Outline

Plane Rotations

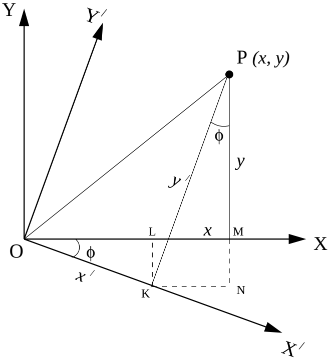

Figure: Rotation of axes and change of basis

$$x = OL + LM = OL + KN = x' \cos\phi + y' \sin\phi$$
$$y = PN - MN = PN - LK = y' \cos\phi - x' \sin\phi$$

Plane Rotations

Orthogonal change of basis:

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \Re \mathbf{r}'$$

Mapping of position vectors with

$$\Re^{-1} = \Re^T = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

In three-dimensional (ambient) space,

$$\Re_{xy} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Re_{xz} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \text{ etc.}$$

Plane Rotations

Generalizing to n-dimensional Euclidean space (R^n),

$$\mathbf{P}_{pq} = \begin{bmatrix} 1 & & & & 0 & & & & 0 & & \\ & 1 & & & 0 & & & & 0 & & \\ & & \ddots & & \vdots & & & & \vdots & & \\ & & & 1 & 0 & & & & 0 & & \\ 0 & 0 & \cdots & 0 & c & 0 & \cdots & 0 & s & \cdots & 0 \\ & & & & 0 & 1 & & & 0 & & \\ & & & & \vdots & & \ddots & & \vdots & & \\ & & & & 0 & & & 1 & 0 & & \\ 0 & 0 & \cdots & 0 & -s & 0 & \cdots & 0 & c & \cdots & 0 \\ & & & & \vdots & & & & \vdots & \ddots & \\ & & & & 0 & & & & 0 & & 1 \end{bmatrix}$$

Matrix $\mathbf{A}$ is transformed as

$$\mathbf{A}' = \mathbf{P}_{pq}^{-1}\mathbf{A}\mathbf{P}_{pq} = \mathbf{P}_{pq}^{T}\mathbf{A}\mathbf{P}_{pq},$$

only the p-th and q-th rows and columns being affected.

Jacobi Rotation Method

$$\begin{aligned} a'_{pr} = a'_{rp} &= c a_{rp} - s a_{rq} \text{ for } p \neq r \neq q, \\ a'_{qr} = a'_{rq} &= c a_{rq} + s a_{rp} \text{ for } p \neq r \neq q, \\ a'_{pp} &= c^2 a_{pp} + s^2 a_{qq} - 2sc a_{pq}, \\ a'_{qq} &= s^2 a_{pp} + c^2 a_{qq} + 2sc a_{pq}, \text{ and} \\ a'_{pq} = a'_{qp} &= (c^2 - s^2) a_{pq} + sc(a_{pp} - a_{qq}) \end{aligned}$$

In a Jacobi rotation,

$$a'_{pq} = 0 \Rightarrow \frac{c^2 - s^2}{2sc} = \frac{a_{qq} - a_{pp}}{2a_{pq}} = k \text{ (say)}.$$

Left side is $\cot 2\phi$: solve this equation for ϕ.

Jacobi rotation transformations $\mathbf{P}_{12}, \mathbf{P}_{13}, \cdots, \mathbf{P}_{1n}$; $\mathbf{P}_{23}, \cdots, \mathbf{P}_{2n}$; $\cdots$; $\mathbf{P}_{n-1,n}$ complete a full sweep.

Note: The resulting matrix is far from diagonal!

Jacobi Rotation Method

Sum of squares of off-diagonal terms before the transformation

$$
\begin{aligned}
S &= \sum_{r \neq s} |a_{rs}|^2 = 2\left[\sum_{r \neq p} a_{rp}^2 + \sum_{p \neq r \neq q} a_{rq}^2\right] \\
&= 2\left[\sum_{p \neq r \neq q} (a_{rp}^2 + a_{rq}^2) + a_{pq}^2\right]
\end{aligned}
$$

and that afterwards

$$
\begin{aligned}
S' &= 2\left[\sum_{p \neq r \neq q} (a_{rp}'^2 + a_{rq}'^2) + a_{pq}'^2\right] \\
&= 2\sum_{p \neq r \neq q} (a_{rp}^2 + a_{rq}^2)
\end{aligned}
$$

differ by

$$\Delta S = S' - S = -2a_{pq}^2 \leq 0; \quad \text{and } S \to 0.$$

Givens Rotation Method

While applying the rotation $\mathbf{P}_{pq}$, demand $a'_{rq} = 0$: $\tan\phi = -\frac{a_{rq}}{a_{rp}}$

$r = p - 1$: Givens rotation

- Once $a_{p-1,q}$ is annihilated, it is never updated again!

Sweep $\mathbf{P}_{23}, \mathbf{P}_{24}, \cdots, \mathbf{P}_{2n}$; $\mathbf{P}_{34}, \cdots, \mathbf{P}_{3n}$; $\cdots$; $\mathbf{P}_{n-1,n}$ to annihilate $a_{13}, a_{14}, \cdots, a_{1n}$; $a_{24}, \cdots, a_{2n}$; $\cdots$; $a_{n-2,n}$.

Symmetric tridiagonal matrix

How do eigenvectors transform through Jacobi/Givens rotation steps?

$$\tilde{\mathbf{A}} = \cdots \mathbf{P}^{(2)^T} \mathbf{P}^{(1)^T} \mathbf{A} \mathbf{P}^{(1)} \mathbf{P}^{(2)} \cdots$$

Product matrix $\mathbf{P}^{(1)}\mathbf{P}^{(2)} \cdots$ gives the basis.

To record it, initialize $\mathbf{V}$ by identity and keep multiplying new rotation matrices on the right side.

Givens Rotation Method

Contrast between Jacobi and Givens rotation methods

- What happens to intermediate zeros?
- What do we get after a complete sweep?
- How many sweeps are to be applied?
- What is the *intended* final form of the matrix?
- How is size of the matrix relevant in the choice of the method?

Fast forward ...

- Householder method accomplishes 'tridiagonalization' more efficiently than Givens rotation method.
- But, with a half-processed matrix, there come situations in which Givens rotation method turns out to be more efficient!

Points to note

Rotation transformation on symmetric matrices

- Plane rotations provide orthogonal change of basis that can be used for diagonalization of matrices.
- For small matrices (say $4 \leq n \leq 8$), Jacobi rotation sweeps are competitive enough for diagonalization upto a reasonable tolerance.
- For large matrices, one sweep of Givens rotations can be applied to get a symmetric tridiagonal matrix, for efficient further processing.

Necessary Exercises: **2,3,4**