

So, (B) is correct

(C) $Rf'(2) = -1$ and $Lf'(2) = 24$

so, not differentiable at x = 2.

Hence, (C) is correct.

(D) we know f(x) is increasing on [–1, 2) and decreasing on (2, 3],

Thus maximum at x = 2,

Hence, (D) is correct.

$\therefore$ **(A)**, **(B)**, **(C)**, **(D)** all are correct.

Ex. 59 Show that $f(x) = (x^3 - 6x^2 + 12x - 8)$ does not have any point of local maxima or minima. Hence draw graph

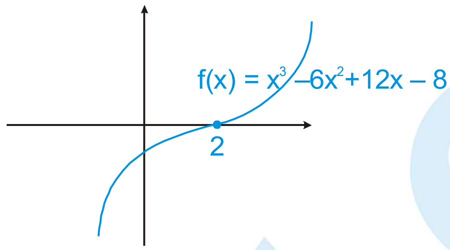
Sol. $f(x) = x^3 - 6x^2 + 12x - 8$

$f'(x) = 3(x^2 - 4x + 4)$

$f'(x) = 3(x-2)^2$

$f'(x) = 0 \quad \Rightarrow \quad x = 2$

but clearly $f'(x)$ does not change sign about $x = 2$. $f'(2^+) > 0$ and $f'(2^-) > 0$. So $f(x)$ has no pointof maxima or minima. In fact $f(x)$ is a monotonically increasing function for $x \in R$.



Ex. 60 Find the global maximum and global minimum of $f(x) = \dfrac{e^x + e^{-x}}{2}$ in $[-\log_e 2, \log_e 7]$.

Sol. $f(x) = \dfrac{e^x + e^{-x}}{2}$ is differentiable at all x in its domain.

Then $f'(x) = \dfrac{e^x - e^{-x}}{2}$, $f''(x) = \dfrac{e^x + e^{-x}}{2}$

$f'(x) = 0 \Rightarrow \dfrac{e^x - e^{-x}}{2} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$

$f''(0) = 1 \; \therefore \quad x = 0$ is a point of local minimum

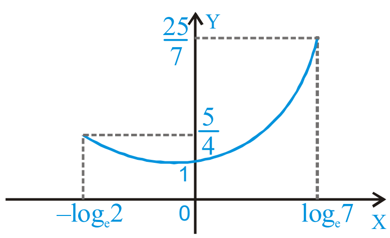
Points $x = -\log_e 2$ and $x = \log_e 7$ are extreme points.

Now, check the value of f(x) at all these three points $x = -\log_e 2, 0, \log_e 7$

$\Rightarrow \quad f(-\log_e 2) = \dfrac{e^{-\log_e 2} + e^{+\log_e 2}}{2} = \dfrac{5}{4}$

$f(0) = \dfrac{e^0 + e^{-0}}{2} = 1$

$f(\log_e 7) = \dfrac{e^{\log_e 7} + e^{-\log_e 7}}{2} = \dfrac{25}{7}$



$\therefore \quad x = 0$ is absolute minima & $x = \log_e 7$ is absolute maxima

Hence, absolute/global minimum value of f(x) is 1 at x = 0

and absolute/global maximum value of f(x) is $\dfrac{25}{7}$ at $x = \log_e 7$

Ex. 61 Find the possible points of Maxima/Minima for $f(x) = |x^2 - 2x|$ $(x \in R)$

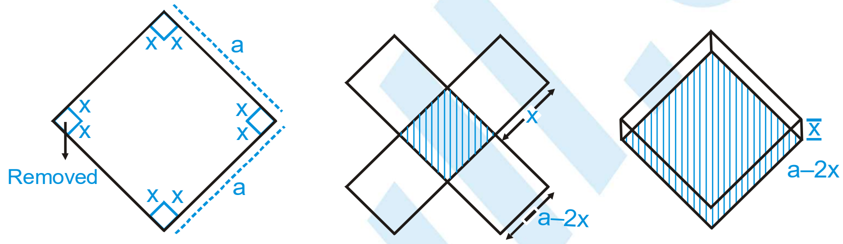
Sol.. $$f(x) = \begin{cases} x^2 - 2x & x \geq 2 \\ 2x - x^2 & 0 < x < 2 \\ x^2 - 2x & x \leq 0 \end{cases}$$

$$f'(x) = \begin{cases} 2(x-1) & x > 2 \\ 2(1-x) & 0 < x < 2 \\ 2(x-1) & x < 0 \end{cases}$$

$f'(x) = 0$ at $x = 1$ and $f'(x)$ does not exist at $x = 0, 2$. Thus these are critical points.

Ex. 62 A box of maximum volume with top open is to be made by cutting out four equal squares from four corners of a square tin sheet of side length a ft, and then folding up the flaps. Find the side of the square base cut off.

Sol. Volume of the box is, $V = x(a - 2x)^2$ i.e., squares of side x are cut out then we will get a box with a square base of side $(a - 2x)$ and height x.



$$\therefore \quad \frac{dV}{dx} = (a - 2x)^2 + x \cdot 2(a - 2x)(-2)$$

$$\frac{dV}{dx} = (a - 2x)\,(a - 6x)$$

For V to be extremum $\frac{dV}{dx} = 0 \quad \Rightarrow \quad x = a/2,\ a/6$

But when $x = a/2$; $V = 0$ (minimum) and we know minimum and maximum occurs alternately in a continuous function. Hence, V is maximum when $x = a/6$.

Ex. 63 Let $f(x) = \begin{cases} x^3 + x^2 - 10x & x < 0 \\ 3\sin x & x \geq 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

Sol. f(x) is continuous at x = 0.

$$f'(x) = \begin{cases} 3x^2 + 2x - 10 & x < 0 \\ 3\cos x & x > 0 \end{cases}$$

$f'(0^+) = 3$ and $f'(0^-) = -10$ thus f(x) is non-differentiable at x = 0

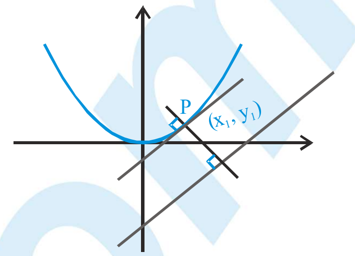
$\Rightarrow$ x = 0 is a critical point.

Also derivative changes sign from negative to positive, so x = 0 is a point of local minima.

Ex. 64 Find the co-ordinates of the point on the curve $x^2 = 4y$, which is at least distance from the line $y = x - 4$.

Sol. Let $P(x_1 y_1)$ be a point on the curve $x^2 = 4y$

at which normal is also a perpendicular to the line $y = x - 4$.



Slope of the tangent at (x_1, y_1) is $2x = 4\dfrac{dy}{dx} \Rightarrow \left.\dfrac{dy}{dx}\right|_{(x_1, y_1)} = \dfrac{x_1}{2}$

$\therefore \quad \dfrac{x_1}{2} = 1 \Rightarrow x_1 = 2$

✈ $\quad x_1^2 = 4y_1 \Rightarrow y_1 = 1$

Hence required point is (2, 1)

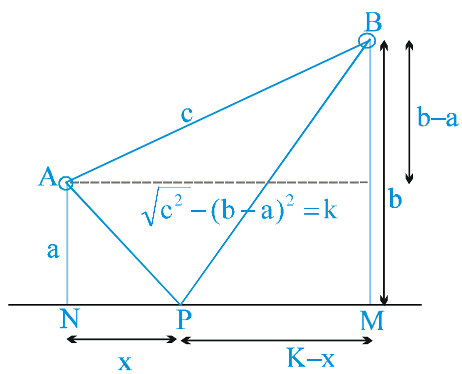
Ex. 65 Two towns located on the same side of the river agree to construct a pumping station and filteration plant at the river's edge , to be used jointly to supply the towns with water. If the distance of the two towns from the river are 'a' & 'b' and the distance between them is 'c', show that the pipe lines joining them to the pumping station is atleast as great as $\sqrt{c^2 + 4ab}$.

Sol. $L = AP + PB$

$L = \sqrt{a^2 + x^2} + \sqrt{(K - x)^2 + b^2}$

$\dfrac{dL}{dx} = \dfrac{2x}{2\sqrt{a^2 + x^2}} - \dfrac{2(K - x)}{2\sqrt{(K - x)^2 + b^2}}$

for maximum or minimum $\dfrac{dL}{dx} = 0$



$\dfrac{x}{\sqrt{a^2 + x^2}} = \dfrac{K - x}{\sqrt{(K - x)^2 + b^2}}$

$x^2 [(K-x)^2 + b^2)] = (K-x)^2 (a^2 + x^2)$

$b^2 x^2 = a^2 (K-x)^2$

$\therefore \quad bx = a (K - x)$

$x (a + b) = aK$

$x = \dfrac{Ka}{a + b} \qquad (K - x = \dfrac{Kb}{a + b})$

$L|_{x = \frac{Ka}{a+b}} = \sqrt{a^2 + \dfrac{K^2 a^2}{(a+b)^2}} + \sqrt{b^2 + \dfrac{K^2 b^2}{(a+b)^2}} \Rightarrow \dfrac{a}{a+b}\sqrt{(a+b)^2 + K^2} + \dfrac{b}{a+b}\sqrt{(a+b)^2 + K^2}$

$= \dfrac{\sqrt{(a+b)^2 + K^2}}{a+b}(a+b) = \sqrt{(a+b)^2 - c^2 - (a-b)^2} = \sqrt{c^2 + 4ab}$

Ex. 66 Find the critical points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$ if

(i) $x \in [0, 3]$ **(ii)** $x \in [-3, 3]$ **(iii)** $x \in [-1, 2]$.

Sol. $f'(x) = 12(x^2 - x - 2)$

$= 12(x - 2)(x + 1)$

$f'(x) = 0 \quad \Rightarrow \quad x = -1$ or 2

(i) if $x \in [0, 3]$, $x = 2$ is critical point.

(ii) if $x \in [-3, 3]$, then we have two critical points $x = -1, 2$.

(iii) If $x \in [-1, 2]$, then no critical point as both $x = 1$ and $x = 2$ become boundary points.

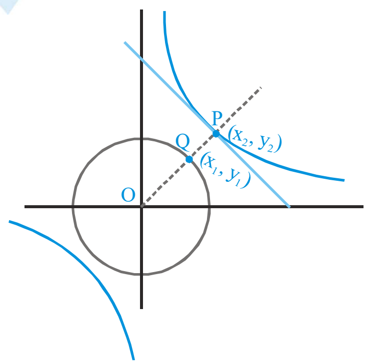
Ex. 67 Find the minimum value of $(x_1 - x_2)^2 + \left(\sqrt{2 - x_1^2} - \frac{9}{x_2}\right)^2$ where $x_1 \in (0, \sqrt{2})$ and $x_2 \in R^+$

Sol. $d^2 = (x_1 - x_2)^2 + \left(\sqrt{2 - x_1^2} - \frac{9}{x_2}\right)^2$

The above expression is the square of the distance between the points $\left(x_1, \sqrt{2 - x_1^2}\right)$, $\left(x_2, \frac{9}{x_2}\right)$ which lie on the curves $x^2 + y^2 = 2$ and $xy = 9$ respectively.

Now, the minimum value of the expression means square of the shortest distance between the two curves. Slope of the normal at $P(x_2, y_2)$ on the curve $xy = 9$

$\frac{dy}{dx} = \frac{-9}{x^2}$



Slope of OP $= \frac{x_2^2}{9} = \frac{y_2}{x_2} \Rightarrow x_2 y_2 = 9$

$x_2^4 = 81 \quad \Rightarrow \quad x_2 = \pm 3$

$\therefore \quad y_2 = \pm 3$

$(x_2, y_2) = (3, 3)$

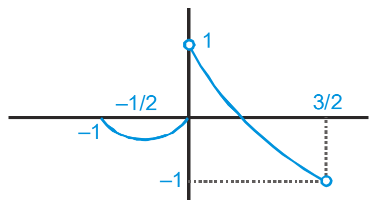
Now, shortest distance = PQ = OP – OQ = $3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$

$\therefore \quad d^2 = 8$

Ex. 68 Let $f(x) = \begin{cases} x^2 + x & ; \quad -1 \le x < 0 \\ \lambda & ; \quad x = 0 \\ \log_{1/2}\left(x + \frac{1}{2}\right) & ; \quad 0 < x < \frac{3}{2} \end{cases}$

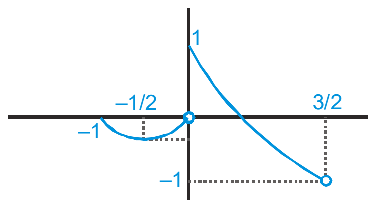
Discuss global maxima, minima for $\lambda = 0$ and $\lambda = 1$. For what values of λ does f(x) has global maxima

Sol. Graph of $y = f(x)$ for $\lambda = 0$



No global maxima, minima

Graph of $y = f(x)$ for $\lambda = 1$



Global maxima is 1, which occurs at $x = 0$

Global minima does not exists

$\lim\limits_{x \to 0^-} f(x) = 0,\ \lim\limits_{x \to 0^+} f(x) = 1,\ f(0) = \lambda$

For global maxima to exists

$f(0) \geq 1 \quad \Rightarrow \quad \lambda \geq 1.$

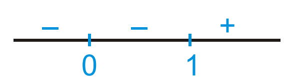
Ex. 69 Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to maxima, minima and concavity.

Sol. $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$f'(x) = 12x^2(x - 1)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1$



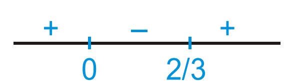
examining sign change of f'(x)

thus $x = 1$ is a point of local minima

$f''(x) = 12(3x^2 - 2x)$

$f''(x) = 12x(3x - 2)$

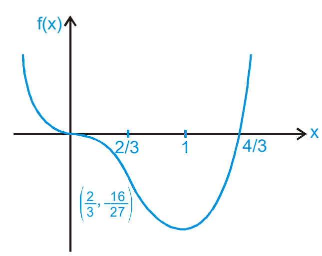
$f''(x) = 0 \quad \Rightarrow \quad x = 0, \dfrac{2}{3}$



Again examining sign of f''(x)

thus $x = 0, \dfrac{2}{3}$ are the inflection points

Hence the graph of f(x) is



Ex. 70 Rectangles are inscribed inside a semicircle of radius r. Find the rectangle with maximum area.

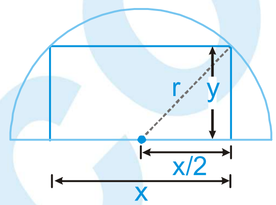
Sol. Let sides of rectangle be x and y (as shown in figure).

$\Rightarrow$ A = xy.

Here x and y are not independent variables and are related by Pythogorus theorem with r.

$$\frac{x^2}{4} + y^2 = r^2 \quad \Rightarrow \quad y = \sqrt{r^2 - \frac{x^2}{4}}$$

$$\Rightarrow \quad A(x) = x\sqrt{r^2 - \frac{x^2}{4}}$$



$$\Rightarrow \quad A(x) = \sqrt{x^2r^2 - \frac{x^4}{4}}$$

Let $f(x) = r^2x^2 - \frac{x^4}{4}$; $x \in (0, r)$

A(x) is maximum when f(x) is maximum

Hence $f'(x) = x(2r^2 - x^2) = 0$

$\Rightarrow$ $x = r\sqrt{2}$

also $f'(r\sqrt{2^+}) < 0$ and $f'(r\sqrt{2^-}) > 0$

confirming at f(x) is maximum when $x = r\sqrt{2}$ & $y = \frac{r}{\sqrt{2}}$.

Ex. 71 Find all the values of a for which the function $f(x) = (a^2 - 3a + 2)\cos\left(\frac{x}{2}\right) + (a-1)x$ possesses critical points.

Sol. Given $f(x) = (a^2 - 3a + 2)\cos\left(\frac{x}{2}\right) + (a-1)x$

$$\therefore \quad f'(x) = -\frac{(a^2 - 3a + 2)}{2}\sin\left(\frac{x}{2}\right) + (a-1) \quad = (a-1)\left\{1 - \frac{1}{2}(a-2)\sin\left(\frac{x}{2}\right)\right\}$$

Put f'(x) = 0 then a = 1 and $\sin\left(\frac{x}{2}\right) = \frac{2}{a-2}$

but $-1 \le \sin\left(\frac{x}{2}\right) \le 1$

or $\left|\sin\left(\frac{x}{2}\right)\right| \le 1 \quad \Rightarrow \quad \left|\frac{2}{a-2}\right| \le 1 \Rightarrow |a-2| \ge 2$

$\Rightarrow$ $a - 2 \ge 2$ and $a - 2 \le -2$

$\therefore$ $a \ge 4$ and $a \le 0$ $\Rightarrow$ $a \in (-\infty, 0] \cup [4, \infty)$

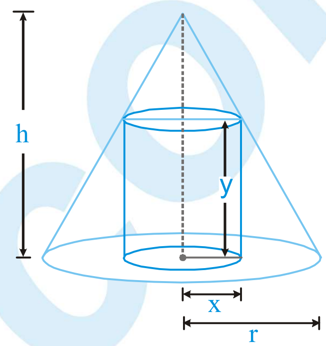
Hence $a \in (-\infty, 0] \cup \{1\} \cup [4, \infty)$

Ex. 72 If a right circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Sol. Let x be the radius of cylinder and y be its height

$$v = \pi x^2 y$$

x, y can be related by using similar triangles (as shown in figure).



$$\frac{y}{r-x} = \frac{h}{r}$$

$$\Rightarrow \quad y = \frac{h}{r}(r-x)$$

$$\Rightarrow \quad v(x) = \pi x^2 \frac{h}{r}(r-x) \qquad x \in (0, r)$$

$$\Rightarrow \quad v(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

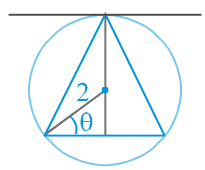
$$v'(x) = \frac{\pi h}{r} x(2r - 3x)$$

$$v'\left(\frac{2r}{3}\right) = 0 \quad \text{and} \quad v''\left(\frac{2r}{3}\right) < 0$$

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

Ex. 73 A point P is given on the circumference of a circle of radius *r*. Chords QR are parallel to the tangent at P . Determine the maximum possible area of the triangle PQR.

Sol. $\text{Ar. of } \Delta PQR = \frac{2r\cos\theta(r + r\sin\theta)}{2}$



$A(\theta) = r^2 \cos\theta\,(1 + \sin\theta)$

$A'(\theta) = r^2 [\cos\theta \,.\, \cos\theta - (1 + \sin\theta)\sin\theta]$

for maximum or minimum $A'(\theta) = 0$

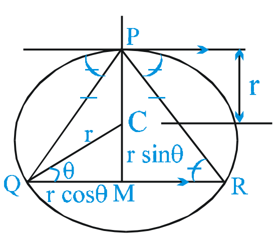
$\cos^2\theta - \sin\theta\,(1 + \sin\theta) = 0 \quad = \quad 1 - \sin^2\theta - \sin\theta - \sin^2\theta = 0$

$= 2\sin^2\theta + \sin\theta - 1 = 0 \quad = \quad (\sin\theta + 1)(2\sin\theta - 1) = 0$

$\sin\theta = -1$ (not possible)

$\sin\theta = 1/2$ hence, $\theta = \pi/6$ or $5\pi/6$

$$\frac{d^2A}{d\theta^2} = -2\cos\theta\sin\theta - \cos\theta - 2\sin\theta\cos\theta$$

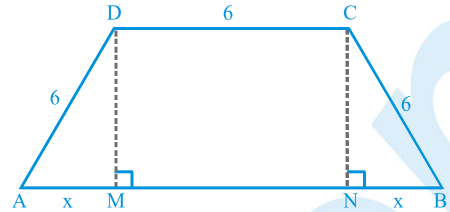


$$\left.\frac{d^2A}{d\theta^2}\right]_{\theta=\pi/6} = -\text{ve} \Rightarrow \text{A is maximum}$$

$$A_{max} = 2r^2 \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right)\frac{1}{2} \Rightarrow A_{max} = \frac{3\sqrt{3}}{4} r^2$$

Ex. 74 The three sides of a trapezium are equal each being 6cm long; find the area of trapezium when it is maximum.

Sol. Let ABCD be the given trapezium.



Let $AM = BN = x$ cm

then $DM = CN = \sqrt{(36 - x^2)}$

$\therefore$ Area of trapezium ABCD is

$$S = \frac{1}{2}(6 + x + 6 + x) \times \sqrt{(36 - x^2)}$$

$$= (6 + x)\sqrt{(36 - x^2)}$$

or $S^2 = (6 + x)^2(36 - x^2)$

Let $y = (6 + x)^2(36 - x^2)$

$$\therefore \quad \frac{dy}{dx} = (6 + x)^2(-2x) + (36 - x^2).2(6 + x)$$

$$= 2(6 + x)^2(6 - 2x) = 4(3 - x)(6 + x)^2$$

and $$\frac{d^2y}{dx^2} = -12x(6 + x)$$

For max. or min. of y, $\frac{dy}{dx} = 0$ then $x = 3$ $(x \neq -6)$

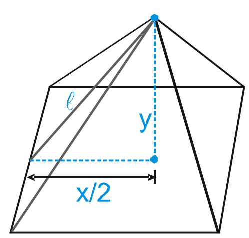
$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{x=3} = -324 < 0$$

$\therefore$ y is maximum at $x = 3$ then S is also maximum at $x = 3$

$$\therefore \quad S = (6 + 3)\sqrt{(36 - 9)} = 27\sqrt{3} \text{ cm}^2$$

Ex. 75 Among all regular square pyramids of volume $36\sqrt{2}$ cm³. Find dimensions of the pyramid having least lateral surface area.

Sol. Let the length of a side of base be x cm and y be the perpendicular height of the pyramid (see figure).



$$V = \frac{1}{3} \times \text{ area of base x height}$$

$$\Rightarrow \quad V = \frac{1}{3}x^2y = 36\sqrt{2} \Rightarrow y = \frac{108\sqrt{2}}{x^2}$$

and $$S = \frac{1}{2} \times \text{perimeter of base x slant height}$$

$$= \frac{1}{2}(4x).\bullet$$

but $$\bullet = \sqrt{\frac{x^2}{4} + y^2}$$

$$\Rightarrow \quad S = 2x\sqrt{\frac{x^2}{4} + y^2} = \sqrt{x^4 + 4x^2y^2} \Rightarrow S = \sqrt{x^4 + 4x^2\left(\frac{108\sqrt{2}}{x^2}\right)^2}$$

$$S(x) = \sqrt{x^4 + \frac{8.(108)^2}{x^2}}$$

Let $f(x) = x^4 + \frac{8.(108)^2}{x^2}$ for minimizing f(x)

$f'(x) = 4x^3 - \frac{16(108)^2}{x^3} = 0 \Rightarrow f'(x) = 4\frac{(x^6 - 6^6)}{x^3} = 0 \Rightarrow$ x = 6, which a point of minima

Hence x = 6 cm and $y = 3\sqrt{2}$.

Ex. 76 Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.

Sol. Area $= \frac{1}{2}$ (Base) × (Height)

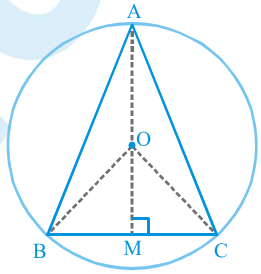
For maximum area height must be maximum. Height will be maximum if triangle is an isosceles triangle.

Let ΔABC is isosceles.

Let AB = AC

Let $\angle B = \angle C = \theta$ then $\angle A = \pi - 2\theta$

$\therefore$ $\angle COM = \angle BOM = \pi - 2\theta$



If r be the radius of circle

$\therefore$ $OM = r\cos(\pi - 2\theta)$ and $MC = r\sin(\pi - 2\theta)$

$\therefore$ Area of $\Delta ABC = \frac{1}{2} \times BC \times AM$

$= \frac{1}{2} \times 2r\sin(\pi - 2\theta) \times \{r + r\cos(\pi - 2\theta)\} \Rightarrow r^2\sin2\theta(1 - \cos2\theta)$

Let $S = r^2\left\{\sin 2\theta - \frac{1}{2}\sin 4\theta\right\}$

$\therefore$ $\frac{dS}{d\theta} = r^2\{2\cos 2\theta - 2\cos 4\theta\}$

and $\frac{d^2S}{d\theta^2} = r^2\{-4\sin 2\theta + 8\sin 4\theta\}$

For max. or min. of S, $\frac{dS}{d\theta} = 0$

or $\cos 2\theta = \cos 4\theta$ or $4\theta = 2\pi - 2\theta$

$\Rightarrow$ $\theta = \frac{\pi}{3}$ and $\left.\frac{d^2S}{d\theta^2}\right|_{\theta=\pi/3} = -6\sqrt{3}r^2 < 0$

$\therefore$ $\theta = \frac{\pi}{3}$ is point of maxima

$\therefore$ $\angle B = \angle C = \frac{\pi}{3}$ and $\angle A = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$\therefore$ Area of triangle is maximum if triangle is equilateral.

Exercise # 1

[Single Correct Choice Type Questions]

1. If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x and y axis respectively, then the value of a^2b is -

(A) $27c^3$ **(B)** $\frac{4}{27}c^3$ **(C)** $\frac{27}{4}c^3$ **(D)** $\frac{4}{9}c^3$

2. Equation of normal drawn to the graph of the function defined as $f(x) = \frac{\sin x^2}{x}$, $x \neq 0$ and $f(0) = 0$ at the origin is

(A) $x + y = 0$ **(B)** $x - y = 0$ **(C)** $y = 0$ **(D)** $x = 0$

3. Using differentials, find the approximate value of $\sqrt{25.2}$.

(A) 5.02 **(B)** 5.01 **(C)** 5.03 **(D)** 5.04

4. The angle at which the curve $y = Ke^{Kx}$ intersects the y-axis is :

(A) $\tan^{-1}k^2$ **(B)** $\cot^{-1}(k^2)$ **(C)** $\sec^{-1}\left(\sqrt{1+k^4}\right)$ **(D)** none

5. If tangents are drawn from the origin to the curve $y = \sin x$, then their points of contact lie on the curve

(A) $x - y = xy$ **(B)** $x + y = xy$ **(C)** $x^2 - y^2 = x^2y^2$ **(D)** $x^2 + y^2 = x^2y^2$

6. A kite is 300 m high and there are 500 m of cord out. If the wind moves the kite horizontally at the rate of 5 km/hr. directly away from the person who is flying it, find the rate at which the cord is being paid ?

(A) 4 **(B)** 8 **(C)** 3 **(D)** cannot be determined

7. The tangent to the curve $3xy^2 - 2x^2y = 1$ at (1,1) meets the curve again at the point -

(A) $\left(\frac{16}{5}, \frac{1}{20}\right)$ **(B)** $\left(-\frac{16}{5}, -\frac{1}{20}\right)$ **(C)** $\left(\frac{1}{20}, \frac{16}{5}\right)$ **(D)** $\left(-\frac{1}{20}, \frac{16}{5}\right)$

8. Equation of the normal to the curve $y = -\sqrt{x} + 2$ at the point of its intersection with the curve $y = \tan(\tan^{-1} x)$ is

(A) $2x - y - 1 = 0$ **(B)** $2x - y + 1 = 0$ **(C)** $2x + y - 3 = 0$ **(D)** none

9. A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. The value of $\frac{dx}{dt}$ when $x = 3$ is -

(A) 4 **(B)** $\frac{9}{2}$ **(C)** $\frac{3\sqrt{3}}{2}$ **(D)** none of these

10. The angle between the tangent lines to the graph of the function $f(x) = \int_2^x (2t-5)\,dt$ at the points where the graph cuts the x-axis is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$

11. The approximate change in the volume of a cube of side x meters caused by increasing the side by 4% is

(A) $0.06x^3m^3$ **(B)** $0.09x^3m^3$ **(C)** $0.12x^3m^3$ **(D)** $0.15x^3m^3$

12. Suppose f and g both are linear function with $f(x) = -2x + 1$ and $f(g(x)) = 6x - 7$, then slope of line $y = g(x)$ is -

(A) 3 **(B)** –3 **(C)** 6 **(D)** –2

13. The curve $y - e^{xy} + x = 0$ has a vertical tangent at

(A) (1, 1) **(B)** (0, 1) **(C)** (1, 0) **(D)** no point

14. A point is moving along the curve $y^3 = 27x$. The interval in which the abscissa changes at slower rate than ordinate, is -

(A) (–3, 3) **(B)** $(-\infty,\infty)$ **(C)** (–1, 1) **(D)** $(-\infty, -3) \cup (3, \infty)$

15. The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to -

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$

16. The ordinate of $y = (a/2)(e^{x/a} + e^{-x/a})$ is the geometric mean of the length of the normal and the quantity:

(A) a/2 **(B)** a **(C)** e **(D)** none of these.

17. If the tangent to the curve $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ at $\theta = \frac{\pi}{3}$ makes an angle α $(0 \le \alpha < \pi)$ with x-axis, then α =

(A) $\frac{\pi}{3}$ **(B)** $\frac{2\pi}{3}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{5\pi}{6}$

18. The angle of intersection of $x = \sqrt{y}$ and $x^3 + 6y = 7$ at (1, 1) is -

(A) $\frac{\pi}{5}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$

19. All points on curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ at which tangents are parallel to the axis of x, lie on a

(A) circle **(B)** parabola **(C)** line **(D)** none of these

20. Water is poured into an inverted conical vessel of which the radius of the base is 2m and height 4m, at the rate of 77 litre/minute. The rate at which the water level is rising at the instant when the depth is 70 cm is - (use π = 22/7)

(A) 10 cm/min **(B)** 20 cm/min **(C)** 40 cm /min **(D)** none

21. At the point $P(a, a^n)$ on the graph of $y = x^n$ $(n \in N)$ in the first quadrant a normal is drawn. The normal intersects the y-axis at the point (0, b). If $\lim_{a\to 0} b = \frac{1}{2}$, then *n* equals

(A) 1 **(B)** 3 **(C)** 2 **(D)** 4

22. Number of tangents drawn from the point (–1/2, 0) to the curve $y = e^{\{x\}}$. (Here { } denotes fractional part function).

(A) 2 **(B)** 1 **(C)** 3 **(D)** 4

23. The length of the subnormal of the curve $y^2 = 8ax$ $(a > 0)$ is -

(A) 2a **(B)** 4a **(C)** 6a **(D)** 8a

24. If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is

(A) 1/2 **(B)** 1/3 **(C)** 2 **(D)** 3

25. The coordinates of the point of the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are

(A) $(2, -4)$ **(B)** $(18, -12)$ **(C)** $(2, 4)$ **(D)** none of these

26. Function $f(x) = x^2 (x-2)^2$ is -

(A) increasing in $(0,1)\cup(2,\infty)$ **(B)** decreasing in $(0,1)\cup(2,\infty)$

(C) decreasing function **(D)** increasing function

27. The function $\frac{|x-1|}{x^2}$ is monotonically decreasing at the point

(A) $x = 3$ **(B)** $x = 1$ **(C)** $x = 2$ **(D)** none of these

28. Consider $f(x) = \int_1^x \left(t + \frac{1}{t}\right) dt$ and $g(x) = f'(x)$ for $x \in \left[\frac{1}{2}, 3\right]$

If P is a point on the curve $y = g(x)$ such that the tangent to this curve at P is parallel to a chord joining the points $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$ and $(3, g(3))$ of the curve, then the coordinates of the point P

(A) can't be found out **(B)** $\left(\frac{7}{4}, \frac{65}{28}\right)$ **(C)** $(1, 2)$ **(D)** $\left(\sqrt{\frac{3}{2}}, \frac{5}{\sqrt{6}}\right)$

29. Least value of the function, $f(x) = 2^{x^2} - 1 + \frac{2}{2^{x^2} + 1}$ is:

(A) 0 **(B)** 3/2 **(C)** 2/3 **(D)** 1

30. The function $f(x) = \tan x - x$

(A) always increases **(B)** always decreases

(C) never decreases **(D)** sometimes increases and sometimes decreases

31. The value of 'a' for which the function $f(x) = \sin x - \cos x - ax + b$ decreases for all real values of x, is -

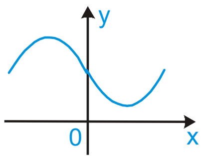
(A) $a \geq -\sqrt{2}$ **(B)** $a \leq -\sqrt{2}$ **(C)** $a \leq \sqrt{2}$ **(D)** $a \geq \sqrt{2}$

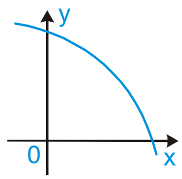
32. $f(x) = \int \left(2 - \frac{1}{1 + x^2} - \frac{1}{\sqrt{1 + x^2}}\right) dx$ then f is -

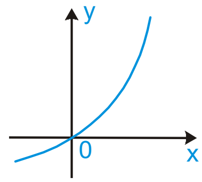
(A) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$ **(B)** increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$

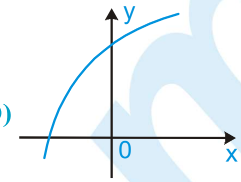
(C) increasing in $(-\infty, \infty)$ **(D)** decreasing in $(-\infty, \infty)$

33. The curve $y = f(x)$ which satisfies the condition $f'(x) > 0$ and $f''(x) < 0$ for all real x, is:

(A) 

(B) 

(C) 

(D) 

34. Let f(x) be a quadratic expression which is positive for all real values of x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x

(A) $g(x) < 0$ **(B)** $g(x) > 0$ **(C)** $g(x) = 0$ **(D)** $g(x) \geq 0$

35. The values of p for which the function $f(x) = \left(\frac{\sqrt{p+4}}{1-p} - 1\right) x^5 - 3x + \ln 5$ decreases for all real x is

(A) $(-\infty, \infty)$ **(B)** $\left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$

(C) $\left[-3, \frac{5-\sqrt{27}}{2}\right] \cup (2, \infty)$ **(D)** $[1, \infty)$

36. Let $f(x) = \begin{cases} \frac{4-x}{2-\sqrt{x}} & \text{for } 0 < x < 4 \\ 4 & \text{for } x = 4 \\ 16 - 3x & \text{for } 4 < x < 6 \end{cases}$

Which of the following properties does f have on the interval (0, 6) ?

I. ●n f(x) exists ; **II.** f is continuous **III.** f is monotonic

(A) I only **(B)** II only **(C)** III only **(D)** none

37. Let f be a function which is continuous and differentiable for all real x. If $f(2) = -4$ and $f'(x) \geq 6$ for all $x \in [2, 4]$ then

(A) $f(4) < 8$ **(B)** $f(4) \geq 8$ **(C)** $f(4) \geq 12$ **(D)** none of these

38. The true set of real values of x for which the function, $f(x) = x \ln x - x + 1$ is positive is

(A) $(1, \infty)$ **(B)** $(1/e, \infty)$ **(C)** $[e, \infty)$ **(D)** $(0,1) \cup (1, \infty)$

39. Consider the function for $x = [-2, 3]$, $f(x) = \begin{cases} \frac{x^3 - 2x^2 - 5x + 6}{x-1} & \text{if } x \neq 1 \\ -6 & \text{if } x = 1 \end{cases}$, then

(A) f is discontinuous at $x = 1 \Rightarrow$ Rolle's theorem is not applicable in $[-2, 3]$

(B) $f(-2) \neq f(3) \Rightarrow$ Rolle's theorem is not applicable in $[-2, 3]$

(C) f is not derivable in $(-2, 3) \Rightarrow$ Rolle's theorem is not applicable

(D) Rolle's theorem is applicable as f satisfies all the conditions and c of Rolle's theorem is 1/2

40. For which values of 'a' will the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ will be concave upward along the entire real line

(A) $a \in [0, \infty)$ (B) $a \in (-2, \infty)$ (C) $a \in [-2, 2]$ (D) $a \in (0, \infty)$

41. If the function $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at $x = 2$ on the closed interval [1, a] then the value of 'a' is equal to -

(A) 3 (B) 4 (C) 6 (D) 1

42. The length of largest continuous interval in which function $f(x) = 4x - \tan 2x$ is monotonic, is -

(A) $\pi/2$ (B) $\pi/4$ (C) $\pi/8$ (D) $\pi/16$

43. The function $f(x) = x(x + 3)e^{-x/2}$ satisfies all the conditions of Rolle's theorem on [–3, 0]. The value of c which verifies Rolle's theorem, is

(A) 0 (B) – 1 (C) – 2 (D) 3

44. Suppose that f is differentiable for all x and that $f'(x) \le 2$ for all x. If $f(1) = 2$ and $f(4) = 8$ then $f(2)$ has the value equal to

(A) 3 (B) 4 (C) 6 (D) 8

45. Consider $f(x) = |1 - x|, 1 \le x \le 2$ and $g(x) = f(x) + b\sin\frac{\pi}{2}x, 1 \le x \le 2$

then which of the following is correct ?

(A) Rolles theorem is applicable to both f, g and $b = \frac{3}{2}$

(B) LMVT is not applicable to f and Rolles theorem if applicable to g with $b = \frac{1}{2}$

(C) LMVT is applicable to f and Rolles theorem is applicable to g with b = 1

(D) Rolles theorem is not applicable to both f, g for any real b.

46. If the point (1, 3) serves as the point of inflection of the curve $y = ax^3 + bx^2$ then the value of 'a' and 'b' are:

(A) $a = 3/2$ & $b = -9/2$ (B) $a = 3/2$ & $b = 9/2$

(C) $a = -3/2$ & $b = -9/2$ (D) $a = -3/2$ & $b = 9/2$

47. Number of solution of the equation $3\tan x + x^3 = 2$ in $\left(0, \frac{\pi}{4}\right)$ is

(A) 0 (B) 1 (C) 2 (D) 3

48. Rolle's theorem in the indicated intervals will not be valid for which of the following function-

(A) $f(x) = \begin{cases} \frac{\sin x}{x} & x \ne 0 \\ 1 & x = 0 \end{cases}$; $x \in [-1, 1]$

(B) $g(x) = \begin{cases} \frac{1-\cos x}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$; $x \in [-2\pi, 2\pi]$

(C) $h(x) = \begin{cases} \frac{1-\cos x}{x^2} & x \ne 0 \\ \frac{1}{2} & x = 0 \end{cases}$; $x \in [-2\pi, 2\pi]$

(D) $k(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \ne 0 \\ 0 & x = 0 \end{cases}$; $x \in \left[-\frac{1}{\pi}, \frac{1}{2\pi}\right]$

49. If f(x) satisfies the requirements of Lagrange's mean value theorem on [0, 2] and if f(0) = 0 and $f'(x) \le \frac{1}{2}\ \forall\ x \in [0, 2]$, then

(A) $|f(x)| \le 2$ **(B)** $f(x) \le 1$

(C) $f(x) = 2x$ **(D)** f(x) = 3 for at least one x in [0, 2]

50. If $f(x) = x^3 + 7x - 1$ then f(x) has a zero between x = 0 and x = 1. The theorem which best describes this, is -

(A) Rolle's theorem **(B)** mean value theorem

(C) maximum-minimum value theorem **(D)** intermediate value theorem

51. Four points A, B, C, D lie in that order on the parabola $y = ax^2 + bx + c$. The coordinates of A, B & D are known as A(–2, 3); B(–1, 1) and D(2, 7). The coordinates of C for which the area of the quadrilateral ABCD is greatest, is

(A) (1/2, 7/4) **(B)** (1/2, – 7/4) **(C)** (–1/2, 7/4) **(D)** none

52. On the interval [0,1] the function $x^{25}(1-x)^{75}$ takes its maximum value at the point -

(A) 0 **(B)** 1/3 **(C)** 1/2 **(D)** 1/4

53. If $f(x) = \sin^3 x + \lambda \sin^2 x$; $-\pi/2 < x < \pi/2$, then the interval in which λ should lie in order that f(x) has exactly one minima and one maxima

(A) (–3/2, 3/2) – {0} **(B)** (–2/3, 2/3) – {0} **(C)** R **(D)** $\left[-\frac{3}{2}, 0\right)$

54. The real number x when added to it's reciprocal gives the minimum value of the sum at x equal to -

(A) 1 **(B)** – 1 **(C)** – 2 **(D)** 2

55. In a regular triangular prism the distance from the centre of one base to one of the vertices of the other base is ●. The altitude of the prism for which the volume is greatest, is :

(A) $\frac{1}{2}$ **(B)** $\frac{1}{\sqrt{3}}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{4}$

56. If f(x) = a ●n |x| + bx² + x has its extremum values at x = – 1 and x = 2, then

(A) a = 2, b = – 1 **(B)** a = 2, b = – 1/2 **(C)** a = – 2, b = 1/2 **(D)** none of these

57. If $a < b < c < d$ & $x \in R$ then the least value of the function,

$f(x) = |x - a| + |x - b| + |x - c| + |x - d|$ is

(A) c – d + b – a **(B)** c + d – b – a **(C)** c + d – b + a **(D)** c – d + b + a

58. The greatest, the least values of the function, $f(x) = 2 - \sqrt{1 + 2x + x^2}$, $x \in [-2, 1]$ are respectively

(A) 2, 1 **(B)** 2, – 1 **(C)** 2, 0 **(D)** none

59. If $f(x) = 1 + 2x^2 + 4x^4 + 6x^6 + ... + 100x^{100}$ is a polynomial in a real variable x, then f(x) has -

(A) neither a maximum nor a minimum **(B)** only one maximum

(C) only one minimum **(D)** none

60. The lower corner of a leaf in a book is folded over so as to just reach the inner edge of the page. The fraction of width folded over if the area of the folded part is minimum is:

(A) 5/8 **(B)** 2/3 **(C)** 3/4 **(D)** 4/5

61. The difference between the greatest and the least value of $f(x) = \cos^2\frac{x}{2}\sin x,\ x \in [0, \pi]$ is -

(A) $\frac{3\sqrt{3}}{8}$ **(B)** $\frac{\sqrt{3}}{8}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2\sqrt{2}}$

62. Let f (x) and g (x) be two continuous functions defined from $R \longrightarrow R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall\ x_1 > x_2$, then solution set of $f\left(g(\alpha^2 - 2\alpha)\right) > f\left(g(3\alpha - 4)\right)$ is

(A) R **(B)** ϕ **(C)** (1, 4) **(D)** R – [1, 4]

63. A rectangle has one side on the positive y-axis and one side on the positive x-axis. The upper right hand vertex on the curve $y = \frac{\ln x}{x^2}$. The maximum area of the rectangle is -

(A) e^{-1} **(B)** $e^{-1/2}$ **(C)** 1 **(D)** $e^{1/2}$

64. The dimensions of the rectangle of maximum area that can be inscribed in the ellipse $(x/4)^2 + (y/3)^2 = 1$ are

(A) $\sqrt{8}, \sqrt{2}$ **(B)** 4, 3 **(C)** $2\sqrt{8}, 3\sqrt{2}$ **(D)** $\sqrt{2}, \sqrt{6}$

65. The set of value (s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)\ x + 2$ possess a negative point of inflection -

(A) $(-\infty, -2) \cup (0, \infty)$ **(B)** $\{-4/5\}$ **(C)** (–2,0) **(D)** empty set

66. If x_1 and x_2 are abscissa of two points on the curve $f(x) = x - x^2$ in the interval [0, 1], then maximum value of the expression $(x_1 + x_2) - (x_1^2 + x_2^2)$ is

(A) $\frac{1}{2}$ **(B)** $\frac{1}{4}$ **(C)** 1 **(D)** 2

67. If the function $f(x) = \frac{t + 3x - x^2}{x - 4}$, where 't' is a parameter has a minimum and a maximum then the range of values of 't' is

(A) (0, 4) **(B)** $(0, \infty)$ **(C)** $(-\infty, 4)$ **(D)** $(4, \infty)$

68. The radius of a right circular cylinder of greatest curved surface which can be inscribed in a given right circular cone is

(A) one third that of the cone **(B)** $1/\sqrt{2}$ times that of the cone
(C) 2/3 that of the cone **(D)** 1/2 that of the cone

69. A solid rectangular brick is to be made from 1 cu feet of clay. The brick must be 3 times as long as it is wide. The width of brick for which it will have minimum surface area is a. Then a^3 is -

(A) $\left(\frac{2}{9}\right)^{1/3}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{8}{729}$ **(D)** $\frac{3}{2}$

70. The maximum area of the rectangle whose sides pass through the angular points of a given rectangle of sides a and b is

(A) 2 (ab) **(B)** $\frac{1}{2}(a+b)^2$ **(C)** $\frac{1}{2}(a^2+b^2)$ **(D)** none of these

71. Which of the following statements is true for the general cubic function $f(x) = ax^3 + bx^2 + cx + d \ (a \neq 0)$

I. If the derivative $f'(x)$ has two distinct real roots then cubic has one local maxima and one local minima.

II. If the derivative $f'(x)$ has exactly one real root then the cubic has exactly one relative extremum.

III. If the derivative $f'(x)$ has no real roots, then the cubic has no relative extrema

(A) only I & II **(B)** only II and III **(C)** only I and III **(D)** all I, II, III are correct.

72. P and Q are two points on a circle of centre C and radius α, the angle PCQ being 2θ then the radius of the circle inscribed in the triangle CPQ is maximum when

(A) $\sin\theta = \frac{\sqrt{3}-1}{2\sqrt{2}}$ **(B)** $\sin\theta = \frac{\sqrt{5}-1}{2}$ **(C)** $\sin\theta = \frac{\sqrt{5}+1}{2}$ **(D)** $\sin\theta = \frac{\sqrt{5}-1}{4}$

73. Let $f(x) = \begin{cases} \sin\frac{\pi x}{2}, & 0 \le x < 1 \\ 3 - 2x, & x \ge 1 \end{cases}$ then :

(A) f(x) has local maxima at x = 1 **(B)** f(x) has local minima at x = 1

(C) f(x) does not have any local extrema at x = 1 **(D)** f(x) has a global minima at x = 1

74. The maximum distance of the point (k, 0) from the curve $2x^2 + y^2 - 2x = 0$ is equal to

(A) $\sqrt{1+2k-k^2}$ **(B)** $\sqrt{1+2k+2k^2}$ **(C)** $\sqrt{1-2k+2k^2}$ **(D)** $\sqrt{1-2k+k^2}$

75. Function f(x), g(x) are defined on [–1, 3] and $f''(x) > 0$, $g''(x) > 0$ for all $x \in [-1, 3]$, then which of the following is always true ?

(A) f(x) – g(x) is concave upwards on (–1, 3) **(B)** f(x) g(x) is concave upwards on (–1, 3)

(C) f(x) g(x) does not have a critical point on (–1, 3) **(D)** f(x) + g(x) is concave upwards on (–1, 3)

Exercise # 2

Part # I [Multiple Correct Choice Type Questions]

1. Which of the following pair(s) of curves is/are orthogonal.

(A) $y^2 = 4ax$; $y = e^{-x/2a}$
(B) $y^2 = 4ax$; $x^2 = 4ay$
(C) $xy = a^2$; $x^2 - y^2 = b^2$
(D) $y = ax$; $x^2 + y^2 = c^2$

2. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then.

(A) h is increasing whenever f is increasing
(B) h is increasing whenever f is decreasing
(C) h is decreasing whenever f is decreasing
(D) nothing can be said in general

3. Let S be the set of real values of parameter λ for which the function $f(x) = 2x^3 - 3(2+\lambda)x^2 + 12\lambda x$ has exactly one local maxima and exactly one local minima. Then the subset of S is -

(A) $(5, \infty)$
(B) $(-4, 4)$
(C) $(3, 8)$
(D) $(-\infty, -1)$

4. If tangent to curve $2y^3 = ax^2 + x^3$ at point (a, a) cuts off intercepts α, β on co-ordinate axes, where $\alpha^2 + \beta^2 = 61$, then the value of 'a' is equal to

(A) 20
(B) 25
(C) 30
(D) – 30

5. If P is a point on the curve $5x^2 + 3xy + y^2 = 2$ and O is the origin, then OP has

(A) minimum value $\frac{1}{2}$
(B) minimum value $\frac{2}{\sqrt{11}}$
(C) maximum value $\sqrt{11}$
(D) maximum value 2

6. The angle at which the curve $y = Ke^{Kx}$ intersects the y-axis is -

(A) $\tan^{-1}k^2$
(B) $\cot^{-1}(k^2)$
(C) $\sin^{-1}\left(\frac{1}{\sqrt{1+k^4}}\right)$
(D) $\sec^{-1}\left(\sqrt{1+k^4}\right)$

7. For the function $f(x) = x\cot^{-1}x,\ x \geq 0$

(A) there is atleast one $x \in (0, 1)$ for which $\cot^{-1}x = \frac{x}{1+x^2}$

(B) for atleast one x in the internal $(0, \infty)$, $f\left(x + \frac{2}{\pi}\right) - f(x) < 1$

(C) number of solution of the equation f(x) = sec x is 1

(D) f'(x) is strictly decreasing is the internal $(0, \infty)$

8. If $f(x) = \frac{x}{1 + x\tan x}$, $x \in \left(0, \frac{\pi}{2}\right)$, then

(A) f(x) has exactly one point of minima
(B) f(x) has exactly one point of maxima
(C) f(x) is increasing in $\left(0, \frac{\pi}{2}\right)$
(D) maxima occurs at x_0 where $x_0 = \cos x_0$

9. If $\frac{x}{a} + \frac{y}{b} = 1$ is a tangent to the curve $x = Kt$, $y = \frac{K}{t}$, $K > 0$ then :

(A) $a > 0,\ b > 0$
(B) $a > 0,\ b < 0$
(C) $a < 0,\ b > 0$
(D) $a < 0,\ b < 0$

10. $f''(x) > 0$ for all $x \in [-3, 4]$, then which of the following are always true?

(A) $f(x)$ has a relative minimum on $(-3, 4)$

(B) $f(x)$ has a minimum on $[-3, 4]$

(C) $f(x)$ is concave upwards on $[-3, 4]$

(D) if $f(3) = f(4)$ then $f(x)$ has a critical point on $[-3, 4]$

11. The function $y = \frac{2x-1}{x-2}$ $(x \neq 2)$ -

(A) is its own inverse

(B) decreases for all values of x

(C) has a graph entirely above x-axis

(D) is bound for all x

12. The curve $y = \frac{x+1}{x^2+1}$ has

(A) $x = 1$, as point of inflection

(B) $x = -2 + \sqrt{3}$, as point of inflection

(C) $x = -1$, as point of minimum

(D) $x = -2 - \sqrt{3}$, as point of inflection

13. The coordinates of the point(s) on the graph of the function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cut off intercepts from the coordinate axes which are equal in magnitude but opposite in sign, is -

(A) (2,8/3) **(B)** (3,7/2) **(C)** (1,5/6) **(D)** none

14. The co-ordinates of point(s) on the graph of the function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cut off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign, is

(A) (2, 8/3) **(B)** (3, 7/2) **(C)** (1, 5/6) **(D)** none

15. Let $f(x) = \int e^x (x-1)(x-2)dx$. Then 'f' increases in the interval -

(A) $(-\infty, -2)$ **(B)** $(-2, -1)$ **(C)** $(1, 2)$ **(D)** $(2, \infty)$

16. If $f(x) = \begin{cases} -\sqrt{1-x^2} & , \quad 0 \le x \le 1 \\ -x & , \quad x > 1 \end{cases}$, then

(A) Maximum of f(x) exist at x = 1

(B) Maximum of f(x) doesn't exists

(C) Minimum of $f^{-1}(x)$ exist at $x = -1$

(D) Minimum of $f^{-1}(x)$ exist at $x = 1$

17. The abscissa of the point on the curve $\sqrt{xy} = a + x$, the tangent at which cuts off equal intercepts from the co-ordinate axes is $(a > 0)$

(A) $\frac{a}{\sqrt{2}}$ **(B)** $-\frac{a}{\sqrt{2}}$ **(C)** $a\sqrt{2}$ **(D)** $-a\sqrt{2}$

18. The value of 'a' for which the function $f(x)=\begin{cases}-x^3+\cos^{-1}a, & 0<x<1\\ x^2, & x\ge 1\end{cases}$ has a local minimum at x = 1, is -

(A) –1 **(B)** 1 **(C)** 0 **(D)** $-\frac{1}{2}$

19. If f(0) = f(1) = f(2) = 0 & function f(x) is twice differentiable in (0, 2) and continuous in [0, 2]. Then which of the following is/are definitely true -

(A) $f''(c)=0;\ \forall\ c\in(0,2)$

(B) $f'(c)=0$; for atleast two $c\in(0,2)$

(C) $f'(c)=0$; for exactly one $c\in(0,2)$

(D) $f''(c)=0$; for atleast one $c\in(0,2)$

20. Which of the following inequalities always hold good in (0, 1)

(A) $x>\tan^{-1}x$

(B) $\cos x<1-\frac{x^2}{2}$

(C) $1+x\,ln\left(x+\sqrt{1+x^2}\right)>\sqrt{1+x^2}$

(D) $x-\frac{x^2}{2}<ln(1+x)$

21. Equation of a tangent to the curve $y\cot x=y^3\tan x$ at the point where the abscissa is $\frac{\pi}{4}$ is -

(A) $4x+2y=\pi+2$

(B) $4x-2y=\pi+2$

(C) x = 0

(D) y = 0

22. For the curve $x=t^2+3t-8$, $y=2t^2-2t-5$, at point (2, – 1)

(A) length of subtangent is 7/6.

(B) slope of tangent = 6/7

(C) length of tangent = $\sqrt{(85)}/6$

(D) none of these

23. For the curve represented parametrically by the equations, x = 2 ●n cot t + 1 and y = tan t + cot t

(A) tangent at t = π/4 is parallel to x-axis

(B) normal at t = π/4 is parallel to y-axis

(C) tangent at t = π/4 is parallel to the line y = x

(D) normal at t = π/4 is parallel to the line y = x

24. The co-ordinates of a point on the parabola $2y=x^2$ which is nearest to the point (0, 3) is

(A) (2, 2) **(B)** $\left(-\sqrt{2},\ 1\right)$ **(C)** $\left(\sqrt{2},1\right)$ **(D)** (– 2, 2)

25. If $f(x)=\begin{cases}3x^2+12x-1, & -1\le x\le 2\\ 37-x, & 2<x\le 3\end{cases}$ then -

(A) f(x) is increasing on (–1, 2)

(B) f(x) is continuous on [–1, 3]

(C) f'(2) does not exist

(D) f(x) has the maximum value at x = 2

26. If $f(x)=2x+\cot^{-1}x+$ ●n $\left(\sqrt{1+x^2}-x\right)$, then f(x) :

(A) increases in [0, ∞)

(B) decreases in [0, ∞)

(C) neither increases nor decreases in [0, ∞)

(D) increases in (– ∞, ∞)

27. Let $\phi(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3\sin x + 4\cos x \ \forall\ x \in R$, then -

(A) ϕ is increasing whenever f is increasing
(B) ϕ is increasing whenever f is decreasing
(C) ϕ is decreasing whenever f is decreasing
(D) ϕ is decreasing if $f'(x) = -11$

28. A function f is defined by $f(x) = \int_0^{\pi} \cos t \cos(x - t)dt$, $0 \le x \le 2\pi$ then which of the following hold(s) good ?

(A) $f(x)$ is continuous but not differentiable in $(0, 2\pi)$
(B) Maximum value of f is π
(C) There exists atleast one $c \in (0, 2\pi)$ s.t. $f'(c) = 0$.
(D) Minimum value of f is $-\dfrac{\pi}{2}$.

29. If y = f(x) be the equation of a parabola which is touched by the line y = x at the point where x = 1. Then -

(A) $f'(1) = 1$
(B) $f'(0) = f'(1)$
(C) $2f(0) = 1 - f'(0)$
(D) $f(0) + f'(0) + f''(0) = 1$

30. The function $f(x) = \int_0^x \sqrt{1 - t^4}\ dt$ is such that :

(A) it is defined on the interval $[-1,1]$
(B) it is an increasing function
(C) it is an odd function
(D) the point (0,0) is the point of inflection

31. Consider the curve $f(x) = x^{1/3}$, then -

(A) the equation of tangent at (0, 0) is x = 0
(B) the equation of normal at (0, 0) is y = 0
(C) normal to the curve does not exist at (0, 0)
(D) f(x) and its inverse meet at exactly 3 points.

32. Which of the following statements is/are correct ?

(A) x + sinx is increasing function
(B) sec x is neither increasing nor decreasing function
(C) x + sinx is decreasing function
(D) sec x is an increasing function

33. Let f and g be two functions defined on an interval I such that $f(x) \ge 0$ and $g(x) \le 0$ for all $x \in I$ and f is strictly decreasing on I while g is strictly increasing on I then -

(A) the product function fg is strictly increasing on I
(B) the product function fg is strictly decreasing on I
(C) fog(x) is monotonically increasing on I
(D) fog(x) is monotonically decreasing on I

34. Let f and g be two differentiable functions defined on an interval I such that $f(x) \ge 0$ and $g(x) \le 0$ for all $x \in I$ and f is strictly decreasing on I while g is strictly increasing on I then

(A) The product function fg is strictly increasing on I
(B) The product function fg is strictly decreasing on I
(C) fog(x) is monotonically increasing on I
(D) fog (x) is monotonically decreasing on I

35. Let $g(x) = 2f(x/2) + f(1 - x)$ and $f''(x) < 0$ in $0 \le x \le 1$ then g (x) -

(A) decreases in [0,2/3)
(B) decreases in (2/3,1]
(C) increases in [0,2/3)
(D) increases in (2/3,1]

36. The function $f(x) = x^{1/3}(x-1)$

(A) has 2 inflection points.

(B) is strictly increasing for $x > 1/4$ and strictly decreasing for $x < 1/4$.

(C) is concave down in $(-1/2, 0)$.

(D) Area enclosed by the curve lying in the fourth quadrant is 9/28.

37. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then -

(A) $a > 0, b > 0$ (B) $a > 0, b < 0$ (C) $a < 0, b > 0$ (D) $a < 0, b < 0$

38. Let $\phi(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3\sin x + 4\cos x \ \forall x \in R$, then

(A) ϕ is increasing whenever f is increasing

(B) ϕ is increasing whenever f is decreasing

(C) ϕ is decreasing whenever f is decreasing

(D) ϕ is decreasing if $f'(x) = -11$

39. If a continuous function $f(x)$ has a local maximum at $x = a$, then -

(A) $f'(a^+)$ may be 0

(B) $f'(a^+)$ may be $-\infty$

(C) $f'(a^+)$ may be non-zero finite real number

(D) $f'(a^-)$ may be $-\infty$

40. Let $f(x) = 8x^3 - 6x^2 - 2x + 1$, then -

(A) $f(x) = 0$ has no root in (0, 1)

(B) $f(x) = 0$ has at least one root in (0, 1)

(C) $f'(c)$ vanishes for some $c \in (0,1)$

(D) none

41. Let $f(x) = (x^2 - 1)^n (x^2 + x + 1)$. f(x) has local extremum at x = 1 if

(A) n = 2 (B) n = 3 (C) n = 4 (D) n = 6

42. The function $\frac{\sin(x+a)}{\sin(x+b)}$ has no maxima or minima if -

(A) $b - a = n\pi, \quad n \in I$

(B) $b - a = (2n+1)\pi \ n \in I$

(C) $b - a = 2n\pi, \quad n \in I$

(D) none of these.

43. Equation of common tangent(s) of $x^2 - y^2 = 12$ and $xy = 8$ is (are) -

(A) $y = 3x + 4\sqrt{6}$ (B) $y = -3x + 4\sqrt{6}$ (C) $3y = x + 4\sqrt{6}$ (D) $y = -3x - 4\sqrt{6}$

44. Which of the following statements are true :

(A) $|\tan^{-1} x - \tan^{-1} y| \leq |x - y|$, where x, y are real numbers.

(B) The function $x^{100} + \sin x - 1$ is strictly increasing in [0, 1]

(C) If a, b, c are is A.P, then at least one root of the equation $3ax^2 - 4bx + c = 0$ is positive

(D) Curve $y^2 = 4ax$ and $y = e^{-\frac{x}{2a}}$ are orthogonal curves.

45. Equation $\frac{1}{(x+1)^3} - 3x + \sin x = 0$ has -

(A) no real root

(B) two real and distinct roots

(C) exactly one negative root

(D) exactly one root between –1 and 1

46. Let $f(x) = \dfrac{x-1}{x^2}$ then which of the following is correct.

(A) $f(x)$ has minima but no maxima.

(B) $f(x)$ increases in the interval $(0, 2)$ and decreases in the interval $(-\infty, 0) \cup (2, \infty)$.

(C) $f(x)$ is concave down in $(-\infty, 0) \cup (0, 3)$.

(D) $x = 3$ is the point of inflection.

47. The coordinates of the point P on the graph of the function $y = e^{-|x|}$, where area of triangle made by tangent and the coordinate axis has the greatest area, is -

(A) $\left(1, \dfrac{1}{e}\right)$ **(B)** $\left(-1, \dfrac{1}{e}\right)$ **(C)** (e, e^{-e}) **(D)** none

48. Let $f(x) = 40/(3x^4 + 8x^3 - 18x^2 + 60)$. Which of the following statement(s) about f(x) is (are) correct ?

(A) f(x) has local minima at $x = 0$.

(B) f(x) has local maxima at $x = 0$.

(C) Absolute maximum value of f(x) is not defined.

(D) f(x) is local maxima at $x = -3$, $x = 1$.

49. Let $g(x) = 2f(x/2) + f(1-x)$ and $f''(x) < 0$ in $0 \le x \le 1$ then g(x)

(A) decreases in $\left[0, \dfrac{2}{3}\right]$

(B) decreases $\left[\dfrac{2}{3}, 1\right]$

(C) increases in $\left[0, \dfrac{2}{3}\right]$

(D) increases in $\left[\dfrac{2}{3}, 1\right]$

50. If $f(x) = \tan^{-1}x - (1/2)$ ●n x. Then

(A) the greatest value of f(x) on $\left[1/\sqrt{3}, \sqrt{3}\right]$ is $\pi/6 + (1/4)$ ●n 3

(B) the least value of f(x) on $\left[1/\sqrt{3}, \sqrt{3}\right]$ is $\pi/3 - (1/4)$ ●n 3

(C) f(x) decreases on $(0, \infty)$

(D) f(x) increases on $(-\infty, 0)$

Part # II [Assertion & Reason Type Questions]

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.
(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.
(C) Statement-I is true, Statement-II is false.
(D) Statement-I is false, Statement-II is true.

1. **Statement-I :** The ratio of length of tangent to length of normal is proportional to the ordinate of the point of tangency at the curve $y^2 = 4ax$.

Statement-II : Length of normal and tangent to a curve y = f(x) is $\left|y\sqrt{1+m^2}\right|$ and $\left|\frac{y\sqrt{1+m^2}}{m}\right|$, where $m = \frac{dy}{dx}$.

2. **Statement I :** The curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{1+a^2} + \frac{y^2}{1-b^2} = 1$ are orthogonal, for $b \in (-1,1)$.

Statement II : $ax^2 + by^2 = 1$ and $Ax^2 + By^2 = 1$ are orthogonal iff $ab(A-B) = AB(a-b)$.

3. **Statement-I :** The quadratic equation $10x^2 - 28x + 17 = 0$ has atleast one root in [1,2].

Statement-II : $f(x) = e^{10x}(x-1)(x-2)$ satisfies all the conditions for Rolle's theorem in [1,2]

4. Let $f(x) = x^{50} - x^{20}$

Statement-I : Global maximum of f(x) in [0, 1] is 0.

Statement-II : x = 0 is a stationary point of f(x).

5. **Statement-I :** The greatest of the numbers 1, $2^{1/2}$, $3^{1/3}$, $4^{1/4}$, $5^{1/5}$, $6^{1/6}$, $7^{1/7}$ is $3^{1/3}$.

Statement-II : $x^{1/x}$ is increasing for 0 < x < e and decreasing for x > e.

6. Consider the graph of the function $f(x) = x + \sqrt{|x|}$

Statement–I : The graph of $y = f(x)$ has only one critical point

Statement-II: $f'(x)$ vanishes only at one point

7. **Statement-I :** Any tangent to the curve $y = x^7 + 8x^3 + 2x + 1$ makes an acute angle with the positive x-axis.

Statement-II : Any tangent to the curve $y = a_0x^{2n+1} + a_1x^{2n-1} + a_2x^{2n-3} + \ldots\ldots + a_nx + 1$ makes an acute angle with the positive x-axis where $a_1, \ldots\ldots, a_{n-1} \geq 0$; $a_0, a_n > 0$ and $n \in N$.

8. **Statement-I :** Consider the function, $f(x) = \begin{cases} -\frac{x}{2}, & x < 0 \\ 7x + 8 & x \geq 0 \end{cases}$; f(x) has local minima at x = 0

Statement-II : If f(a) < f(a – h) & f(a) < f(a + h)
where 'h' is sufficiently small; then f(x) has local minima at x = a.

9. **Statement-I :** If f(x) is increasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

Statement-II : If f(x) is decreasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

10. Consider the polynomial function $f(x) = \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x$

Statement-I : The equation $f(x) = 0$ can not have two or more roots.
Statement-II : Rolles theorem is not applicable for $y = f(x)$ on any interval [a, b] where $a, b \in R$

11. **Statement-I :** The product of the ordinates to the point of tangency to the curve $(1 + x^2)y = 2 - x$, where the tangent makes equal intercept with coordinate axes is equal to 1.
Statement-II : Slope of straight line making equal intercept with coordinate axis is equal to 1.

12. **Statement-I :** e^{π} is bigger than π^e.
Statement-II : $f(x) = x^{1/x}$ is a increasing function when $x \in [e, \infty)$

13. **Statement-I :** The largest term in the sequence $a_n = \frac{n^2}{n^3 + 200}$, $n \in N$ is the 7^{th} term.
Statement-II : The function $f(x) = \frac{x^2}{x^3 + 200}$ attains local maxima at $x = 7$.

14. **Statement I :** ABC is given triangle having respective sides a,b,c. D,E,F are points of the sides BC,CA,AB respec tively so that AFDE is a parallelogram. The maximum area of the parallelogram is $\frac{1}{4}$ bcsinA.

Statement II : Maximum value of $2kx - x^2$ is at $x = k$.

15. Let $f : R \to R$, $f(x) = x^3 + x^2 + 3x + \sin x$.
Statement-I : f(x) is one-one.
Statement-II : f(x) is decreasing function.

16. Let $f(x) = ln(2 + x) - \frac{2x + 2}{x + 3}$.

Statement-I: The equation $f(x) = 0$ has a unique solution in the domain of $f(x)$.
Statement-II: If $f(x)$ is continuous in [a, b] and is strictly monotonic in (a, b) then f has a unique root in (a, b)

17. **Statement-I :** $e^{\pi} > \pi^e$.
Statement-II : The function $f(x) = x^{1/x}$ attains global maxima at $x = e$.

18. Let $y = f(x)$ be a thrice derivable function such that $f(a) f(b) < 0$, $f(b) f(c) < 0$, $f(c) f(d) < 0$ where $a < b < c < d$. Also the equations $f(x) = 0$ & $f''(x) = 0$ have no common roots.
Statement-I : The equation $f(x)(f''(x))^2 + f(x) f'(x) f'''(x) + (f'(x))^2 f''(x) = 0$ has atleast 5 real roots.
Statement-II : The equation $f(x) = 0$ has atleast 3 real distinct roots & if $f(x) = 0$ has k real distinct roots, then $f'(x) = 0$ has atleast $k - 1$ distinct roots.

Exercise # 3 Part # I [Matrix Match Type Questions]

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with one or more statement(s) in **Column-II**.

1.

Column-I		Column-II	
(A)	The angle of intersection of $y^2 = 4x$ and $x^2 = 4y$ is 90° and $\tan^{-1}\left(\frac{m}{n}\right)$ then $\|m + n\|$ is equal to (m and n are coprime)	(p)	0
		(q)	$\frac{1}{2}$
(B)	The area of triangle formed by normal at the point (1, 0) on the curve $x = e^{\sin y}$ with axes is	(r)	7
(C)	If the angle between curves $x^2y = 1$ and $y = e^{2(1-x)}$ at the point (1, 1) is θ then tanθ is equal to		
(D)	The length of sub-tangent at any point on the curve $y = be^{x/3}$ is equal to	(s)	3

2.

Column-I		Column-II	
(A)	If θ is angle between the curves $y = [\|\sin x\| + \|\cos x\|]$, ([·] denote GIF) and $x^2 + y^2 = 5$ then $\text{cosec}^2\theta$ is	(p)	$\frac{5}{4}$
(B)	Length of subnormal to $x = \sqrt{2}\cos t$, $y = -3\sin t$ at $t = \frac{-\pi}{4}$ is	(q)	2
(C)	If [a, b], (b < 1) is largest interval in which $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 19$ is strictly increasing then $\frac{a}{b}$ is	(r)	$\frac{8}{3}$
(D)	If a + b = 8, a, b > 0 then minimum value of $\frac{a^3 + b^3}{48}$ is	(s)	$\frac{9}{2}$

3.

Column-I		Column-II	
(A)	The equation x log x = 3 – x has at least one root in	(p)	[0, 1]
(B)	If 27a + 9b + 3c + d = 0, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in	(q)	[1, 3]
(C)	If $c = \sqrt{3}$ & $f(x) = x + \frac{1}{x}$ then interval in which LMVT is applicable for f(x) is	(r)	[0, 3]
(D)	If $c = \frac{1}{2}$ & $f(x) = 2x - x^2$, then interval in which LMVT is applicable for f(x) is	(s)	[–1, 1]

4.

Column – I		Column – II	
(A)	$f(x)=\dfrac{\sin x}{e^{x}}, x\in[0,\pi]$	**(p)**	Conditions in Rolle's theorem are satisfied.
(B)	$f(x)=\operatorname{sgn}((e^{x}-1)\ \ell n x), x\in\left[\dfrac{1}{2},\dfrac{3}{2}\right]$	**(q)**	Conditions in LMVT are satisfied.
(C)	$f(x)=(x-1)^{2/5}, x\in[0,3]$	**(r)**	At least one condition in Rolle's theorem is not satisfied.
(D)	$f(x)=\begin{cases} x\left(\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right), & x\in[-1,1]-\{0\} \\ 0, & x=0 \end{cases}$	**(s)**	At least one condition in LMVT is not satisfied.

5. Four points A, B, C and D lie in the order on the parabola $y = ax^2 + bx + c$ and the coordinates of A, B and D are known A(–2, 3); B(–1, 1); D(2, 7).

On the basis of above information match the following :

Column-I		Column-II	
(A)	The value of $a + b + c =$	**(p)**	–1
(B)	If roots of the equation $ax^2 + bx + c = 0$ are α & β then $\alpha^{19} + \beta^7 =$	**(q)**	8
(C)	If the value of function $(a + 2)x^2 + 2\dfrac{(b+2)}{x} + c$ at minima is L then L – 3 is equal to	**(r)**	3
(D)	If area of quadrilateral ABCD is greatest and co-ordinates of C are (p, q) then 2p + 4q =	**(s)**	7

6.

Column – I		Column – II	
(A)	A rectangle is inscribed in an equilateral triangle of side 4cm. Square of maximum area of such a rectangle is	**(p)**	65
(B)	The volume of a rectangular closed box is 72 and the base sides are in the ratio 1 : 2. The least total surface area is	**(q)**	36
(C)	Maximum value of $\left(\sqrt{-3+4x-x^2}+4\right)^2+(x-5)^2$ (where $1\le x\le 3$) is	**(r)**	12
(D)	The sides of a rectangle of greatest perimeter which is inscribed in a semicircle of radius $\sqrt{5}$ are a and b. Then $a^3+b^3=$	**(s)**	108

7.

Column-I		Column-II	
(A)	The slope of the curve $2y^2=ax^2+b$ at $(1, -1)$ is -1, then	**(p)**	$a - b = 2$
(B)	If (a, b) be the point on the curve $9y^2 = x^3$ where normal to the curve makes equal intercepts with the axes, then	**(q)**	$a - b = 7/2$
(C)	If the tangent at a point $(1, 2)$ on the curve $y = ax^2 + bx + \frac{7}{2}$ be parallel to the normal at $(-2, 2)$ on the curve $y = x^2 + 6x + 10$, then	**(r)**	$a - b = 4/3$
(D)	If the tangent to the curve $xy + ax + by = 0$ at $(1, 1)$ is inclined at an angle $\tan^{-1} 2$ with x-axis, then	**(s)**	$a - b = 3$

8.

Column - I		Column - II	
(A)	$\sin^{-1} x - \cos^{-1} x$ is maximum at	**(p)**	$x = -1$
(B)	$(\sin^{-1} x)^2 + (\cos^{-1} x)^2$ is minimum at	**(q)**	$x = -\frac{1}{\sqrt{2}}$
(C)	$(\tan^{-1} x)^2 + (\cot^{-1} x)^2$ is minimum at	**(r)**	$x = 0$
(D)	$(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ is maximum at	**(s)**	$x = \frac{1}{\sqrt{2}}$
		(t)	$x = 1$

9. For the function $f(x) = x^4(12$●$nx - 7)$ match the following :

Column-I		Column-II	
(A)	If (a, b) is the point of inflection then $a - b$ is equal to	**(p)**	3
(B)	If e^t is a point of minima then $12t$ is equal to	**(q)**	1
(C)	If graph is concave downward in (d, e) then $d + 3e$ is equal to	**(r)**	4
(D)	If graph is concave upward in (p, ∞) then the least value of p is equal to	**(s)**	8

Part # II [Comprehension Type Questions]

Comprehension # 1

Consider the function $f(x) = x^2 f(1) - xf'(2) + f''(3)$ such that $f(0) = 2$

1. The values of $f'(1)$ is -

(A) 0 **(B)** 1 **(C)** 2 **(D)** 1

2. Equation of tangent to $y = f(x)$ at $x = 3$ is -

(A) $y = x - 7$ **(B)** $y = \frac{x}{4} - 7$ **(C)** $y = 4x - 7$ **(D)** none of these

3. The angle of intersection of $y = f(x)$ and $y = 2e^{2x}$ is -

(A) $\tan^{-1}\left(\frac{3}{4}\right)$ **(B)** $\tan^{-1}\left(\frac{4}{3}\right)$ **(C)** 0 **(D)** $\tan^{-1}\left(\frac{6}{7}\right)$

Comprehension # 2

Let a(t) be a function of t such that $\frac{da}{dt} = 2$ for all values of t and a = 0 when t = 0. Further $y = m(t)\,x + c(t)$ is tangent to the curve $y = x^2 - 2ax + a^2 + a$ at the point whose abscissa is 0. Then

1. If the rate of change of distance of vertex of $y = x^2 - 2ax + a^2 + a$ from the origin with respect to t is k, then k =

(A) 2 **(B)** $2\sqrt{2}$ **(C)** $\sqrt{2}$ **(D)** $4\sqrt{2}$

2. If the rate of change of c(t) with respect to t, when t = k, is ●, then ● =

(A) $16\sqrt{2} - 2$ **(B)** $8\sqrt{2} + 2$ **(C)** $10\sqrt{2} + 2$ **(D)** $16\sqrt{2} + 2$

3. The rate of change of m(t), with respect to t, at t = ● is

(A) –2 **(B)** 2 **(C)** – 4 **(D)** 4

Comprehension # 3

Consider the cubic $f(x) = 8x^3 + 4ax^2 + 2bx + a$ where $a, b \in R$.

1. For a = 1 if $y = f(x)$ is strictly increasing $\forall\, x \in R$ then maximum range of values of b is

(A) $\left(-\infty, \frac{1}{3}\right]$ **(B)** $\left(\frac{1}{3}, \infty\right)$ **(C)** $\left[\frac{1}{3}, \infty\right)$ **(D)** $(-\infty, \infty)$

2. For b = 1, if $y = f(x)$ is non monotonic then the sum of all the integral values of $a \in [1, 100]$, is

(A) 4950 **(B)** 5049 **(C)** 5050 **(D)** 5047

3. If the sum of the base 2 logarithms of the roots of the cubic $f(x) = 0$ is 5 then the value of 'a' is

(A) – 64 **(B)** – 8 **(C)** – 128 **(D)** – 256

Comprehension # 4

Suppose you do not know the function f (x), however some information about $f(x)$ is listed below. Read the following carefully before attempting the questions

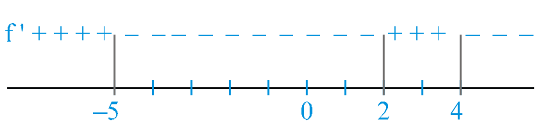
(i) $f(x)$ is continuous and defined for all real numbers

(ii) $f'(-5) = 0$; $f'(2)$ is not defined and $f'(4) = 0$

(iii) (–5, 12) is a point which lies on the graph of f (x)

(iv) $f''(2)$ is undefined, but $f''(x)$ is negative everywhere else.

(v) the signs of $f'(x)$ is given below



1. On the possible graph of $y = f(x)$ we have

(A) $x = -5$ is a point of relative minima.

(B) $x = 2$ is a point of relative maxima.

(C) $x = 4$ is a point of relative minima.

(D) graph of $y = f(x)$ must have a geometrical sharp corner.

2. From the possible graph of $y = f(x)$, we can say that

(A) There is exactly one point of inflection on the curve.

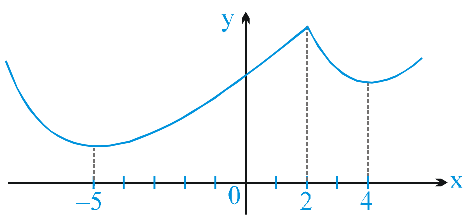
(B) $f(x)$ increases on $-5 < x < 2$ and $x > 4$ and decreases on $-\infty < x < -5$ and $2 < x < 4$.

(C) The curve is always concave down.

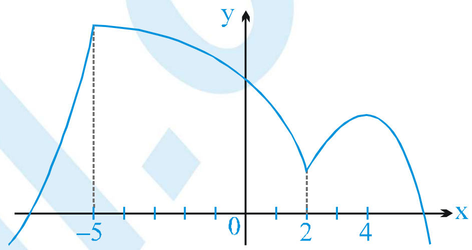
(D) Curve always concave up.

3. Possible graph of $y = f(x)$ is

(A)



(B)



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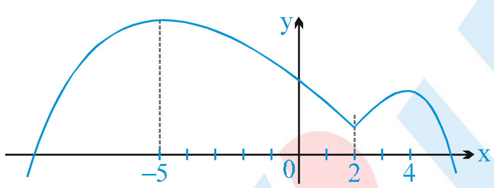
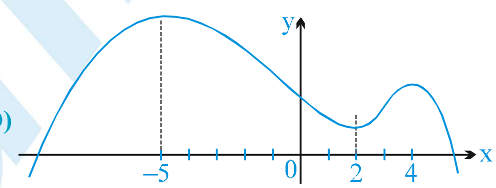


(D)



Comprehension # 5

Let y = f(x) be a differentiable function which satisfies $f'(x) = f^2(x)$ and $f(0) = -\frac{1}{2}$. The graph of the differentiable function y = g(x) contains the point (0, 2) and has the property that for each number 'P', the line tangent to y = g(x) at (P, g(P)) intersects x-axis at P + 2.

On the basis of above information, answer the following questions :

1. If the tangent is drawn to the curve y = f(x) at a point where it crosses the y-axis then its equation is -

(A) $x - 4y = 2$ **(B)** $x + 4y = 2$ **(C)** $x + 4y + 2 = 0$ **(D)** none of these

2. The function y = g(x) is given by -

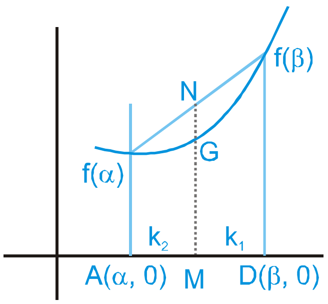
(A) $\frac{e^{-x/2}}{2}$ **(B)** $e^{-x/2}$ **(C)** $2.e^{-x/2}$ **(D)** $e^{-x/2} + 2$

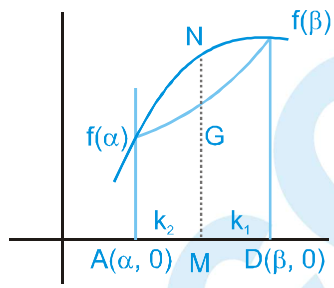
3. The number of point of intersection of y = f(x) and y = g(x) -

(A) 4 **(B)** 0 **(C)** 2 **(D)** 1

Comprehension # 6

For a double differentiable function f(x) if $f''(x) \ge 0$ then f(x) is concave upward and if $f''(x) \le 0$ then f(x) is concave downward





Here $M\left(\frac{k_1\alpha+k_2\beta}{k_1+k_2}, 0\right)$

If f(x) is a concave upward in [a, b] and $\alpha, \beta \in [a, b]$ then $\frac{k_1 f(\alpha)+k_2 f(\beta)}{k_1+k_2} \ge f\left(\frac{k_1\alpha+k_2\beta}{k_1+k_2}\right)$, where $k_1, k_2 \in R^+$

If f(x) is a concave downward in [a, b] and $\alpha, \beta \in [a, b]$ then $\frac{k_1 f(\alpha)+k_2 f(\beta)}{k_1+k_2} \le f\left(\frac{k_1\alpha+k_2\beta}{k_1+k_2}\right)$, where $k_1, k_2 \in R^+$

then answer the following

1. Which of the following is true

(A) $\frac{\sin\alpha+\sin\beta}{2} > \sin\left(\frac{\alpha+\beta}{2}\right)$; $\alpha, \beta\ (0, \pi)$

(B) $\frac{\sin\alpha+\sin\beta}{2} < \sin\left(\frac{\alpha+\beta}{2}\right)$; $\alpha, \beta \in (\pi, 2\pi)$

(C) $\frac{\sin\alpha+\sin\beta}{2} < \sin\left(\frac{\alpha+\beta}{2}\right)$; $\alpha, \beta \in (0, \pi)$

(D) none of these

2. Which of the following is true

(A) $\frac{2^\alpha+2^{\beta+1}}{3} \le 2^{\frac{\alpha+2\beta}{3}}$

(B) $\frac{2\ln\alpha+\ln\beta}{3} \ge \ln\left(\frac{2\alpha+\beta}{3}\right)$

(C) $\frac{\tan^{-1}\alpha+\tan^{-1}\beta}{2} \le \tan^{-1}\left(\frac{\alpha+\beta}{2}\right)$ a, b $\in R^-$

(D) $\frac{e^\alpha+2e^\beta}{3} \ge e^{\frac{\alpha+2\beta}{3}}$

3. Let α, β and γ are three distinct real numbers and $f''(x) < 0$. Also f(x) is increasing function and let $A = \frac{f^{-1}(\alpha)+f^{-1}(\beta)+f^{-1}(\gamma)}{3}$ and $B = f^{-1}\left(\frac{\alpha+\beta+\gamma}{3}\right)$, then order relation between A and B is - (given $f^{-1}(x)$)

(A) A > B **(B)** A < B **(C)** A = B **(D)** none of these

Comprehension # 7

Consider the polynomial function

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. This function has monotonicity as given below :

in $(-\infty, a_1)$	decreasing
in (a_1, a_2)	increasing
in (a_2, a_3)	decreasing
in (a_3, ∞)	increasing

A rectangle ABCD is formed such that

● (AB)= portion of the tangent to the curve $y = f(x)$ at $x = a_1$, intercepted between the lines $x = a_1$ & $x = a_3$.

● (BC)= portion of the line $x = a_3$ intercepted between the curve & x-axis.

1. Triplet (a_1, a_2, a_3) is given by -

(A) (–1, 0, 2) **(B)** (0, –1, 2) **(C)** (2, –1, 0) **(D)** (2, 0, –1)

2. Area of rectangle ABCD -

(A) 51 **(B)** 57 **(C)** 87 **(D)** 81

3. The equation $f(x) = 0$ has -

(A) 2 real, 2 imaginary roots **(B)** 2 complex, 2 irrational roots

(C) 4 real & distinct roots **(D)** 2 real coincident roots & 2 irrational roots

Comprehension # 8

Let $f(x) = \left(1+\frac{1}{x}\right)^x \; (x > 0)$ and $g(x) = \begin{cases} x\ln\left(1+(1/x)\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$

1. $\lim\limits_{x\to 0^+} g(x)$

(A) is equal to 0 **(B)** is equal to 1 **(C)** is equal to e **(D)** is non existent

2. The function f

(A) has a maxima but no minima **(B)** has a minima but no maxima

(C) has both a maxima and minima **(D)** is monotonic

3. $\lim\limits_{n\to\infty}\left\{f\left(\frac{1}{n}\right)\cdot f\left(\frac{2}{n}\right)\cdot f\left(\frac{3}{n}\right)\ldots\ldots\ldots f\left(\frac{n}{n}\right)\right\}^{1/n}$ equals

(A) $\sqrt{2}\,e$ **(B)** $\sqrt{2e}$ **(C)** $2\sqrt{e}$ **(D)** $\sqrt{e}$

Comprehension # 9

Consider a function f defined by $f(x) = \sin^{-1}\sin\left(\frac{x+\sin x}{2}\right)$, $\forall\, x \in [0, \pi]$, which satisfies $f(x) + f(2\pi - x) = \pi$, $\forall\, x \in [\pi, 2\pi]$ and $f(x) = f(4\pi - x)$ for all $x \in [2\pi, 4\pi]$, then

1. If α is the length of the largest interval on which f(x) is increasing, then $\alpha =$

(A) $\frac{\pi}{2}$ **(B)** π **(C)** 2π **(D)** 4π

2. If f(x) is symmetric about x = β, then β =

(A) $\frac{\alpha}{2}$ **(B)** α **(C)** $\frac{\alpha}{4}$ **(D)** 2α

3. Maximum value of f(x) on [0, 4π] is :

(A) $\frac{\beta}{2}$ **(B)** β **(C)** $\frac{\beta}{4}$ **(D)** 2β

Comprehension # 10

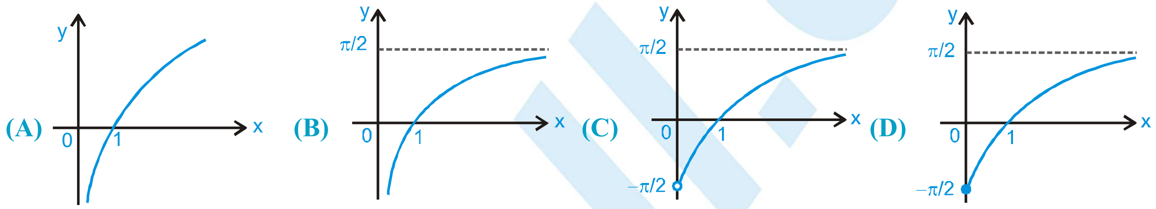
$f:(0,\infty)\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ be defined as, $f(x)=\tan^{-1}(\bullet nx)$

1. The above function can be classified as

(A) injective but not surjective **(B)** surjective but not injective

(C) neither injective nor surjective **(D)** both injective as well as surjective

2. The graph of y = f(x) is best represented as -



3. If x_1, x_2 and x_3 are the points at which g(x) = [f(x)] is discontinuous where [.] denotes greatest integer function, then $x_1+x_2+x_3$ is -

(A) equal to 2 **(B)** equal to 3 **(C)** greater than 3 **(D)** greater than 2 but less than 3

Comprehension # 11

Suppose f(x) is a real valued polynomial function of degree 6 satisfying the following condition

(A) 'f' has minimum value at x = 0 & 2

(B) 'f' has maximum value at x = 1

(C) for all x, $\lim_{x\to 0}\frac{1}{x}\ln\begin{vmatrix} f(x)/x & 1 & 0 \\ 0 & 1/x & 1 \\ 1 & 0 & 1/x \end{vmatrix}=2$

1. Number of solutions of the equation 8f(x) – 1 = 0 is -

(A) one **(B)** two **(C)** three **(D)** four

2. Range of f(x) is -

(A) $\left[-\frac{32}{15},\infty\right)$ **(B)** $\left[-\frac{4}{15},\infty\right)$ **(C)** $\left(-\infty,\frac{2}{15}\right]$ **(D)** none of these

3. If the area bounded by y = f(x), x-axis, x = ± 1; is $\frac{a}{b}$, where a & b are relatively prime then the value of $\tan^{-1}(a-b)$ is -

(A) π/4 **(B)** –π/4 **(C)** π/3 **(D)** π/6

Exercise # 4 [Subjective Type Questions]

1. Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$.

2. Find the set of all values of the parameter 'a' for which the function
$f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for all $x \in R$.

3. The curve $y = ax^3 + bx^2 + cx + 5$, touches the x-axis at $P(-2, 0)$ and cuts the y axis at a point Q, where its gradient is 3. Find a, b, c.

4. Find the points of local maxima/minima of following functions

(A) $f(x) = 2x^3 - 21x^2 + 36x - 20$

(B) $f(x) = -(x - 1)^3(x + 1)^2$

(C) $f(x) = x$●nx

(D) $y = \dfrac{3x^2 + 4x + 4}{x^2 + x + 1}$

5. A function is defined parametrically by the equations

$$x = \begin{cases} 2t + t^2 \sin\dfrac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \quad \text{and} \quad y = \begin{cases} \dfrac{1}{t}\sin t^2 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Find the equation of the tangent and normal at the point for $t = 0$ if they exist.

6. For $x \in \left(0, \dfrac{\pi}{2}\right)$ identify which is greater $(2\sin x + \tan x)$ or $(3x)$. Hence find $\lim\limits_{x \to 0}\left[\dfrac{3x}{2\sin x + \tan x}\right]$, where [.] denote the greatest integer function.

7. A man 1.5 m tall walks away from a lamp post 4.5 m high at a rate of 4 km/hr.

(i) How fast is his shadow lengthening ?

(ii) How fast is the farther end of shadow moving on the pavement ?

8. Find the absolute maxima/minima value of following functions

(A) $f(x) = 4x - \dfrac{x^2}{2}$; $x \in \left[-2, \dfrac{9}{2}\right]$

(B) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $x \in [0, 3]$

(C) $f(x) = \sin x + \dfrac{1}{2}\cos 2x$; $x \in \left[0, \dfrac{\pi}{2}\right]$

9. If p_1 and p_2 be the lengths of the perpendiculars from the origin on the tangent and normal respectively at any point (x, y) on a curve, then show that $\begin{bmatrix} p_1 = |x\sin\Psi - y\cos\Psi| \\ p_2 = |x\cos\Psi + y\sin\Psi| \end{bmatrix}$ where $\tan\Psi = \dfrac{dy}{dx}$. If in the above case, the curve be $x^{2/3} + y^{2/3} = a^{2/3}$ then show that : $4p_1^2 + p_2^2 = a^2$

10. If f, ϕ, ψ are continuous in [a,b] and derivable in]a,b[then show that there is a value of c lying between a & b such

that, $\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix} = 0$

11. If $y = \dfrac{ax+b}{(x-1)(x-4)}$ has a turning value at (2, –1) find a & b and show that the turning value is a maximum.

12. Show that the angle between the tangent at any point 'A' of the curve $\ell n(x^2+y^2) = C\tan^{-1}\dfrac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

13. Using LMVT prove that : **(A)** $\tan x > x$ in $\left(0, \dfrac{\pi}{2}\right)$, **(B)** $\sin x < x$ for $x > 0$.

14. Show that the locus of point of inflection of the curve $y = x\sin x$ is $y^2(4+x^2) = 4x^2$

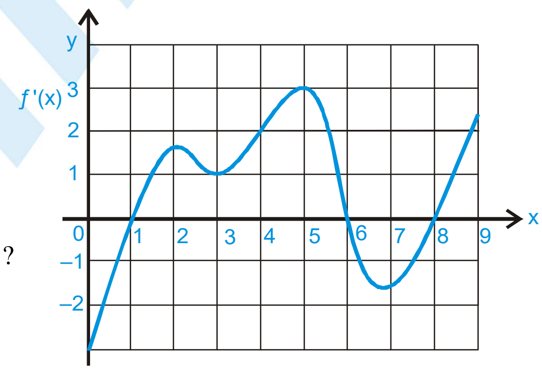
15. Suppose f(x) is a function satisfying the following conditions -

(i) $f(0) = 2, f(1) = 1$ **(ii)** If f(x) has a minimum value at $x = \dfrac{5}{2}$ and

(iii) for all x $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Where a, b are some constants. Determine the constants a, b, & the function f(x).

16. The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$



(i) On what intervals is f increasing or decreasing ?

(ii) At what values of x does f have a local maximum or minimum ?

(iii) On what intervals is f concave upward or downward ?

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that $f(0) = 0$, sketch a graph of f .

17. **(A)** Find the condition that the curves $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ & $\dfrac{x^2}{a'} + \dfrac{y^2}{b'} = 1$ may cut orthogonally.

(B) Show that the curves $\dfrac{x^2}{a^2+K_1} + \dfrac{y^2}{b^2+K_1} = 1$ & $\dfrac{x^2}{a^2+K_2} + \dfrac{y^2}{b^2+K_2} = 1$ intersect orthogonally ($K_1 \neq K_2$).

18. Find the greatest & the least values of the following functions in the given interval if they exist.

(A) $f(x) = 12x^{4/3} - 6x^{1/3}, x \in [-1, 1]$ **(B)** $y = x^5 - 5x^4 + 5x^3 + 1$ in $[-1, 2]$

(C) $y = \sin 2x - x$ $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ **(D)** $y = 2\tan x - \tan^2 x$ $\left[0, \dfrac{\pi}{2}\right)$

19. A window of perimeter P(including the base of the arch) is in the form of a rectangle surmounded by a semi circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square meter as the coloured glass does. What is the ratio for the sides of the rectangle so that the window transmits the maximum light ?

20. Let f(x) and g(x) be two functions which cut each other orthogonally at their common point of intersection (x_1). Both f(x) and g(x) are equal to 0 at $x = x_1$. Also $|f'(x_1)| = |g'(x_1)|$, then find $\lim_{x\to x_1}$ [f(x) . g(x)], where [.] denotes greatest integer functions.

21. If $ax^2 + \frac{b}{x} \geq c$ for all positive x where a > 0 and b > 0, then show that $27ab^2 \geq 4c^3$.

22. Find the points on the curve $ax^2+2bxy+ay^2 = c$; c>b>a>0, whose distance from the origin is minimum.

23. Tangent at a point P_1 {other than (0, 0)} on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissae of $P_1, P_2, P_3, \ldots, P_n$, form a G.P. Also find the ratio [area $(\Delta P_1P_2P_3)$]/[area $(\Delta P_2P_3P_4)$].

24. A swimmer S is in the sea at a distance d km from the closest point A on a straight shore. The house of the swimmer is on the shore at a distance L km from A. He can swim at a speed of u km/hr and walk at a speed of v km/hr (v > u). At what point on the shore should he land so that he reaches his house in the shortest possible time ?

25. Using monotonicity prove that $\frac{\tan x_2}{\tan x_1} > \frac{x_2}{x_1}$ for $0 < x_1 < x_2 < \frac{\pi}{2}$

26. Let $f(x) = \sin^3 x + \lambda \sin^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in order that f(x) has exactly one minimum and exactly one maximum.

27. John has 'x' children by his first wife and Anglina has 'x + 1' children by her first husband. They both marry and have their own children. The whole family has 24 children. It is given that the children of the same parents don't fight. Then find then maximum number of fights that can take place in the family.

28. Investigate the behaviour of the function $y = (x^3 + 4)/(x + 1)^3$ and construct its graph. How many solutions does the equation $(x^3 + 4)/(x + 1)^3 = c$ possess ?

29. Let $A(p^2, -p)$, $B(q^2, q)$, $C(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram AFDE is drawn with vertices D, E and F on the line segments BC, CA and AB respectively. Using calculus, show that maximum area of such a parallelogram is $\frac{1}{4}(p + q)(q + r)(p - r)$

30. What normal to the curve $y = x^2$ forms the shortest chord ?

31. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point (0, –2).

32. If the equation $a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1}x = 0$ has a positive root α, prove that the equation $na_0x^{n-1} + (n-1)a_1x^{n-2} + \ldots\ldots\ldots + a_{n-1} = 0$ also has a positive root smaller that α.

33. Show that $1 + x\ln\left(x + \sqrt{x^2+1}\right) \geq \sqrt{1 + x^2}$ for all x ∈ R.

34. Determine the points of maxima and minima of the function $f(x) = \frac{1}{8}\ln x - bx + x^2$, x > 0 where b ≥ 0 is a constant.

35. For $x \in \left(0, \frac{\pi}{2}\right)$ identify which is greater (2sinx + tanx) or (3x). Hence find $\lim_{x \to 0^+} \left[\frac{3x}{2\sin x + \tan x}\right]$ where [.] denote the GIF.

36. Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the length of the sides of the quadrilateral, prove that $2 \le a^2 + b^2 + c^2 + d^2 \le 4$.

37. If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$

38. f(x) is a differentiable function and g(x) is a double differentiable function such that $|f(x)| \le 1$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$. Prove that there exists some $c \in (-3, 3)$ such that $g(c).\ g''(c) < 0$

39. Let f(x) be differentiable function and g(x) be twice differentiable function. Zeros of f(x), $g'(x)$ be a, b respectively (a < b). Show that there exists at least one root of equation $f'(x)\, g'(x) + f(x)\, g''(x) = 0$ on (a, b).

40. Find the co-ordinates of all the points P on the ellipse $(x^2/a^2)+(y^2/b^2) = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N the foot of the perpendicular from O to the tangent at P.

Exercise # 5 Part # I [Previous Year Questions] [AIEEE/JEE-MAIN]

1. f(x) and g(x) are two differentiable function in [0, 2] such that $f''(x) - g''(x) = 0$, $f'(1) = 2$, $g'(1) = 4$, $f(2) = 3$, $g(2) = 9$, then $f(x) - g(x)$ at $x = 3/2$ is- **[AIEEE-2002]**

(1) 0 **(2)** 2 **(3)** 10 **(4)** –5

2. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals- **[AIEEE-2003]**

(1) 1/2 **(2)** 3 **(3)** 1 **(4)** 2

3. The real number x when added to its inverse gives the minimum value of the sum at x equal to- **[AIEEE-2003]**

(1) –2 **(2)** 2 **(3)** 1 **(4)** –1

4. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ then the difference between the maximum and minimum values of u^2 is given by- **[AIEEE-2004]**

(1) $2(a^2 + b^2)$ **(2)** $2\sqrt{a^2 + b^2}$ **(3)** $(a + b)^2$ **(4)** $(a - b)^2$

5. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2,1) and at that point the tanget to the graph is $y = 3x - 5$, then the function, is- **[AIEEE-2004]**

(1) $(x - 1)^2$ **(2)** $(x - 1)^3$ **(3)** $(x + 1)^3$ **(4)** $(x + 1)^2$

6. The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at 'θ' always passes through the fixed point- **[AIEEE-2004]**

(1) (a, 0) **(2)** (0, a) **(3)** (0, 0) **(4)** (a, a)

7. A function is matched below against an interval where it is supposed to be increasing. which of the following pairs is incorrectly matched ? **[AIEEE-2005]**

interval	function
(1) $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(2) $[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(3) $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(4) $(-\infty, -4)$	$x^3 + 6x^2 + 6$

8. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that- **[AIEEE-2005]**

(1) it passes through the origin **(2)** it makes angle $\left(\frac{\pi}{2} + \theta\right)$ with the x-axis

(3) it passes through $\left(a\frac{\pi}{2}, -a\right)$ **(4)** it is a constant distance from the origin

9. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is- **[AIEEE-2006]**

(1) $\frac{\pi}{2}$ **(2)** $\frac{\pi}{6}$ **(3)** $\frac{\pi}{4}$ **(4)** $\frac{\pi}{3}$

10. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at- **[AIEEE-2006]**

(1) $x = -2$ (2) $x = 0$ (3) $x = 1$ (4) $x = 2$

11. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is- **[AIEEE-2006]**

(1) $\sqrt{\frac{x^3}{8}}$ (2) $\frac{1}{2}x^2$ (3) πx^2 (4) $\frac{3}{2}x^2$

12. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in- **[AIEEE-2007]**

(1) $(\pi/4, \pi/2)$ (2) $(-\pi/2, \pi/4)$ (3) $(0, \pi/2)$ (4) $(-\pi/2, \pi/2)$

13. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p + q)$ is- **[AIEEE-2007]**

(1) 2 (2) $\frac{1}{2}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\sqrt{2}$

14. Suppose the cubic $x^3 - px + q$ has three real roots where $p > 0$ and $q > 0$. Then which of the following holds ? **[AIEEE-2008]**

(1) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$ (2) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(3) The cubic has minima at both $-\sqrt{\frac{p}{3}}$ & $\sqrt{\frac{p}{3}}$ (4) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ & $-\sqrt{\frac{p}{3}}$

15. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x=0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$:-

(1) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P. **[AIEEE-2009]**

(2) Neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

(3) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(4) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

16. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is :- **[AIEEE-2009]**

(1) $\frac{3\sqrt{2}}{5}$ (2) $\frac{\sqrt{3}}{4}$ (3) $\frac{3\sqrt{2}}{8}$ (4) $\frac{2\sqrt{3}}{8}$

17. Let $f : R \to R$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$

Statement–1 : $f(c) = \frac{1}{3}$, for some $c \in R$.

Statement–2 : $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all $x \in R$. **[AIEEE-2010]**

(1) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

18. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is :- **[AIEEE-2010]**

(1) $y = 0$ **(2)** $y = 1$ **(3)** $y = 2$ **(4)** $y = 3$

19. Let $f : R \to R$ be defined by $f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$ **[AIEEE-2010]**

If f has a local minimum at x = – 1, then a possible value of k is :

(1) 1 **(2)** 0 **(3)** $-\frac{1}{2}$ **(4)** –1

20. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then f has :- **[AIEEE-2011]**

(1) local minimum at π and local maximum at 2π

(2) local maximum at π and local minimum at 2π

(3) local maximum at π and 2π

(4) local minimum at π and 2π

21. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is :- **[AIEEE-2011]**

(1) $\frac{8}{3\sqrt{2}}$ **(2)** $\frac{4}{\sqrt{3}}$ **(3)** $\frac{\sqrt{3}}{4}$ **(4)** $\frac{3\sqrt{2}}{8}$

22. Let f be a function defined by $f(x) = \begin{cases} \frac{\tan x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

Statement - 1: x = 0 is point of minima of f.

Statement - 2: $f'(0) = 0$. **[AIEEE-2011]**

(1) Statement-1 is false, statement-2 is true.

(2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.

(3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.

(4) Statement-1 is true, statement-2 is false.

23. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[AIEEE-2012]**

(1) 9/2 **(2)** 9/7 **(3)** 7/9 **(4)** 2/9

24. Let a, b $\in$ R be such that the function f given by $f(x) = \ln|x| + bx^2 + ax$, $x \ne 0$ has extreme values at $x = -1$ and $x = 2$.

Statement–1 : f has local maximum at x = –1 and at x = 2.

Statement–2 : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$. **[AIEEE-2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

25. The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\, dt$, $x \in R$, which are parallel to the line y = 2x, are equal to **[JEE-MAIN 2013]**

(1) ± 1 **(2)** ± 2 **(3)** ± 3 **(4)** ± 4

26. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1] **[JEE-MAIN 2013]**

(1) lies between 1 and 2. **(2)** lies between 2 and 3.

(3) lies between –1 and 0 **(4)** does not exist

27. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1) : **[JEE Main 2015]**

(1) meets the curve again in the thrid quadrant.

(2) meets the curve again in the fourth quardant.

(3) does not meets the curve again.

(4) meets the curve again in the second quardant.

28. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then : **[JEE Main 2016]**

(1) $(4 - \pi)x = \pi r$ **(2)** $x = 2r$ **(3)** $2x = r$ **(4)** $2x = (\pi + 4)r$

Part # II [Previous Year Questions][IIT-JEE ADVANCED]

1. If the normal to the curve, y = f(x) at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis. Then f'(3) **[JEE 2000]**

(A) –1 **(B)** $-\frac{3}{4}$ **(C)** $\frac{4}{3}$ **(D)** 1

2. **(A)** For all $x \in (0, 1)$:

(A) $e^x < 1 + x$ **(B)** $\log_e(1+x) < x$ **(C)** $\sin x > x$ **(D)** $\log_e x > x$

(B) Consider the following statements S and R : **[JEE 2000]**

S : Both sin x & cos x are decreasing functions in the interval $(\pi/2, \pi)$.

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b)

Which of the following is true ?

(A) both S and R are wrong

(B) both S and R are correct, but R is not the correct explanation for S

(C) S is correct and R is the correct explanation for S

(D) S is correct and R is wrong.

3. Let $f(x) = \begin{cases} |x| & \text{for } 0 < |x| \le 2 \\ 1 & \text{for } x = 0 \end{cases}$. Then at x = 0, 'f' has - **[JEE 2000]**

(A) a local maximum **(B)** no local maximum

(C) a local minimum **(D)** no extremum.

4. Let $f(x) = \int e^x (x-1)(x-2)dx$ then f decreases in the interval - **[JEE 2000]**

(A) $(-\infty, 2)$ **(B)** $(-2, -1)$ **(C)** $(1, 2)$ **(D)** $(2, \infty)$

5. If $f(x) = xe^{x(1-x)}$, then f(x) is - **[JEE 2001]**

(A) increasing on $\left(-\frac{1}{2}, 1\right)$ **(B)** decreasing on R

(C) increasing on R **(D)** decreasing on $\left[-\frac{1}{2}, 1\right]$

6. Let $-1 \le p \le 1$. Show that the equations $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it. **[JEE 2001]**

7. The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is - **[JEE 2002]**

(A) $\frac{\pi}{3}$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{3\pi}{2}$ **(D)** π

8. The point(s) on the curve $y^3+3x^2 = 12y$ where the tangent is vertical, is(are) -

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ **(B)** $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$ **(C)** $(0, 0)$ **(D)** $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$ **[JEE 2002]**

9. **(A)** Using the relation $2(1 - \cos x) < x^2$, $x \ne 0$ or otherwise, prove that $\sin(\tan x) \ge x$ $\forall$ $x \in \left[0, \frac{\pi}{4}\right]$.

(B) Let $f : [0, 4] \to R$ be a differentiable function. **[JEE 2003]**

(i) Show that there exist $a, b \in [0, 4]$, $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$

(ii) Show that there exist α, β with $0 < \alpha < \beta < 2$ such that $\int_0^4 f(t)dt = 2(\alpha f(\alpha^2) + \beta f(\beta^2))$

10. The minimum value of $f(x) = x^2 + 2bx + 2c^2$ is more than the maximum value of $g(x) = -x^2 - 2cx + b^2$, x being real, for - **[JEE 2003]**

(A) $|c| > |b|\sqrt{2}$ **(B)** $0 < c < b\sqrt{2}$ **(C)** $b\sqrt{2} < c < 0$ **(D)** no values of b and c

11. For every $\alpha \in \left(0, \frac{\pi}{2}\right)$, the value of $\sqrt{x^2 + x} + \frac{\tan^2 \alpha}{\sqrt{x^2 + x}}$, $x > 0$ is greater than or equal to - **[JEE 2003]**

(A) 2 **(B)** $2\tan\alpha$ **(C)** $\frac{5}{2}$ **(D)** $\sec\alpha$

12. For a circle $x^2+y^2 = r^2$, find the value of 'r' for which the area enclosed by the tangents drawn from the point P (6, 8) to the circle and the chord of contact is maximum. **[JEE 2003]**

13. Let $f(x)=\begin{cases} x^{\alpha}\ln x, & x>0 \\ 0, & x=0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if $\alpha =$ **[JEE 2004]**

(A) –2 **(B)** –1 **(C)** 0 **(D)** 1/2

14. If $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$, using Rolle's theorem prove that atleast one root of $p(x) = 0$ lies between. $\left(45^{\frac{1}{100}},\ 46\right)$. **[JEE 2004]**

15. Prove that $\sin x + 2x \geq \frac{3x\ (x+1)}{\pi}\ \ \forall\ \ x \in \left[0,\ \frac{\pi}{2}\right]$ (Justify the inequality, If any used.) **[JEE 2004]**

16. If $f(x) = x^3 + bx^2 + cx + d$, $0 < b^2 < c$, then f(x) **[JEE 2004]**

(A) is strictly increasing **(B)** has a local maxima **(C)** has a local minima **(D)** has bounded area

17. If f(x) is twice differentiable function $f(1) = 1, f(2) = 4, f(3) = 9$ **[JEE 2005]**

(A) $f''(x) = 2$, for atleast one in $x \in (1, 3)$

(C) $f''(x) = f'(x) = 5$, for some $x \in (2, 3)$

(C) $f''(x) = 3, \forall \in (2, 3)$

(D) $f''(x) = 2$, for some $x \in (1, 2)$

18. Tangent to the curve $y = x^2 + 6$ at a point P(1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are - **[JEE 2005]**

(A) (–6, –11) **(B)** (–9, –13) **(C)** (–10, –15) **(D)** (–6, –7)

19. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1, 2). **[JEE 2005]**

20. If p(x) be a polynomial of degree 3 satisfying $p(-1) = 10$, $p(1) = -6$ and p(x) has maximum at $x = -1$ and p'(x) has minima at $x = 1$. Find the distance between the local maximum and local minimum of the curve. **[JEE 2005]**

21. f(x) is cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18$, $f(1) = -1$ and f'(x) has local minima at $x = 0$ then **[JEE 2006]**

(A) the distance between (–1, 2) and (a, f(a)), where x = a is the point of local minima is $2\sqrt{5}$

(B) f(x) is increasing for $x \in (1,\ 2\sqrt{5}\,]$

(C) f(x) has local minima at $x = 1$

(D) the value of $f(0) = 5$

22. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, $x \in [0, 3]$ then g(x) has

(A) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$ **[JEE 2006]**

(B) local maxima at $x = 1$ and local minima at $x = 2$

(C) no local maxima **(D)** no local minima

23. If f(x) is a twice differentiable function such that f(a) = 0, f(b) = 2, f(c) = –1, f(d) = 2 , f(e) = 0 where a < b < c < d < e then the minimum number of zeros of $g(x) = (f'(x))^2 + f''(x) f(x)$ in the interval [a, e] is. **[JEE 2006]**

24. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ - **[JEE 2007]**

(A) on the left of x = c **(B)** on the right of x = c

(C) at no point **(D)** at all points

25. Let f(x) = 2 + cos x for all real x.

Statement-I : For each real t, there exists a point c in [t, t + π] such that f'(c) = 0 because

Statement-II : f(t) = f(t + 2π) for each real t. **[JEE 2007]**

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

Comprehension (Q. 26 to 28)

If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation f(x) = 0 has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation f(x) = 0 has a root in R.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

26. The line y = x meets $y = ke^x$ for $k \le 0$ at :- **[2007]**

(A) no point **(B)** one point **(C)** two point **(D)** more than two points

27. The positive value of k for which $ke^x - x = 0$ has only one root is :- **[2007]**

(A) $\frac{1}{e}$ **(B)** 1 **(C)** e **(D)** $\log_e 2$

28. For k > 0, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is :- **[2007]**

(A) $\left(0, \frac{1}{e}\right)$ **(B)** $\left(\frac{1}{e}, 1\right)$ **(C)** $\left(\frac{1}{e}, \infty\right)$ **(D)** (0, 1)

29. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is -

(A) even and is strictly increasing in (0, ∞) **[JEE 2008]**

(B) odd and is strictly decreasing in (–∞, ∞)

(C) odd and is strictly increasing in (–∞, ∞)

(D) neither even nor odd, but is strictly increasing in (–∞, ∞)

30. Let $f(x)$ be a non-constant twice differentiable function defined on (–∞, ∞) such that $f(x) = f(1 - x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then, **[JEE 2008]**

(A) $f''(x)$ vanishes at least twice on [0, 1] **(B)** $f'\left(\frac{1}{2}\right) = 0$

(C) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$ **(D)** $\int_0^{1/2} f(t) e^{\sin \pi t} dt = \int_{1/2}^{1} f(1 - t) e^{\sin \pi t} dt$

31. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3} & -1 < x < 2 \end{cases}$ is :-

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **[JEE 2008]**

32. Match the column : **[JEE 2008]**

Column I		Column II	
(A)	The minimum value of $\dfrac{x^2+2x+4}{x+2}$ is	**(p)**	0
(B)	Let A and B be 3 × 3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and (A+B)(A–B) = (A–B)(A+B). If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible value of k are	**(q)**	1
(C)	Let $a = \log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	**(r)**	2
(D)	If $\sin\theta = \cos\phi$, then the possible values of $\dfrac{1}{\pi}\left(\theta \pm \phi - \dfrac{\pi}{2}\right)$ are	**(s)**	3

Paragraph for Question 33 to 35

Consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by

$f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}$, $0 < a < 2$.

33. Which of the following is true ? **[JEE 2008]**

(A) $(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$ **(B)** $(2-a)^2 f''(1) - (2+a)^2 f''(-1) = 0$

(C) $f'(1) f'(-1) = (2-a)^2$ **(D)** $f'(1) f'(-1) = -(2+a)^2$

34. Which of the following is true ? **[JEE 2008]**

(A) $f(x)$ is decreasing on (–1, 1) and has a local minimum at x = 1

(B) $f(x)$ is increasing on (–1, 1) and has a local maximum at x = 1

(C) $f(x)$ is increasing on (–1, 1) but has neither a local maximum nor a local minimum at x = 1

(D) $f(x)$ is decreasing on (–1, 1) but has neither a local maximum nor a local minimum at x = 1

35. Let $g(x) = \displaystyle\int_0^{e^x} \frac{f'(t)}{1+t^2}\,dt$ **[JEE 2008]**

Which of the following is true ?

(A) g'(x) is positive on (–∞, 0) and negative on (0, ∞) **(B)** g'(x) is negative on (–∞, 0) and positive on (0, ∞)

(C) g'(x) charges sign on both (–∞, 0) and (0, ∞) **(D)** g'(x) does not change sign on (–∞, ∞)

36. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is **[JEE 2009]**

37. Let p(x) be a polynomial of degree 4 having extremum at x=1, 2 and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right)$ =2, then the value of p(2) is **[JEE 2009]**

38. For the function $f(x) = x\cos\frac{1}{x},\ x \ge 1$, **[JEE 2009]**

(A) for at least one x in the interval $[1,\infty)$, $f(x+2)-f(x)<2$

(B) $\lim_{x\to\infty} f'(x) = 1$

(C) for all x in the interval $[1,\infty)$, $f(x+2)-f(x)>2$

(D) f '(x) is strictly decreasing in the interval $[1,\infty)$

39. Let $f(x) = (1+b^2)x^2+2bx+1$ and let m (B) the minimum value of f(x). As b varies, the range of m(b) is - **[JEE 2010]**

(A) [0, 1] **(B)** $\left(0,\frac{1}{2}\right]$ **(C)** $\left[\frac{1}{2},1\right]$ **(D)** $(0,\ 1]$

40. A straight line L with negative slope passes through the point (8,2) and cuts the positive coordinates axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin. **[JEE 2010]**

41. Let f be a real-valued function defined on the interval (0, ∞) by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true ? **[JEE 2010]**

(A) f''(x) exists for all $x \in (0, \infty)$

(B) f'(x) exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)|+|f'(x)| \le \beta$ for all $x \in (0, \infty)$

42. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2}+e^{-x^2}$, $g(x) = xe^{x^2}+e^{-x^2}$ and $h(x) = x^2e^{x^2}+e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then **[JEE 2010]**

(A) a = b and c ≠ b **(B)** a = c and a ≠ b **(C)** a ≠ b and c ≠ b **(D)** a = b = c

43. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010\,(x-2009)\,(x-2010)^2\,(x-2011)^3\,(x-2012)^4$, for all x ∈ R.

If g is a function defined on R with values in the interval (0,∞) such that

$f(x) = \ln(g(x))$, for all x ∈ R,

then the number of points in R at which g has a local maximum is - **[JEE 2010]**

44. Let $f : (0,1) \to R$ be defined by $f(x) = \dfrac{b-x}{1-bx}$, **[JEE 2011]**

where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on (0,1)

(B) $f \neq f^{-1}$ on (0,1) and $f'(b) = \dfrac{1}{f'(0)}$

(C) $f = f^{-1}$ on (0,1) and $f'(b) = \dfrac{1}{f'(0)}$

(D) f^{-1} is differentiable on (0,1)

45. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[JEE 2011]**

Paragraph for Question 46 and 47

Let $f(x) = (1-x)^2\sin^2 x + x^2$ for all $x \in \mathbb{R}$, and let $g(x) = \int_1^x \left(\dfrac{2(t-1)}{t+1} - \ln t\right) f(t)dt$ for all $x \in (1,\infty)$.

46. Consider the statements :

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then **[JEE 2012]**

(A) both P and Q are true

(B) P is true and Q is false

(C) P is false and Q is true

(D) both P and Q are false

47. Which of the following is true ? **[JEE 2012]**

(A) g is increasing on $(1,\infty)$

(B) g is decreasing on $(1,\infty)$

(C) g is increasing on (1,2) and decreasing on $(2,\infty)$

(D) g is decreasing on (1,2) and increasing on $(2,\infty)$

48. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0,\infty)$, then - **[JEE 2012]**

(A) f has a local maximum at $x = 2$

(B) f is decreasing on (2,3)

(C) there exists some $c \in (0,\infty)$ such that $f''(c) = 0$

(D) f has a local minimum at $x = 3$

49. Let $f : R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[JEE 2012]**

50. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then p'(0) is **[JEE 2012]**

51. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio of 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are **[JEE 2013]**

(A) 24 **(B)** 32 **(C)** 45 **(C)** 60

52. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x = **[JEE 2013]**

(A) -2 **(B)** $\frac{-2}{3}$ **(C)** 2 **(D)** $\frac{2}{3}$

53. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is **[JEE 2013]**

(A) 6 **(B)** 4 **(C)** 2 **(D)** 0

54. Let $f(x) = x \sin \pi x$, $x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at - **[JEE 2013]**

(A) a unique point in the interval $\left(n, n+\frac{1}{2}\right)$

(B) a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(C) a unique point in the interval (n, n + 1)

(D) two points in the interval (n, n + 1)

55. Let $f:(0, \infty) \to R$ be given by $f(x) = \int_{\frac{1}{2}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$ Then **[JEE Ad. 2014]**

(A) f(x) is monotonically increasing on $[1, \infty]$ **(B)** f(x) is monotonically decreasing on (0, 1)

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$ **(D)** $f(2^x)$ is an odd function of x on R

56. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at point (1, 3) is **[JEE Ad. 2014]**

57. A cylindrical container is to be made from certain solid material with the following constraints : It has a fixed inner volume of V mm3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular dise of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is **[JEE Ad. 2015]**

58. Let $f:(0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then

(A) $\lim_{x\to 0+} f'\left(\frac{1}{x}\right) = 1$ **(B)** $\lim_{x\to 0+} f\left(\frac{1}{x}\right) = 2$ **(C)** $\lim_{x\to 0+} x^2 f'(x) = 0$ **(D)** $|f(x)| \leq$ for all $x \in (0, 2)$

MOCK TEST

SECTION - I : STRAIGHT OBJECTIVE TYPE

1. A particle moving on a curve has the position at time t given by $x = f'(t)\sin t + f''(t)\cos t$, $y = f'(t)\cos t - f''(t)\sin t$, where f is a thrice differentiable function. Then the velocity of the particle at time t is :

(A) $f'(t) + f''(t)$ **(B)** $f'(t) - f'''(t)$ **(C)** $f'(t) + f'''(t)$ **(D)** $f'(t) - f''(t)$

2. If $f(x) = \dfrac{a\sin x + b\cos x}{c\sin x + d\cos x}$ is monotonically increasing, then

(A) $ad \geq bc$ **(B)** $ad < bc$ **(C)** $ad \leq bc$ **(D)** $ad > bc$

3. The beds of two rivers (within a certain region) are a parabola $y = x^2$ and a straight line $y = x - 2$. These rivers are to be connected by a straight canal. The coordinates of the ends of the shortest canal can be:

(A) $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $\left(-\frac{11}{8}, \frac{5}{8}\right)$ **(B)** $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $\left(\frac{11}{8}, -\frac{5}{8}\right)$

(C) $(0, 0)$ & $(1, -1)$ **(D)** none of these

4. If $\dfrac{a_0}{n+1} + \dfrac{a_1}{n} + \dfrac{a_2}{n-1} + \ldots\ldots\ldots + \dfrac{a_{n-1}}{2} + a_n = 0$, then the equation

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0$ has

(A) exactly one root in (0, 1) **(B)** at least one root in (0, 1)

(C) no root in (0, 1) **(D)** at the most one root in (0, 1)

5. The longest interval in which $f(x) = x\sqrt{4ax - x^2}$ $(a < 0)$ is decreasing is –

(A) $[4a, 0]$ **(B)** $[3a, 0]$ **(C)** $(-\infty, 3]$ **(D)** none of these

6. The set of values of the parameter 'a' for which the function ;

$f(x) = 8ax - a\sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in R$, is

(A) $[-1, 1]$ **(B)** $(-\infty, -6)$ **(C)** $(6, +\infty)$ **(D)** $[6, +\infty)$

7. The values of 'a' and 'b' for which all the extrema of the function $f(x) = a^2x^3 - \frac{a}{2}x^2 - 2x - b$ are positive and the minimum is at the point $x_0 = \frac{1}{3}$

(A) when $a = -2 \Rightarrow b < \frac{-11}{27}$ and when $a = 3 \Rightarrow b < -\frac{1}{2}$

(B) when $a = 3 \Rightarrow b < \frac{-11}{27}$ and when $a = 2 \Rightarrow b < -\frac{1}{2}$

(C) when $a = -2 \Rightarrow b < \frac{-11}{27}$ and when $a = 2 \Rightarrow b < -\frac{1}{2}$

(D) None of these

8. A truck is to be driven 300 km on a highway at a constant speed of x kmph. Speed rules of the highway required that $30 \le x \le 60$. The fuel costs Rs. 10 per litre and is consumed at the rate of $2 + \frac{x^2}{600}$ liters per hour. The wages of the driver are Rs. 200 per hour. The most economical speed to drive the truck, in kmph, is

(A) 30 **(B)** 60 **(C)** $30\sqrt{3.3}$ **(D)** $20\sqrt{3.3}$

9. The equation $x^3 - 3x + [a] = 0$, where [.] denotes the greatest integer function, will have three real and distinct roots if

(A) $a \in (-\infty, 2)$ **(B)** $a \in (0, 2)$ **(C)** $a \in (\infty, -2) \cup (0, \infty)$ **(D)** $a \in [-1, 2)$

10. **S_1 :** $f(x) = x\,e^{x(1-x)}$ is increasing on $\left[-\frac{1}{2}, 1\right]$

S_2 : Critical points for $f(x) = (x-2)^{2/3}(2x+1)$ are $x = 1$ & $x = 2$.

S_3 : $f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$, then f(x) is monotonic for $x \in R$.

S_4 : Let f is a real valued differentiable function such that f(x). $f'(x) < 0$ for all real x. Then $f^2(x)$ is decreasing function.

(A) TTTT **(B)** TFFT **(C)** FTFT **(D)** FFTT

SECTION - II : MULTIPLE CORRECT ANSWER TYPE

11. Let $g'(x) > 0$ and $f'(x) < 0$, $\forall\, x \in R$, then

(A) $g(f(x+1)) > g(f(x-1))$ **(B)** $f(g(x-1)) > f(g(x+1))$

(C) $g(f(x+1)) < g(f(x-1))$ **(D)** $g(g(x+1)) < g(g(x-1))$

12. If p, q, r be real, then the intervals in which, $f(x) = \begin{vmatrix} x+p^2 & pq & pr \\ pq & x+q^2 & qr \\ pr & qr & x+r^2 \end{vmatrix}$,

(A) increase is $x < -\frac{2}{3}(p^2+q^2+r^2)$, $x > 0$ **(B)** decrease is $\left(-\frac{2}{3}(p^2+q^2+r^2), 0\right)$

(C) decrease is $x < -\frac{2}{3}(p^2+q^2+r^2)$, $x > 0$ **(D)** increase is $\left(-\frac{2}{3}(p^2+q^2+r^2), 0\right)$

13. If $f(x) = x^3 - x^2 + 100x + 1001$, then

(A) $f(2000) > f(2001)$ **(B)** $f\left(\frac{1}{1999}\right) > f\left(\frac{1}{2000}\right)$ **(C)** $f(x+1) > f(x-1)$ **(D)** $f(3x-5) > f(3x)$

14. Which of the following inequalities are valid –

(A) $|\tan^{-1}x - \tan^{-1}y| \le |x-y|\ \forall\, x, y \in R$ **(B)** $|\tan^{-1}x - \tan^{-1}y| \ge |x-y|$

(C) $|\sin x - \sin y| \le |x-y|$ **(D)** $|\sin x - \sin y| \ge |x-y|$

15. Let $f(x) = (x-1)^4(x-2)^n$, $n \in N$. f(x) has

(A) Local minimum at x = 2 if n is even **(B)** Local minimum at x = 1 if n is odd

(C) Local maximum at x = 1 if n is odd **(D)** Local minimum at x = 1 if n is even

SECTION - III : ASSERTION AND REASON TYPE

16. **Statement-I :** The ratio of length of tangent to length of normal is inversely proportional to numerical value of the ordinate of the point of tangency at the curve $y^2 = 4ax$.

Statement-II : Length of normal & tangent to a curve $y = f(x)$ is $\left|y\sqrt{1+m^2}\right|$ and $\left|\dfrac{y\sqrt{1+m^2}}{m}\right|$, where $m = \dfrac{dy}{dx}$

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

17. **Statement-I :** $\dfrac{2e^{x_1} + e^{x_2}}{3} > e^{\left(\frac{2x_1+x_2}{3}\right)}$, where e is Napier's constant.

Statement-II : If $f'(x)$ and $f''(x)$ is positive $\forall\ x \in R$, then f(x) increases with concavity up $\forall\ x \in R$ and any chord lies above the curve.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

18. **Statement-I :** If f(x) is increasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

Statement-II : If f(x) is decreasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

19. **Statement-I :** Let $f(x) = 5 - 4(x-2)^{2/3}$, then at $x = 2$ the function f(x) attains neither least value nor greatest value.

Statement-II : $x = 2$ is the only critical point of f(x).

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

20. **Statement-I :** The largest term in the sequence $a_n = \dfrac{n^2}{n^3+200}$, $n \in N$ is $\dfrac{(400)^{2/3}}{600}$.

Statement-II : $f(x) = \dfrac{x^2}{x^3+200}$, $x > 0$, then at $x = (400)^{1/3}$, f(x) is maximum.

(A) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I
(B) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I
(C) Statement-I is True, Statement-II is False
(D) Statement-I is False, Statement-II is True

SECTION - IV : MATRIX - MATCH TYPE

21. **Column – I** | **Column – II**

(A) If portion of the tangent at any point on the curve $x = at^3$, $y = at^4$ between the axes is divided by the point of contact in the ratio m : n externally, then $|n + m|$ is equal to (m and n are coprime) — **(p)** 1

(B) The area of triangle formed by normal at the point (1, 0) on the curve $x = e^{\sin y}$ with axes is — **(q)** $\frac{1}{2}$

(C) If the angle between curves $x^2y = 1$ and $y = e^{2(1-x)}$ at the point (1, 1) is θ then tan θ is equal to — **(r)** 7

(D) The length of sub-tangent at any point on the curve $y = be^{x/3}$ is equal to — **(s)** 3

(t) 0

22. **Column – I** | **Column – II**

(A) Number of points which are local extrema of $f(x) = \begin{cases} (2+x)^3 & ; \ -3 \le x \le -1 \\ x^{2/3} & ; \ -1 < x < 2 \end{cases}$ — **(p)** 1

(B) If $a + b = 1$; $a > 0, b > 0$, then the minimum value of $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ is — **(q)** 2

(C) The maximum value attained by $y = 10 - |x - 10|$, $-1 \le x \le 3$, is — **(r)** 3

(D) If $P(t^2, 2t)$, $t \in [0, 2]$ is an arbitrary point on parabola $y^2 = 4x$ and Q is foot of perpendicular from focus S on the tangent at P, then maximum area of triangle PQS is — **(s)** 4

(t) 5

SECTION - V : COMPREHENSION TYPE

23. **Read the following comprehension carefully and answer the questions.**

If a continuous function f defined on the real line R, assumes positive and negative values in R then the zequation f(x) = 0 has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation f(x) = 0 has a root in R.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

1. The line $y = x$ meets $y = ke^x$ for $k \le 0$ at

(A) no point **(B)** one point **(C)** two points **(D)** more than two points

2. The positive value of k for which $ke^x - x = 0$ has only one root is

(A) $\frac{1}{e}$ **(B)** 1 **(C)** e **(D)** $\log_e 2$

3. For k > 0, the set of all values of 'k' for which $ke^x - x = 0$ has two distinct roots is

(A) $\left(0, \frac{1}{e}\right)$ **(B)** $\left(\frac{1}{e}, 1\right)$ **(C)** $\left(\frac{1}{e}, \infty\right)$ **(D)** (0, 1)

24. Read the following comprehension carefully and answer the questions.

Let f and g are two functions such that f(x) & g(x) are continuous in [a, b] and differentiable in (a, b)

Then at least one $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

(i) If $f(a) = f(b)$, then $f'(c) = 0$ (RMVT)

(ii) If $f(a) \neq f(b)$ and $a \neq b$, (LMVT)

(iii) If $g'(x) \neq 0$, then $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ (cauchy theorem)

1. The set of values of k, for which equation $x^3 - 3x + k = 0$ has two distinct roots in (0, 1) is

(A) (1, 4) **(B)** $(0, \infty)$ **(C)** (0,1) **(D)** ϕ

2. Which of the following is true ?

(A) $|\tan^{-1}x - \tan^{-1}y| \leq |x - y| \ \forall\, x, y \in R$ **(B)** $|\tan^{-1}x - \tan^{-1}y| \geq |x - y| \ \forall\, x, y \in R$

(C) $|\sin x - \sin y| \geq |x - y| \ \forall\, x, y \in R$ **(D)** none of these

3. Let $0 < \alpha < \theta < \beta < \dfrac{\pi}{2}$, then $\dfrac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta}$ is equal to

(A) $\tan\theta$ **(B)** $-\tan\theta$ **(C)** $\cot\theta$ **(D)** $-\cot\theta$

25. Read the following comprehension carefully and answer the questions.

A function f(x) having the following properties;

(i) f(x) is continuous except at x = 3

(ii) f(x) is differentiable except at x = – 2 and x = 3

(iii) $f(0) = 0, \ \lim\limits_{x\to 3} f(x) \to -\infty, \ \lim\limits_{x\to -\infty} f(x) = 3, \ \lim\limits_{x\to\infty} f(x) = 0$

(iv) $f'(x) > 0 \ \forall\, x \in (-\infty, -2) \cup (3, \infty)$ and $f'(x) \leq 0 \ \forall\, x \in (-2, 3)$

(v) $f''(x) > 0 \ \forall\, x \in (-\infty, -2) \cup (-2, 0)$ and $f''(x) < 0 \ \forall\, x \in (0, 3) \cup (3, \infty)$

then answer the following questions

1. Maximum possible number of solutions of $f(x) = |x|$ is

(A) 2 **(B)** 1 **(C)** 3 **(D)** 4

2. Graph of function $y = f(-|x|)$ is

(A) differentiable for all x, if $f'(0) = 0$

(B) continuous but not differentiable at two points, if $f'(0) = 0$

(C) continuous but not differentiable at one points, if $f'(0) = 0$

(D) discontinuous at two points, if $f'(0) = 0$

3. $f(x) + 3x = 0$ has five solutions if

(A) $f(-2) > 6$ **(B)** $f'(0) < -3$ and $f(-2) > 6$

(C) $f'(0) > -3$ **(D)** $f'(0) > -3$ and $f(-2) > 6$

SECTION - VI : INTEGER TYPE

26. The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point $x = 2$ and is bisected by that point . Find 'a' .

27. Let α be the angle in radians between $\frac{x^2}{36} + \frac{y^2}{4} = 1$ and the circle $x^2 + y^2 = 12$ at their points of intersection. If $\alpha = \tan^{-1}\frac{k}{2\sqrt{3}}$, then find the value of k^2.

28. Find positive real numbers ‘a’ and ‘b’ such that $f(x) = ax - bx^3$ has four extrema on $[-1, 1]$ at each of which $|f(x)| = 1$

29. A cubic f(x) vanishes at $x = -2$ and has relative minimum/maximum at $x = -1$ and $x = \frac{1}{3}$. If $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$, the cubic $f(x) = \lambda_1x^3 + \lambda_2x^2 - x + 2$, then find $(\lambda_1 + \lambda_2)$

30. For any acute angled ΔABC , find the maximum value of $\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}$

ANSWER KEY

EXERCISE - 1

1. C **2.** A **3.** A **4.** B **5.** C **6.** A **7.** B **8.** A **9.** A **10.** D **11.** C **12.** B **13.** C
14. C **15.** D **16.** B **17.** D **18.** D **19.** B **20.** B **21.** C **22.** B **23.** B **24.** B **25.** A **26.** A
27. A **28.** D **29.** D **30.** A **31.** D **32.** C **33.** D **34.** B **35.** B **36.** B **37.** B **38.** D **39.** D
40. C **41.** A **42.** B **43.** C **44.** B **45.** C **46.** D **47.** B **48.** D **49.** B **50.** D **51.** A **52.** D
53. A **54.** A **55.** B **56.** B **57.** B **58.** C **59.** C **60.** B **61.** A **62.** C **63.** A **64.** C **65.** A
66. A **67.** C **68.** D **69.** B **70.** B **71.** C **72.** B **73.** A **74.** C **75.** D

EXERCISE - 2 : PART # I

1. ACD **2.** AC **3.** ACD **4.** CD **5.** BD **6.** BC **7.** BD **8.** BD **9.** AD
10. BCD **11.** AB **12.** ABD **13.** AB **14.** AB **15.** ABD **16.** ACD **17.** AB **18.** AD
19. BD **20.** ACD **21.** ABD **22.** ABC **23.** AB **24.** AD **25.** ABCD **26.** AD **27.** AD
28. CD **29.** AC **30.** ABCD **31.** ABD **32.** AB **33.** AD **34.** AD **35.** BC **36.** ABCD
37. BC **38.** AD **39.** ABC **40.** BC **41.** ACD **42.** ABC **43.** BD **44.** ABCD **45.** BCD
46. BCD **47.** AB **48.** ACD **49.** BCD **50.** ABC

PART - II

1. A **2.** A **3.** A **4.** B **5.** A **6.** D **7.** A **8.** D **9.** D **10.** A **11.** C **12.** C **13.** C
14. A **15.** C **16.** C **17.** A **18.** A

EXERCISE - 3 : PART # I

1. $A\to r$ $B\to q$ $C\to p$ $D\to s$ **2.** $A\to p$ $B\to s$ $C\to q$ $D\to r$ **3.** $A\to q,r$ $B\to r$ $C\to q$ $D\to p$
4. $A\to p,q$ $B\to r,s$ $C\to r,s$ $D\to r,s$ **5.** $A\to r$ $B\to p$ $C\to s$ $D\to q$ **6.** $A\to r$ $B\to s$ $C\to q$ $D\to p$
7. $A\to p$ $B\to r$ $C\to q$ $D\to s$ **8.** $A\to t$ $B\to s$ $C\to t$ $D\to p$ **9.** $A\to s$ $B\to r$ $C\to p$ $D\to q$

PART - II

Comprehension # 1 : **1.** A **2.** C **3.** D
Comprehension # 2 : **1.** B **2.** D **3.** C
Comprehension # 3 : **1.** C **2.** B **3.** D
Comprehension # 4 : **1.** D **2.** C **3.** C
Comprehension # 5 : **1.** A **2.** C **3.** D
Comprehension # 6 : **1.** C **2.** D **3.** A
Comprehension # 7 : **1.** A **2.** D **3.** D
Comprehension # 8 : **1.** A **2.** D **3.** D
Comprehension # 9 : **1.** C **2.** B **3.** A
Comprehension # 10 : **1.** D **2.** C **3.** C
Comprehension # 11 : **1.** D **2.** A **3.** B

EXERCISE - 5 : PART # I

1. 4 **2.** 4 **3.** 3 **4.** 4 **5.** 2 **6.** 1 **7.** 3 **8.** 2,4 **9.** 1 **10.** 4 **11.** 2 **12.** 2 **13.** 4
14. 1 **15.** 4 **16.** 3 **17.** 1 **18.** 4 **19.** 4 **20.** 2 **21.** 4 **22.** 3 **23.** 4 **24.** 3 **25.** 1 **26.** 4
27. 2 **28.** 2

PART - II

1. D **2.** **(A)** B **(B)** D **3.** A **4.** C **5.** A **6.** $\cos\left(\frac{1}{3}\cos^{-1}p\right)$ **7.** A

8. D **10.** A **11.** B **12.** 5 **13.** D **16.** A **17.** A **18.** D **19.** $y-2=0$

20. Distance between (–1, 10) and (3, –22) is $4\sqrt{65}$ units **21.** BC **22.** A **23.** 6 **24.** A

25. B **26.** B **27.** A **28.** A **29.** C **30.** ABCD **31.** C

32. **A** $\rightarrow$ r **B** $\rightarrow$ q, s **C** $\rightarrow$ r, s **D** $\rightarrow$ p, r **33.** A **34.** A **35.** B **36.** 7 **37.** 0

38. BCD **39.** D **40.** 18 **41.** BC **42.** D **43.** 1 **44.** A **45.** 2 **46.** C

47. B **48.** ABCD **49.** 5 **50.** 9 **51.** AC **52.** AB **53.** C **54.** BC **55.** ACD

56. 8 **57.** 4 **58.** A

MOCK TEST

1. C **2.** A **3.** B **4.** B **5.** D **6.** D **7.** A **8.** B **9.** D

10. A **11.** BC **12.** AB **13.** BC **14.** AC **15.** ACD **16.** D **17.** A **18.** D

19. D **20.** D **21.** $A\rightarrow r$ $B\rightarrow q$ $C\rightarrow t$ $D\rightarrow s$ **22.** $A\rightarrow q$ $B\rightarrow r$ $C\rightarrow r$ $D\rightarrow t$

23. **1.** B **2.** A **3.** A **24.** **1.** D **2.** A **3.** D **25.** **1.** C **2.** B **3.** D

26. $a=1$ **27.** 16 **28.** $a=3, b=4$ **29.** 2 **30.** $\frac{9\sqrt{3}}{2\pi}$