Vector Analysis Notes

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Autumn 2006 Lectures

Contents

Lecture 0:3/10/06

0 Introduction

0.1 What is vector analysis

In analysis differentiation and integration were mostly considered on $\mathbb{R}$ or on rectangles (between points a and b). However a function on a circle is as valid as on a straight line. Vector analysis generalises these results onto curves, surfaces and volumes in $\mathbb{R}^n$

Example 0.1. The normal way to calculate an integral is to find an anti-derivative of the function and use the fundamental theorem of calculus (FTC)

$$f(x) + F^{(1)}(x) \longrightarrow \int_a^b f(x)dx = \int_a^b F(x)dx = F(b) - F(a) \tag{0.1}$$

The value of $\int_a^b f(x)dx$ can be computed by looking at the boundary points, a and b.

This can be generalised to $\mathbb{R}^n$, by Gauss' Theorem. Gauss' Theorem says that we can find the area with just the boundary lines.

0.2 Notation

There are many different notations you may use, especially in Physics/Engeneering [1]

0.2.1 Vectors

$$x \in \mathbb{R}^n,\ x = (x_1, x_2, \ldots, x_n)$$

alternatives $\bar{x},\ \vec{x},\ \underline{x},\ \boldsymbol{x}$ I normally use $\boldsymbol{x}$, physicists generally use $\vec{r} = (x, y, z)$ where $r = \sqrt{x^2 + y^2 + z^2}$, however this is no good with n dimensions as you soon run out of letters!

0.2.2 Functions

$f : \mathbb{R}^m \to \mathbb{R}^n$ with component functions $f_1, \ldots, f_n$ $\mathbb{R}^m \to \mathbb{R}$, alternate ways of showing functions are $\overrightarrow{f},\ \underline{f}, \ldots$

0.2.3 Inner Product

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i \text{ for } x, y \in \mathbb{R}^n \;{}^{2} \tag{0.2}$$

alternate $(xy),\ x \cdot y,\ x^T y$

[1] The biggest challenge is getting LaTeX to write them all... ;)
[2] In this case this is actually the scalar (dot) product

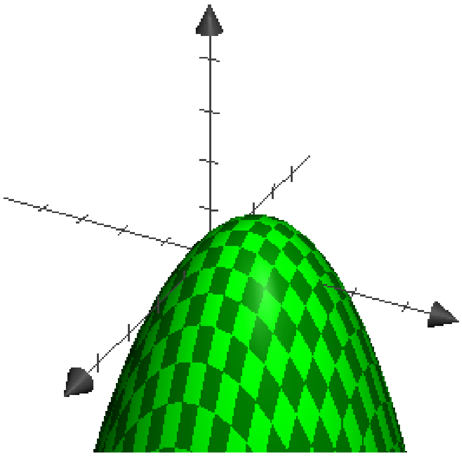

Figure 1: Graph showing a peak

0.2.4 Partial derivatives

for $f : \mathbb{R}^n \to \mathbb{R},\ x \in \mathbb{R}^n$

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + he^i) + f(x)}{h}$$

alternates, $\partial f(x),\ \frac{\partial f}{\partial x_i}(x),\ d_{x_i}, \ldots$

$$f_t(t,x) = \frac{\partial}{\partial t} f(t,x)$$

$$f_x(t,x) = \frac{\partial}{\partial x} f(t,x)$$

1 Lecture 1 The real thing

1.1 Gradients and Directional derivatives

How does a function $f : \mathbb{R}^n \to \mathbb{R}$ change when we move from a point $x \in \mathbb{R}^n$ in some direction $y \in \mathbb{R}^n$? This can be seen in figure 1 We can reduce the problem to one dimension. Consider $\varphi : \mathbb{R} \to \mathbb{R},\ \varphi \mapsto f(x + \lambda y)$ the change of f at a point x in direction of y equals the change of φ at point $\lambda = 0$ and thus is $\varphi(0)$.

Definition 1.1 (The directional derivative). *$f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ in the direction of $y \in \mathbb{R}^n$*

$$D_y f(x) = \lim_{\lambda \to 0} \frac{f(x + \lambda y)}{\lambda} \tag{1.1}$$

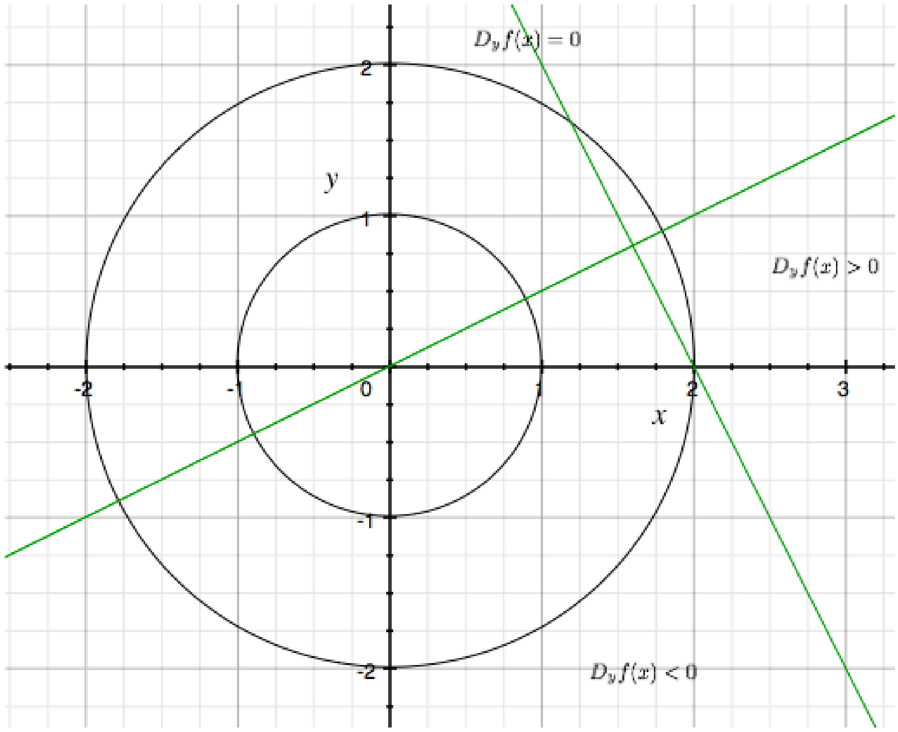

Figure 2: Graph showing how the directional derivative varies

Example 1.1.

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x) = x_1^2 + x_2^2$$

$$f(x + \lambda y) = (x_1 + \lambda y_1)^2 + (x_1 + \lambda y_2)^2$$

$$= x_1^2 + x_2^2 + 2\lambda x_1 y_1 + 2\lambda x_2 y_2 + \lambda^2 y_1^2 + \lambda^2 y_2^2 \longrightarrow D_y f(x) = 2x_1 y_1 + 2x_2 y_2 = \langle 2x, y \rangle$$

This is shown in figure 2. [3] In this case this gives:

$$\varphi'(\lambda) = \sum_{i=1}^{n} \delta_i f(x_i + \lambda y_i)$$

$$\to D_y f(x) = \varphi'(0) = \sum_{i=1}^{n} \delta_i f(x) y_i$$

where $\delta_i f(x)$ is the gradient part (of ∇f).

1.2 Directional Derivatives

$$D_y f(x) = \lim_{\lambda \to 0} \frac{f(x + \lambda y) - f(x)}{\lambda}$$

Do you really need to calculate this for every $D_y f(x)$ for all directions y?

[3] It's probably worth looking over section 1.2 first as thats the order we did it in lectures.

We have to calculate the derivative of $\varphi : \lambda \mapsto x + \lambda y \mapsto f(x + \lambda y)$

$$\mathbb{R} \longrightarrow^{g} \mathbb{R}^{n} \longrightarrow^{f} \mathbb{R}$$

In general this is shown in the next two subsections.

1.2.1 Generally in one dimension

$$\big(f(g(\lambda)\big)' = f'\big(g(\lambda)\big) \cdot g'(\lambda)$$

1.2.2 Generally in n dimensions

$$\big(f(g(\lambda)\big)' = \sum_{i=1}^{n} \delta_i f'\big(g(\lambda)\big) \cdot g_i'(\lambda)$$

Definition 1.2. *For $\mathbb{R}^n \to \mathbb{R}$ the vector $\nabla f = (\delta_i f, \dots, \delta_n f)$ is called the gradient of f. Alternative notations include* $\mathrm{grad} f, \dots \underline{\nabla f}$

Definition 1.3 (Cauchy-Schwarz inequality)**.** *If $x, y \in \mathbb{C}$ then*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle \quad [1] \tag{1.2}$$

Another form of this, which we use here is:

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \tag{1.3}$$

Remark 1.1. *From the Cauchy Schwarz inequality (equation 1.3) we get:*

$$|D_y f(x)| = |\langle \nabla f(x), y\rangle| \leq \|\nabla f(x)\| \|y\| \tag{1.4}$$

Where $||y|| = 1$ we get:

$$||\nabla f(x)|| \leq ||D_y f(x)|| \leq ||\nabla f(x)||$$

And for: $y = \frac{\nabla f(x)}{||\nabla f(x)||}$ we get

$$||D_y f(x)| = \left\langle \nabla f(x), \frac{\nabla f(x)}{\|\nabla f(x)\|} \right\rangle$$

This then implies that y is maximal if y points in direction of the gradient.

2 Visualisation of a function $f : \mathbb{R}^n \to \mathbb{R}^n$

Graphs of scalar fields.

Definition 2.1. $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}^m$

Example 2.1. $m = n = 1$ as in Analysis I, $f(x) = \sin(x)$, this is shown in figure 3

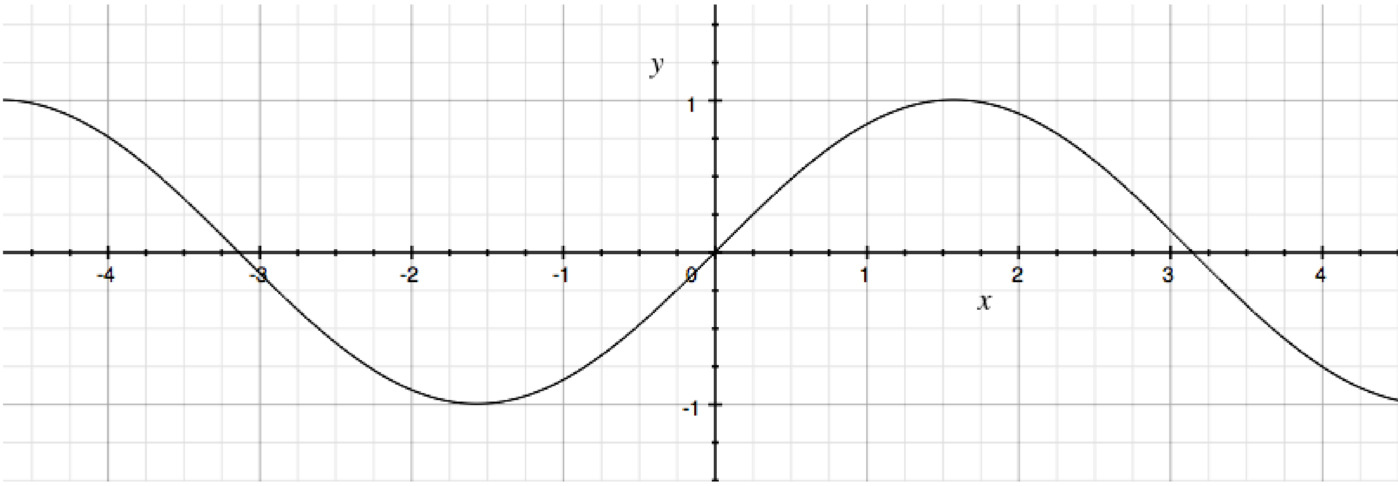

Figure 3: Graph of $f(x) = \sin x$

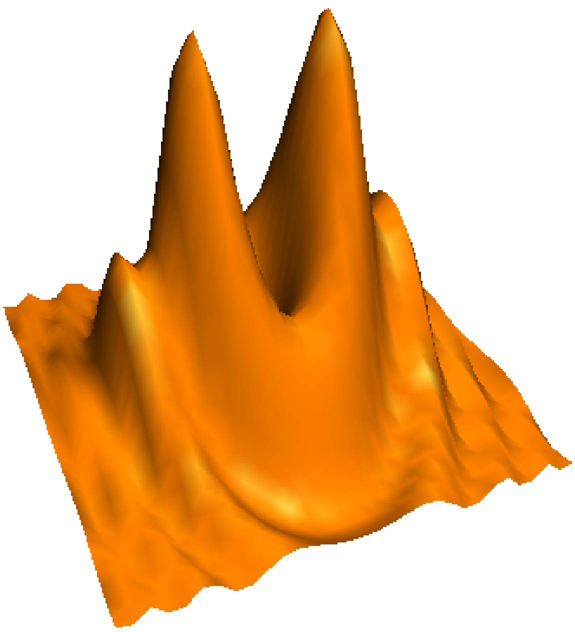

Figure 4: Graph of the function: $z = \frac{\sin(x^2+3y^2)}{0.1+r^2} + (x^2+5y^2)\cdot\frac{\exp(1-r^2)}{2}$, $r = \sqrt{x^2+y^2}$

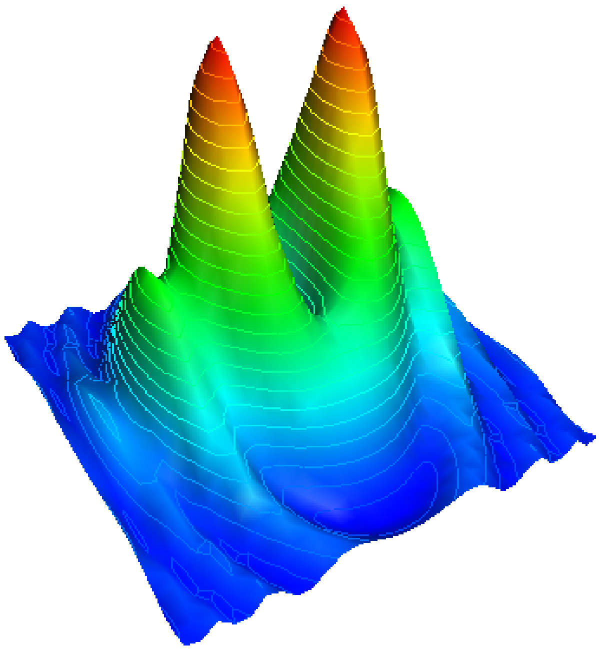

Figure 5: This graph is the original Grapher example, it is included because it looks cool, and not a boring uniform orange, if it confuses you ignore it, it adds nothing to figure 4 in terms of vector analysis. In the image the colour gradient represents the height (z value, on x, y, z axis), but could be use to represent some extra criteria.

Example 2.2. For m=2,n=1 we can draw something as shown in figure 4 [4]

For $m \geq 3, n = 1$ it is difficult, colour coding could be used to represent a variable, in a similar way to how figure 5 does for the z direction. Applications of this include temperatures distributions on 3D bodies or pressure in a liquid.

$$f^{-1}(c) = \{x \in \mathbb{R}^n \mid f(x) = 0\}^5$$

$$f(x, y) = x^2 + y^2$$

2.1 Curves

These can be described in two ways:

1. Implicitly giving the graph $\mathcal{C} \subseteq \mathbb{R}^n$
2. Explicitly as parametric curves $\varphi : \mathbb{R} \to \mathbb{R}^n$

Example 2.3.

$$1 = x^2 + y^2 \tag{2.1}$$

Then equation 2.1 can be written parametrically as:

$$\varphi(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \tag{2.2}$$

For a curve $\varphi : \mathbb{R} \to \mathbb{R}^n$ where φt is the position at "time" t.

$$\varphi'(t) = \lim_{h \to 0} \frac{\varphi(t + h) - \varphi(t)}{h}$$

If $\varphi \neq 0$ then $\varphi'(t)$ is a tangent vector and the tangent line is given by $\lambda \mapsto \varphi(t) + \varphi'(t)$

Definition 2.2. *A vector $x \in \mathbb{R}^n$ is orthogonal to a curve $\varphi : \mathbb{R} \to \mathbb{R}^n$ at the point $\varphi(t)$ if $\langle x, \varphi(t) \rangle = 0$ i.e. if it is orthogonal to the tangent line.*

Lemma 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function, $a \in \mathbb{R}^n$. Then $\nabla f(a) : \perp \{x \in \mathbb{R}^n | f(x) - f(a)\}$ this means the gradient is orthogonal to the tangent lines.*

Proof. Let $\varphi : \mathbb{R} \to \{x \mid f(x) = f(a)\}$ be a curve with $\varphi(0) = a$.

$$= \langle \nabla f(\varphi(t), \varphi'(t) \rangle \big|_{t=0} = \langle \nabla f(a), \varphi'(t) \rangle$$

$$= \sum_{i=1}^{m} \varphi_i'(t) \delta_i f'(\varphi(t))$$

Then

$$0 = \frac{d}{dt} f(\varphi(t)) \big|_{t=0}$$

∎

[4] Sorry for the excessively complex example. Blame Apple for making such a cool graph.[/ apple/maths geek]

[5] It seems something is missing here

3 Line Integrals

We want to take integrals along a curve $\mathcal{C} \in \mathbb{R}^n$, there are two methods of integrating line integrals.

3.1 Integrating by scalar fields

Definition 3.1 (Length of curve). *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve, then the scalar line of $u : \mathbb{R}^n \to \mathbb{R}$ is:*

$$\int_\gamma u = \int_a^b u\big(\gamma(t)\big)||\gamma'(t)||dt$$

other forms of line integrals $\int_\gamma uds$ the[6] *has no useful role.*

3.2 Integrating vector fields

Definition 3.2. *Ley γ be a curve and f a vector field the tangent line integral of f along $\gamma[a, b] \to \mathbb{R}^n$ a vector field. Then the tangent line integral of f along γ is given by:*

$$\int_\gamma f = \int_a^b \langle f\big(\gamma(t)\big), \gamma'(t)\rangle dt$$

alternative notations are:

$$\int_\gamma f \cdot \hat{T}\overrightarrow{ds}, \quad \int_\gamma \overrightarrow{f}\,\overrightarrow{ds}, \quad \int_C f\overrightarrow{ds}$$

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Example 3.1. Length of the circle line, let

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad \forall t \in [0, 2\pi]$$

$$\Rightarrow \gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \quad \forall t \in [0, 2\pi]$$

$$||\gamma'(t)|| = \sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{1} = 1$$

$$\int_C 1 = \int_0^{2\pi} 1 \cdot 1dt = 2\pi$$

This answer is the same as you get from the old fashioned method of finding the circumference using circumference $= 2\pi r$

Remark 3.1. *The tangent line integral can be written as:*

$$\int_C f = \int_a^b \left\langle f\big(\gamma(t)\big), \frac{\gamma'(t)}{||\gamma'(t)||} \right\rangle ||\gamma'(t)||dt$$

with an inner product.[7] *$\int_C f$ is the scalar line integral of the component of f along the tangent line.*

[6] a piece of LaTeX is missing here check original notes

[7] doesn't make much sense to me now, maybe I was distracted ;)

Example 3.2. Work done when moving a mass along the line $-\cos x$

$$\gamma(t) = \begin{pmatrix} t \\ -\cos t \end{pmatrix}$$

$$f(x,y) = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$$

$$-\int_\gamma f = -\int_0^\pi \left\langle \begin{pmatrix} 0 \\ -mg \end{pmatrix}, \begin{pmatrix} 1 \\ \sin t \end{pmatrix} \right\rangle dt = \int_0^\pi mg \sin t dt = \Big[-mg\cos t\Big]_0^\pi = mg+mg = \underline{\underline{2mg}} \tag{3.1}$$

This example is continued as example 4.1 in the next chapter.

4 Gradient Vector Fields

Definition 4.1. *A gradient vector field is a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f = \nabla V$ for some $V : \mathbb{R}^n \to \mathbb{R}$, V is called the potential of f.*

Remark 4.1. *1. ∇V is not unique as $\nabla V = \nabla(V + C)$, $\forall c \in \mathbb{R}$, this means the potential is not unique.*

2. Not every vector field is a gradient vector field

Theorem 4.1 (Fundamental Theorem of Calculus for gradient vector field)**.** *Let $V : \mathbb{R} \to \mathbb{R}$? be a scalar field $f = \nabla V$? and $\gamma : [a,b] \to \mathbb{R}^n$? a curve then*

$$\int_\gamma f = V\big(\gamma(b)\big) - V\big(\gamma(a)\big)$$

Proof. The ??? $\mapsto V\big(\gamma(t)\big)$ has $\mathbb{R} \to \mathbb{R}$ derivative.

$$\Big(V\big(\gamma(t)\big)\Big)' = \sum_{i=1}^{n} \delta_i V'\big(\gamma(t)\big)\gamma'(t) = \Big\langle \nabla V\big(\gamma(t)\big), \gamma'(t)\Big\rangle \tag{4.1}$$

Therefore [8]

$$\int_\gamma f = \int_a^b \Big\langle f\big(\gamma(t)\big), \gamma'(t)\Big\rangle dt = \int_a^b \Big(V\big(\gamma(t)\big)\Big)' dt = V\big(\gamma(b)\big) - V\big(\gamma(a)\big) \tag{4.2}$$

□

As we now know:

$$\int_\gamma f = V\big(\gamma(b)\big) - V\big(\gamma(a)\big)$$

[8] More destractedness I think ;)

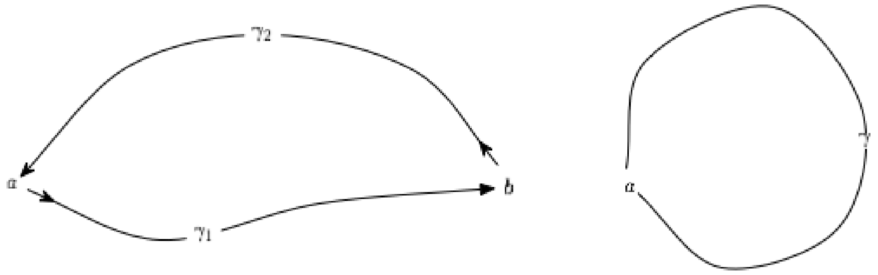

Figure 6: Diagram showing routes between two points a and b, and a loop about a point a

Example 4.1.

$$f = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$$

can be written as ∇V_γ and $V = -mgy \Rightarrow$ for every curve, $\gamma(t) = (x(t), y(t))$ we get:

$$-\int_\gamma f = -V(\gamma(b)) + V(\gamma(a)) = mgy(b) - mgy(a)$$

but in this case $y(b) = 1$, $y(a) = -1 = mg + mg = 2mg$ [9] f is a vector field. γ is a curve. $\Rightarrow \int_\gamma f$ which is the integral of f over the curve γ.

$$\Rightarrow \int_\gamma f = V\big(\gamma(b)\big) - V\big(\gamma(a)\big)$$

Remark 4.2. *If f[10] is a vector field and γ is a line then $\int_\gamma f$ does <u>not</u> depend on the path γ but only the two end points.*

Remark 4.3. *If $f = \nabla V$ then figure 6 implies that*

$$\int_{\gamma_1} f = V(b) - V(a) \text{ and } \int_{\gamma_2} f = V(a) - V(b)$$

this means that $\int_{\gamma_1} f = -\int_{\gamma_2} f$.

Definition 4.2. *A loop is where the end point is the start point as shown in figure 6.*

Example 4.2.

$$f(x,y) = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \ \forall\, t \in [0, 2\pi] \Longrightarrow \int_\gamma f = \int_0^{2\pi} \left\langle \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle dt$$

$$\int_0^{2\pi} \sin^2 t + \cos^2 t dt = \int_0^{2\pi} 1\, dt = 2\pi$$

this means that f is <u>not</u> a gradient vector field.

[9] $2mg$ is what we got before

[10] not 100% on whether this is f

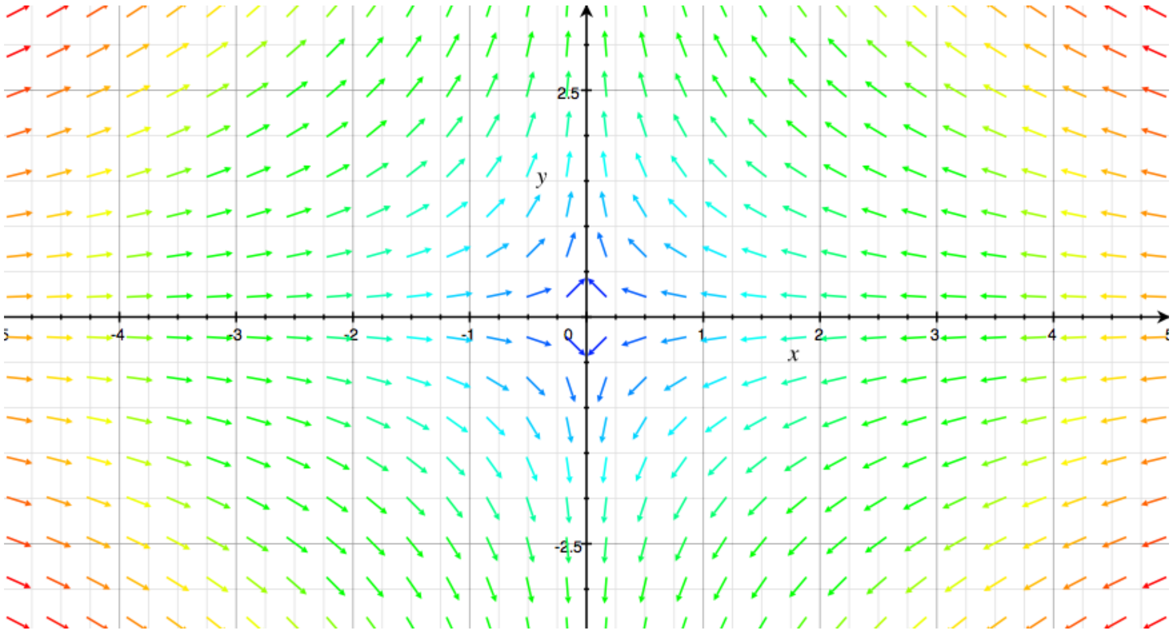

Figure 7: Graph of a vector field

$$V(x) = V(0) + \int_{\gamma_x} f$$

where γ_x is the straight line from 0 to x for every potential V. If V_1 and V_2 are two different potentials, then:

$$V_1(x) - V_2(x) = V_1(0) + \cancel{\int_{\gamma} f} - V_2(0) - \cancel{\int_{\gamma} f}$$

Is then constant.

Example 4.3 (Finding a potential). Let $f(x,y) = \frac{1}{2}\begin{pmatrix} -x \\ y \end{pmatrix}$ (this vector field is shown in figure 7) Suppose $f = \nabla V$ then this implies that

$$\frac{\partial V}{\partial x} = -\frac{1}{2}x \Leftrightarrow V = -\frac{x^2}{4} + C_1$$

$$\frac{\partial V}{\partial y} = \frac{1}{2}y \Leftrightarrow V = -\frac{y^2}{4} + C_2$$

These two equations imply that:

$$V = \frac{y^2}{4} - \frac{x^2}{4} + C$$

so $V(x,y) = \frac{y^2}{4} - \frac{x^2}{4}$ is a valid solution.

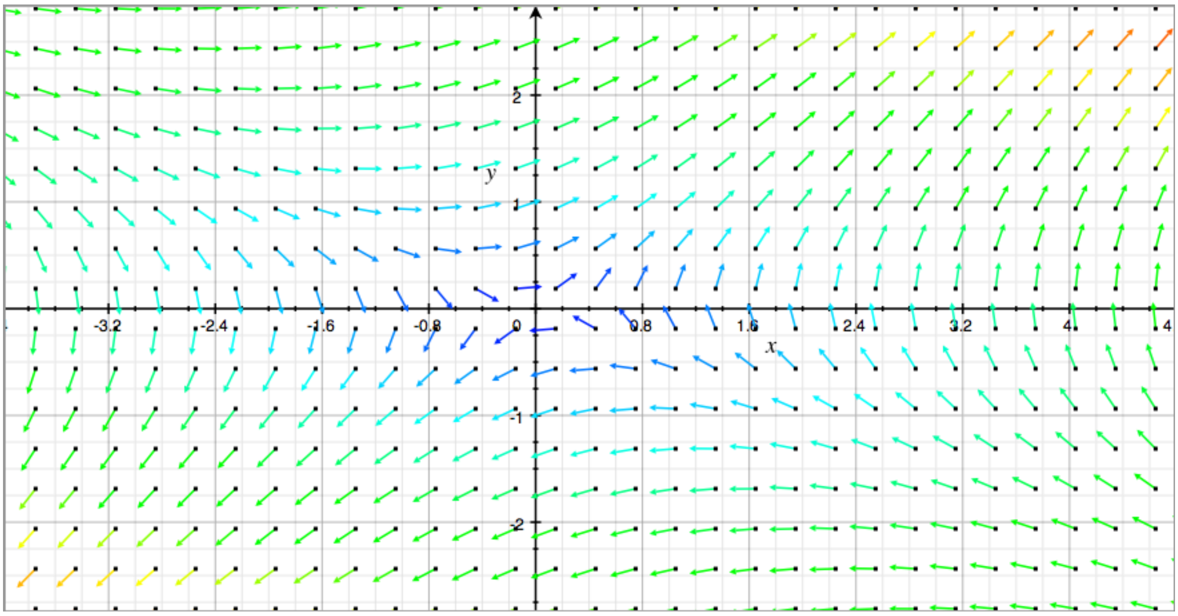

Figure 8: Graph of a vector field

Example 4.4. Let $f(x,y) = \begin{pmatrix} 2y \\ x+y \end{pmatrix}$ (this vector field is shown in figure 8) Suppose $f = \nabla V$ then

$$\frac{\partial V}{\partial x} = 2y \Leftrightarrow V = 2xy + C_1(y)$$
$$\frac{\partial V}{\partial y} = x + y \Leftrightarrow V = xy + \frac{1}{2}y^2 + C_2(x)$$

This has no solution which implies that f cannot be a gradient field.

Definition 4.3. *$f : \mathbb{R} \to \mathbb{R}$ is called a radial vector field if*

$$f(x) = \begin{cases} g(\|x\|) & \frac{x}{\|x\|} \text{ if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

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$$\int_\gamma f = \int_\gamma \nabla V = V(\gamma(b)) - V(\gamma(a))$$

(if f is a gradient)

Example 4.5. Radial vector fields: Let:

$$f(x) = \begin{cases} g(\|x\|) & \frac{x}{\|x\|} \text{ if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

, where $g : (0,\infty) \to \mathbb{R}$ We always find $\phi(0,\infty) \to \mathbb{R}$ with $\phi' = g$. Let $v(x) = \phi(\|x\|) = \phi\left(\sqrt{x_1^2 + \cdots + x_n^2}\right)$ We then get:

$$\frac{\partial}{\partial x} v(x) = \phi(\|x\|) = \frac{1}{2\sqrt{x_1^2 + \cdots + x_n^2}} 2x_i = \phi'(\|x\|)\frac{x_i}{\|x\|} = g\frac{x_i}{\|x\|}$$

$$= f_i(x) \to \nabla V = f$$

Thus we now know that radial vector fields are always gradients.

5 Surface Integrals

There are two methods of describing a surface in $\mathbb{R}^3$

1. Level set of $f : \mathbb{R}^3 \to \mathbb{R}$
2. Parameterisation $r : A \to \mathbb{R}^3$ where $A \subseteq \mathbb{R}^3$

$$r(s,t) = \begin{pmatrix} r_1(s,t) \\ r_2(s,t) \\ r_3(s,t) \end{pmatrix}$$

Find the surface $\mathcal{C}$'s normal vectors for level sets:

$$\nabla f \perp \mathcal{C} \Rightarrow \hat{N} = \frac{\nabla f}{\|\nabla f\|}$$

for parameterisations [11] A plane is defined by two vectors and a point. Or a point and a vector orthogonal to the plane. At $r(s,t)$ the vectors $\frac{\partial r}{\partial s}$ and $\frac{\partial r}{\partial t}$ are tangent vectors of the surface:

$$\mathcal{C} \Rightarrow N = \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}$$

is therefore normal to $\mathcal{C}$ this implies that

$$\hat{N} = \frac{\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}}{\left\|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right\|}$$

[12]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

Definition 5.1. *Let $r : A \to \mathbb{R}^3,\ A \subseteq \mathbb{R}^2$ be a parameterisation of some surface C. Then the scalar surface integral of $f : \mathbb{R}^3 \to \mathbb{R}$ is:*

$$\int_r f = \iint_A f\big(r(s,t)\big) \left\|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right\| dsdt$$

Remark 5.1. *$\int_r 1$ is the surface area of C.*

Remark 5.2. *Alternative notations $\int_C f_1$, $\int_C f ds$, $\int_C f dA$ (some diagrams I'm not copying)*

$$\int_r f \approx \sum_{s,t} f\big(r(s,t)\big) \left\|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right\| dsdt$$

[11] (see paper notes)

[12] next bit unclear

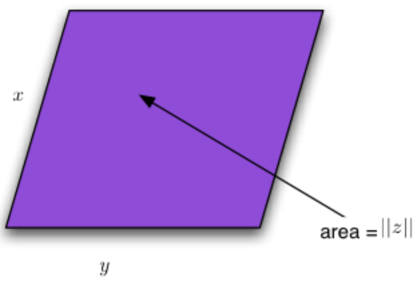

Figure 9: Diagram showing how the area of a parallelogram is found.

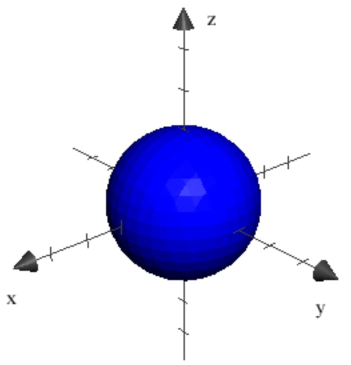

Figure 10: Diagram showing a sphere.

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$$\int_\gamma f = \iint_A f\big(r(s,t)\big)\left\|\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right\| dsdt \tag{5.1}$$

Properties of x, let $z = xy$ then:

1. $z \perp x$ and $z \perp y$
2. $\|z\|$ is the area of the parallelogram spanned by x and y (shown if figure **??**.
3. The orientation of z is given by the right hand rule.[13]

Example 5.1 (Spherical Cap)**.** First we parameterise it.

$$r(s,t) = \begin{pmatrix} \cos s\cos t \\ \cos s\sin t \\ \sin s \end{pmatrix}$$

$$\frac{\partial r}{\partial s} = \begin{pmatrix} -\sin s\cos t \\ -\sin s\sin t \\ \cos s \end{pmatrix}$$

[13] `http://en.wikipedia.org/wiki/Right_hand_rule`

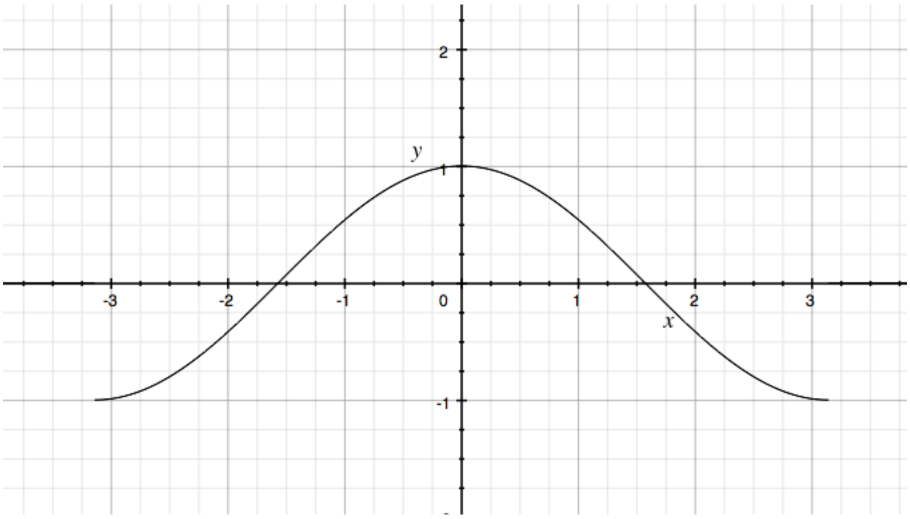

Figure 11: Diagram showing a $y = \cos x$ for $x \in [-\pi, \pi]$

So therefore

$$\left\| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right\| = \sqrt{\cos^4 s + \cos^2 s \sin^2 s} = \cos s \sqrt{\cos^2 s + \sin^2 s} = \cos s \tag{5.2}$$

Therefore the area of the cap with radius 1, is:

$$\int_\gamma 1 = \int_{-\pi}^{\pi} \int_0^{\frac{\pi}{2}} 1 \cdot \cos s ds dt$$

$$= \int_0^{\frac{\pi}{2}} 2\pi \cos s ds$$

$$= [2\pi \sin s]_0^{\frac{\pi}{2}}$$

$$= 2\pi - 2\pi \sin\theta$$

$$= 2\pi\big(1 - \sin\theta\big)$$

If $\theta = -\frac{\pi}{2}$, $\sin\theta = -1$ So therefore

$$2\pi(1 - -1) = 4\pi \tag{5.3}$$

Which is the surface area of a sphere of radius 1.

Example 5.2 (Newton's kissing problem)**.** An example of this in one and two dimesions is shown in figure 12 For $\mathbb{R}^3$ how many simultaneously touching balls can touch a given ball? [14] To find an upper bound by calculating the area "shadowed" surface area taken by each ball, this is shown in figure 13.

$$\frac{1}{2} = \sin\alpha, \text{ as we can define } \frac{\pi}{2} - \alpha = \theta$$

[14] J Leech proved that the correct number is 12, in Math Gazette 40 (1956) p22/23

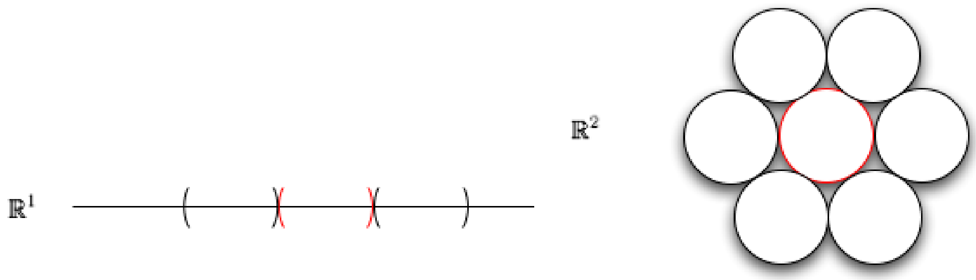

Figure 12: Diagram showing Newton's kissing problem in one and two dimensions.

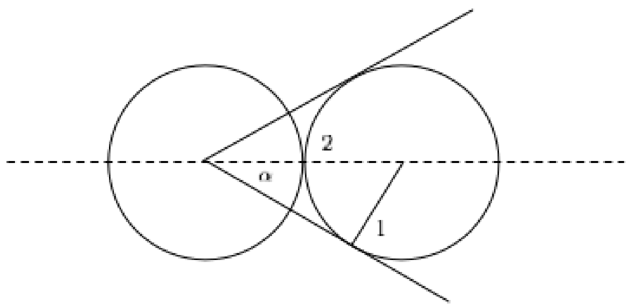

Figure 13: Diagram showing the "shadowed area in Newton's kissing problem

$$\frac{1}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{3}$$

Shadowed surface has area $2\pi(1 - \sin\theta)$ as we found in example 5.1.

$$2\pi\left(1 - \sin\frac{\pi}{3}\right) = 2\pi\left(1 - \frac{\sqrt{3}}{2}\right) = 2\pi - \frac{\cancel{2}\pi\sqrt{3}}{\cancel{2}} = (2 - \sqrt{3})\pi$$

This therefore gives an upper bound of

$$\frac{4\cancel{\pi}}{(2-\sqrt{3})\cancel{\pi}} = 14.93 \text{ (2dp)}$$

This gives between 12 and 14 balls. Below are the results for various values of n

n=1	2
n=2	6
n=3	12
n=4	24*
n=8	240
n=24	196560

For $n = 4$ the answer is probably 24 but a strict upper bound of 25 is as far as we know for sure. [2][15]

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$$\int_\gamma \nabla V = V(\gamma(b)) - V(\gamma(a))$$

6 Divergence of Vector Fields

Definition 6.1. *Let $S \subseteq \mathbb{R}^3$ be a surface with unit normals $\hat{N} : S \to \mathbb{R}^3$ then the flux of $f : \mathbb{R}^3 \to \mathbb{R}^3$ across S in the direction of $\hat{N}$ is $\int_S \langle f, \hat{N}\rangle$. The inner product tells you how much of f is pointing in the direction $\hat{N}$ and this is shown in figure 14. Alternative notations include $\int_S f \cdot \hat{N}$ and $\int_S f d\overrightarrow{s}$.*

Example 6.1 (Flux out of a box)**.** If we take the box $\Omega = [a_x, b_x] \times [a_y, b_y] \times [a_z, b_z]$, this is shown in figure 15

The flux through the top

Using the parameterisation

$$r_t = (x, y) = \begin{pmatrix} x \\ y \\ t_z \end{pmatrix}, x \in [a_x, b_x],\ y \in [a_y, b_y]$$

$$\langle f, \hat{N}\rangle = \int_{a_x}^{b_x}\int_{a_y}^{b_y} f_3 \left\| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right\| dydx$$

[15]For this citation the ability to open .ps and .gz files required, i.e. Linux/Mac OS X, if you use Windows you're basically screwed, you can use a command line tool to open the .gz but then you need a copy of ghostscript or Acrobat Professional (well for the cost of that you might as well buy a Mac ;)), drop me an email and I can send you a copy.

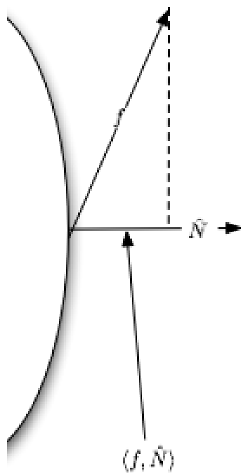

Figure 14: Diagram showing how the inner product works

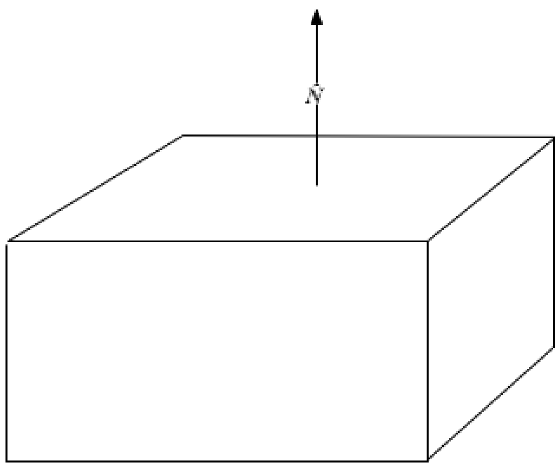

Figure 15: Flux Box

$$\left\|\frac{\partial r}{\partial x}\times\frac{\partial r}{\partial x}\times\frac{\partial r}{\partial y}\right\| = \left\|\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right\| = \left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\right\| = 1 \tag{6.1}$$

$$\int_{s_t}\langle f,\hat{N}\rangle = \int_{a_x}^{b_x}\int_{a_y}^{b_y} f_3(x,y,t_z)dydx$$

<u>The flux through the bottom</u>

$$\hat{N} = \begin{pmatrix}0\\0\\-1\end{pmatrix}$$

$\langle f,\hat{N}\rangle = f_3$ b_2 changed a_2

$$\int_{S_{bottom}}\langle f,\hat{N}\rangle = -\int_{a_x}^{b_x}\int_{a_y}^{b_y} f_3(x,y,t_z)dydx$$

<u>Top+Bottom</u>

$$\begin{aligned}&\int_{a_x}^{b_x}\int_{a_y}^{b_y} f_3(x,y,b_2) - f_3(x,y,a_2)dydx\\ &= \int_{a_x}^{b_x}\int_{a_y}^{b_y}\underbrace{\int_{a_z}^{b_z}\frac{\partial f_3}{\partial z}(x,y,z)dz}_{\text{by FTC}}\,dydx \end{aligned}\tag{6.2}$$

Similarly

$$\int_{S_{\text{left}}\cup S_{\text{right}}}\langle f,\hat{N}\rangle = \int_{a_x}^{b_x}\int_{a_y}^{b_y}\int_{a_z}^{b_z}\frac{\partial f_1}{\partial x}(x,y,z)dzdydx \tag{6.3}$$

$$\int_{S_{\text{front}}\cup S_{\text{back}}}\langle f,\hat{N}\rangle = \int_{a_x}^{b_x}\int_{a_y}^{b_y}\int_{a_z}^{b_z}\frac{\partial f_2}{\partial y}dxdydz \tag{6.4}$$

Taking the sum of equations 6.2, 6.3 and 6.4 we then get the total flux which is:

$$\int_{\delta\Omega}\langle f,\hat{N}\rangle = \int_{a_x}^{b_x}\int_{a_y}^{b_y}\int_{a_z}^{b_z}\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\frac{\partial f_3}{\partial z}dzdydx \tag{6.5}$$

Definition 6.2 (Divergence)**.** *The divergence of a vector field* $f:\mathbb{R}^n\to\mathbb{R}^n$ *is* $\text{div}f:\mathbb{R}^n\to\mathbb{R}$

$$\text{div}f = \sum_{i=0}^{n}\frac{\partial f_i}{\partial x}$$

Alternative notations: $\nabla\cdot V$, $\nabla\overrightarrow{V}$ *and* $\underline{\nabla}\cdot\underline{V}$

Remark 6.1. *Scalar field to vector field*

$$V:\mathbb{R}^n\to\mathbb{R}\to\text{grad}V:\mathbb{R}^n\to\mathbb{R}^n$$

$$f:\mathbb{R}^n\to\mathbb{R}^n\Leftrightarrow f:\mathbb{R}^n\to\mathbb{R}$$

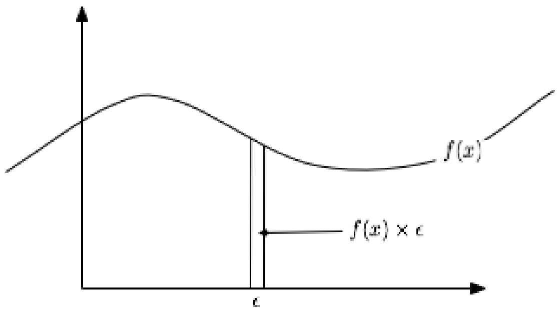

Figure 16: Diagram showing $f(x) \times \epsilon$

Definition 6.3. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a scalar field. The Laplacian of V is $\Delta V = \text{div grad}V$ (Scalar field)*

$$\text{grad}V = \begin{pmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_n} \end{pmatrix} \Rightarrow \text{div grad}V = \sum_{i=1}^{n} \frac{\partial V_i}{\partial k_i}$$

Alternative notations include: $\nabla \cdot \nabla V$, ΔV, $\nabla^2 V$.

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$$\text{div}f(x) = \sum \frac{\partial f}{\partial x}(x)$$

$\Omega = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

$$\int_\Omega \langle f, \hat{N} \rangle = \int \text{div}f(x)dx \tag{6.6}$$

Remark 6.2. *If Ω is a small box around $a \in \mathbb{R}^3$ then $\Omega = [a_1 - \epsilon, a_1 + \epsilon] \times [a_2 - \epsilon, a_2 + \epsilon] \times [a_3 - \epsilon, a_3 + \epsilon]$ Then*

$$\int_\Omega \text{div}(x)dx \approx \text{div}f(\alpha) - V_a(\Omega)$$

$$\Rightarrow \text{div}f(a) \cong \frac{\int_{\delta\Omega} \langle f, \hat{N} \rangle}{V_0(\Omega)}$$

Thus the divergence gives the outward flux per unit volume.

Remembering that $\text{div}f(x) = \nabla \cdot f$

7 Gauss Divergence Theorem

Definition 7.1 (Divergence)**.** *The divergence of a C^1 vector field $f \subseteq \mathbb{R}^3$ is:*

$$\text{div}V = \nabla \cdot V = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} \tag{7.1}$$

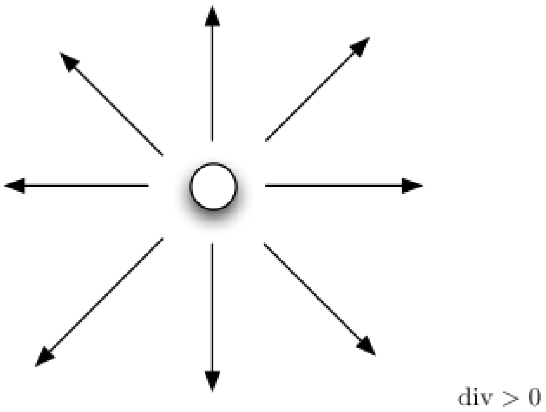

Figure 17: Positive Divergence

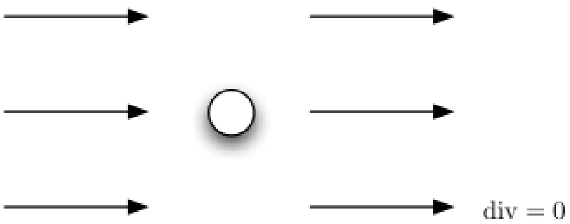

Figure 18: Zero Divergence

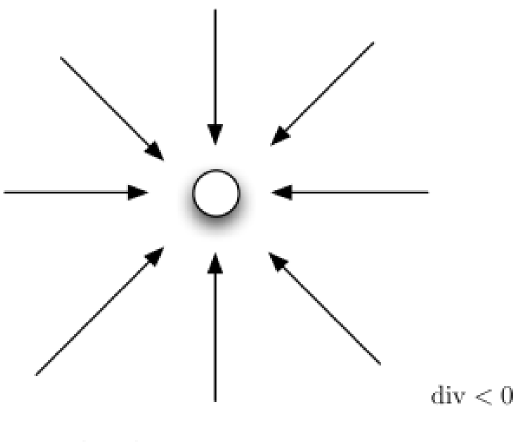

Figure 19: Negative Divergence

Theorem 7.1 (Divergence Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded with a surface $\delta\Omega$ and outward unit normals $\hat{N}$. Let $V : \Omega \cup \delta\Omega \to \mathbb{R}^3$ be continuously differentiable, then:*

$$\int_{\delta\Omega} \langle V, \hat{N} \rangle = \int_{\Omega} \mathrm{div} V(x) dx \tag{7.2}$$

Remark 7.1. *The theorem also works for n $\neq 3$ but one has to define the surface integral, $n = 2\delta\Omega$ is a line and $\int_{\delta\Omega}$ is a (scalar) line integral.*

Remark 7.2. $\mathrm{div} f(x)dx$ *is just an iterated integral, alternative notation is:* $\int_\Omega \mathrm{div} f dV$ $(n = 3)$*,* $\int_\Omega \mathrm{div} f dA$ $(n = 2)$*.*

Example 7.1. Box in $\mathbb{R}^3$ already done in example 6.1

Example 7.2. Ball of radius R in $\mathbb{R}^n$

$$\Omega = B(0, R) \subseteq \mathbb{R}^n$$

$$\delta\Omega = \text{Shell of Sphere, } \hat{N}(x) = \frac{x}{R}$$

Let:

$$\begin{aligned} f(x) = x \Rightarrow \mathrm{div} f(x) &= \frac{\partial x_1}{\partial x_1} + \cdots + \frac{\partial x_n}{\partial x_n} \\ &= 1 + \cdots + 1 = n \\ \Rightarrow \int_{B(0,R)} n dx &= \int \delta B(0, R) \left\langle x, \frac{x}{R} \right\rangle \end{aligned}$$

$$\int_{\delta B(0,R)} R \Rightarrow n \times \text{Volume of Ball} = R \times \text{Surface Area of Ball}$$

For $n = 2$ then $\mathrm{Vol} B(0, R) = \pi R^2$ This means that the length of $\delta B(0, R) = 2\pi R$, for $n = 3$ the volume of the ball is $\frac{4}{3}\pi R^3$ And the surface area is $\frac{3}{R}$ times that.

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Sketch Proof of Divergence theorem.

1. As we covered in example 6.1, it is already true for boxes.
2. Therefore we can use the fact that it holds for boxes to suppose the theorem holds for Ω_1, Ω_2 as shown in figure 20, what about the region $\Omega_1 \cup \Omega_2$? Then:

$$\int_{\delta\Omega} \left\langle V, \hat{N} \right\rangle = \int_{\delta\Omega_1} \left\langle V, \hat{N} \right\rangle + \int_{\delta\Omega_2} \left\langle V, \hat{N} \right\rangle$$

   Since the contribution form the shared boundary cancels as they are in opposite directions, this is equal to:

$$= \int_{\Omega_1} \nabla V + \int_{\Omega_2} \nabla V = \int_{\Omega_1 \cup \Omega_2} \nabla V$$

   [16]

[16] I'm using nabla instead of div here as its easier

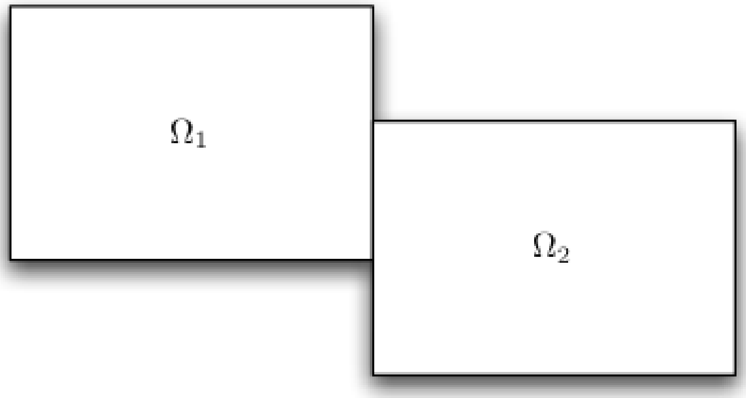

Figure 20: Diagram of 2 boxes for sketch proof of divergence theorem

3. For simple regions (in $\mathbb{R}^2$)

$$\Omega \text{ is simple} \Leftrightarrow \Omega = \left\{(x,y)\middle| f(y) \leq x \leq g(y) \ \forall\, y \in [a,b]\right\}$$
$$= \left\{(x,y)\middle| \tilde{f}(x) \leq y \leq \tilde{g}(x) \ \forall\, x \in [c,d]\right\}$$

So a circle with a hole in the middle of it (like a wheel with a missing axle) isn't simple but if you cut it in half into a semi circle then it would be simple.[17].

4. If we show Ω is simple then:

$$\int_\Omega \frac{\partial V_1}{\partial x} dA = \int_a^b \int_{f(y)}^{g(y)} \frac{\partial V}{\partial x_1}(x,y)dxdy = \int_a^b V_1(f(y),y) - \int_{V_1} \big(f(y),y\big)dy \quad (7.3)$$

Flux of $\begin{pmatrix} V_1 \\ 0 \end{pmatrix}$ through the boundary, the region is shown in figure 21 The flux

$$\gamma_k(\psi) = \begin{pmatrix} g(y) \\ y \end{pmatrix} \text{ for } y \in [a,b]$$

$$\gamma_k'(\psi) = \begin{pmatrix} g'(y) \\ 1 \end{pmatrix}$$

Generally we know that $\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} -y \\ x \end{pmatrix}$

$$\hat{N} = \frac{1}{\|\gamma_R\|} \begin{pmatrix} g'(y) \\ 1 \end{pmatrix} = \frac{1}{\sqrt{1+\big(g'(y)\big)^2}} \begin{pmatrix} g'(y) \\ 1 \end{pmatrix}$$

$$\int_{\gamma_k} \left\langle \begin{pmatrix} V_1 \\ 0 \end{pmatrix}, \hat{N} \right\rangle$$

[17]In metric spaces language it is simple if it is topologically equivalent to a square

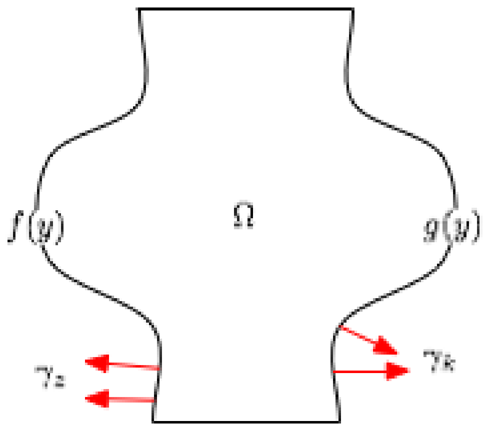

Figure 21: Diagram showing the region Ω with two boundary functions and two flux functions.

$$\frac{1}{\|\gamma'(y)\|}\left\langle \begin{pmatrix} V_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -g'(y) \end{pmatrix} \right\rangle$$

$$\int_a^b V_1 \frac{1}{\cancel{\|\gamma_R'(y)\|}} \cancel{\|\gamma_R'(y)\|} dy$$

$$= \int_a^b V_1\big(g(y), y\big) dy$$

Similarly for the LHS:

$$\int_{\gamma_L} \left\langle \begin{pmatrix} V_1 \\ 0 \end{pmatrix}, \hat{N} \right\rangle = -\int_a^b V_1\big(f(y), y\big) dy$$

As

$$\int_{\gamma_T} \left\langle \begin{pmatrix} V_1 \\ 0 \end{pmatrix}, \hat{N} \right\rangle = \int_{\gamma_B} \left\langle \begin{pmatrix} V_1 \\ 0 \end{pmatrix}, \hat{N} \right\rangle = 0$$

$$\Rightarrow \int_a^b V_1\big(g(y), y\big) - V_1\big(f(y), y\big) dy$$

$$= \int_\Omega \nabla \begin{pmatrix} V_1 \\ 0 \end{pmatrix} dA$$

This then shows that the theorem holds for $V = \begin{pmatrix} V_1 \\ 0 \end{pmatrix}$. If we then use $\tilde{f}$ and $\tilde{g}$ instead of f and g, and if we interchange x and y we get that:

$$\int_{\delta\Omega} \left\langle \begin{pmatrix} 0 \\ V_2 \end{pmatrix}, \hat{N} \right\rangle = \int_\Omega \nabla \begin{pmatrix} 0 \\ V_2 \end{pmatrix} dA$$

Adding these results together completes the proof.

$\square$

8 Integration by Parts

If the following hold:

$$\Omega \subset \mathbb{R},\ \overline{\Omega}^{18} = \Omega \cup \delta\Omega$$

Proposition 8.1. *Then for* $f : \overline{\Omega} \to \mathbb{R}$ *where* f *is* C^1. *Then:*

$$\int_\Omega \frac{\partial f}{\partial x} = \int_{\delta\Omega} f \cdot \hat{N}^{19} \tag{8.1}$$

Proof. Let $v = (f, 0, 0)$ then:

$$\div v = \frac{\partial f}{\partial x_1} \langle v, \hat{N} \rangle = f \hat{N}_1$$

$\Rightarrow$ If the divergence thoerem is applied to the special vector field v given in equation 8.1, then using $(0, f, 0)$ and $(0, 0, f)$ we can also show it for $i = 2, 3$. $\square$

Remark 8.1. *The full divergence theorem (theorem 7.1) can be derived from equation 8.1*

Proposition 8.2. *Integration by parts:*

$$\int_a^b uv'dx = \big[uv\big]_a^b - \int_a^b u'vdx \tag{8.2}$$

Then applying equation 8.1 to $f = gh$ *we get:*

$$\frac{\partial f}{\partial x_i} = \frac{\partial g}{\partial x_i} h + g \frac{\partial h}{\partial x_i}$$

$$\Rightarrow \int_\Omega \frac{\partial g}{\partial x_i} = -\int_\Omega g \frac{\partial h}{\partial x_i} + \int_{\delta\Omega} gh\hat{N}_i$$

Proposition 8.3. *Let* $g : \overline{\Omega} \to \mathbb{R}$ *be twice continuously differentiable, i.e. apply proposition 8.1to* $f = \frac{\partial g}{\partial x_i}$

$$\Rightarrow \int_\Omega \frac{\partial^2 g}{\partial x_i^2} = \int_{\delta\Omega} \frac{\partial y}{\partial x_i} \hat{N}_i$$

Summing up over i then gives

$$\int_\Omega \Delta g = \int_{\delta\Omega} \langle \nabla g, \hat{N} \rangle \tag{8.3}$$

Proposition 8.4. *Applying proposition 8.2 with* $g = \frac{\partial f}{\partial x_i}$ *and sum over i we get:*

$$\int_\Omega \Delta f h = -\int_\Omega \langle \nabla f, \nabla h \rangle + \int_{\delta\Omega} g \langle \nabla f, \hat{N} \rangle$$

$$\langle \div \mathrm{grad} f, h \rangle - \langle \mathrm{grad} f, \mathrm{grad} h \rangle \tag{8.4}$$

[18] This means the closure of Ω as in metric spaces

[19] in the lectures described as $\hat{N}_i$ for $i = 1, 2, 3$.

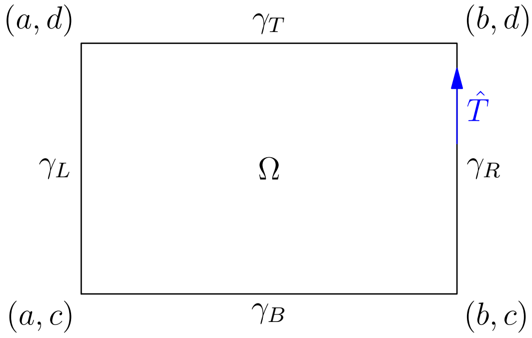

Figure 22: A rectangle Ω

8.1 Application

Temperature distribution in steady state. $\Omega \subseteq \mathbb{R}^3$ a piece of material. Fix temperature $f(x)\ \forall\, x \in \delta\Omega$. In steady state the temperature solves:

$$\begin{cases} \Delta T(x) = 0 & \forall\, x \in \Omega \\ T(x) = f(x) & \forall\, x \in \delta\Omega \end{cases}$$

The existence of a solution s of this is a difficult question.

8.2 Uniqueness

Assume T and $\tilde{T}$ are solutions, then $D = T - \tilde{T}$ solves

$$\Delta D = \Delta T - \Delta\tilde{T} = 0 - 0 = 0\ \forall\, x \in \delta\Omega$$

$$D(x) = T(x) - \tilde{T}(x) = f(x) - f(x) = 0\ \forall\, x \in \delta\Omega$$

Using proposition 8.4 we get:

$$\underbrace{\int_\Omega \Delta D}_{=0} = -\int_\Omega \langle \nabla D, \nabla D\rangle + \int_{\delta\Omega} D\langle \nabla D, \hat{N}\rangle$$

$$\int_\Omega \|\nabla D\|^2 = 0$$

$$\Rightarrow \nabla D(x) = 0\ \forall\, x \in \Omega$$

$\Rightarrow$ D is constant and $D = 0$ at the boundary, therefore $T = \tilde{T}$ so T is a unqiue solution.

9 Green's Theorem

[20]

[20]This section is done by Matthew Pusey

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a rectangle $\Omega = [a, b] \times [c, d]$ with boundary unit tangent vectors $\hat{T}$ in the anticlockwise (positive) direction. This is shown in Figure 22.

On the right side, γ_R. which is the line segment from (b, c) to (b, d):

$$\int_{\gamma_R} \left\langle f, \hat{T} \right\rangle = \int_c^d \left\langle f(b, y), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle dy = \int_c^d f_2(b, y) dy$$

The remaining sides, $\gamma_L, \gamma_B, \gamma_T$ are similar (but take care with signs):

$$\int_{\gamma_L} \left\langle f, \hat{T} \right\rangle = -\int_c^d f_2(a, y) dy$$

$$\int_{\gamma_B} \left\langle f, \hat{T} \right\rangle = \int_a^b f_1(x, c) dx$$

$$\int_{\gamma_T} \left\langle f, \hat{T} \right\rangle = -\int_a^b f_1(x, d) dx$$

Summing these gives:

$$\int_{\partial\Omega} \left\langle f, \hat{T} \right\rangle = \int_c^d f_2(b, y) - f_2(a, y) dy + \int_a^b f_1(x, c) - f_1(x, d) dx$$

The Fundamental Theorem of Calculus then gives:

$$= \int_c^d \int_a^b \frac{\partial f_2}{\partial x}(x, y) dx dy - \int_a^b \int_c^d \frac{\partial f_1}{\partial y}(x, y) dy dx$$

$$= \int_c^d \int_a^b \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} dy dx$$

Definition 9.1 (2-D curl). *For $f : \mathbb{R}^2 \to R^2$, the <u>curl</u> of f is given by:*

$$\operatorname{curl} f(x) = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$$

Remark 9.1. $\int_{\partial\Omega} \left\langle f, \hat{T} \right\rangle$ *is the circulation around Ω. By talking small boxes we find that* $\operatorname{curl} f(x)$ *is the circulation of f around x.*

Sometimes you can see what the curl is:

Example 9.1.

$$v(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

This is shown in figure 23.

$$\operatorname{curl} v(x, y) = 1 - (-1) = 2$$

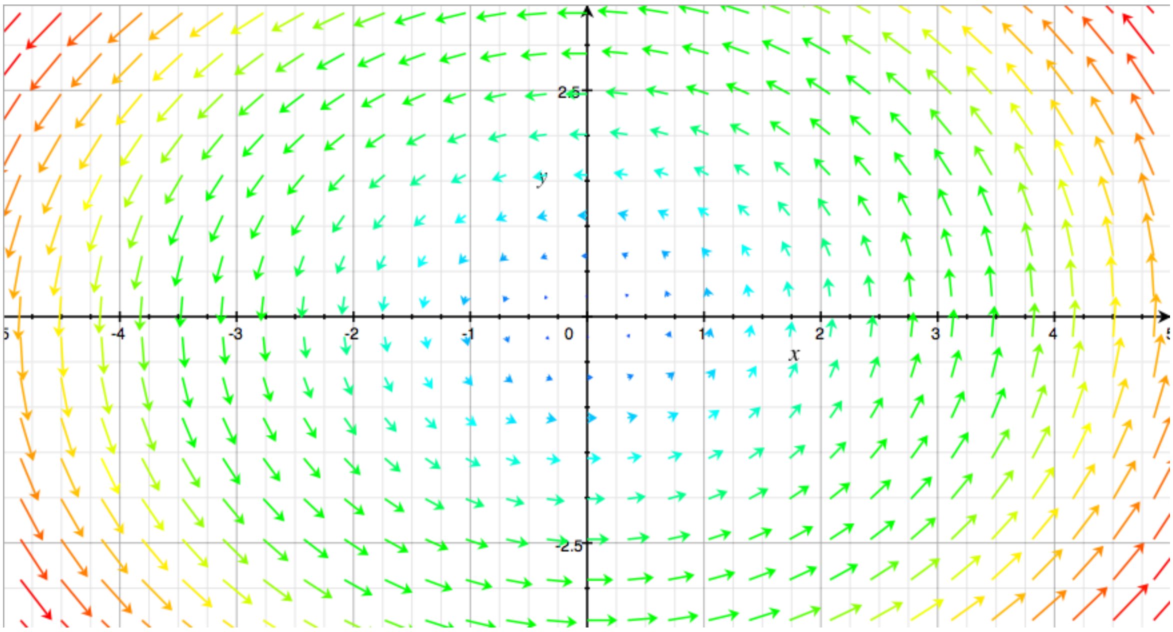

Figure 23: Diagram showing a vector field.

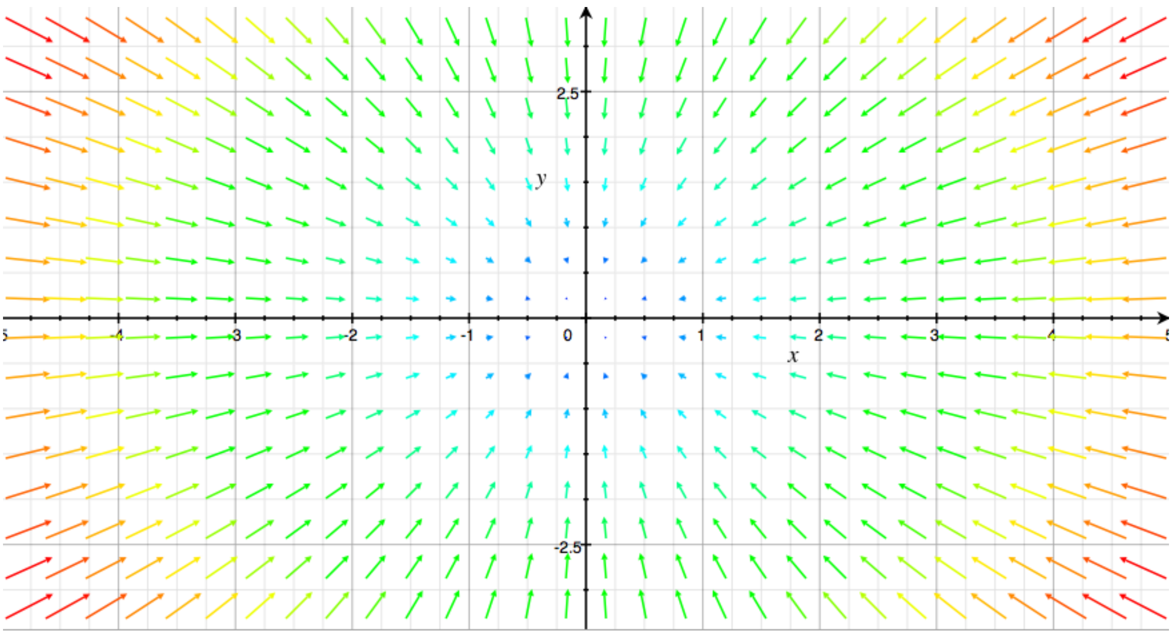

Figure 24: Diagram showing a vector field.

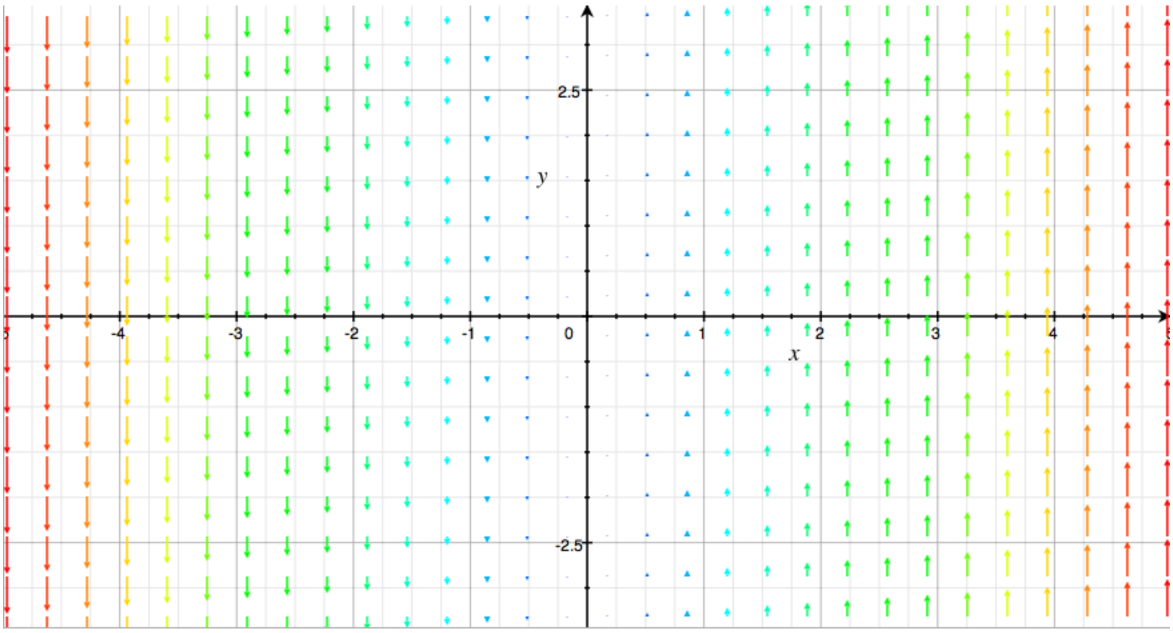

Figure 25: Diagram showing a vector field.

Example 9.2.

$$v(x,y) = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

This is shown in figure 24.

$$\operatorname{curl} v(x,y) = 0 - 0 = 0$$

Example 9.3.

$$v(x,y) = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

This is shown in figure 25.

$$\operatorname{curl} v(x,y) = 1 - 0 = 1$$

Theorem 9.1 (Green's Theorem or Stokes' Theorem in $\mathbb{R}^2$)**.** *Let Ω be a bounded region in $\mathbb{R}^2$, and $\hat{T}$ be positively oriented tangent vectors for $\partial\Omega$. If $f : \bar{\Omega} \to \mathbb{R}^2$ is continously differentiable then:*

$$\int_{\partial\Omega} \left\langle f, \hat{T} \right\rangle = \int_{\Omega} \operatorname{curl} f$$

Proof. Define $g = \begin{pmatrix} f_2 \\ -f_1 \end{pmatrix}$.

By the Divergence Theorem:

$$\int_{\partial\Omega} \left\langle g, \hat{N} \right\rangle = \int_{\Omega} \operatorname{div} g$$

By considering the relationship between $\hat{N}$ and $\hat{T}$, and the definition of $\operatorname{div} g$ this becomes:

$$\int_{\partial\Omega} \left\langle f, \hat{T} \right\rangle = \int_{\Omega} \operatorname{curl} f$$

□

10 Stokes Theorem (curls in $\mathbb{R}^3$)

6/11/06

Theorem 10.1. *Let $\mathcal{S} \in \mathbb{R}^3$ be a bounded surface with $\hat{N}$ as unit normals of $\mathcal{S}$, and $\hat{T}$ unit tangent vectors to the boundary line $\delta\mathcal{S}$. Let $f : \mathcal{S} \cup \delta\mathcal{S} \to \mathbb{R}^3$ be continuously differentiable. If $\left\langle \hat{N}, \hat{T} \right\rangle$ is positively oriented, then:*

$$\int_{\mathcal{S}} \left\langle \mathrm{curl} f, \hat{N} \right\rangle - \int_{\delta\mathcal{S}} \langle f, \hat{T} \rangle = 0$$

Remark 10.1. *Positive oriented means that if $\hat{N}$ points upwards, the tangent vectors are anti-clockwise,and if $\hat{N}$ points downwards, the tangent vectors are clockwise. (i.e. the right hand rule applies).*

Remark 10.2. *In $\mathbb{R}^3$,* $\mathrm{curl} = \nabla \times f$

$$\begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix}$$

Remark 10.3. *In $\mathbb{R}^2$* $\mathrm{curl} g = \partial_1 g_2 - \partial_2 g_1$

Example 10.1. For $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x, y, z) = \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \\ 0 \end{pmatrix}$$

$$\mathrm{curl} f = \begin{pmatrix} 0 - 0 \\ 0 - 0 \\ \mathrm{curl}\ g \end{pmatrix} = \mathrm{curl}\ g$$

For $\Omega \subseteq \mathbb{R} \times \mathbb{R} \times \{0\}$ and $\Omega' \subset \mathbb{R}^2$ (the x-y plane), we then get

$$\int_{\Omega'} \mathrm{curl}\ g = \int_{\Omega} \left\langle \mathrm{curl}\ f, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

And so by Stoke's theorem

$$= \int_{\delta\Omega} \left\langle f, \hat{T} \right\rangle$$

$$= \int_{\delta\Omega'} \left\langle g, \hat{T} \right\rangle$$

Example 10.2. $\mathcal{S} = \{(x, y, z) \big| z = x^2 + y^2, z \leq 4\}$ This is then of the form

$$r(x, y) = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}$$

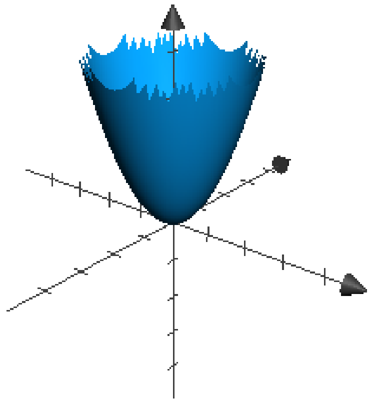

Figure 26: Graph showing a hemispherical bowl, the top should be smooth at the point of the peaks, but grapher can't draw it correctly.

$$N(x,y,z) = \nabla(2 - x^2 - y^2) = \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix} = \delta\mathcal{S}$$

Is then parameterised by:

$$\gamma(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \\ 4 \end{pmatrix}, t \in [0, 2\pi]$$

A diagram is shown in figure 26 Now let

$$f := \begin{pmatrix} y \\ x^2 \\ 3z^2 \end{pmatrix}$$

$$\int_{\delta\mathcal{S}} \left\langle f, \hat{T}, = \right\rangle \int_0^{2\pi} \left\langle \begin{pmatrix} 2\sin t \\ 4\cos^2 t \\ 3 \cdot 16 \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ -2\cos t \\ 0 \end{pmatrix} \right\rangle = -4\pi$$

$$\text{curl } f = \begin{pmatrix} 0-0 \\ 0-0 \\ 2x-1 \end{pmatrix}$$

$$\int_{\mathcal{S}} \langle \text{curl } f, \hat{N} \rangle = \iint \begin{pmatrix} 0 \\ 0 \\ 2x-1 \end{pmatrix} \cdot \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix} dsdt = \iint 2x - 1dsdt$$

$$\iint_{B(0,2)} 2x - 1dxdy$$

$$\iint_{B(0,2)} 2xdxdy - \iint_{B(0,2)} 1dxdy$$

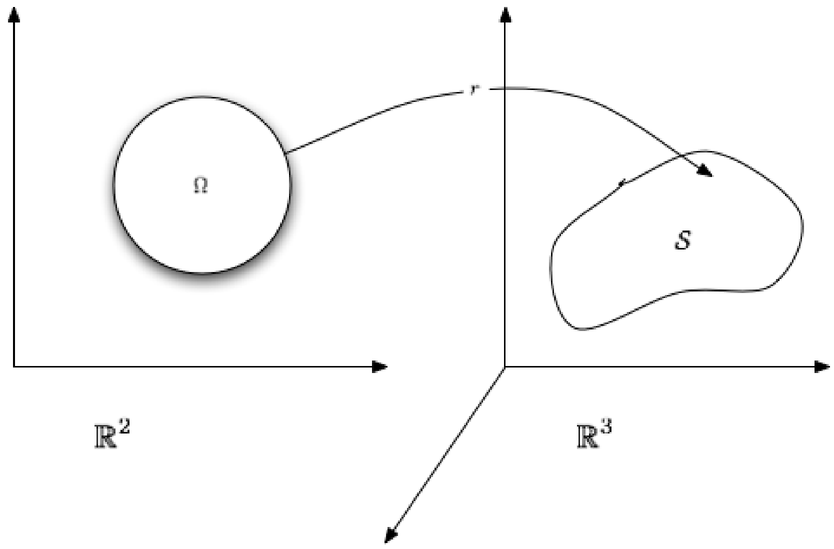

Figure 27: Graph showing region being "lifted" to $\mathbb{R}^3$

$$0 - 4\pi = -4\pi \text{ (as we got before!)}$$

$$\int_{\mathcal{S}} \left\langle \text{curl } f, \hat{N} \right\rangle = \int_{\delta\mathcal{S}} \left\langle f, \hat{T} \right\rangle$$

Proof of theorem 10.1. We "lift" Green's theorem from $\mathbb{R}^2$ to $\mathbb{R}^3$, this is shown in figure 27. Now we parameterise $\delta\Omega$ by $\gamma : [a, b] \to \mathbb{R}^2$. $\Rightarrow \delta\mathcal{S}$ is parameterised by $r\big(\gamma(u)\big) = u \in [a, b]$. Tangent $T_{\mathcal{S}}\bigg(r\big(\gamma(u)\big)\bigg)'$

$$\frac{\partial r}{\partial s}\frac{\partial \gamma_1}{\partial u} + \frac{\partial r}{\partial t}\frac{\partial \gamma_2}{\partial u}$$

$$\Rightarrow \int_{\partial\mathcal{S}} \langle f, \hat{T} \rangle$$

$$= \int_a^b \left\langle f\big(\gamma(u)\big), \frac{\partial r}{\partial s}\frac{\partial \gamma_1}{\partial u} + \frac{\partial r}{\partial t}\frac{\partial \gamma_2}{\partial u} \right\rangle$$

$$= \int_a^b \left\langle \begin{pmatrix} \left\langle f, \frac{\partial r}{\partial s} \right\rangle \\ \left\langle f, \frac{\partial r}{\partial t} \right\rangle \end{pmatrix}, \gamma'(u) \right\rangle du$$

$$\int_{\delta\Omega} \left\langle \begin{pmatrix} \left\langle f, \frac{\partial r}{\partial s} \right\rangle \\ \left\langle f, \frac{\partial r}{\partial t} \right\rangle \end{pmatrix}, \hat{T}_{\Omega} \right\rangle \tag{10.1}$$

Similarly

$$\int_{\mathcal{S}} \left\langle \text{curl } f, \hat{N} \right\rangle$$

$$= \left\langle \text{curl } f\big(r(s,t)\big), \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right\rangle dsdt$$

Omitting the middle (Chain rule + hard work)

$$= \iint_\Omega \text{curl } \begin{pmatrix} \left\langle f, \frac{\partial r}{\partial s} \right\rangle \\ \left\langle f, \frac{\partial r}{\partial t} \right\rangle \end{pmatrix} \tag{10.2}$$

By Green's theorem equations 10.1 and 10.2 are equal. When showing the equality of 10.2 we have to keep track of many (about 48) terms, in later courses we find differential forms useful for this. $\square$

Remark 10.4. *If $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a gradient then:*

$$\int_S \left\langle \text{curl } f, \hat{N} \right\rangle \underbrace{=}_{Stokes} \int_{\delta\Omega} \left\langle f, \hat{T} \right\rangle = 0$$

$$\int_\gamma \left\langle f, \hat{T} \right\rangle = V(b) - V(a)$$

by the FTC for gradient vector fields.

$$\text{curl grad } v = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \end{pmatrix} = \left(\frac{\partial^2 v}{\partial y \partial z} - \frac{\partial^2 v}{\partial z \partial y} \right) = 0$$

Curl of gradient vector fields is always zero. Similarly $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f = \text{curl } v$ we get $\int_{\delta\Omega} \left\langle f, \hat{N} \right\rangle = \int_\Omega \text{div } f$, $\text{div curl } v = \cdots = 0$

11 Spherical Coordinates

Skipped, will be a PDF of most of this topic included later.

12 Complex Differentation

21

The aim of this section is to understand calculus for functions $f : \mathbb{C} \to \mathbb{C}$, and its link to vector analysis.

Definition 12.1 (Complex Numbers)**.** *The complex numbers are defined by:*

$$\mathbb{C} = \left\{ x + iy \;\middle|\; x, y \in \mathbb{R}, i^2 = -1 \right\}$$

Clearly, there is a one-to-one correspondence between $\mathbb{C} \to \mathbb{C}$ functions and $\mathbb{R}^2 \to \mathbb{R}^2$ functions:

$$\{f : \mathbb{C} \to \mathbb{C}\} \leftrightarrow \{(u, v) : \mathbb{R}^2 \to \mathbb{R}^2\}$$

Where $f(x + iy) = u(x, y) + iv(x, y)$.

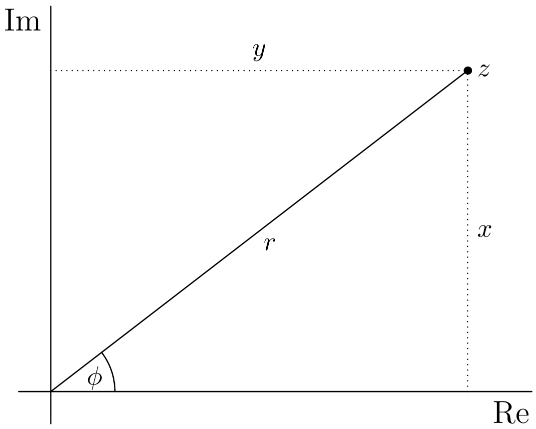

Figure 28: Planar representation of a real number z

12.1 Basic properties of Complex Numbers

$$\begin{aligned} z &= x + iy \\ z &= r\cos\phi + ir\sin\phi \\ r &= \sqrt{x^2+y^2} = |z| \\ \phi &= \arctan\left(\frac{y}{x}\right) = \arg(z) \end{aligned}$$

Addition

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y1_y2) + i(y_1x_2 + x_1y_2)$$

The meaning of this is clearer in polars, for example if $z = r\cos\phi + ir\sin\phi$ and $w = s\cos\psi + is\sin\psi$ then:

$$zw = rs\cos(\phi+\psi) + irs\sin(\phi+\psi)$$

Complex conjugation

$$z = x + iy \iff \bar{z} = x - iy$$

12.2 Limits

Definition 12.2 (Limits)**.** *For $z_n, z \in \mathbb{C}$:*

$$z_n \to z \iff |z_n - z| \to 0$$

[21]The next two sections are done by Matthew Pusey

By the definition of $|z|$:

$$z_n \to z \iff \sqrt{(x_n - x)^2 + (y_n - y)^2} \to 0 \iff x_n \to x \text{ and } y_n \to y$$

Lemma 12.1. *If $z_n \to z$ and $w_n \to w$ then:*

$$\begin{aligned} z_n + w_n &\to z + w \\ z_n w_n &\to zw \\ \frac{z_n}{z_w} &\to \frac{z}{w} \textit{ if } w \neq 0 \end{aligned}$$

Sketch proof. Exactly as for $\mathbb{R}$, but need to avoid inequalities in $\mathbb{C}$, which make no sense. Still have that:

$$|zw| = |z||w|$$

$$|z + w| \leq |z| + |w|$$

And can define an open ball:

$$B(z, \epsilon) = z' \in \mathbb{C} : |z - z'| \leq \epsilon$$

So that:

$$z_n \to z \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, z_n \in B(z, \epsilon)$$ □

12.3 Continuity & Differentiation

Definition 12.3 (Continuity)**.** *A function $f : D \to \mathbb{C}$ for $D \subset \mathbb{C}$ is continuous at $z \in D$ if: $B(z, \epsilon) \subseteq D$ for some $\epsilon > 0$ and:*

$$z_n \to z \implies f(z_n) \to f(z)$$

Note: This must hold for all sequences (z_n) with $z_n \to z$. We say f is continuous on D if f is continuous at every $z \in D$.

Remark 12.1. *f is continuous at $z \iff \forall \epsilon > 0, \exists \delta > 0$ such that:*

$$|z_n - z| < \delta \implies |f(z_n) - f(z)| < \epsilon$$

Definition 12.4 (Differentiation)**.** *A function $f : D \to \mathbb{C}$ with $D \subset \mathbb{C}$ is differentiable at $z \in D$ if*

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists. Note that this means

$$f'(z) = \lim_{n \to \infty} \frac{f(z + h_n) - f(z)}{h_n}$$

exists for <u>any</u> $h_z \to 0$.

Example 12.1.

$$f(z) = z$$

It's clear that

$$z_n \to z \implies f(z_n) \to f(z)$$

So f is continuous.

$$\frac{f(z+h) - f(z)}{h} = \frac{z+h-z}{h} = 1 \to 1 \implies f'(z) \equiv 1$$

Example 12.2.

$$f(z) = \bar{z}$$

$$\begin{aligned} z_n \to z &\implies x_n \to x, y_n \to y \\ &\implies x_n \to x, -y_n \to -y \\ &\implies f(z_n) \to f(z) \end{aligned}$$

So f is continuous. But it is not differentiable, since the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}$$

does not exist. For example, with $h_n = \frac{1}{n}$ it tends to 1 but with $h_n = \frac{i}{n}$ it tends to -1.

Lemma 12.2. *Let $f, g : D \to \mathbb{C}$, with $D \subseteq \mathbb{C}$, be continuous (and differentiable) at z. Then $f + g$, fg and $\frac{f}{g}$ $(g \neq 0)$ are also continuous (and differentiable).*

Let $f(x + iy) = u(x, y) + iv(x, y)$. Then certainly:

$$f \text{ continuous} \iff uv \text{ continuous}$$

But:

$$f \text{ differentiable} \iff uv \text{ differentiable}$$

does not hold in general.

13 Complex power series

Definition 13.1. $\sum_{n=0}^{\infty} c_n$ *converges (to c) if:*

$$S_N = \sum_{n=0}^{N} c_n$$

converges (to c).

As in $\mathbb{R}$, we have the Cauchy criterion for a sequence (z_n):

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N, |z_m - z_n| < \epsilon$$

Loosely speaking, a sequence is Cauchy if "$|z_m - z_n| \to 0$ as $m, n \to \infty$". It is easy to show that if (z_n) is Cauchy its real and imaginary parts are Cauchy, so the sequence converges in $\mathbb{C}$ since the parts must converge in $\mathbb{R}$.

Lemma 13.1. *If $|f_n(z)| \leq M_n \forall z \in D$ and $\sum_{n=0}^{\infty} M_n < \infty$ then*

$$f(z) = \sum_{n=0}^{\infty} f_n(z)$$

converges for all $z \in D$. Also, if all the f_n are continuous then so is f.

Proof. Let $S_N = \sum_{k=0}^{N} f_k(z)$. Then, assuming without loss of generality that $m > n$:

$$\begin{aligned} |S_m(z) - S_n(z)| &= \left| \sum_{k=n+1}^{m} f_k(z) \right| \\ &\leq \left| \sum_{k=n+1}^{m} M_k \right| \\ &\leq \left| \sum_{k=n+1}^{\infty} M_k \right| \to 0 \text{ as } n \to \infty \end{aligned}$$

□

Theorem 13.2 (Power series)**.** *Let (c_n) be a sequence in $\mathbb{C}$, and define:*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

Then there exists some $R \in [0, \infty]$ such that $f(z)$ converges if $|z| < R$, and doesn't converge if $|z| > R$.

Notes:

1. f may or may not converge when $|z| = R$.
2. A similar theorem holds in $\mathbb{R}$ and the proof carries over.
3. The theorem can be applied repeatedly.
4. f is C^∞ on $B(0, R)$, with:

$$\frac{\partial^k}{\partial z^k} f(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n z^{n-k}$$

5. If $f(z)$ converges on $B(0, R)$ then $g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ converges on $B(a, R)$.

To calculate R we can use the ratio test.

Lemma 13.3 (Ratio test)**.** *If*

$$\frac{|z_{n+1}|}{|z_n|} \to L \in [0, \infty]$$

then $\sum_{n=0}^{\infty} z_n$ converges if $L < 1$ and diverges if $L > 1$.

Sketch proof. Observe that
$$\left|\sum_{k=n+2}^{m} z_k\right| \le \sum_{k=n+1}^{m} |z_k|$$
and use the result on $\mathbb{R}$ + Cauchy $\square$

Example 13.1.
$$f(z) = \sum_{n=0}^{\infty} \underbrace{(3+i)(2i)^n}_{c_n}(z+i)^n$$
$$\begin{aligned}\frac{|z_{n+1}|}{|z_n|} &= \left|\frac{(2i)^{n+1}(z+i)^{n+i}}{(2i)^n(z+i)^n}\right| = |2i(z+i)| = |2i||z+i| \\ &= 2|z+i| \to 2|z+i| = L\end{aligned}$$
So when $|z+i| < \frac{1}{2}$, $f(z)$ converges, and when $|z+i| > \frac{1}{2}$, $f(z)$ doesn't converge. This gives $R = \frac{1}{2}$.

Example 13.2.
$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$
$$\frac{|z_{n+1}|}{|z_n|} = \left|\frac{z^{n+1}n}{(n+1)z^n}\right| = \frac{n}{n+1}|z| \to |z| = L$$
This gives $R = 1$.

What about when $|z| = 1$? $f(1)$ diverges, $f(-1)$ converges to $\log 2$. In general this is a <u>hard</u> problem.

Definition 13.2 (Common power series)**.**
$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\cos(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}(-1)^n$$
$$\cosh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$$
$$\sin(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}(-1)^n$$
$$\sinh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}$$

Lemma 13.4.
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sinh(z) = \frac{e^z - e^{-z}}{2} \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}$$

Proof. Use the power series. For example, to prove the one for $\cos(z)$:

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{(iz)^n}{n!}+\frac{(-iz)^n}{n!}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n((i)^n+(-i)^n)}{n!}$$

The numerator here is $2i^n$ if n is even, and 0 is n is odd. Therefore it equals:

$$\sum_{k=0}^{\infty}\frac{z^2k}{2k!}(-1)^k = \cos(z) \qquad \square$$

Example 13.3.

$$\sin(iy) = \frac{e^{-y}-e^{y}}{2i} = i\frac{e^{y}-e^{-y}}{2} = i\sinh(y)$$

14 Holomorphic Functions

Let $f(x+iy) = u(x,y)+iv(x,y)$, for $h = \epsilon + i\cdot 0$

$$\lim_{\epsilon\to 0}\frac{u(x+\epsilon,y)-u(x,y)}{\epsilon} + i\lim_{\epsilon\to 0}\frac{v(x+\epsilon,y)-v(x,y)}{\epsilon} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

For $h = 0 + i\epsilon$.

$$f'(z) = (-i)\lim_{\epsilon\to 0}\frac{u(x,y+\epsilon)-u(x,y)}{(-i)\epsilon} + \cancel{i}\lim_{\epsilon\to 0}\frac{v(x,y+\epsilon)-v(x,y)}{\cancel{i}\epsilon}$$

f is only differentiable if

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Definition 14.1 (The Cauchy Riemann equations)**.** *A complex function $f(x+iy) = u(x,y)+iv(x,y)$ is differentiable if and only if:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Theorem 14.1. *Consider $f : D \to \mathbb{C},\ D \subseteq \mathbb{C}$ with:*

$$f(x+iy) = u(x,y)+iv(x,y)$$

1. *If f is differentiable at (x_0,y_0) then$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist at (x_0,y_0) and the Cauchy Riemann equations (definition 14.1) hold at (x_0,y_0).*

2. If $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and are continuous in a small Ball around (x_0, y_0) then f is differentiable at (x_0, y_0) with $z_0 = x_0 + iy_0$ with

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Example 14.1 (Example to show the difference between points 1 and 2)**.** $f(z) = z^3$ must be differentiable everywhere.

$$f(x + iy) = (x + iy)^3$$

$$= x^3 + 3x(iy)^2 + 3x^2(iy) + (iy)^3$$

$$= \underbrace{x^3 - 3xy^2}_{u(x,y)} + i\,\underbrace{(3yx^2 - y^3)}_{v(x,y)}$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \tag{14.1}$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \tag{14.2}$$

As you can see equations 14.1 and 14.2 are the same.

$$\frac{\partial u}{\partial y} = -6xy \tag{14.3}$$

$$\frac{plv}{\partial x} = 6xy \tag{14.4}$$

As you can see equations 14.3 and 14.4 are the same.

21/11/06

Example 14.2 (Hard example)**.**

$$f(x + iy) = x^2 + iy^2$$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 2y$, (these are in general not equal so the function isn't differentiable. $\frac{\partial v}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$ (these are equal and have to be equal for differentiability.)

Therefore f is only differentiable if $x = y$.

Definition 14.2. *If a function $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$ is holomorphic at $z_0 \in D$ if f is differentiable for all $z \in B(z_0, \epsilon)$ for some $\epsilon > 0$. f is holomorphic on D if it is holomorphic $\forall\, z \in D$.*

Remark 14.1. *The aim is to apply Vector Analysis to this problem.*

If $f(x+iy) = u + iv$ is holomorphic then we define:

$$f(x,y) = \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix}$$

For $\mathbb{R}^2 \to \mathbb{R}^2$, then

$$\nabla \cdot F(x,y) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \underbrace{=}_{\text{by Cauchy Riemann}} 0$$

and

$$\text{curl } F(x,y) = -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

$\frac{\partial F_2}{\partial x} - \frac{\partial F}{\partial y}$[22]

15 Complex Integration

Theorem 15.1. *Consider a parameterised curve $\gamma[a,b] \to \mathbb{C}$*

$$\gamma(t) = x(t) + iy(t)$$
$$\Rightarrow \gamma'(t) = x'(t) + iy'(t)$$

Remark 15.1. *The Divergence theorem and Green's theorem might be useful here.*

Definition 15.1. *For $F : D \to \mathbb{C}$:*

$$\int_\gamma f = \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt$$

$$= \int_a^b \text{Re}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)dt + i\int_a^b \text{Im}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)dt$$

Example 15.1.

$$f(x+iy) = x^2 + iy$$
$$\gamma(t) = t(1+i) \text{ for } t \in [0,1]$$
$$\int_\gamma f = \int_0^1 t^2 + it^2(1+i)dt$$
$$= \int_0^1 \cancel{t^2} + it^2 + it^2 - \cancel{t^2}dt$$
$$\int_0^1 2it^2 dt$$
$$= 2i\int_0^1 t^2 dt$$
$$= 2i\left[\frac{t^3}{3}\right]_0^1$$
$$= \frac{2i}{3}$$

[22] I don't understand this.

Remark 15.2. *If γ and $\tilde{\gamma}$ parameterise the same path in the same direction, then if: $\int_\gamma f = \int_{\tilde{\gamma}} f$. If the direction is reversed then $\int_\gamma f = -\int_{\tilde{\gamma}} f$.*

Example 15.2. Integrate $f(z) = \bar{z}$ around $\delta B(i, 2)$. Since $e^{it} = \cos t + i \sin t$, we can use $\gamma(t) = 2e^{it} + i$ to parameterise $\delta B(i, 2)$.

$$\int_\gamma f = \int \left(\overline{2e^{it} + i}\right) 2ie^{it} dt$$

$$\int_0^{2\pi} (\cancel{-i} + 2e^{-it} + \cancel{i}) 2ie^{it} dt$$

$$= \int_0^{2\pi} 2e^{it} + 4i dt$$

$$= \left[\frac{2e^{it}}{i}\right]_0^{2\pi} + 8\pi i$$

$$= 8\pi i$$

$$\int_\gamma f = \int_a^b f\big(\gamma(t)\big)\gamma'(t) dt$$

$$\int_0^{2\pi} e^{it} dt = \left[\frac{e^{it}}{i}\right]_{t=0}^{t=2\pi}$$

[23]

Theorem 15.2 (Fundamental Theorem of Calculus for Complex Integrals)**.** *Let $f : D \to \mathbb{C}$ be holomorphic for $D \subseteq \mathbb{C}, \gamma : [a, b] \to \mathbb{C}$ then*

$$\int_\gamma f' = f\big(\gamma(b)\big) - f\big(\gamma(a)\big)$$

Proof.

$$\int_\gamma \frac{\partial f}{\partial z} = \int_a^b \frac{\partial f}{\partial z}\big(\gamma(t)\big)\frac{\partial \gamma}{\partial t}(t) dt$$

$$\int_a^b \frac{\partial}{\partial t}\Big(f\big(\gamma(t)\big)\Big) dt = f\big(\gamma(b)\big) - f\big(\gamma(a)\big)$$

This last statement is true by applying the real Fundamental theorem of calculus to $\mathrm{Re}\big(f(\gamma(t)\big)$ and $\mathrm{Im}\Big(f\big(\gamma(t)\big)\Big)$ $\square$

Remark 15.3. *If $f' = 0$ and f is over a connected region[24], then f is constant.*

[23] This doesn't make much sense tbh

[24] As in metric spaces

16 Cauchy's theorem

For $\gamma : [a, b] \to \mathbb{C}$

$$\gamma(t) = x(t) + iy(t)$$
$$f(x+iy) = u(x,y) + iv(x,y)$$

We get that

$$\int_\gamma f = \int_a^b u\big(x(t), y(t)\big) + iv\big(x(t), y(t)\big)\big(x'(t) + iy'(t)\big)dt$$
$$= \int_a^b (ux' - vy')dt + i\int_a^b (uy' + vx')dt$$
$$= \int_a^b \begin{pmatrix} u \\ -v \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} dt + i\int_a^b \begin{pmatrix} u \\ -v \end{pmatrix} \cdot \begin{pmatrix} y' \\ -x' \end{pmatrix} dt$$
$$= \int_\gamma \left\langle F, \hat{T} \right\rangle + i\int_{\gamma?????} \left\langle F, \hat{N} \right\rangle$$

where $F(x,y) = \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix}$ Now by using Green's theorem (9.1) and the divergence theorem (7.1)

Theorem 16.1 (Cauchy)**.** *Let a function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ be holomorphic and $\Omega \subseteq D$ a region with a boundary of $\delta\Omega$. If γ is a parameterisation of $\delta\Omega$ then:*

$$\int_\gamma f = 0^{25}$$

Proof.

$$\int_\gamma \underbrace{f}_{\mathbb{C}} = \int_\gamma \underbrace{\left\langle F, \hat{T} \right\rangle}_{\mathbb{R}^2} + i\int_\gamma \underbrace{\left\langle F, \hat{N} \right\rangle}_{\mathbb{R}^2}$$

$$\int_\gamma \left\langle F, \hat{T} \right\rangle \underbrace{=}_{\text{Green}} \pm \int_\Omega \text{curl } F \qquad \underbrace{=}_{\text{Cauchy-Riemann}} \qquad 0$$

$$\int_{\gamma?????} \left\langle F, \hat{N} \right\rangle \underbrace{=}_{\text{Divergence}} \pm \int_\Omega \text{div } F \qquad \underbrace{=}_{\text{Cauchy-Riemann}} \qquad 0$$

□

Remark 16.1. *The theorem holds for more general curves so as the curve in figure 29 and in this case*

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f = 0$$

However the curve has to be simple, i.e. it must be possible to contract the curve to a point, so for example it wouldn't apply to the curve in figure 30, as $\Omega \nsubseteq D$

[25] This is the main result of the Complex Analysis part if the course

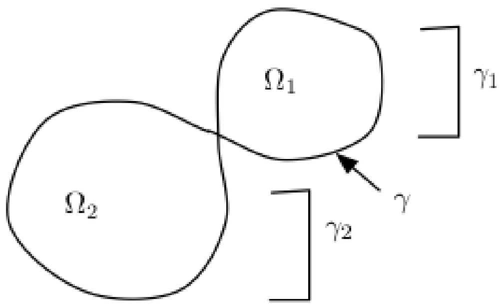

Figure 29: A more general curve to which Cauchy's theorem applies.

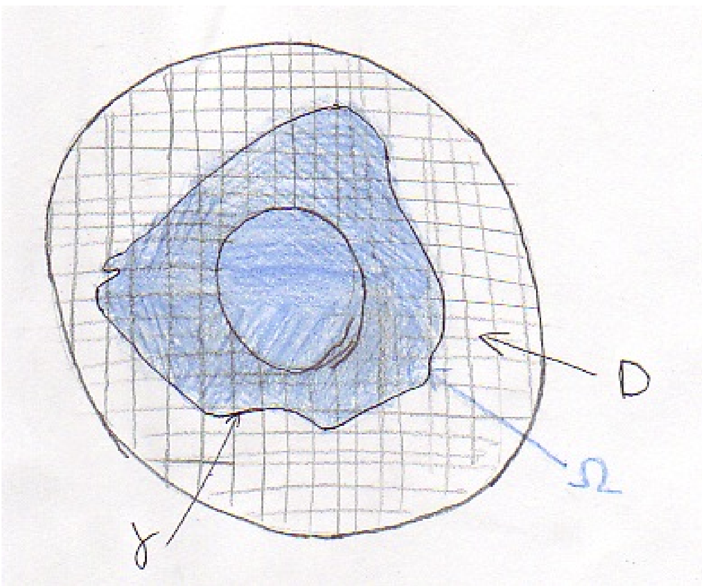

Figure 30: A region and domain which it doesn't apply

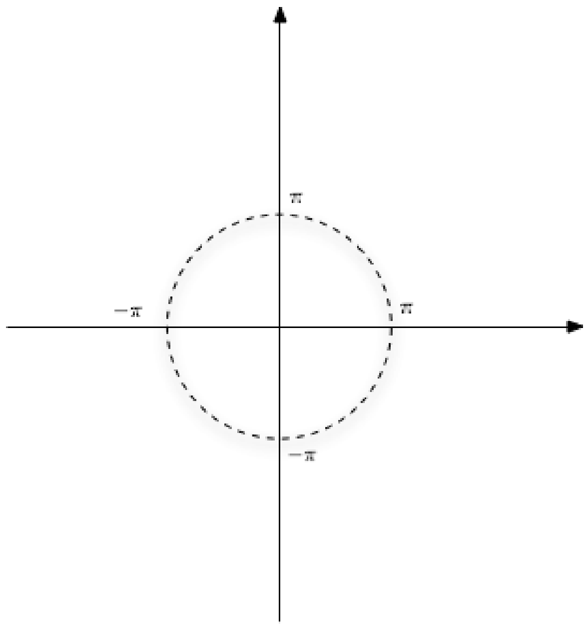

Figure 31: Complex circle of radius π.

Example 16.1.

$$f(z) = \frac{1}{z},\ \gamma(t) = Re^{it}\ t \in [0, 2\pi]$$

$$\Rightarrow \int_\gamma f = \int_0^{2\pi} \frac{1}{\cancel{Re^{it}}} i\cancel{Re^{it}} dt = i\int_0^{2\pi} 1 dt = 2\pi i \neq 0$$

27/11/06

Proposition 16.2. *We have seen that:*

$$\int_{\delta B(0,R)} \frac{1}{1+e^z} dz = 0 \quad \forall\, \textit{Real numbers}$$

Proof.

$$1 + e^z = 0 \Leftrightarrow e^z = -1$$

This is shown in figure 31. Now we know that $|e^z| = e^x$ and $\arg(e^z) = y$ and $e^z = -1$ which implies that $x = 0, y = (2n+1)\pi$, $\forall n \in \mathbb{Z}$. This means that for a ball of radius less than π, i.e $R \in [0, \pi)$ then the value of the integral is zero. This means f is holomorphic on any complex circle with $R < \pi$. □

17 Cauchy Integral Formula

Let γ be the boundary of a connected region $\Omega \subseteq \mathbb{C}$ with positive orientation.

Remark 17.1 (Idea 1)**.** [26] *We can deform γ without changing the integral $\int_\gamma f$. As we*

[26]These are really ideas but I using remarks instead

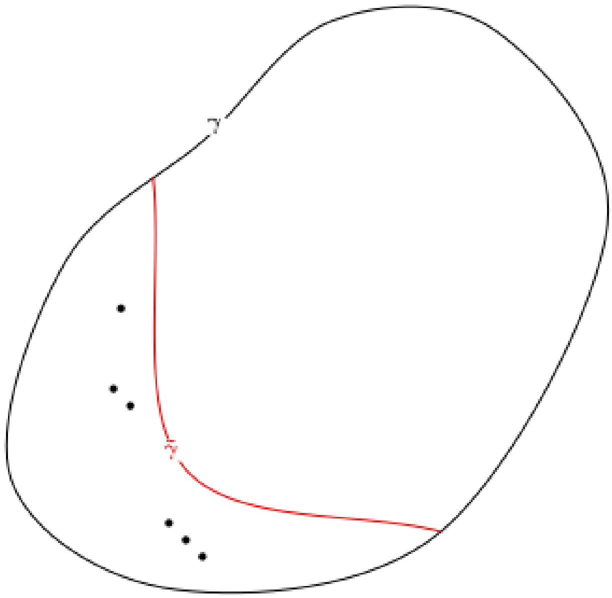

Figure 32: Diagram showing the region bounded by a curve γ, and a second curve $\tilde{\gamma}$ splitting it into two pieces.

can see in figure 32 if there are a few points in the region bounded by gamma which aren't holomorphic we can split the region into two pieces without changing the integral with a split off region, bounded by the curve $\tilde{\gamma}$. If f is holomprohic on and between γ and $\tilde{\gamma}$, then if κ is the part of γ and $\tilde{\gamma}$ bounding this new region then

$$\int_{\kappa} f = 0$$

Also

$$\int_{\gamma} f - \int_{\tilde{\gamma}} f = \int f - \int f$$

[27]

$$\int_{\textit{simple loop}} f = 0$$

Remark 17.2 (Idea 2)**.** *If f is holomorphic on and inside γ except for a finite number of points $z_1, \ldots, z_n$, this is shown in figure 33 and leads to what is shown in 34.*

$$\int_{\gamma} f = \sum_{i=1}^{n} \int_{\delta B(z_i, \epsilon)} f$$

$\forall\, \epsilon > 0$ *small enough so that the balls don't overlap.*

[27]check original notes

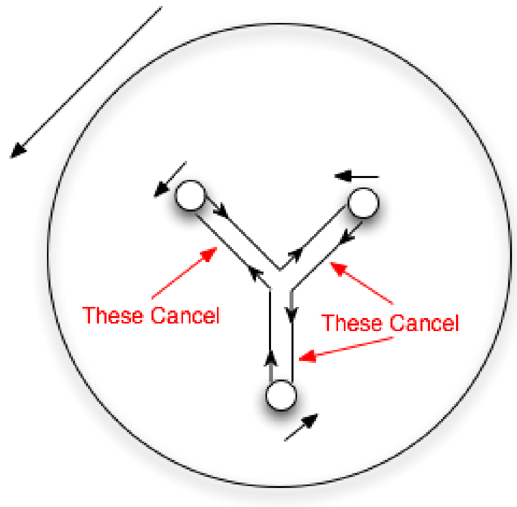

Figure 33: Diagram showing the integral inside a curve γ

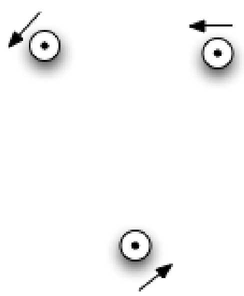

Figure 34: Integrals around the non homomorphic points.

Remark 17.3 (Idea 3). *If:*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

converges on $B(0,R)$, $R > 0$, then:

$$\int_{\delta B(0,\epsilon)} \frac{f(z)}{z} dz = \int_{\delta B(0,\epsilon)} \frac{a_0}{z} + \underbrace{a_1 + a_2 z + \cdots}_{Holomorphic} dz$$

$$= a_0 \underbrace{2\pi i}_{As\ in\ example} + \underbrace{0}_{\text{Cauchy}}$$

Similarly

$$\int_{\delta B(0,\epsilon)} \frac{f(z)}{z^n} dz = \int \underbrace{\frac{a_0}{z^n} + \cdots}_{zero\ as\ primitive\ by\ FTC} + \frac{a_{n-1}}{z} + \underbrace{a_n + a_{n+1} z + \cdots}_{Holomorphic}$$

$$\frac{\partial}{\partial z} z^{1-k} = (1-k) z^k \quad (except\ for\ k = 1)$$

$$\Rightarrow \int \frac{a_0}{z^n} + \cdots + \frac{a_{n-2}}{z^2} dz \underbrace{=}_{by\ the\ FTC} 0$$

$$\Rightarrow \int_{\delta B(0,R)} \frac{f(z)}{z^n} = \underbrace{0}_{FTC} + a_{n-1} 2\pi i + \underbrace{0}_{Cauchy}$$

28/11/06

Theorem 17.1 (Cauchy Integral Formula). *Let γ be the boundary of a connected region. Let f be holomorphic on and inside γ.*

$$\int_\gamma \frac{f(z)}{z - z_0} = 2\pi i f(z_0) \ \forall\, z_0 \ inside\ \gamma$$

Remark 17.4. *Holomorphic functions are special, the values of f along γ completely determine the values of f inside γ.*

Lemma 17.2.

$$\left| \int_a^b x(t) + iy(t) dt \right| \le \int_a^b |x(t) + y(t)| dt$$

Proof. If:

$$\int_a^b x(t) + iy(t) dt \in \mathbb{R}$$

$$\left| \int_a^b x(t) + iy(t) \right| dt = \left| \int_a^b x(t) + \right| dt \underbrace{\le}_{\text{by Analysis 3}} \int_a^b |x(t)| dt \le \int_a^b |x(t) + iy(t)| dt$$

Generally

$$\int_a^b x(t)+iy(t)dt = re^{i\theta} \ forr, \in \mathbb{R}, \theta \in [0,2\pi]$$

$$r = e^{-i\theta}\int_a^b x(t)+iy(t)dt = \int_a^b e^{i\theta}\big(x(t)+iy(t)\big)dt$$

$$= \left|\int_a^b e^{i\theta}\big(x(t)+iy(t)\big)dt\right| \leq \int_a^b \left|e^{i\theta}\right| \left|x(t)+iy(t)\right|dt = \int_a^b 1\cdot\left|x(t)+iy(t)\right|dt$$

□

Remark 17.5. *If $|f(z) \leq M$ on a curve γ then:*

$$\left|\int_\gamma f\right| = \left|\int_a^b \Big(f\big(\gamma(t)\big)\gamma'(t)\Big)\right| \leq \int_a^b < M|\gamma'(t)|dt = M\int_a^b |\gamma'(t)|dt$$

Therefore this means that M is the length of γ.

Proof of theorem 17.1. By deforming γ

$$\int_\gamma \frac{f(z)}{z-z_0} = \int_{\delta B(z_0,\epsilon)} \frac{f(z)}{z-z_0}dz$$

$$= \underbrace{\int_{\delta B(z_0,\epsilon)} \frac{f(z_0)}{z+z_0}dz}_{(a)} + \underbrace{\int_{\delta B(z_0,\epsilon)} \frac{f(z)-f(z_0)}{z-z_0}\,dz}_{(b)} \tag{17.1}$$

Then part a of equation 17.1 is equal to:

$$(a) = f(z_0)\int_{\delta B(0,\epsilon)} \frac{1}{z}dz = f(z_0)2\pi i$$

Then the absolute value of part b of equation 17.1 is:

$$|(b)| \leq \max\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \cdot 2\pi\epsilon \leq \big(|f'(z_0)|+1\big)2\pi\epsilon \tag{17.2}$$

then for small epsilon the right hand side of equation 17.2 tends to zero as $\epsilon \downarrow 0$, therefore part b of equation 17.1 tends to zero. [28] □

Example 17.1.

$$\int \frac{\sin z}{z-i}dz = \begin{cases} 0 & \text{If } i \text{ is outside } \gamma \\ 2\pi i \sin i & \text{If } i \text{ is inside } \gamma \\ ? & \text{If } i \text{ is on the curve though we are lost.} \end{cases}$$

This is shown in figure 35

[29]

[28] Not convinced complete.
[29] Check complete.

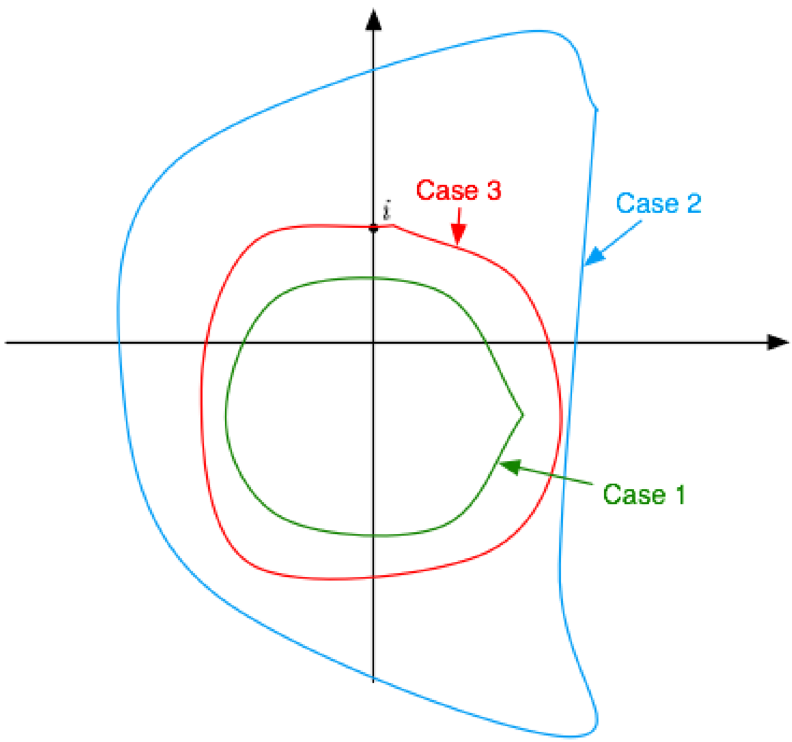

Figure 35: Diagram showing the three possible cases for the curve γ

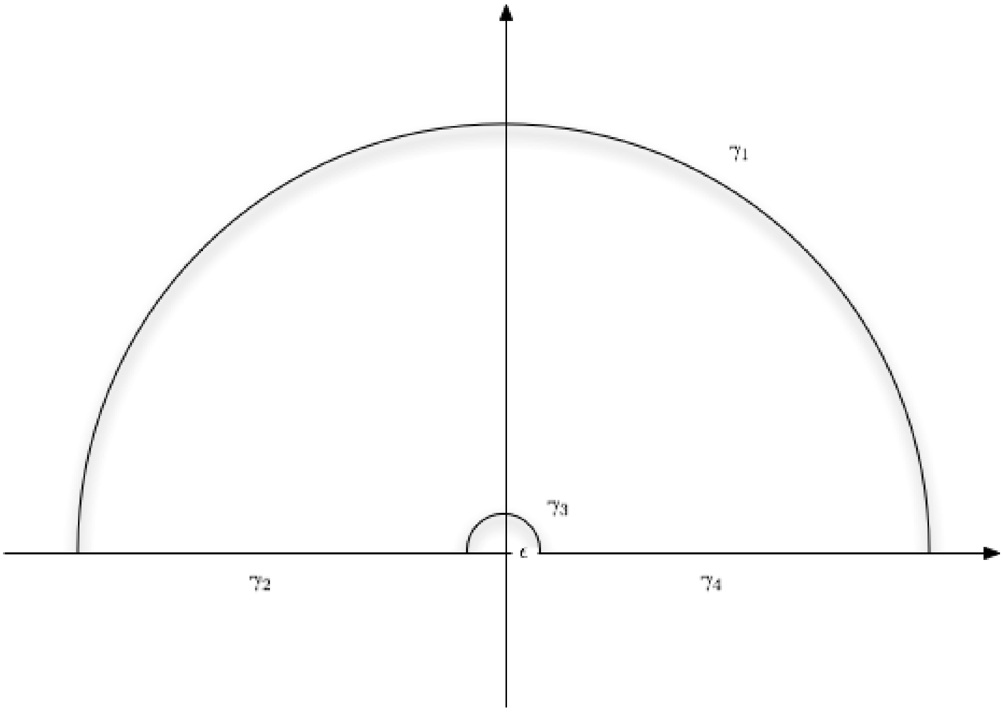

Figure 36: Diagram showing the integral of γ split into four pieces.

18 Real Integrals

Aim to find the integral of:

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx \tag{18.1}$$

As $\frac{\sin x}{x} \to 1$ as $x \to 0$ (by L'Hopital as $\frac{\cos x}{1} \to 1$) So the function is defined everywhere. A diagram of the curve γ which we will use to integrate equation 18.1 is shown in figure 36. The strategy for solving this integral is to Integrate f along γ by Cauchy's theorem (16.1). As we known that f is holomorphic in the region enclosed by γ in figure 36 then we know that:

$$\int_{\gamma_1} f + \int_{\gamma_3} f + \underbrace{\int_{\gamma_2} f + \int_{\gamma_4} f}_{\text{what we want}} = 0 \tag{18.2}$$

30/11/06 To solve equation 18.1 we find f over γ_1 and γ_3.

$$f(z) = \frac{e^{iz}}{z}$$

Equation 18.2 oviously then leads to:

$$\int_{\gamma_2} f + \int_{\gamma_4} f = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx$$

$$= \int_{-R}^{-\epsilon} \frac{\cos x + i \sin x}{x} dx + \int_{\epsilon}^{R} \frac{\cos x + i \sin x}{x} dx$$

As since cos is an even function, and x is odd.

$$\frac{\cos(-x)}{-x} = -\frac{\cos x}{x}$$

$$\underset{\epsilon \downarrow 0}{\longrightarrow} i \int_{-R}^{R} \frac{\sin x}{x} dx$$

From Cauchy's theorem as we showed in equation 18.2

$$\int_{\gamma_1} f + \int_{\gamma_2} f + int_{\gamma_3} f + \int_{\gamma_4} f = 0$$

$$\to i \int_{-R}^{R} \frac{\sin x}{x} dx = \lim_{\epsilon \downarrow 0} \left(\int_{\gamma_1} f + \int_{\gamma_3} f \right)$$

So:

$$\int_{\gamma_3} f = \underbrace{\int_{\gamma_3} \frac{1}{z} dz}_{(a)} + \underbrace{\int_{\gamma_3} \frac{e^{iz} - 1}{z} dz}_{(b)} \tag{18.3}$$

Now $(a) = -i\pi$ by question 2.2 of sheet 4.

$$|(b)| \leq \max_{\gamma_3} \left| \frac{e^{iz} - 1}{z} \right| \text{ Length of } \gamma_3$$

$\leq C\pi\epsilon$ where C is a constant. Which tends to zero as $\epsilon \to 0$.

$$\frac{e^{iz} - 1}{z} = i \frac{e^{iz} - e^0}{iz} \Rightarrow ie'(0)$$

[30] This $\int_{\gamma_3} f \to -\pi i$ as $\epsilon \downarrow 0$.

Now we claim that $\int_{\gamma_1} \to 0$ as $R \to \infty$ This is because $\gamma_1(t) = Re^{it},\ t \in [0, \pi]$ $= R(\cos t + i \sin t)$

$$\left| \int_0^{\pi} \frac{e^{iR(\cos t + i \sin t)}}{\cancel{Re^{it}}} i\cancel{Re^{it}} \right|$$

$$\leq \int_0^{\pi} \left| e^{iR(\cos t + i \sin t)} \right| dt = \int_0^{\pi} e^{-R \sin t} dt \tag{18.4}$$

So as $\sin t$ is positive over $0, \pi$ and it is symmetric along $\frac{\pi}{2}$. Equation 18.4 then becomes:

$$= 2 \int_0^{\frac{\pi}{2}} e^{-R \sin t} dt \tag{18.5}$$

As on $\left[0, \frac{\pi}{2}\right]$ $\sin t \geq \frac{t}{\frac{\pi}{2}}$ So therefore equation 18.5 then is less than

$$(18.5) \leq 2 \int_0^{\frac{\pi}{2}} e^{\frac{-Rt}{\pi/2}} = 2 \int_0^{\frac{\pi}{2}} e^{-2Rt} \pi$$

[30] not totally sure why last point holds

$$= \frac{-\pi}{\cancel{2}R}\cdot\cancel{2}\left(e^{-\frac{\cancel{2}R\cancel{\pi}/\cancel{2}}{\cancel{\pi}}} - e^{\frac{-2R\cdot 0}{\pi}}\right) = -\frac{\pi}{R}\big(e^{-R}-1\big)$$

$= \frac{\pi}{R}\big(1-e^{-R}\big) \to 0$ as $R \to \infty$ By Cauchy's Theorem as $R \to \infty$

$$\underbrace{i\int_{-\infty}^{+\infty}\frac{\sin x}{x}dx}_{\gamma_2,\gamma_4} + \underbrace{0}_{\gamma_1} - \underbrace{i\pi}_{\gamma_3} = 0$$

19 Power Series for holomorphic functions

Theorem 19.1. *Suppose $f : D \to \mathbb{C}$ and $D \subseteq \mathbb{C}$ is holomorphic, then:*

$$f = \sum_{r=0}^{\infty} a_r z^r$$

where $\sum_{r=0}^{\infty} a_r$ is a convergent power series on any ball $B(a,R) \subseteq D$

Remark 19.1. *If f is holomorphic implies that f is a power series so f is C^∞*

Theorem 19.2.

$$\int_\gamma \frac{f(z)}{z}dz = 2\pi f(0)$$

This is assuming that $f : D \to \mathbb{C}$ is holomorphic, $B(a,R) \subseteq D$ (An image of this is shown in figure 37). Together they imply that f is equal to a power series on $B(a,R)$

Proof. We can assume (without loss of generality) that $a = 0$ by shifting everything to the origin. Then for every $0 < r < R$

$$f(0) = \frac{1}{2\pi i}\int_{\delta B(0,r)} \frac{f(z)}{z}dz \ \forall z_0 \in B(0,R) \tag{19.1}$$

If $z_0 \in \delta B(0,r)$ then:

$$\left|\frac{z_0}{z}\right| < 1 \Rightarrow \frac{1}{z-z_0} = \frac{1}{z}\frac{1}{1-\frac{z_0}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z_0}{z}\right)^n$$

$$\Rightarrow f(z_0) = \frac{1}{2\pi i}\int_{\delta B(0,R)} f(z)\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z_0}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\delta B(0,r)} f(z)\frac{1}{z^{n+1}}dz\right)z_0^n$$

Thus f equals a power series on $B(0,r)$ for all $r < R$, this implies the radius of convergence is greater than or equal to R. $\square$

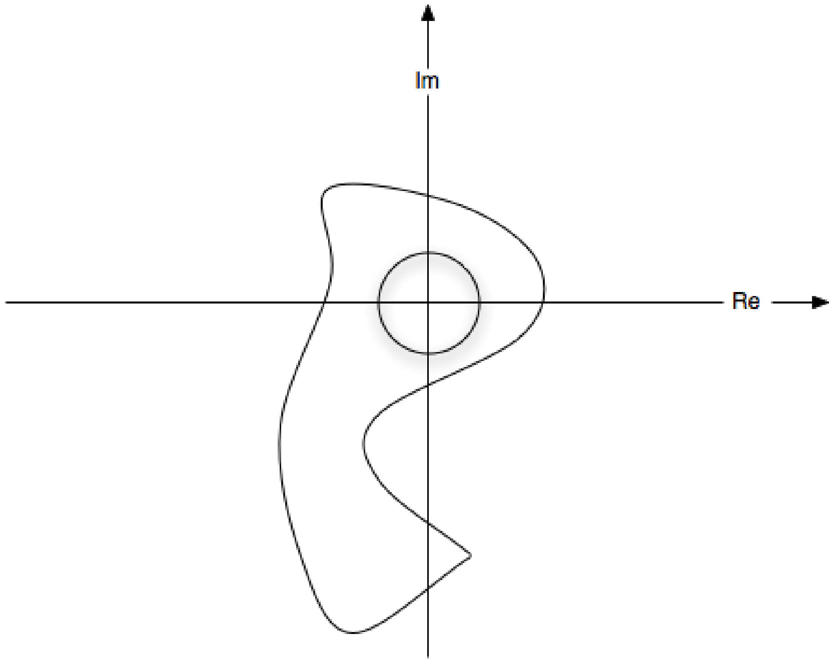

Figure 37: A region containing a ball

Remark 19.2. *Let f be holomorphic and $g \in D$. By the theorem, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$*

$$\Rightarrow f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$$

$$\Rightarrow \frac{\partial^k}{\partial z^k} f(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k)c_n(z-a)^{n-k}$$

$$\frac{\partial^k}{\partial z^k} f(a) = k!c_k(0^0 + 0^1 + \cdots)$$

Thus we get:

$$f(z) = \sum_{n=0}^{\infty} \frac{\partial^n}{\partial z^n} f(a0 \frac{(z-a)^n}{n!}$$

Remark 19.3 (Taylor's formula)**.**

$$f(x) = f(a) + f'(a)(x-n) + \frac{1}{2} f''(a)(x-a)^2 + \cdots \tag{19.2}$$

Corollary 19.3. *If γ is a simple loop, f is holomorphic on and inside γ then:*

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz = 2\pi i c_n = \frac{2\pi i}{n!} \frac{\partial^n}{\partial z^n} f(a)$$

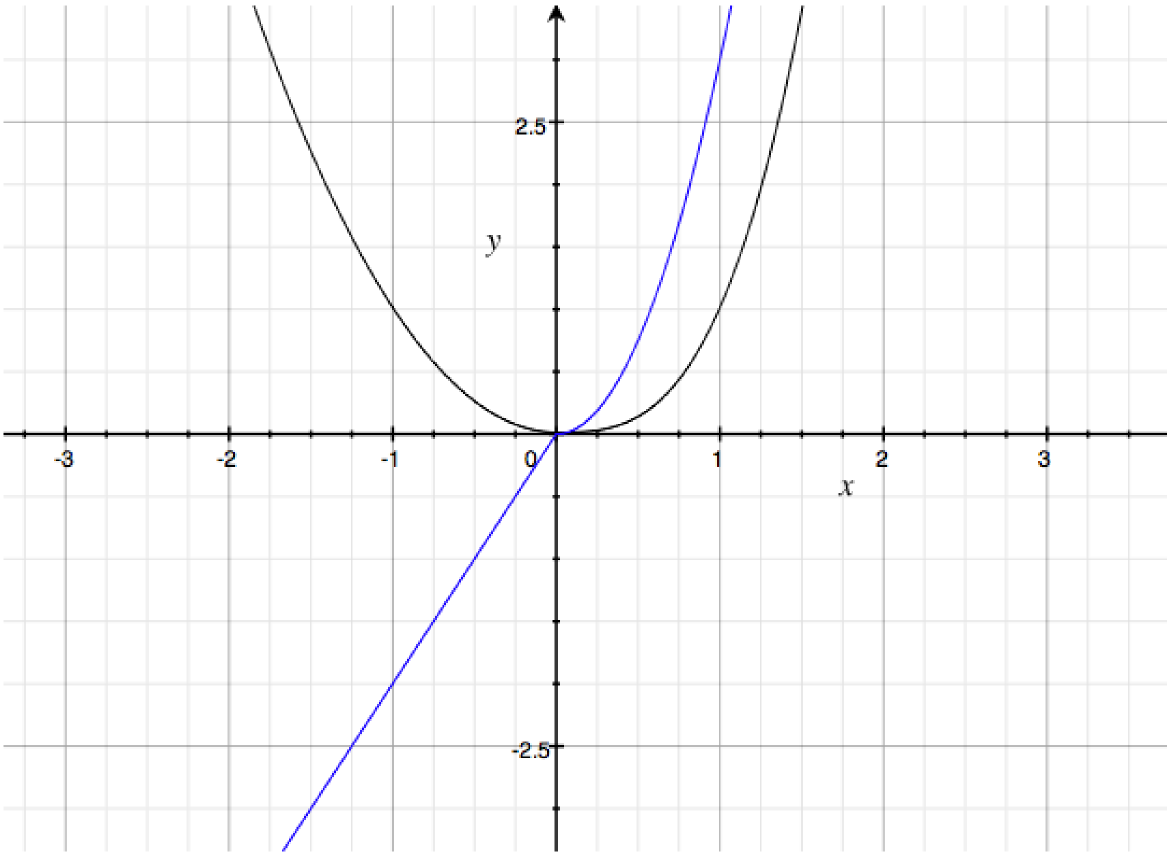

Figure 38: As you can see the black function is C^1, the blue function is the differential of it.

$$\frac{f(z)}{z^{n+1}} = \underbrace{\frac{c_0}{z^{n+1}} + \cdots +}_{=0 \ \ by\ FTC} \frac{c_n}{z} + \underbrace{c_{n+1} + \cdots}_{=0 \ \ by\ Cauchy} \tag{19.3}$$

For $n = 0$, this is the Cauchy Integral formula (CIF).

Definition 19.1. *A function is called Analytic if it can be expanded into a power series everywhere, around every point. We have just seen that analytic $\Leftrightarrow$ holomorphic in complex analysis.*

Lecture 28 [31] In $\mathbb{C}$ f is $C^1 \Rightarrow f$ is C^∞ f is $C^\infty \Rightarrow f$ can be expanded as a power series. In $\mathbb{R}$ if f is C^1 it doesn't imply that f is C^∞. e.g:

$$f(x) := \begin{cases} x^3 & x > 0 \\ x^2 & \end{cases}$$

is C^1 as you can see in figure 38 In $\mathbb{R}$ if f is C^∞ this doesn't imply that f can be expanded as a power series.

[31] As (well it seems like usual), I have borrowed Jack Heal's notes for this lecture, I was very very tired so I missed it

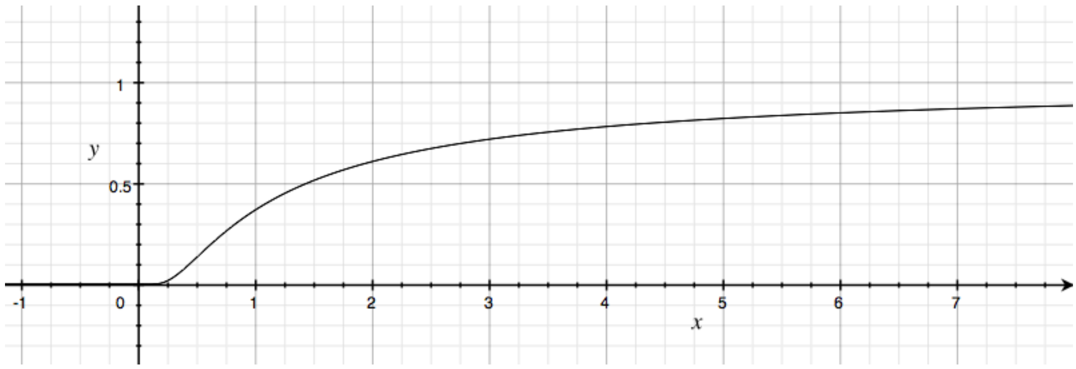

Figure 39: $f(x) = e^{-\frac{1}{x}},\ x > 0$ and $f(x) = 0,\ x \leq 0$.

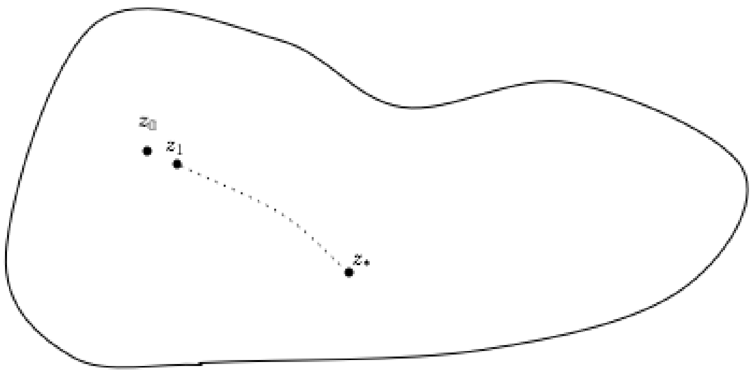

Figure 40: A diagram shown the sequence z_i tending to a point z_*.

Example 19.1.

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

f is drawn in figure 39. There is no power series for this function because $\frac{\partial^n f}{\partial x^n}(0) = 0$[32]

Proposition 19.4. *Let $(z_n) \subset D$ be a sequence, such that $\lim_{n\to\infty} z_n = z_* \in D$ If f and g are holomorphic, and $f(z_n) = g(z_n)$ for every n then $f = g$. This is shown in figure 40.*

Proof. Let $h = f - g$, since h is holomorphic, one has:

$$h(z) = a_0 + a_1(z - z_*) + a_2(z - z_*)^2 + \cdots$$

Take limits as $n \to \infty$, this implies that $a_0 = 0$. If you then divide by $z - z_*$ then:

$$\frac{h(z)}{z - z_*} = a_1 + a_2(z - z_*) + \cdots$$

[32]As $x \to 0$, $f(x) \to 0$ faster than any polynomial

If you evaluate at $z = z_n$ and take limits as $n \to \infty$ we get $a_1 = 0$. By induction, you then get $a_k = 0 \; \forall k$. $\square$

Remark 19.4. *If (z_n) doesn't converge the above doesn't hold. e.g.* $f(x) = \sin x$, $g(x) = 0$ $[(z_n) = k\pi k \in \mathbb{Z}]$ *e.g.* $f(x) = \sin^2 x + \cos^2 x$ $g(x) = 1$ *One has* $\sin^2 x + \cos^2 x = 1 \;\; \forall x \in \mathbb{C}$.

Theorem 19.5. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic, if their exists M such that $|f(z)| < M$ $\forall z \in \mathbb{C}$ then f is constant.*

Proof. We have:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with

$$a_n = \frac{1}{2\pi i} \int_{B(0,R)} \frac{f(z)}{z^{n+1}} dz$$

$$|a_n| = \frac{1}{2\pi} \left| \int_{B(0,R)} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \int_{B(0,R)} \left| \frac{f(z)}{z^{n+1}} \right| dz$$

$$\leq \frac{1}{2\pi} \int_{B(0,R)} \frac{M}{R^{n+1}} dz$$

$$|a_n| \leq \frac{1}{2\pi} \frac{M}{R^{n+\cancel{1}}} 2\pi \cancel{R} = \frac{M}{R^n}$$

This holds for every $R > 0$, therefore $a_n = 0$ for $n \geq 1$. This means that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0$$

(i.e. $f(z)$ is constant.) $\square$

Theorem 19.6. *Every non-constant polynomial P on $\mathbb{C}$ has at least one root. ($\exists\, z \in \mathbb{C}$ s.t $P(z) = 0$)*

Proof. Suppose f has no root. Then $f(z) = \frac{1}{P(z)}$ is holomorphic in $\mathbb{C}$. $P = a_0 + a_1 + \cdots + a_n z^n$ There exists an $R > 0$ and $C > 0$ s.t. $|P(t)| \geq C|z|^n$ for $|z| > R$. Therefore $|f(z)| \leq \frac{1}{CR^n}$ for $|z| > R$. Since P has no root, f has no C pole (i.e. 1/0 is undefined)[3], this means that their exists M s.t $|f(z)| < M$ for $|z| < R$, by the previous theorem f must be constant, which implies that P is also constant. If P is a polynomial of degree n and $P(a) = 0$, then $P(z) = (z - a)\hat{P}(z)$ where $\hat{P}$ is a polynomial of degree $n - 1$. $\square$

20 Real Sums

The aim is to find the solution of:

$$\sum_{k=-\infty}^{\infty} \tag{20.1}$$

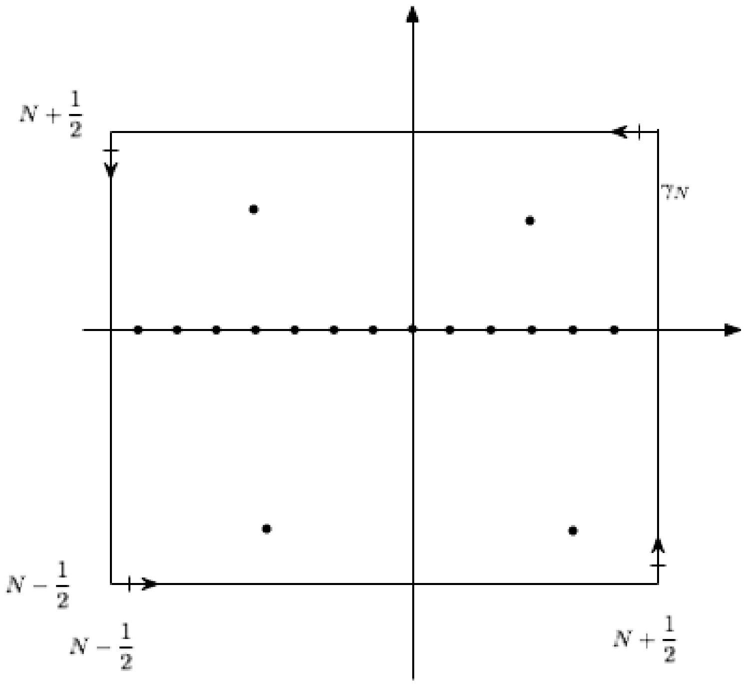

Figure 41: A diagram of the box γ_N

Idea: calculate

$$\int_{\gamma_N} f(z)\frac{\cos(\pi z)}{\sin(\pi z)}dz$$

f is holomorphic except at $z_1, z_2, \ldots, z_M$.

$$\int_{\gamma_N} f(z)\frac{\cos(\pi z)}{\sin(\pi z)}dz \tag{20.2}$$

$$\underbrace{=}_{CIF} \sum_{k=1}^{M} \int_{\delta B(z_N,\epsilon)} \cdots + \sum_{k=-N}^{N} 2\cancel{\pi} i\frac{f(k)}{\cancel{\pi}}$$

This allows us to compute $\sum_{k=-N}^{N} f(k)$.

Remark 20.1 (Details)**.**

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} = 0$$

$$\Rightarrow e^{i\pi z} - e^{-i\pi z} \Rightarrow z \in \mathbb{Z}$$

Using:

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

As we know that $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$$\sin(\pi z) = \sin\big(\pi(z-k) + \pi k\big) = \sin\big(\pi(z-k)\big)\cos \pi k + \cos\big(\pi(z-k)\big)\underbrace{\sin(\pi k)}_{=0}$$

$$= \pi(z-k)\underbrace{\left(1 - \frac{\pi^2(z-k)^2}{3!} + \cdot\right)}_{=g(z)\to z \text{ as } z\to k}\cos(\pi k)$$

$$\Rightarrow \int_{\delta B(k,\epsilon)} f(z)\frac{\cos(\pi z)}{\sin(\pi z)} = \int \frac{1}{z-k} f(z)\frac{\cos\big(\pi k\big)}{\pi g(z)\cos(pik)}$$

$$\underbrace{=}_{CIF} 2\pi i f(k)\frac{\cos(\pi k)}{\pi - \cos(\pi k)} = 2if(k)$$

Example 20.1.

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$$\text{At } i: \int_{\delta B(i,\epsilon)} f(z)\frac{\cos(\pi z)}{\sin(\pi z)} \underbrace{=}_{CIF} 2\pi i\frac{1}{i+i}\frac{\cos(\pi i)}{\sin(\pi i)} = \pi\frac{\cos(\pi i)}{\sin(\pi i)}$$

$$\text{At } -i: \; 2\pi i\frac{1}{-i-i}\frac{\cos(-\pi i)}{\sin(-\pi i)} = \pi\frac{\cos(\pi i)}{\sin(\pi i)}$$

, $\frac{\cos(\pi z)}{\sin(\pi z)}$ has poles at $\frac{1}{\pi(z-k)}$ for $z \in \mathbb{Z}$. The integral along the box γ_N (as shown if figure 41) as $N \to \infty$.

$$\left|\int_{\gamma_N} f(z)\frac{\cos(\pi z)}{\sin(\pi z)}dz\right| \le \max_{z\in\gamma_N}|\cdots| \cdot (\text{Length of } \gamma_N) \le \frac{C}{N^2}\cdot 4(2N+1) \to 0$$

With $N \to \infty$

$$0 = 2\pi \frac{\cos(\pi i)}{\sin(\pi i)} + 2i \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \frac{-\pi}{i} \frac{\cos(\pi i)}{\sin(\pi i)}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \frac{\pi \cosh \pi}{\sinh \pi} = \pi \tanh(\pi) \approx \underline{\underline{3.15}}$$

References

[1] Cauchy Schwarz inequality on Wikipedia
`http://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality`

[2] Kissing Number Problem
`www.lix.polytechnique.fr/~liberti/kissing-ctw.ps.gz`

[3] Pole (Complex Analysis)
`http://en.wikipedia.org/wiki/Pole_(complex_analysis)`