Chapter 5

Introduction to Euclid's Geometry

In the Chapter

- Axioms or postulates are assumptions which are obvious universal truths. They are not proved.
- Theorems are statements which are proved, using definitions, axioms, previously statements and deductive reasoning.
- **Euclid's Postulates** :
  - **Postulate 1 :** A straight line may be drawn from any one point to any other point.
  - **Postulate 2 :** A terminated line can be produced indefinitely.
  - **Postulate 3 :** A circle can be drawn with any centre and any radius.
  - **Postulate 4 :** All right angles are equal to one another.
  - **Postulate 5 :** If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.
- Two equivalent versions of Euclid's fifth postulate are :
  (a) "For every line l and for every point p not lying on l, there exists a unique line m passing through p and parallel to l. (Playfair's axiom).
  (b) Two distinct intersecting lines cannot be parallel to the same line.
- All the attempts to prove Euclid's fifth postulate using the first 4 postulates failed. But they led to the discovery of several other geometries called, non-euclidean geometries.
- **Some of Euclid's Axioms :**
  - Things which are halves of the same things are equal to one another.
  - Things which are double of the same things are equal to one another.
  - The whole is greater than the part.
  - Things which coincid with one another are equal to one another.
  - If equals are subtracted from equals, the remainders are equal.
  - If equals are added to equals, the wholes are equal.
  - Things which are equal to the same things are equal to one another.
- **Euclid's Definitions :**

  Solid : A solid has shape, size, position and can be moved from one place to another.

  Its boundaries are called surfaces.

  They separate one part of the space from another and are said to have no thickness. The boundaries of the surfaces are curves or lines. The boundaries of these lines are called points which have no magnitude, but only the position.

  A solid has three dimensions, a surface has two, a line has one and a point has none. Some expositions of Euclid are given below :

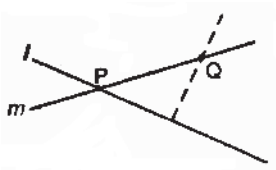
  (a) A line has breadthless length.

(b) A point has no part.
(c) A plane surface is a surface which lies evenly with the straight lines on itself.
(d) The edges of a surface are lines.
(e) A surface is that which has length and breadth only.
(f) A straight line is a line which lies evenly with the points on itself.
(g) The ends of a line are points.

- **Independent Axioms :** An indivisible axiom is said to be independent, if it cannot be logically deduced from the other axioms or statements in the system.

  Theorems/Propositions : The provided statements are called theorems or propositions. Euclid deduced 465 propositions in a logical chain using his axioms, postulates and definitions.

  Theorem 1. Two distinct lines cannot have more than one point in common.



  Sol. Given : Two distinct lines l and m.

  To prove : Lines l and m have only one point P in common.

  Proof : We shall prove it by contradiction.

  Let us consider that the two lines intersect in two distinct points P and Q.

  But this assumption clashes with the axiom. "Given two dinstinct points, there is a unique line that passes through them."

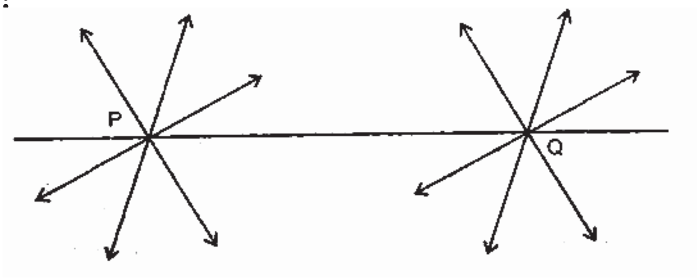
  Hence our assumption that the two lines can pass through two distinct points is wrong.

  Hence two distinct lines cannot have more than one point in common.

- A system of axioms is called consistent, if it is impossible to deduce from these axioms a statement that contradicts any axiom or previously proved statement.
- Euclid's Postulate 1 can also be stated as below :

  Axiom 5.1. Given two distinct points there is a unique line that passes through them.

  Illustration :



Only one live PQ passes through two distinct points P and Q. Thus, the statement above is self-evident and so is taken as an axiom.

EXERCISE 5.1

Q.1. Which of the following statements are true and which are false ?

Give reasons for your answers.

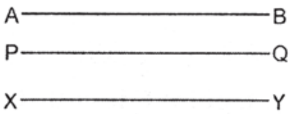
(i) Only one line can pass through a single point.

(ii) There are infinite number of lines which passes through two distinct points.

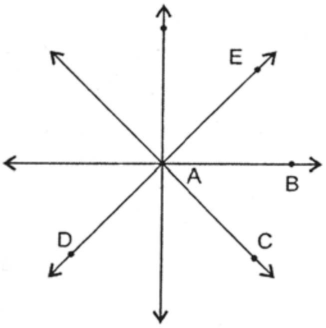
(iii) A terminated line can be produced indefinitely on both the sides.

(iv) If two circles are equal then their radii are equal.

(v) In Fig. if $AB = PQ$ and $PQ = XY$, than $AB = XY$.

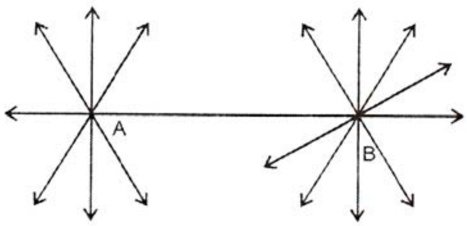


Ans. (i) False. Through a single point, infinite number of lines can pass through it.



(ii) False. For two distinct points only one straight line is passing.

(iii) True.
A terminated line or line segement can be produced infinitely on both sides.



(iv) This statement is true.
If you superimpose the region bounded by one circle on the other, then they coincide. So, their centres and boundaries coincide. Therefore, their radii will coincide.

(v) True, From Axiom 1 : Things which are equal to the same things are equal to one another.

Q.2. Give a definition for each of the following terms. Are there other terms that need to be defined first? What are they, and how might you define them?

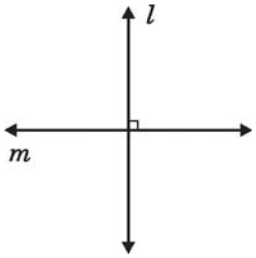
(i) Parallel Lines
(ii) Perpendicular Lines
(iii) Line Segment
(iv) Radius of a circle
(v) Square

Ans. (i) Parallel Lines : Two lines in a plane are said to be parallel, if they have no point in common.

In figure. *a* and *b* are said to be parallel because they have no point in common and we write, $a \parallel b$.

Here, the term point is undefined.

(ii) Perpendicular Lines : Two lines (ray or line segements) are said to be perpendicular, if they intersect at a right angle.



In other words : If one of the two parallel lines is turned by 90° then the two lines become perpendicular to each other. So we need to define rotation and parallel line.

(iii) Line Segment : A line with two end points is a line segement. 'Line' and 'point' have been defined before.

(iv) Radius of a circle : The line segment with one end point at the centre and the other at any point on the circle.

'Centre' may be defined (assuming inside) as a point inside the circle which is at the same distance from all points on the circle.

(v) Square : A quadrilateral with all sides equal and all angles right angles, is a square.

Undefined terms are : Figure, side, angles.

Q.3. Consider two 'postulates' given below:

(i) Given any two distinct points A and B, there exists a third point C which is in between A and B.

(ii) There exist at least three points that are not on the same line.

Do these postulates contain any undefined terms? Are these postulates consistent? Do they follow from Euclid's postulates? Explain.

Ans. There are several undefined terms which the student should list. They are consistent because they deal with two different situations :

(i) Says that given two points A and B, there is a point C lying on the line in between them.

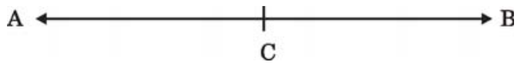
(ii) Says that given A and B, and you take C not lying on the line through A and B. These 'postulates' do not follow from Euclid's postulates.

However, they follow from Axiom 5.1 (Given two dinstinct points, there is a unique line passing through them.)

Q.4. If a point C lies between two points A and B such that AC = BC, then prove that AC = $\frac{1}{2}$ AB.

Explain by drawing the figure.

Ans. Given : AC = BC



$\therefore$ AC + AC = BC + AC

(Equals are added to equals)

or 2AC = AB

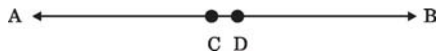
(AC + BC coincide with AB)

Hence $AC = \frac{1}{2}$ AB.

Q.5. In Question 4, point C is called a mid-point of line segment AB. Prove that every line segment has one and only one mid-point.

Ans. If possible, let D be another mid-point of AB.

$\therefore$ AD = DB ...(i)



But it is given that C is the mid-point of AB.

$\therefore$ AC = CB ... (ii)

Subtracting (i) and (ii), we get

AC – AD = CB – DB

$\Rightarrow$ DC = – DC

$\Rightarrow$ 2DC = 0

$\Rightarrow$ DC = 0

$\therefore$ C and D coincides.

Thus, every line segement has one and only one mid-point.

Alternatively :

Let there be two such mid-points C and D. Then from above theorem.

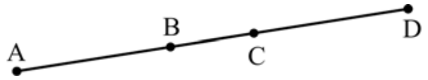
$$AC = \frac{1}{2} AB$$

and. $AD = \frac{1}{2} AB$

$\therefore$ AC = AD

But, this is possible only if D coincides with C. Therefore, C is the unique mid-point. Hence Proved.

Q.6. In Fig. , if AC = BD, then prove that AB = CD.



Ans. Proof AC = BD (given) ... (i)

AC = AB + BC ... (ii)

[Point B lies between A and C]

Also BD = BC + CD ... (iii)

[Point C lies between B and D]

Substituting (ii) and (iii) in (i), we get

AB + BC = BC + CD

Hence AB = CD

[Subtracting equals from equals]

Proved.

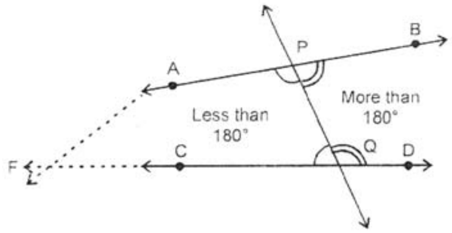
Q.7. Why is Axiom 5, in the list of Euclid's axioms, considered a 'universal truth'? (Note that the question is not about the fifth postulate.)

Ans. Axiom 5 in the list of Euclid's axioms, is true for any thing in any part of universe so this is a universal truth.

EXERCISE 5.2

Q.1. How would you rewrite Euclid's fifth postulate so that it would be easier to understand?

Ans. When two lines are cut by a third line (say transversal), such that the sum of interior angles is less than 180° then the first two lines intersect on that side on which the sum fo angles is less than 180°.



Q.2. Does Euclid's fifth postulate imply the existence of parallel lines? Explain.

Ans. If a straight line *l* falls on two straight lines *m* and *n* such that the sum of the interior angles on one side of *l* is two right angles, then by Euclid's fifth postulate the line will not meet on this side of *l*. Next, we know that the sum of the interior angles on the other side of line *l* will also be two right angles. Therefore, they will not meet on the other side also. So, the lines *m* and *n* never meet on the other side also. So, the lines *m* and *n* never meet and are, therefore, parallel.

Additional Questions

Q.1. Attempts to prove Euclid's fifth postulate using the other postulates and axioms led to the discovery of several other geometries.

Ans. True, these geometries are different from Euclidean geometry.

Q.2. The edges of a surface are curves.

Ans. False, the edges of surfaces are lines.

Q.3. Euclidean geometry is valid only for curved surfaces.

Ans. True, it is valid only for the figures in the plane.

Q. 4. The things which are double of the same thing are equal to one another.

Ans. True, one of the Euclid's axioms. (Axiom 6)

Q. 5. The boundaries of the solids are curves.

Ans. False, boundaries of the solids are surfaces.

Q.6. Two distinct intersecting lines cannot be parallel to the same line.

Ans. True, it is an equivalent version of Euclid's fifth postulate.

Q. 7. If a quantity B is a part of another quantity A, then A can be written as the sum of B and some third quantity C.

Ans. True, because of one of Euclid's axioms.

Q. 8. The statements that are proved are called axioms.

Ans. False, statements that are proved are theorems.

Q.9. "For every line *l* and for every point P not lying on a given line *l*, there exists a unique line *m* passing through P and parallel to *l*" is known as Playfair's axiom.

Ans. True, it is an equivalent version of Euclid's fifth postulate.

Q.10. Ram and Ravi have the same weight, If they each gain weight by 2 kg, how will their new weights be compared ?

Ans. Let x kg be the weight each of Ram and Ravi. On gaining 2 kg, weight of Ram and Ravi will be $(x + 2)$ each. According to Euclid's second axiom, when equals are added to equals, the wholes are equal. So, weight of Ram and Ravi are again equal.

Q.11. Solve the equation a – 15 = 25 and state which axiom do you use here.

Ans. $a - 15 = 25$. On adding 15 to both sides, we have

$$a - 15 + 15 = 25 + 15 = 40.$$

(Using Euclid's second axiom)

or $a = 40$.

Q.12. Read the following statement :

"A square is a polygon made up of four line segements, out of which, length of three line segments are equal to the length of fourth one and all its angles are right angles."

Define the terms used in this definition which you feel necessary. Are there any undefined terms in this? Can you justify that all angles and sides of a square are equal?

Ans. The terms need to be defined are :

Line Segement : Part of a line with two end points.

Line : Undefined term.

Polygon : A simple closed figure made up of three or more line segments.

Angle : A figure formed by two rays with a common initial point.

Point : Undefined term.

Ray : Part of a line with one end point.

Right angle : Angle whose measure is 90°.

Undefined terms used are : line, point. Euclid's fourth postulate says that "All right angles are equal to one another."

In a squre, all angles are right angles, therefore, all angles are equal (From Euclid's fourth postulate).

Three line segments are equal to fourth line segement. (Given).

Therefore, all the four sides of a square are equal. (by Euclid's first axiom "things which are equal to the same thing are equal to one another.")

Q.13. Two salesmen make equal sales during the month of August. In September, each salesman doubels his sale of the month of August. Compare their sales in September.

Ans. Let the sales of salesmen in August be x.

If we add the same quantity in the share of both salesmen, then we get sales $2 \times (x + x)$.

According to Euclid's axioms, when the things are doubled to equals the wholes are also equal.

So, sales in the month of September = x.

Multiple Choice Questions

Q.1. Euclid's fifth postulate is :
(a) The whole is greater than the part.
(b) A circle may be described with any centre and any radius.
(c) All rights angles are equal to one another.
(d) If a straight line falling on two straight lines make the interior angles on the same side of it taken together less than two right angles, then the two straight lines if produced indefintely, meet on that side on which the sam of angles is less than two right angles.

Ans. (d)

Q.2. The things which are double of the same thing are :
(a) equal
(b) unequal
(c) halves of the same thing
(d) double of the same thing

Ans. (a)

Q.3. How many number of lines do(es) pass through two distinct points ?
(a) 1 (b) 2
(c) 3 (d) 4

Ans. (a)

Q.4. Euclid's second axiom (as per order given in the text book for class IX) is :
(a) The things which are equal to the same thing are equal to one another.
(b) Things which coincide with one another are equal to one another.
(c) If equal be added to equals, the whole are equal.
(d) If equal be subtracted from equals, the remainders are equals.

Ans. (c)

Q.5. Select the statement which is not true :
(a) Only one line can pass through a single point.
(b) Only one line can pass through two distinct points
(c) A terminated line can be produced indefinitely on both the sides.
(d) If two circles are equal, then their radii are equal.

Ans. (a)

Q.6. Axioms are assumed :
(a) definitions
(b) theorms
(c) universal truths in all branches of mathematics
(d) universal truths specific to geometry

Ans. (c)

Q.7. The things which coincide with one another are :
(a) unequal
(b) equal to another
(c) double of same thing
(d) triple of same thing

Ans. (b)

Q.8. Two intersecting lines cannot be parallel to the same line' is stated in the form of :
(a) a proof
(b) a postulate
(c) an axiom
(d) a definition

Ans. (b)

Q.9. Two planes intersect each other to form a :
(a) plane
(b) point
(c) angle
(d) straight line

Ans. (a)

Q.10. Ashok is the same age as Mihir. Bhuvan is also of the same as Mihir. State the Euclid's axiom that illustrates the relative ages of Ashok and Bhuvan :
(a) First Axiom
(b) Second Axiom
(c) Third Axiom
(d) Fourth Axiom

Ans. (a)

Q.11. Number of planes can be made pass through three non-collinear points are :
(a) 1 (b) 2
(c) 3 (d) Infinite

Ans. (c)

Q.12. The number of line segments determined by three collinear points is :
(a) Two (b) Three
(c) Only one (d) four

Ans. (b)