# Outline

# Matrices

**Question:** What is a "matrix"?
**Answers:**

- a rectangular array of numbers/elements ?
- a mapping $f : M \times N \to F$, where $M = \{1, 2, 3, \cdots, m\}$, $N = \{1, 2, 3, \cdots, n\}$ and $F$ is the set of real numbers or complex numbers ?

**Question:** What does a matrix **do**?
**Explore:** With an $m \times n$ matrix $\mathbf{A}$,

$$\left.\begin{array}{ccl} y_1 & = & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 & = & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots & \vdots & \vdots \\ y_m & = & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{array}\right\} \quad \text{or} \quad \mathbf{Ax} = \mathbf{y}$$

# Matrices

Consider these definitions:

- $y = f(x)$
- $y = f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$
- $y_k = f_k(\mathbf{x}) = f_k(x_1, x_2, \cdots, x_n), \quad k = 1, 2, \cdots, m$
- $\mathbf{y} = \mathbf{f}(\mathbf{x})$
- $\mathbf{y} = \mathbf{A}\mathbf{x}$

**Further Answer:**

*A matrix is the* definition *of a linear vector function of a vector variable.*

*Anything deeper?*

**Caution:** Matrices *do not* define vector functions whose components are of the form

$$y_k = a_{k0} + a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n.$$

# Geometry and Algebra



Let vector $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$ denote a point $(x_1, x_2, x_3)$ in 3-dimensional space in frame of reference $OX_1X_2X_3$.

**Example:** With $m = 2$ and $n = 3$,

$$\left.\begin{array}{rcl} y_1 & = & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 & = & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{array}\right\}.$$

Plot $y_1$ and $y_2$ in the $OY_1Y_2$ plane.

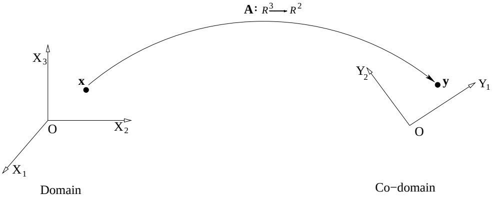


Figure: Linear transformation: schematic illustration

*What is matrix* **A** *doing?*

# Geometry and Algebra



*Operating* on point $\mathbf{x}$ in $R^3$, matrix $\mathbf{A}$ *transforms* it to $\mathbf{y}$ in $R^2$.

Point $\mathbf{y}$ is the *image* of point $\mathbf{x}$ under the mapping defined by matrix $\mathbf{A}$.

Note *domain* $R^3$, *co-domain* $R^2$ with reference to the figure and verify that $\mathbf{A} : R^3 \to R^2$ fulfils the requirements of a *mapping*, by definition.

> *A matrix gives* **a** *definition of a* **linear transformation** *from one vector space to another.*

# Linear Transformations

Operate $\mathbf{A}$ on a large number of points $\mathbf{x}_i \in R^3$.
Obtain corresponding images $\mathbf{y}_i \in R^2$.

*The linear transformation represented by* $\mathbf{A}$ *implies the totality of these correspondences.*

We decide to use a different *frame of reference* $OX_1'X_2'X_3'$ for $R^3$.
[And, possibly $OY_1'Y_2'$ for $R^2$ at the same time.]

*Coordinates* change, i.e. $\mathbf{x}_i$ changes to $\mathbf{x}_i'$ (and possibly $\mathbf{y}_i$ to $\mathbf{y}_i'$).
Now, we need a different matrix, say $\mathbf{A}'$, to get back the correspondence as $\mathbf{y}' = \mathbf{A}'\mathbf{x}'$.

| A matrix: just **one** description. |
|---|

**Question:** How to get the new matrix $\mathbf{A}'$?

# Matrix Terminology



- ... ...
- Matrix product
- Transpose
- Conjugate transpose
- Symmetric and skew-symmetric matrices
- Hermitian and skew-Hermitian matrices
- Determinant of a square matrix
- Inverse of a square matrix
- Adjoint of a square matrix
- ... ...