**Factorisation**

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## Introduction

### Negative or Positive Factors of the Constant Term

In the given expression $x^2$ +mx + n, where ab = n and a + b = m, we should follow some rules regarding positive or negative factors of the constant terms. The rule is following: -

1. If ‘m is positive and ‘n’ is also positive, then both the factors ‘a’ and ‘b’ of ‘n’ will be positive.

2. If ‘m’ is positive and ‘n’ is negative, then both the factors ‘a’ and ‘b of ‘n’ will have opposite signs. It means one factor will be positive and the other will be negative.

3. If ‘m’ is negative and ‘n’ is positive, then the factors ‘a’ and ‘b’ of ‘n’ will be negative sign.

4. If ‘m’ and ‘n’ are both negative, then the factors ‘a’ and ‘b’ of ‘n’ will have opposite sign.

### Factors of algebraic expressions

When a number is written as a product of its factors, it is said factorization of the number. Let us see some example:

$$36 = 3 \times 12$$

We can also write 36 as a product of its different factors, such as

$$36 = 4 \times 9$$

Or $$36 = 18 \times 2$$

Or $$36 = 2 \times 2 \times 3 \times 3$$

In this example, numbers, 2, 3, 4, 6, 9, 12 and 18 are all factors of number 36.

Number 2 and 3 are called the **prime factor** of 36

Therefore, 36 = 2 × 2 × 3 × 3 is called **prime factorization** of 36.

So, prime factors are those factors which cannot be further expressed as a product of factors. It is also called Irreducible factors.

**Remarkable fact:** The number 36 can be written as 1 × 36 also. So, 1 and 36 are also factors of the numbers 36.

For any number, 1 and the number itself are factors of the number. But, unless specifically asked to mention, we never mention these as factors.

## What is Factorisation?

**Factorization of Algebraic Expressions:**

Let us understand what is factorization of Algebraic Expression? It means writing it as a product of prime or irreducible factors. These factors may be algebraic Expression, constant or variable. Let us consider on an algebraic expression.

As we know that an algebraic expression consists of terms and terms are product of factors.

**For example:** 7ab + 4c

In this expression, the term 7ab is obtained by multiplying 7, a, b.

Hence 7, ‘a’ and ‘b’ is factors of 7ab.

We can write it, 7ab = 7 × a × b

Similarly, in other expression, 24ab + 6c, the term 24ab can be written as the product of its factors in the following three different ways.

a. 24ab = 3 × 8 × a × b

b. 24ab = 2 × 2 × 3 × 3 × a × b

c. 24ab = 2 × 2 × 3 × 3 × ab

The factorization in (b) is the factorization of the term 24ab into irreducible factors. It means the term 24ab cannot be further expressed as a product of more irreducible factors. This factorization is called prime factorization or the irreducible factorization or irreducible form of the expression 24ab.

The factorization in (a) is not an irreducible of expression 24ab, since 8 can be factorized further.

Similarly, (c) is not an irreducible form of expression 24ab, since ab can be factorized further and written as a × b.

**Remarkable fact: -** 1 is a factor of every term of an algebraic expression, for instant, the term 7ab can written as 1 × 7 × a × b.

But, as in factorization of numbers, we never write 1 as a factor of the terms of an algebraic expression unless we required it for some reason.

Let us consider one more expression:

**For Example:** 6p (2q + 5r).

This expression can be written as the product of its irreducible factors as:-

$$6p\,(2q + 5r) = 2 \times 3 \times p \times (2q + 5r)$$

Some expression like 9xy, 2x (4y +7), 15abc (2a + b) (3a – 4b) are easy to factorize. But all algebraic expressions are not easy to factorize.

Let us learn how to factorize algebraic expressions systematically:

## Factorization by finding common factors

Let us consider the expression 4ab + 6b.

First of all, we need to write the terms of a given expression as product of their irreducible factors.

**First step:** Here, the irreducible factors of the terms 4ab and 6b are

$$4ab = 2 \times 2 \times a \times b$$

$$6b = 2 \times 3 \times b$$

**Second step:** Search common factor:-

The common factor to both terms is 2 and b, so their HCF is 2b.

Now, we can write

$$4ab + 6b = (2 \times 2 \times a \times b) + (2 \times 3 \times b)$$

$$= (2a \times 2b) + (2b \times 3)$$

$$= 2b\ (2a + 3)$$

[Using distributing property

$$(2b \times 2a) + (2b \times 3) = 2b \times (2a + 3)]$$

So, by finding the common factor 2b we have factorize the expression 4ab + 6b into irreducible factors 2b and (2a + 3)

## Factorisation by regrouping terms

Sometime in algebraic expressing, all the terms of a given expression do not have common factors. In that case, we make two groups of terms which now have a common factor and then factorize the expression.

For example, let us consider on this expression:

$$15ab + 6a + 10b + 4$$

All the four terms have no common factors. However, first and second terms have 3a as a common factor, and the third and fourth terms have 2 as a common factor.

Therefore, we combine first and second terms into one group, and third & fourth terms into another group.

$$15ab + 6a + 10b + 4 = [(3a + 5b) + (3a \times 2)] + [(2 \times 5b) + (2 \times 2)]$$

$$= [3a\ (5b + 2)] + [2(5b + 2)]$$

Now, the expression consists of two terms which have the expression (5b + 2) common. Using the distributive property, we have,

$$15ab + 6a + 10b + 4 = (5b + 2)\ (3a + 2)$$

Let see another example:-

$$12ax - 4ab + 18bx - 6b^2 = [(4a \times 3x) + (4a \times - b)] + [(6b \times 3x) + (6b \times - b)]$$

$$= [4a (3x - b)] + [6b (3x - b)]$$

$$= (4a + 6b) (3x - b)$$

$$= 2(2a + 3b) (3x - b)$$

**Remarkable fact:** Though the regrouping of the terms in an expression can be done in different ways, all regrouping may not lead to a factorization. We need to learn how to regroup the terms in different way by trial and error for getting an expression that can actually be factorized.

**Example:** $18 (a + b)^2 - 6(a + b)$

**Solution:** As it is obvious, 6 (a + b) is the highest common factor of the terms of the given expression, then

$$18 (a + b)^2 - 6(a + b) = [6(a + b) \times 3 (a + b)] - [6(a + b) \times 1]$$

$$= 6(a + b) [3(a + b) - 1]$$

**Remarkable fact:** In the above example, we need to show 1 as a factor since the entire term 6(a + b) is the highest common factor.

## Factorisation using identities

We have used the standard identities to find the product of expression. Now, we shall learn how to use the identities to factorize the given expression.

We shall be using the following three standard identities to factorize expression:-

**a.** $(a + b)^2 = a^2 + 2ab + b^2$

**b.** $(a - b)^2 = a^2 - 2ab + b^2$

**c.** $a^2 - b^2 = (a + b) (a - b)$

While using identities for factorization, we need to compare the given expression with the standard identities. If the right hand side of the given expression is similar to the right hand side of any of the identities, then the expression corresponding to the left hand side of that identity is the required factorization. This method has been illustrated through the following examples.

**Example:** $9x^2 - 6x + 1$

**Solution:** Since, the given expression has three terms; it is not comparable with identity. Comparing the expression with identities (a) and (b), we can see that the two terms of the expression are perfect squares and its middle term is negative.

Therefore, it is in the form of the identity (b) with a = 3x (since square root of 9x2 = 3x) and b =1,

It means,

$$a^2 - 2ab + b^2 = (3x)^2 - 2(3x)(1) + (1)^2$$

Hence, the expression can be written in the form $(a - b)^2$ as

$$9x^2 - 6x + 1 = (3x - 1)^2$$

$$= (3x - 1)(3x - 1)$$

This is required factorization

**Example:** $25x^2 + 30x + 9$

**Solution:** The given expression has three terms. Let us compare it with the identities, we can see that it is in the form of identities (i),

Here, a = 5x ($\sqrt{25}$ = 5x) and b = 9 ($\sqrt{9}$ = 3)

Therefore, $a^2 + 2ab + b^2 = (5x)^2 + 2(5x)(3)^2$

Since, $\mathbf{a^2 + 2ab + b^2 = (a + b)^2}$

We have,

$$25x^2 + 30x + 9 = (5x + 3)^2$$

$$= (5x + 3)(5x + 3)$$

That is required factorization.

**Example:** $81a^2 - 4b^2$

**Solution:** Let us compare the expression with identities, we get that it is in the form of

$$a^2 - b^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (9a)^2 - (2b)^2$$

Since, $\mathbf{a^2 - b^2 = (a - b)(a + b)}$

So, $(9a)^2 - (2b)^2 = (9a + 2b)(9a - 2b)$

This is required factorization.

**Example:** $(3x + 2y)^2 - (2 + a - 2b)^2$

**Solution:** It's in the form of identities (iii),

Here we have,

$$a = (3x + 2y) \text{ and } b = (2 + a - 2b)$$

Since, $a^2 - b^2 = (a + b)(a - b)$

So, $(3x + 2y)^2 - (2 + a - 2b)^2 = \{(3x + 2y) + (2 + a - 2b)\}\{(3x + 2y) - (2 + a - 2b)\}$

$$= \{3x + 2y + 2 + a - 2b\}\{3x + 2y - 2 - a + 2b\}$$

Let us see some example before moving further:-

**Example:** $75a^3 b^2 - 108ab^4$

**Solution:** The given expression is not in the form of any identity. However, its two terms have common factors. Let us rewrite the expression, we have

$$(3ab^2 \times 25a^2) - (3ab^2 \times 36b^2) = 3ab^2 (25a^2 - 36b^2)$$

$$= 3ab^2 \{(5a)^2 - (6b)^2\}$$

$$= 3ab^2 (5a + 6b)(5a - 6b)$$

By using identity, $\mathbf{a^2 - b^2 = (a + b)(a - b)}$

**Example:** $9 - x^6 + 2x^3y^3 - y^6$

**Solution:**

$$9 - x^6 + 2x^3y^3 - y^6 = 9 - (x^6 - 2x^3y^3 + y^6)$$

$$= 9 - \{(x^3)^2 - 2(x)^3(y)^3 + (y^3)^2$$

$$= (3)^2 - (x^3 - y^3)^2$$

[On substituting $x^3 = a$, $y^3 = b$ in the identity $a^2 - 2ab + b^2 = (a - b)^2$,

We have

$$(x^3)^2 - 2(x^3)(y^3) + (y^3)^2 = (x^3 - y^3)^2]$$

$$= \{3 + (x^3 - y^3)\}\{3 - (x^3 - y^3)\}$$

Using, $\mathbf{a^2 - b^2 = (a + b)(a - b)}$

$$= (3 + x^3 - y^3)(3 - x^3 + y^3)$$

## Factors of the form ( x + a) ( x + b)

An expression of the type $x^2 + mx + n$ is not a perfect square, therefore, cannot be factorized by identities (i) and (ii) used in the preceding topic. This expression also does not fit the identity type $(a^2 - b^2)$ either.
However, we can use identity

$$\mathbf{(x + a)(x + b) = x^2 + (a + b)x + ab}$$

To factorize this type of expression, we need to find two factor 'a' and b of the constant term n such that ab = n and a + b = m (the coefficient of the middle term).

Then, the given expression is factorized in the following way:-

$$x^2 + mx + n = x^2 + (a + b)x + ab$$

$$= x^2 + ax + bx + ab$$

$$= x(x + a) + b(x + a)$$

$$= (x + a)(x + b)$$

This is required factorization.

**Example:** $x^2 + 14x + 45$

**Solution:**

The constant term 45 can be factorized as $45 = 15 \times 3$ or $9 \times 5$,
We'll consider only those factors of 45, whose sum is equal to 14 (the coefficient of middle term).

Hence, factor 15 + 3 is not required as $15 + 3 \neq 14$. We have to take other pair of factors;

i.e $9 + 5 = 14$

Therefore, the required factor of 45 is 9 and 5.

Putting these factors in the given expression we have,

$$x^2 + 14x + 45 = x^2 + (9 + 5)x + (9)(5)$$

$$= x^2 + 9x + 5x + (9)(5)$$

$$= x^2 + 14x + 45$$

**Example:** $x^2 - 22x - 48$

**Solution:** The constant term 48 can be factorized as

$48 = 8 \times 6$ or $12 \times 4$ or $16 \times 3$ or $24 \times 2$

Since, $2 - 24 = -22$, 2 and – 24 are the required factors of – 48.

By using identities,

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

So, $$x^2 - 22x - 48 = x^2 + (2 - 24)x + (2)(-24)$$

$$= (x + 2)(x - 24)$$

**Example:** $x^2 - 13x + 36$

**Solution**: The constant term 36 can be factorized as

$36 = 6 \times 6$ or $9 \times 4$ or $12 \times 3$ or $18 \times 2$

Since, $(-9) + (-4) = 1 - 13$, - 9 and – 4 are required factors of 36.

Hence, $$x^2 - 13x + 36 = x^2 + (-4 - 9)x + (-9)(-4)$$

$$= x^2 - 4x - 9x + (-9)(-4)$$

$$= x(x - 4) - 9(x - 4)$$

$$= (x - 4)(x - 9)$$

This is a required factorization.

## Factorization by splitting the middle term

There are some expressions which can be factorized by splitting middle term.

Let us consider this expression

$$Px^2 + qx + r \text{ and } Px^2 + qxy + ry^2$$

Both are neither a square nor in the term of $(x + a)(x + b) = x^2 + (a + b)x + ab$. Such expressions are factorized by splitting their two middle terms.

viz the middle terms qx of the expression $px^2 + qx + r$ is split into two terms ax and bx such that their sum is qx and their product is equal to the product of the first and the third term.
Let us consider on following example:-

**Example:** $6x^2 - 5xy - 6y^2$

**Solution:**

**1st step:** Get the product of first and third terms

i.e., $6x^2 \times 6y^2 = 36x^2y^2$

**2nd step:** Split middle terms such as it product would be equal to product of first and third terms.

Here, middle term = $-5xy$

So, we can write it as $-5xy = 4xy - 9xy$ $(36x^2y^2)$

Therefore,

$$6x^2 - 5xy - 6y^2 = 6x^2 + 4xy - 9xy - 6y^2$$

$$= 2x(3x + 2y)(2x - 3y)$$

**Example:** $7x^2 + 23xy + 6y^2$

**Solution:**
Product of the first and the third terms = $42x^2Y^2$

Therefore, the middle term 23xy can be writing as 21xy + 2xy

Hence, $7x^2 + 23xy + 6y^2 = 7x^2 + 21xy + 2xy + 6y^2$

$= 7x(x + 3y) + 2y(x + 3y)$

$= (x + 3y)(7x + 2y)$

**Example:** $14a^2 + 11ab - 15b^2$

**Solution:** Product of the first and the third terms = $-210a^2b^2$

Therefore it can be writing as 21ab – 10ab

Hence, 1 $4a^2 + 11ab - 15b^2 = 14a^2 - 21ab - 10ab - 15b^2$

$= 7a(2a - 3b) - 5b(2a - 3b)$

$= (2a - 3b)(7a - 5b)$

Note: it is sometimes necessary to change the order of the terms for getting the expression that is common in terms of the given expression.

## Division of Algebraic Expressions

Division of algebraic expressions is similar to the division of numbers. However, when we divide numbers, we use our knowledge of factors of numbers through multiplication tables but to divide algebraic expression we need to factorize their terms.

Let us consider on following algebraic expression:

1. $4a^2 \times 3a^3 = 12a^5$

=> $12a^5 \div 4a^2 = 3a^3$

Also, $12a5 \div 3a^3 = 4a^2$

2. $15m(4m + 3n) = 60m^2 + 45mn$

=> $(60m^2 + 45mn) \div (4m + 3n)$

Also, $(60m^2 + 45mn) \div (4m + 3n) = 15m$

## Division of a monomial by another monomial

Let us consider $24a^4 \div 6a$

Factorizing $24a^4$ completely, we have

$$24a^4 = 2 \times 2 \times 2 \times 3 \times a \times a \times a \times a$$

$$24a^4 \div 6a = \frac{2 \times 2 \times 2 \times 3 \times a \times a \times a \times a}{2 \times 3 \times a}$$

$$= 2 \times 2 \times a \times a \times a$$

$$= 4a^3$$

The above division can also be done by factorizing $24a^4$ into two terms.

$$24a^4 = (2 \times 3 \times a) \times (2 \times 2 \times a \times a \times a)$$

$$= 6a \times 4a^3$$

$$24a^4 \div 6a = \frac{6a \times 4a^3}{6a}$$

$$= 4a^3$$

## Division of a polynomial by a monomial

To divide a polynomial by a monomial, we divide each of its term by the monomial and add the quotient. Let us consider on the following expression:

$$(12a^3 + 15a) \div 3a$$

Factorizing $12a^3$ completely, we have

$12a^3 = 2 \times 2 \times 3 \times a \times a \times a$

Divide first terms by 3a, we have

$$12a^3 \div 3a = \frac{2 \times 2 \times 3 \times a \times a \times a}{3 \times a}$$

$$= 2 \times 2 \times a \times a$$

$$= 4a^2$$

Factorizing the second term 15a, we have

$$15a = 3 \times 5 \times a$$

Divide the second term by 3a, we have

$$15a \div 3a = \frac{3 \times 5 \times a}{3 \times a}$$

$$= 5$$

Therefore, combining both the quotients, we have

$$(12a^3 + 5a) \div 3a = 4a^2 + 5$$

In this example, the division of algebraic expressions has been carried out by two methods.

In first method, the dividend is written as a product of factors such that the divisor is one of the factors. Then, the quotient is obtained by cancelling this common factor in both the numerators and the denominator.

In second method, each term of the dividend is divided individually by the divisor and resultant quotients of all the terms are added to get the required quotient.
Alternative Method: in this method, each term of the algebraic expression is divided individually by the divisor.

$$16a^4 + 8a^3 + 12a \div 4a = \frac{16a^4 + 8a^3 + 12a}{4a}$$

$$= \frac{16a^4}{4a} + \frac{8a^3}{4a} + \frac{12a}{4a}$$

$$= 4a^3 + 2a^2 + 3$$

## Division of Algebraic Expressions Continued (Polynomial + Polynomial)

There are two methods to divide a polynomial by another polynomial.

1. **By factorization**
2. **By long division**

1. **Division by factorization:** This method is similar to the division of a polynomial by a monomial. To divide a polynomial by a polynomial, we factorize the dividend and the divisor (if needed) and cancel the factors common to both numerator and denominator.

Let us consider on following expression:-

$$(x^2 + 15x + 56) \text{ by } (x + 8)$$

First we'll factorize the dividend $x^2 + 15x + 56$

$$x^2 + 15x + 56 = x^2 + 8x + 7x + 56$$

$$= x(x + 8) + 7(x + 8)$$

$$= (x + 8)(x + 7)$$

$$x^2 + 15x + 56 \div (x + 8) = \frac{x^2 + 15x + 56}{x + 8}$$

$$= \frac{(x + 8)(8 + 7)}{x + 8}$$

$$= x + 7$$

**2. Division by long method:**

$$x^3 - 6x^2 + x + 8 \text{ by } x + 1$$

**Step1:-** Write the terms of the dividend and divisor in descending order of their degree.

**Step2:-** Divide the first term of the dividend by the first term of the divisor. It gives us the first term of the quotient. Therefore, $x^3 \div x = x^2$ in the first term of the quotient.

**Step3:-** Multiply the divisor $(x + 1)$ by $x^2$. Write the result under proper term of the dividend and subtract the result from the dividend. To subtract, change the sign of each term in the lower row from + to – and from – to +. With the new signs add column wise (the new signs are written below the terms of the lower row. We get 1 – 7 $x^2$

Now bring down +x + 8. We get 1 – $7x^2$ + x + 8. This is the new dividend.

**Step 4:-** We now repeat step 2 to get the next term of the quotient. Dividing the first term – $7x^2$ of the new dividend by the first term of the divisor, we get – $7x^2 \div x = -7x$, which is the next term of the quotient.

**Step 5:-**Multiply the divisor (x + 1) by – 7x and subtract the result from the dividend ($-7x^2 + x + 8$) to get (8x +8), which is the next dividend.

**Step 6:-** Again repeat step 2 to get the next term of the quotient, i.e., $8x \div x = 8$, which is the next term of the quotient.

**Step 7:-** Multiply the divisor (x +1) by 8 and subtract the result from the dividend. We get remainder 0.

$$
\begin{array}{r|l}
 & x^2 - 7x + 8 \\
\hline
x + 1 & x^3 - 6x^2 + x + 8 \\
 & +x^3 + x^2 \\
 & - \quad - \\
\hline
 & -7x^2 + x + 8 \\
 & -7x^2 - 7x \\
 & + \quad + \\
\hline
 & 8x + 8 \\
 & +8x + 8 \\
 & - \quad - \\
\hline
 & 0 \\
\hline
\end{array}
$$

Therefore,

$$x^3 - 6x^2 + x + 8) \div (x + 1) = x^2 - 7x + 8$$

Note: When we divide a number 56 by 7, we get quotient = 8 and remainder = 0, since 7 is a factor of 56.

Similarly, when we divide an algebraic expression by another algebraic expression and our remainder is 0, it implies that the divisor is a factor of the dividend.

In above example, we get remainder 0, therefore $(x + 1)$ is a factor of $x^3 - 6x^2 + x + 8$ and the quotient $(x^2 - 7x + 8)$ is also a factor of the dividend.