Unit Vectors

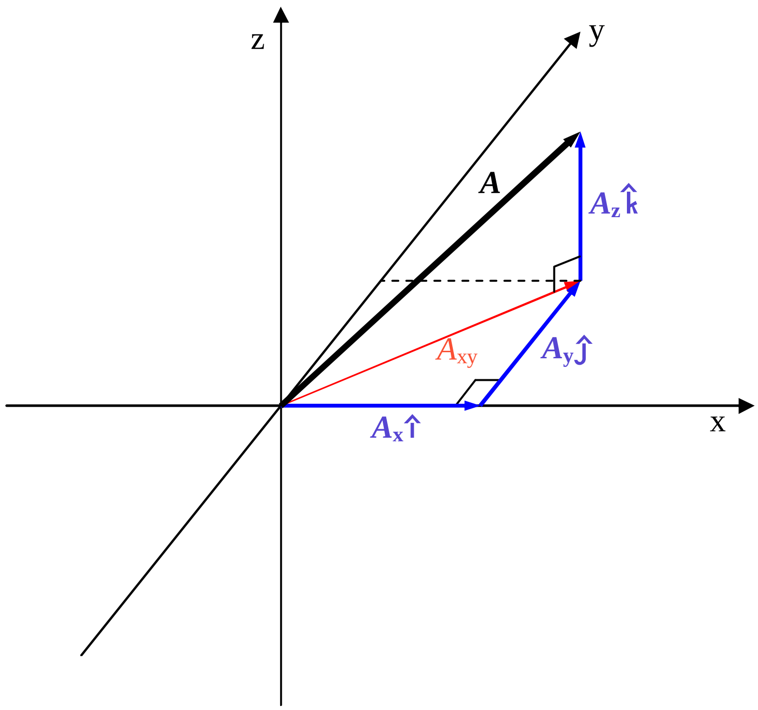
> *What is probably the most common mistake involving unit vectors is simply leaving their hats off. While leaving the hat off a unit vector is a nasty communication error in its own right, it also leads one to worse mistakes such as treating vectors as if they were scalars.*

The mathematicians have come up with a special kind of vector called a unit vector which comes in very handy in physics. By definition a unit vector has magnitude 1, with no units. By convention, a unit vector is represented by a letter marked with a circumflex. The circumflex is an accent mark that appears above the letter. It looks like an inverted "v" and is typically referred to as a "hat". So for instance $\hat{\mathbf{r}}$ (read "*r*-hat") is a unit vector. Let's suppose, just to make this discussion more concrete, that $\hat{\mathbf{r}}$ is at 36.0° (counterclockwise from the +x direction, in the x-y plane). Now the fact that a unit vector has a magnitude 1, with no units, means that if you multiply a unit vector by a scalar, the resulting vector has a magnitude equal to the value-with-units of the scalar. So for instance, if you multiply the vector $\hat{\mathbf{r}}$ by 5.00 m/s, you get a velocity vector $5.00\frac{\text{m}}{\text{s}}\hat{\mathbf{r}}$ which has a magnitude of 5.00 m/s and points in the same direction as the unit vector $\hat{\mathbf{r}}$. Thus, in the case at hand, $5.00\frac{\text{m}}{\text{s}}\hat{\mathbf{r}}$, means 5.00 m/s at 36.0°.

There is a special set of three unit vectors that are exceptionally useful for problems involving vectors, namely the Cartesian coordinate axis unit vectors. There is one of them for each positive coordinate axis direction. These unit vectors are so prevalent that we give them special names. For a two-dimensional x-y coordinate system we have the unit vector $\hat{\imath}$ pointing in the +x direction, and, the unit vector $\hat{\jmath}$ pointing in the +y direction. For a three-dimensional x-y-z coordinate system, we have those two, and one more, namely the unit vector $\hat{k}$ pointing in the +z direction.

Any vector can be expressed in terms of unit vectors. Consider, for instance, a vector $\vec{\mathbf{A}}$ with components A_x, A_y, and A_z. The vector formed by the product $A_x\hat{\imath}$ has magnitude $|A_x|$ and is in the +x direction if A_x is positive and in the –x direction if A_x is negative. This means that $A_x\hat{\imath}$ is the x-component vector of $\vec{\mathbf{A}}$. Similarly, $A_y\hat{\jmath}$ is the y-component vector of $\vec{\mathbf{A}}$, and, $A_z\hat{k}$ is the z-component vector of $\vec{\mathbf{A}}$. Thus $\vec{\mathbf{A}}$ can be expressed as:

$$\vec{\mathbf{A}} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$



The vector $\vec{\mathbf{A}} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ is depicted in the diagram just above, along with the vectors $A_x\hat{\imath}$, $A_y\hat{\jmath}$, and $A_z\hat{k}$ drawn so that is clear that the three of them add up to $\vec{\mathbf{A}}$.

The Magnitude of a Vector in Terms of its Components

Check out (in the diagram above) the right triangle in the x-y plane—the right triangle that has sides of length A_x and A_y, and, a hypotenuse of length A_{xy}.

From Pythagorean's theorem, $A_{xy}{}^2 = A_x{}^2 + A_y{}^2$, so:

$$A_{xy} = \sqrt{A_x{}^2 + A_y{}^2}$$

Now focus your attention on the vertical triangle that has sides of length A_{xy} and A_z , and, a hypotenuse of length A. Applying the Pythagorean theorem to this triangle yields $A^2 = A_{xy}{}^2 + A_z{}^2$ which means that

$$A = \sqrt{A_{xy}{}^2 + A_z{}^2}$$

Substituting the expression for A_{xy} that we just found above, into this expression for A gives us

$$A = \sqrt{\left(\sqrt{A_x{}^2 + A_y{}^2}\right)^2 + A_z{}^2}$$

$$A = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$$

That is, the magnitude of a vector is equal to the square root of the sum of the squares of its components.

Adding Vectors Expressed in Unit Vector Notation

Adding vectors that are expressed in unit vector notation is easy in that individual unit vectors appearing in each of two or more terms can be factored out. The concept is best illustrated by means of an example.

Let

$$\vec{\mathbf{A}} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \quad \text{and} \quad \vec{\mathbf{B}} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\,.$$

Then

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} + B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

which can be rearranged to read

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = A_x\hat{\imath} + B_x\hat{\imath} + A_y\hat{\jmath} + B_y\hat{\jmath} + A_z\hat{k} + B_z\hat{k}$$

Adding parenthesis does not change the sum:

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_x\hat{\imath} + B_x\hat{\imath}) + (A_y\hat{\jmath} + B_y\hat{\jmath}) + (A_z\hat{k} + B_z\hat{k}).$$

Now we can factor out the units vectors:

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$$

We see that the sum of vectors that are expressed in unit vector notation is simply the sum of the x components times $\hat{\imath}$, plus, the sum of the y components times $\hat{\jmath}$, plus, the sum of the z components times $\hat{k}$.

The Position Vector

Consider a particle whose position, on a three-dimensional Cartesian coordinate system, is (x, y, z). The position vector for that particle is a vector that extends from the origin of the coordinate system to the particle. Hence, the position vector for the particle is just

$$\vec{\mathbf{r}} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$$

The Relative Position Vector

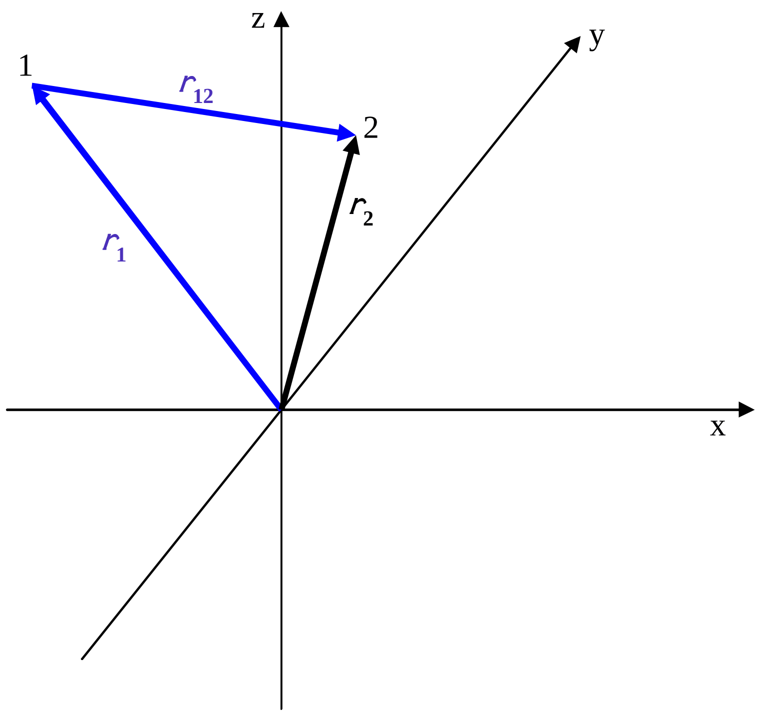
Consider particle 1, at (x_1, y_1, z_1), whose position vector is given by

$$\vec{\mathbf{r}}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{\mathbf{k}}$$

and particle 2, at (x_2, y_2, z_2), whose position vector is given by

$$\vec{\mathbf{r}}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{\mathbf{k}}$$

Now suppose we need to find a vector that extends from particle 1 to particle 2. Graphically depicted, we are looking for the vector $\vec{\mathbf{r}}_{12}$ in the diagram:



From the diagram it is clear that $\vec{\mathbf{r}}_2$ is the vector sum of $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_{12}$:

$$\vec{\mathbf{r}}_1 + \vec{\mathbf{r}}_{12} = \vec{\mathbf{r}}_2$$

We can solve for $\vec{\mathbf{r}}_{12}$ by subtracting the vector $\vec{\mathbf{r}}_1$ from both sides.

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

Substituting our expressions above for $\vec{r}_1$ and $\vec{r}_2$ and solving yields:

$$\vec{r}_{12} = (x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})$$

$$\vec{r}_{12} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

Note that the x-component of the vector $\vec{r}_{12}$ is simply the x-coordinate of particle 2 minus the x-coordinate of particle 1. Likewise for the y and z components. Thus, one can jump directly from the coordinates of the particles (x_1, y_1, z_1) and (x_2, y_2, z_2) to the relative position vector, $\vec{r}_{12} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$, the vector that gives the position of particle 2 relative to particle 1.

Finding a Unit Vector in the Same Direction as a Given Vector

Consider the vector

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

The unit vector $\hat{r}$ in the same direction as the vector $\vec{r}$ is simply the vector $\vec{r}$ divided by its magnitude.

$$\hat{r} = \frac{\vec{r}}{r}$$

We can, as discussed in an earlier section of this chapter, express the magnitude of the vector is given by the square root of the sum of the squares of its components,

$$r = \sqrt{x^2 + y^2 + z^2}$$

and then use it in the expression $\hat{r} = \frac{\vec{r}}{r}$ which can be expanded as:

$$\hat{r} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{r}$$

$$\hat{r} = \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k}$$

The result makes it clear that each component of the unit vector is simply the corresponding component, of the original vector, divided by the magnitude $r = \sqrt{x^2 + y^2 + z^2}$ of the original vector.