Exercise 2.4

Question 1:

Determine which of the following polynomials has $x + 1$ a factor:

(i) $x^3 + x^2 + x + 1$

(ii) $x^4 + x^3 + x^2 + x + 1$

(iii) $x^4 + 3x^3 + 3x^2 + x + 1$

(iv) $x^3 - x^2 - (2 + \sqrt{2})x + \sqrt{2}$

Answer 1:

(i) Let $p(x) = x^3 + x^2 + x + 1$

Putting $x + 1 = 0$, we get, $x = -1$

Using remainder theorem, when $p(x) = x^3 + x^2 + x + 1$ is divided by $x + 1$, remainder is given by $p(-1)$

$= (-1)^3 + (-1)^2 + (-1) + 1$

$= -1 + 1 - 1 + 1$

$= 0$

Since, remainder $p(-1) = 0$, hence $x + 1$ is a factor of $x^3 + x^2 + x + 1$.

(ii) Let $p(x) = x^4 + x^3 + x^2 + x + 1$

Putting $x + 1 = 0$, we get, $x = -1$

Using remainder theorem, when $p(x) = x^4 + x^3 + x^2 + x + 1$ is divided by $x + 1$, remainder is given by $p(-1)$

$= (-1)^4 + (-1)^3 + (-1)^2 + (-1) + 1$

$= 1 - 1 + 1 - 1 + 1$

$= 1$

Since, remainder $p(-1) \neq 0$, hence $x + 1$ is not a factor of $x^4 + x^3 + x^2 + x + 1$.

(iii) Let $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$

Putting $x + 1 = 0$, we get, $x = -1$

Using remainder theorem, when $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$ is divided by $x + 1$, remainder is given by $p(-1)$

$= (-1)^4 + 3(-1)^3 + 3(-1)^2 + (-1) + 1$

$= 1 - 3 + 3 - 1 + 1$

$= 1$

Since, remainder $p(-1) \neq 0$, hence $x + 1$ is not a factor of $x^4 + 3x^3 + 3x^2 + x + 1$.

(iv) Let $p(x) = x^3 - x^2 - (2 + \sqrt{2})x + \sqrt{2}$

Putting $x + 1 = 0$, we get, $x = -1$

Using remainder theorem, when $p(x) = x^3 - x^2 - (2 + \sqrt{2})x + \sqrt{2}$ is divided by $x + 1$, remainder is given by $p(-1)$

$= (-1)^3 - (-1)^2 - \left(2 + \sqrt{2}\right)(-1) + \sqrt{2}$

$= -1 - 1 + 2 + \sqrt{2} + \sqrt{2}$

$= 2\sqrt{2}$

Since, remainder $p(-1) \neq 0$, hence $x + 1$ is not a factor of $x^3 - x^2 - (2 + \sqrt{2})x + \sqrt{2}$.

Question 2:

Use the Factor Theorem to determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = 2x^3 + x^2 - 2x - 1,\ g(x) = x + 1$
(ii) $p(x) = x^3 + 3x^2 + 3x + 1,\ g(x) = x + 2$
(iii) $p(x) = x^3 - 4x^2 + x + 6,\ g(x) = x - 3$

Answer 2:

(i) $p(x) = 2x^3 + x^2 - 2x - 1$ and $g(x) = x + 1$
Putting $x + 1 = 0$, we get, $x = -1$
Using remainder theorem, when $p(x) = 2x^3 + x^2 - 2x - 1$ is divided by $g(x) = x + 1$, remainder is given by $p(-1)$
$= (-1)^3 + (-1)^2 + (-1) + 1$
$= -1 + 1 - 1 + 1$
$= 0$
Since, remainder $p(-1) = 0$, hence $g(x)$ is a factor of $p(x)$.

(ii) $p(x) = x^3 + 3x^2 + 3x + 1$ and $g(x) = x + 2$
Putting $x + 2 = 0$, we get, $x = -2$
Using remainder theorem, when $p(x) = x^3 + 3x^2 + 3x + 1$ is divided by $g(x) = x + 2$, remainder is given by $p(-2)$
$= (-2)^3 + 3(-2)^2 + 3(-2) + 1$
$= -8 + 12 - 6 + 1$
$= -1$
Since, remainder $p(-2) \neq 0$, hence $g(x)$ is not a factor of $p(x)$.

(iii) $p(x) = x^3 - 4x^2 + x + 6$ and $g(x) = x - 3$
Putting $x - 3 = 0$, we get, $x = 3$
Using remainder theorem, when $p(x) = x^3 - 4x^2 + x + 6$ is divided by $g(x) = x - 3$, remainder is given by $p(3)$
$= (3)^3 - 4(3)^2 + (3) + 6$
$= 27 - 36 + 3 + 6$
$= 0$
Since, remainder $p(3) = 0$, hence $g(x)$ is a factor of $p(x)$.

Question 3:

Find the value of k, if x – 1 is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = x^2 + x + k$
(ii) $p(x) = 2x^2 + kx + \sqrt{2}$
(iii) $p(x) = kx^2 - \sqrt{2}x + 1$
(iv) $p(x) = kx^2 - 3x + k$

Answer 3:

(i) $p(x) = x^2 + x + k$
Putting $x - 1 = 0$, we get, $x = 1$
Using remainder theorem, when $p(x) = x^2 + x + k$ is divided by $x - 1$, remainder is given by $p(1)$

$= (1)^2 + (1) + k$
$= 2 + k$
Since $x - 1$ is a factor of $p(x)$, hence remainder $p(1) = 0$
$\Rightarrow 2 + k = 0$
$\Rightarrow k = -2$

(ii) $p(x) = 2x^2 + kx + \sqrt{2}$
Putting $x - 1 = 0$, we get, $x = 1$
Using remainder theorem, when $p(x) = 2x^2 + kx + \sqrt{2}$ is divided by $x - 1$, remainder is given by $p(1)$
$= 2(1)^2 + k(1) + \sqrt{2}$
$= 2 + k + \sqrt{2}$
Since $x - 1$ is a factor of $p(x)$, hence remainder $p(1) = 0$
$\Rightarrow 2 + k + \sqrt{2} = 0$
$\Rightarrow k = -2 - \sqrt{2}$

(iii) $p(x) = kx^2 - \sqrt{2}x + 1$
Putting $x - 1 = 0$, we get, $x = 1$

Using remainder theorem, when $p(x) = kx^2 - \sqrt{2}x + 1$ is divided by $x - 1$, remainder is given by $p(1)$
$= k(1)^2 - \sqrt{2}(1) + 1$
$= k - \sqrt{2} + 1$
Since $x - 1$ is a factor of $p(x)$, hence remainder $p(1) = 0$
$\Rightarrow k - \sqrt{2} + 1 = 0$
$\Rightarrow k = \sqrt{2} - 1$

(iv) $p(x) = kx^2 - 3x + k$
Putting $x - 1 = 0$, we get, $x = 1$

Using remainder theorem, when $p(x) = kx^2 - 3x + k$ is divided by $x - 1$, remainder is given by $p(1)$
$= k(1)^2 - 3(1) + k$
$= 2k - 3$
Since $x - 1$ is a factor of $p(x)$, hence remainder $p(1) = 0$
$\Rightarrow 2k - 3 = 0$
$\Rightarrow k = \frac{3}{2}$

Question 4:
Factorise:
(i) $12x^2 - 7x + 1$
(ii) $2x^2 + 7x + 3$
(iii) $6x^2 + 5x - 6$
(iv) $3x^2 - x - 4$

Answer 4:

(i) $12x^2 - 7x + 1$
$= 12x^2 - (4 + 3)x + 1$
$= 12x^2 - 4x - 3x + 1$
$= 4x(3x - 1) - 1(3x - 1)$
$= (3x - 1)(4x - 1)$

(ii) $2x^2 + 7x + 3$
$= 2x^2 + (6 + 1)x + 3$
$= 2x^2 + 6x + x + 3$
$= 2x(x + 3) + 1(x + 3)$
$= (x + 3)(2x + 1)$

(iii) $6x^2 + 5x - 6$
$= 6x^2 + (9 - 4)x - 6$
$= 6x^2 + 9x - 4x - 6$
$= 3x(2x + 3) - 2(2x + 3)$
$= (2x + 3)(3x - 2)$

(iv) $3x^2 - x - 4$
$= 3x^2 - (4 - 3)x - 4$
$= 3x^2 - 4x + 3x - 4$
$= x(3x - 4) + 1(3x - 4)$
$= (3x - 4)(x + 1)$

Question 5:

Factorise:

(i) $x^3 - 2x^2 - x + 2$

(ii) $x^3 - 3x^2 - 9x - 5$

(iii) $x^3 + 13x^2 + 32x + 20$

(iv) $2y^3 + y^2 - 2y - 1$

Answer 5:

(i) $x^3 - 2x^2 - x + 2$

Let $p(x) = x^3 - 2x^2 - x + 2$

Putting $x = 1$, we get

$p(1) = (1)^3 - 2(1)^2 - (1) + 2 = 1 - 2 - 1 + 2 = 0$

$\Rightarrow x - 1$ is a factor of $p(x)$.

$$
\begin{array}{r|l}
 & x^2 - x - 2 \\
\hline
x - 1 & x^3 - 2x^2 - x + 2 \\
 & x^3 - x^2 \\
 & - \quad + \\
\hline
 & -x^2 - x + 2 \\
 & -x^2 + x \\
 & + \quad - \\
\hline
 & -2x + 2 \\
 & -2x + 2 \\
 & + \quad - \\
\hline
 & 0
\end{array}
$$

So, $p(x) = (x - 1)(x^2 - x - 2)$
$= (x - 1)(x^2 - x - 2) = (x - 1)[x^2 - (2 - 1)x - 2]$
$= (x - 1)[x^2 - 2x + x - 2] = (x - 1)[x(x - 2) + 1(x - 2)]$
$= (x - 1)(x - 2)(x + 1)$

(ii) $x^3 - 3x^2 - 9x - 5$

Let $p(x) = x^3 - 3x^2 - 9x - 5$

Putting $x = 1$, we get

$p(1) = (1)^3 - 3(1)^2 - 9(1) - 5 = 1 - 3 - 9 - 5 = -16 \neq 0$

Putting $x = -1$, we get
$p(-1) = (-1)^3 - 3(-1)^2 - 9(-1) - 5 = -1 - 3 + 9 - 5 = 0$
$\Rightarrow x + 1$ is a factor of $p(x)$.

$$
\begin{array}{r|l}
 & x^2 - 4x - 5 \\ \hline
x + 1 & x^3 - 3x^2 - 9x - 5 \\
 & x^3 + x^2 \\
 & - \quad - \\ \hline
 & \quad -4x^2 - 9x - 5 \\
 & \quad -4x^2 - 4x \\
 & \quad + \qquad + \\ \hline
 & \qquad\quad - 5x - 5 \\
 & \qquad\quad -5x - 5 \\
 & \qquad\quad + \qquad + \\ \hline
 & \qquad\qquad 0
\end{array}
$$

So, $p(x) = (x + 1)(x^2 - 4x - 5)$
$= (x + 1)(x^2 - 4x - 5)$
$= (x + 1)[x^2 - (5 - 1)x - 5]$
$= (x + 1)[x^2 - 5x + x - 5]$
$= (x + 1)[x(x - 5) + 1(x - 5)]$
$= (x + 1)(x - 5)(x + 1)$

(iii) $x^3 + 13x^2 + 32x + 20$
Let $p(x) = x^3 + 13x^2 + 32x + 20$
Putting $x = -1$, we get
$p(-1) = (-1)^3 + 13(-1)^2 + 32(-1) + 20 = -1 + 13 - 32 + 20 = 0$
$\Rightarrow x + 1$ is a factor of $p(x)$.

$$
\begin{array}{r|l}
 & x^2 + 12x + 20 \\ \hline
x + 1 & x^3 + 13x^2 + 32x + 20 \\
 & x^3 + x^2 \\
 & - \quad - \\ \hline
 & \quad 12x^2 + 32x + 20 \\
 & \quad 12x^2 + 12x \\
 & \quad - \qquad - \\ \hline
 & \qquad\qquad 20x + 20 \\
 & \qquad\qquad 20x + 20 \\
 & \qquad\qquad - \qquad - \\ \hline
 & \qquad\qquad 0
\end{array}
$$

So, $p(x) = (x + 1)(x^2 + 12x + 20)$
$= (x + 1)(x^2 + 12x + 20)$
$= (x + 1)[x^2 + (10 + 2)x + 20]$
$= (x + 1)[x^2 + 10x + 2x + 20]$
$= (x + 1)[x(x + 10) + 2(x + 10)]$
$= (x + 1)(x + 10)(x + 2)$

(iv) $2y^3 + y^2 - 2y - 1$
Let $p(y) = y^3 + y^2 - 2y - 1$
Putting $y = -1$, we get
$p(-1) = 2(-1)^3 + (-1)^2 - 2(-1) - 1 = -2 + 1 + 2 - 1 = 0$
$\Rightarrow y + 1$ is a factor of $p(y)$.

$$
\begin{array}{r|l}
 & 2y^2 - y - 1 \\ \hline
y + 1 & 2y^3 + y^2 - 2y - 1 \\
 & 2y^3 + 2y^2 \\
 & - \quad\; - \\ \hline
 & \quad -y^2 - 2y - 1 \\
 & \quad -y^2 - y \\
 & \quad + \quad\; + \\ \hline
 & \qquad\quad -y - 1 \\
 & \qquad\quad -y - 1 \\
 & \qquad\quad + \quad + \\ \hline
 & \qquad\qquad 0
\end{array}
$$

So, $p(y) = (y + 1)(2y^2 - y - 1)$
$= (y + 1)(2y^2 - y - 1)$
$= (y + 1)[2y^2 - (2 - 1)y - 1]$
$= (y + 1)[2y^2 - 2y + y - 1]$
$= (y + 1)[2y(y - 1) + 1(y - 1)]$
$= (y + 1)(y - 1)(2y + 1)$