Solution Paper – I

Mathematical Methods in Engineering & Science

Example 1 Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in $\mathbb{R}^3$, where $P = (2,1,5), Q = (3,5,7), R = (1,-3,-2)$ and $S = (2,1,0)$. Does $\overrightarrow{PQ} = \overrightarrow{RS}$?

Solution: The vector $\overrightarrow{PQ}$ is equal to the vector $\mathbf{v}$ with initial point $(0,0,0)$ and terminal point $Q - P = (3,5,7) - (2,1,5) = (3-2, 5-1, 7-5) = (1,4,2)$.

Similarly, $\overrightarrow{RS}$ is equal to the vector $\mathbf{w}$ with initial point $(0,0,0)$ and terminal point $S - R = (2,1,0) - (1,-3,-2) = (2-1, 1-(-3), 0-(-2)) = (1,4,2)$.

So $\overrightarrow{PQ} = \mathbf{v} = (1,4,2)$ and $\overrightarrow{RS} = \mathbf{w} = (1,4,2)$.

$\therefore \overrightarrow{PQ} = \overrightarrow{RS}$

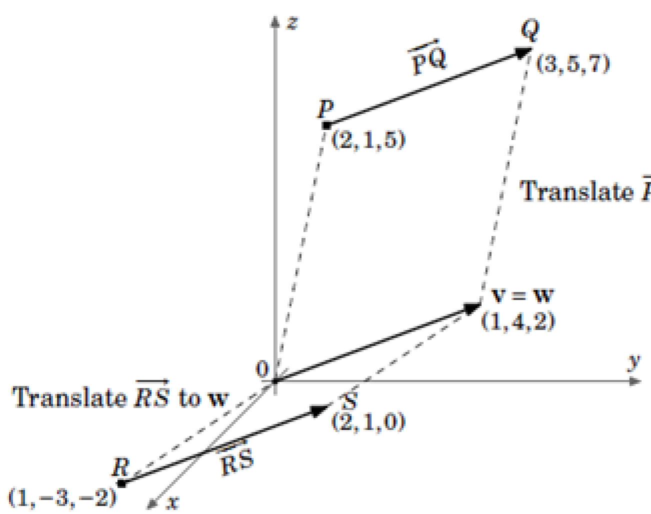

Figure 1.1.7

Recall the distance formula for points in the Euclidean plane:

For points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ in $\mathbb{R}^2$, the distance d between P and Q is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.1}$$

By this formula, we have the following result:

For a vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with initial point $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$, the magnitude of $\overrightarrow{PQ}$ is:

$$\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.2}$$

Finding the magnitude of a vector $\mathbf{v} = (a,b)$ in $\mathbb{R}^2$ is a special case of formula (1.2) with $P = (0,0)$ and $Q = (a,b)$:

For a vector $\mathbf{v} = (a,b)$ in $\mathbb{R}^2$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2} \tag{1.3}$$

To calculate the magnitude of vectors in $\mathbb{R}^3$, we need a distance formula for points in Euclidean space (we will postpone the proof until the next section):

Theorem 1.1. The distance d between points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1.4}$$

The proof will use the following result:

Theorem 1.2. For a vector $\mathbf{v} = (a,b,c)$ in $\mathbb{R}^3$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2} \tag{1.5}$$

Proof: There are four cases to consider:
Case 1: $a = b = c = 0$. Then $\mathbf{v} = \mathbf{0}$, so $\|\mathbf{v}\| = 0 = \sqrt{0^2 + 0^2 + 0^2} = \sqrt{a^2 + b^2 + c^2}$.

Case 2: exactly two of a, b, c *are* 0. Without loss of generality, we assume that $a = b = 0$ and $c \neq 0$ (the other two possibilities are handled in a similar manner). Then $\mathbf{v} = (0,0,c)$, which is a vector of length $|c|$ along the z-axis. So $\|\mathbf{v}\| = |c| = \sqrt{c^2} = \sqrt{0^2 + 0^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$.

Case 3: exactly one of a, b, c *is* 0. Without loss of generality, we assume that $a = 0$, $b \neq 0$ and $c \neq 0$ (the other two possibilities are handled in a similar manner). Then $\mathbf{v} = (0,b,c)$, which is a vector in the yz-plane, so by the Pythagorean Theorem we have $\|\mathbf{v}\| = \sqrt{b^2 + c^2} = \sqrt{0^2 + b^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$.

Case 4: none of a, b, c *are* 0. Without loss of generality, we can assume that a, b, c are all positive (the other seven possibilities are handled in a similar manner). Consider the points $P = (0,0,0)$, $Q = (a,b,c)$, $R = (a,b,0)$, and $S = (a,0,0)$, as shown in Figure 1.1.8. Applying the Pythagorean Theorem to the right triangle $\triangle PSR$ gives $|PR|^2 = a^2 + b^2$. A second application of the Pythagorean Theorem, this time to the right triangle $\triangle PQR$, gives $\|\mathbf{v}\| = |PQ| = \sqrt{|PR|^2 + |QR|^2} = \sqrt{a^2 + b^2 + c^2}$.
This proves the theorem. **QED**

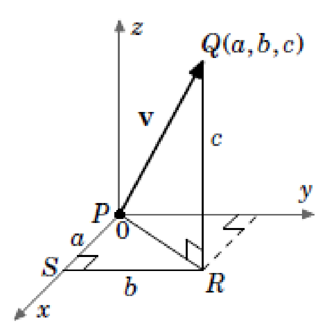

Figure 1.1.8

Example-2 Let $\mathbf{v} = (2,1,-1)$ and $\mathbf{w} = (3,-4,2)$ in $\mathbb{R}^3$.

(a) Find $\mathbf{v} - \mathbf{w}$.
Solution: $\mathbf{v} - \mathbf{w} = (2-3, 1-(-4), -1-2) = (-1,5,-3)$

(b) Find $3\mathbf{v} + 2\mathbf{w}$.
Solution: $3\mathbf{v} + 2\mathbf{w} = (6,3,-3) + (6,-8,4) = (12,-5,1)$

(c) Write $\mathbf{v}$ and $\mathbf{w}$ in component form.
Solution: $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

Example- 3 Prove: $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = \begin{vmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{u} \cdot \mathbf{z} \\ \mathbf{v} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{z} \end{vmatrix}$ for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$ in $\mathbb{R}^3$.

Solution: Let $\mathbf{x} = \mathbf{u} \times \mathbf{v}$. Then

$$
\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) &= \mathbf{x} \cdot (\mathbf{w} \times \mathbf{z}) \\
&= \mathbf{w} \cdot (\mathbf{z} \times \mathbf{x}) \quad \text{(by formula (1.12))} \\
&= \mathbf{w} \cdot (\mathbf{z} \times (\mathbf{u} \times \mathbf{v})) \\
&= \mathbf{w} \cdot ((\mathbf{z} \cdot \mathbf{v})\mathbf{u} - (\mathbf{z} \cdot \mathbf{u})\mathbf{v}) \quad \text{(by Theorem 1.16)} \\
&= (\mathbf{z} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{u}) - (\mathbf{z} \cdot \mathbf{u})(\mathbf{w} \cdot \mathbf{v}) \\
&= (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{z})(\mathbf{v} \cdot \mathbf{w}) \quad \text{(by commutativity of the dot product).} \\
&= \begin{vmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{u} \cdot \mathbf{z} \\ \mathbf{v} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{z} \end{vmatrix}
\end{aligned}
$$

Example - 4 Find the intersection (if any) of the spheres $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + (z - 2)^2 = 16$.

Solution: For any point (x, y, z) on both spheres, we see that

$$
\begin{aligned}
x^2 + y^2 + z^2 = 25 \quad &\Rightarrow \quad x^2 + y^2 = 25 - z^2, \text{ and} \\
x^2 + y^2 + (z - 2)^2 = 16 \quad &\Rightarrow \quad x^2 + y^2 = 16 - (z - 2)^2, \text{ so} \\
16 - (z - 2)^2 = 25 - z^2 \quad &\Rightarrow \quad 4z - 4 = 9 \quad \Rightarrow \quad z = 13/4 \\
&\Rightarrow \quad x^2 + y^2 = 25 - (13/4)^2 = 231/16
\end{aligned}
$$

$\therefore$ The intersection is the circle $x^2 + y^2 = \frac{231}{16}$ of radius $\frac{\sqrt{231}}{4} \approx 3.8$ centered at $(0, 0, \frac{13}{4})$.

Example- - 5 Convert the point $(-2,-2,1)$ from Cartesian coordinates to (a) cylindrical and (b) spherical coordinates.

Solution: (a) $r=\sqrt{(-2)^2+(-2)^2}=2\sqrt{2}$, $\theta=\tan^{-1}\left(\frac{-2}{-2}\right)=\tan^{-1}(1)=\frac{5\pi}{4}$, since $y=-2<0$.
$\therefore\ (r,\theta,z)=\left(2\sqrt{2},\frac{5\pi}{4},1\right)$

(b) $\rho=\sqrt{(-2)^2+(-2)^2+1^2}=\sqrt{9}=3$, $\phi=\cos^{-1}\left(\frac{1}{3}\right)\approx 1.23$ radians.
$\therefore\ (\rho,\theta,\phi)=\left(3,\frac{5\pi}{4},1.23\right)$

Example- 6 Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for the function $f(x,y)=\dfrac{\sin(xy^2)}{x^2+1}$.

Solution: Treating y as a constant and differentiating $f(x,y)$ with respect to x gives

$$\frac{\partial f}{\partial x}=\frac{(x^2+1)(y^2\cos(xy^2))-(2x)\sin(xy^2)}{(x^2+1)^2}$$

and treating x as a constant and differentiating $f(x,y)$ with respect to y gives

$$\frac{\partial f}{\partial y}=\frac{2xy\cos(xy^2)}{x^2+1}\ .$$

Example- 7 Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A+4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1 \lambda_2$ for A and also A^2.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_1 \lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)x = 0$ which is $Ax = \lambda x$:

$$\lambda = 1: \quad (A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3: \quad (A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors as* A. Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \qquad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \qquad A + 4I \text{ has } \begin{matrix} 1+4=5 \\ 3+4=7 \end{matrix}$$

The trace of A^2 is $5 + 5$ which agrees with $1 + 9$. The determinant is $25 - 16 = 9$.

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* since $\lambda_1 \neq \lambda_2$ (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6). A is a *positive definite matrix* (Section 6.5) since $A = A^T$ and the λ's are positive.

Example- 8 Find the eigenvalues and eigenvectors of this 3 by 3 matrix A:

Symmetric matrix
Singular matrix
Trace $1 + 2 + 1 = 4$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of A add to zero, the vector $x = (1, 1, 1)$ gives $Ax = \mathbf{0}$. This is an eigenvector for the eigenvalue $\lambda = 0$. To find λ_2 and λ_3 I will compute the 3 by 3 determinant:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= (1-\lambda)(-\lambda)(3-\lambda). \end{aligned}$$

That factor $-\lambda$ confirms that $\lambda = 0$ is a root, and an eigenvalue of A. The other factors $(1 - \lambda)$ and $(3 - \lambda)$ give the other eigenvalues 1 and 3, adding to 4 (the trace). Each eigenvalue 0, 1, 3 corresponds to an eigenvector:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad Ax_1 = 0x_1 \qquad x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad Ax_2 = 1x_2 \qquad x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad Ax_3 = 3x_3 .$$

I notice again that eigenvectors are perpendicular when A is symmetric.

The 3 by 3 matrix produced a third-degree (cubic) polynomial for $\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 3\lambda$. We were lucky to find simple roots $\lambda = 0, 1, 3$. Normally we would use a command like **eig**(A), and the computation will never even use determinants (Section 9.3 shows a better way for large matrices).

The full command $[S, D] =$ **eig**(A) will produce unit eigenvectors in the columns of the **eigenvector matrix** S. The first one happens to have three minus signs, reversed from $(1, 1, 1)$ and divided by $\sqrt{3}$. The eigenvalues of A will be on the diagonal of the ***eigenvalue matrix*** (typed as D but soon called Λ).

Example- 9 Find all local maxima and minima of $f(x,y)=x^2+xy+y^2-3x$.

Solution: find the critical points, i.e. where $\nabla f = \mathbf{0}$. Since

$$\frac{\partial f}{\partial x}=2x+y-3 \quad \text{and} \quad \frac{\partial f}{\partial y}=x+2y$$

then the critical points (x,y) are the common solutions of the equations

$$\begin{aligned} 2x+\ \ y-3&=0 \\ x+2y\quad\ &=0 \end{aligned}$$

which has the unique solution $(x,y)=(2,-1)$. So $(2,-1)$ is the only critical point.

To use Theorem 2.6, we need the second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2}=2\,, \quad \frac{\partial^2 f}{\partial y^2}=2\,, \quad \frac{\partial^2 f}{\partial y\,\partial x}=1$$

and so

$$D \;=\; \frac{\partial^2 f}{\partial x^2}(2,-1)\frac{\partial^2 f}{\partial y^2}(2,-1)-\left(\frac{\partial^2 f}{\partial y\,\partial x}(2,-1)\right)^2 \;=\; (2)(2)-1^2 \;=\; 3 \;>\; 0$$

and $\frac{\partial^2 f}{\partial x^2}(2,-1)=2>0$. Thus, $(2,-1)$ is a local minimum.

Example- 10 Evaluate $\int_C (x^2+y^2)\,dx + 2xy\,dy$, where:

(a) $C: x = t\,,\quad y = 2t\,,\quad 0 \le t \le 1$

(b) $C: x = t\,,\quad y = 2t^2\,,\quad 0 \le t \le 1$

Solution: Figure 4.1.4 shows both curves.

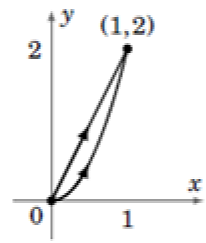

Figure 4.1.4

(a) Since $x'(t) = 1$ and $y'(t) = 2$, then

$$\begin{aligned}\int_C (x^2+y^2)\,dx + 2xy\,dy &= \int_0^1 \big((x(t)^2 + y(t)^2)x'(t) + 2x(t)y(t)\,y'(t)\big)\,dt \\ &= \int_0^1 \big((t^2+4t^2)(1) + 2t(2t)(2)\big)\,dt \\ &= \int_0^1 13t^2\,dt \\ &= \frac{13t^3}{3}\bigg|_0^1 = \frac{13}{3}\end{aligned}$$

(b) Since $x'(t) = 1$ and $y'(t) = 4t$, then

$$\begin{aligned}\int_C (x^2+y^2)\,dx + 2xy\,dy &= \int_0^1 \big((x(t)^2 + y(t)^2)x'(t) + 2x(t)y(t)\,y'(t)\big)\,dt \\ &= \int_0^1 \big((t^2+4t^4)(1) + 2t(2t^2)(4t)\big)\,dt \\ &= \int_0^1 (t^2+20t^4)\,dt \\ &= \frac{t^3}{3} + 4t^5\bigg|_0^1 = \frac{1}{3} + 4 = \frac{13}{3}\end{aligned}$$