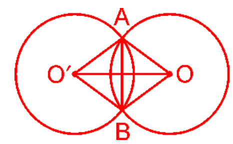
10 CIRCLES

EXERCISE 10.6 (Optional)

Q.1. *Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.*

Sol. **Given :** Two intersecting circles, in which OO′ is the line of centres and A and B are two points of intersection.



To prove : ∠OAO′ = ∠OBO′

Construction : Join AO, BO, AO′ and BO′.

Proof : In ΔAOO′ and ΔBOO′, we have

AO = BO [Radii of the same circle]

AO′ = BO′ [Radii of the same circle]

OO′ = OO′ [Common]

∴ ΔAOO′ ≅ ΔBOO′ [SSS axiom]

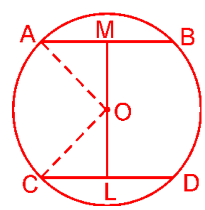
⇒ ∠OAO′ = ∠OBO′ [CPCT]

Hence, the line of centres of two intersecting circles subtends equal angles at the two points of intersection. **Proved.**

Q.2. *Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.*

Sol. Let O be the centre of the circle and let its radius be r cm.

Draw OM ⊥ AB and OL ⊥ CD.



Then, $AM = \frac{1}{2}AB = \frac{5}{2}$ cm

and, $CL = \frac{1}{2}CD = \frac{11}{2}$ cm

Since, AB || CD, it follows that the points O, L, M are collinear and therefore, LM = 6 cm.

Let OL = x cm. Then OM = $(6 - x)$ cm

Join OA and OC. Then OA = OC = r cm.

Now, from right-angled ΔOMA and ΔOLC, we have

$OA^2 = OM^2 + AM^2$ and $OC^2 = OL^2 + CL^2$ [By Pythagoras Theorem]

$$\Rightarrow r^2 = (6 - x)^2 + \left(\frac{5}{2}\right)^2 \quad ..(i) \text{ and } r^2 = x^2 + \left(\frac{11}{2}\right)^2 \quad ... (ii)$$

$\Rightarrow (6 - x)^2 + \left(\frac{5}{2}\right)^2 = x^2 + \left(\frac{11}{2}\right)^2$ [From (i) and (ii)]

$\Rightarrow 36 + x^2 - 12x + \frac{25}{4} = x^2 + \frac{121}{4}$

$\Rightarrow -12x = \frac{121}{4} - \frac{25}{4} - 36$

$\Rightarrow -12x = \frac{96}{4} - 36$

$\Rightarrow -12x = 24 - 36$

$\Rightarrow -12x = -12$

$\Rightarrow \quad x = 1$

Substituting $x = 1$ in (i), we get

$r^2 = (6 - x)^2 + \left(\frac{5}{2}\right)^2$

$\Rightarrow r^2 = (6 - 1)^2 + \left(\frac{5}{2}\right)^2$

$\Rightarrow r^2 = (5)^2 + \left(\frac{5}{2}\right)^2 = 25 + \frac{25}{4}$

$\Rightarrow r^2 = \frac{125}{4}$

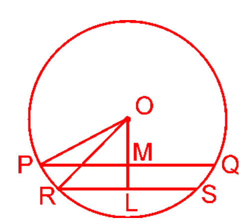
$\Rightarrow r = \frac{5\sqrt{5}}{2}$

Hence, radius $r = \frac{5\sqrt{5}}{2}$ cm. **Ans.**

Q.3. *The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?*

Sol. Let PQ and RS be two parallel chords of a circle with centre O.
We have, PQ = 8 cm and RS = 6 cm.
Draw perpendicular bisector OL of RS which meets PQ in M. Since, PQ || RS, therefore, OM is also perpendicular bisector of PQ.

Also, OL = 4 cm and RL = $\frac{1}{2}$RS $\Rightarrow$ RL = 3 cm



and PM = $\frac{1}{2}$PQ $\Rightarrow$ PM = 4 cm

In ΔORL, we have

$OR^2 = RL^2 + OL^2$ [Pythagoras theorem]

$\Rightarrow OR^2 = 3^2 + 4^2 = 9 + 16$

$\Rightarrow OR^2 = 25 \Rightarrow OR = \sqrt{25}$

$\Rightarrow OR = 5$ cm

$\therefore OR = OP$ [Radii of the circle]

$\Rightarrow OP = 5$ cm

Now, in ΔOPM

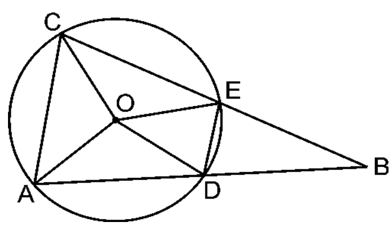
$OM^2 = OP^2 - PM^2$ [Pythagoras theorem]

$\Rightarrow OM^2 = 5^2 - 4^2 = 25 - 16 = 9$

$OM = \sqrt{9} = 3$ cm

Hence, the distance of the other chord from the centre is 3 cm. **Ans.**

Q.4. *Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half the difference of the angles subtended by the chords AC and DE at the centre.*



Sol. Given : Two equal chords AD and CE of a circle with centre O. When meet at B when produced.

To Prove : $\angle ABC = \frac{1}{2}(\angle AOC - \angle DOE)$

Proof : Let $\angle AOC = x$, $\angle DOE = y$, $\angle AOD = z$

$\angle EOC = z$ [Equal chords subtends equal angles at the centre]

$\therefore x + y + 2z = 36°$ [Angle at a point] .. (i)

$OA = OD \Rightarrow \angle OAD = \angle ODA$

$\therefore$ In DOAD, we have

$\angle OAD + \angle ODA + z = 180°$

$\Rightarrow 2\angle OAD = 180° - z$ [$\because \angle OAD = \angle OBA$]

$\Rightarrow \angle OAD = 90° - \frac{z}{2}$ ... (ii)

Similarly $\angle OCE = 90° - \frac{z}{2}$ ... (iii)

$\Rightarrow \angle ODB = \angle OAD + \angle ODA$ [Exterior angle property]

$\Rightarrow \angle OEB = 90° - \frac{z}{2} + z$ [From (ii)]

$\Rightarrow \angle ODB = 90° + \frac{z}{2}$ ... (iv)

Also, $\angle OEB = \angle OCE + \angle COE$ [Exterior angle property]

$\Rightarrow \angle OEB = 90° - \frac{z}{2} + z$ [From (iii)]

$\Rightarrow \angle OEB = 90° + \frac{z}{2}$ ... (v)

Also, $\angle OED = \angle ODE = 90° - \frac{y}{2}$ ... (vi)

O from (iv), (v) and (vi), we have

$$\angle BDE = \angle BED = 90° + \frac{z}{2} - \left(90° - \frac{y}{2}\right)$$

$\Rightarrow \angle BDE = \angle BED = \frac{y+z}{2}$

$\Rightarrow \angle BDE = \angle BED = y + z$ ... (vii)

$\therefore \angle BDE = 180° - (y + z)$

$\Rightarrow \angle ABC = 180° - (y + z)$ ... (viii)

Now, $\frac{y-z}{2} = \frac{360° - y - 2z - y}{2} = 180° - (y + z)$ ... (ix)

From (viii) and (ix), we have

$\angle ABC = \frac{x-y}{2}$ **Proved.**

Q.5. *Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.*

Sol. Given : A rhombus ABCD whose diagonals intersect each other at O.

To prove : A circle with AB as diameter passes through O.

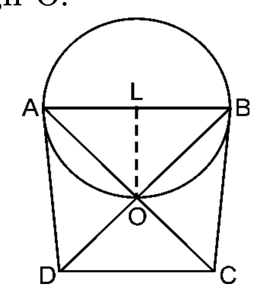
Proof : $\angle AOB = 90°$

[Diagonals of a rhombus bisect each other at 90°]

$\Rightarrow \Delta AOB$ is a right triangle right angled at O.

$\Rightarrow$ AB is the hypotenuse of A B right ΔAOB.

$\Rightarrow$ If we draw a circle with AB as diameter, then it will pass through O. because angle is a semicircle is 90° and $\angle AOB = 90°$ **Proved.**

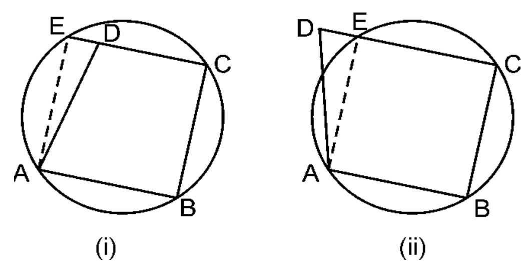


Q.6. *ABCD is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that AE = AD.*

Sol. Given : ABCD is a parallelogram.

To Prove : AE = AD.

Construction : Draw a circle which passes through ABC and intersect CD (or CD produced) at E.



Proof : For fig (i)

$\angle AED + \angle ABC = 180°$ [Linear pair] ... (ii)

But $\angle ACD = \angle ADC = \angle ABC + \angle ADE$

$\Rightarrow \angle ABC + \angle ADE = 180°$ [From (ii)] ... (iii)

From (i) and (iii)

$\angle AED + \angle ABC = \angle ABC + \angle ADE$

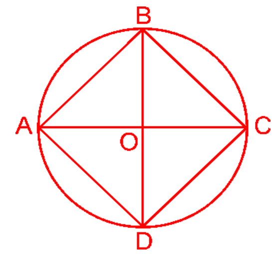
$\Rightarrow \angle AED = \angle ADE$

$\Rightarrow \angle AD = \angle AE$ [Sides opposite to equal angles are equal]

Similarly we can prove for Fig (ii) **Proved.**

Q.7. *AC and BD are chords of a circle which bisect each other. Prove that (i) AC and BD are diameters, (ii) ABCD is rectangle.*

Sol. Given : A circle with chords AB and CD which bisect each other at O.



To Prove : (i) AC and BD are diameters

(ii) ABCD is a rectangle.

Proof : In ΔOAB and ΔOCD, we have

$OA = OC$ [Given]

$OB = OD$ [Given]

$\angle AOB = \angle COD$ [Vertically opposite angles]

$\Rightarrow \Delta AOB \cong \angle COD$ [SAS congruence]

$\Rightarrow \angle ABO = \angle CDO$ and $\angle BAO = \angle BCO$ [CPCT]

$\Rightarrow AB \mid\mid DC$ ... (i)

Similarly, we can prove $BC \mid\mid AD$ ... (ii)

Hence, ABCD is a parallelogram.

But ABCD is a cyclic parallelogram.

$\therefore$ ABCD is a rectangle. [Proved in Q. 12 of Ex. 10.5]

$\Rightarrow \angle ABC = 90°$ and $\angle BCD = 90°$

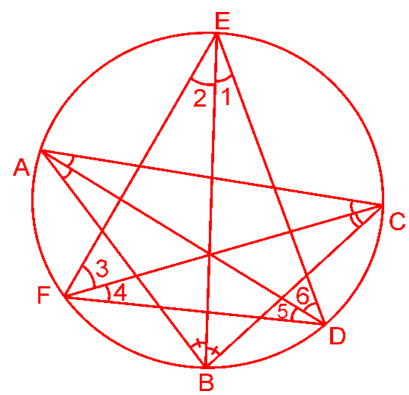
$\Rightarrow$ AC is a diameter and BD is a diameter

[Angle in a semicircle is 90°] **Proved.**

Q.8. *Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at D, E and F respectively. Prove that the angles of the triangle DEF are*

$90° - \frac{1}{2}A$, $90° - \frac{1}{2}B$ *and* $90° - \frac{1}{2}C$.

Sol. Given : ΔABC and its circumcircle. AD, BE, CF are bisectors of $\angle A$, $\angle B$, $\angle C$ respectively.



Construction : Join DE, EF and FD.

Proof : We know that angles in the same segment are equal.

$\therefore \quad \angle 5 = \frac{\angle C}{2}$ and $\angle 6 = \frac{\angle B}{2}$ ..(i)

$\angle 1 = \frac{\angle A}{2}$ and $\angle 2 = \frac{\angle C}{2}$ ..(ii)

$\angle 4 = \frac{\angle A}{2}$ and $\angle 3 = \frac{\angle B}{2}$ ..(iii)

From (i), we have

$$\angle 5 + \angle 6 = \frac{\angle C}{2} + \frac{\angle B}{2}$$

$$\Rightarrow \angle D = \frac{\angle C}{2} + \frac{\angle B}{2} \quad ...(iv)$$

[$\because \angle 5 + \angle 6 = \angle D$]

But $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \angle B + \angle C = 180° - \angle A$

$\Rightarrow \quad \frac{\angle B}{2} + \frac{\angle C}{2} = 90° - \frac{\angle A}{2}$

$\therefore$ (iv) becomes,

$\angle D = 90° - \frac{\angle A}{2}$.

Similarly, from (ii) and (iii), we can prove that

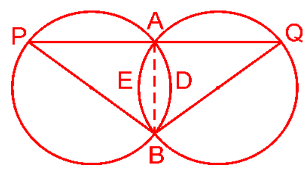
$\angle E = 90° - \frac{\angle B}{2}$ and $\angle F = 90° - \frac{\angle C}{2}$ **Proved.**

Q.9. *Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that BP = BQ.*

Sol. Given : Two congruent circles which intersect at A and B. PAB is a line through A.

To Prove : BP = BQ.

Construction : Join AB.



Proof : AB is a common chord of both the circles.

But the circles are congruent —

$\Rightarrow$ arc ADB = arc AEB

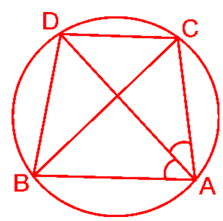
$\Rightarrow \quad \angle APB = \angle AQB$ Angles subtended

$\Rightarrow \quad BP = BQ$ [Sides opposite to equal angles are equal] **Proved.**

Q.10. *In any triangle ABC, if the angle bisector of ∠A and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the triangle ABC.*

Sol. Let angle bisector of $\angle A$ intersect circumcircle of ΔABC at D.

Join DC and DB.



$\angle BCD = \angle BAD$

[Angles in the same segment]

$\Rightarrow \angle BCD = \angle BAD \frac{1}{2} \angle A$

[AD is bisector of $\angle A$] ...(i)

Similarly $\angle DBC = \angle DAC \frac{1}{2} \angle A$ ... (ii)

From (i) and (ii) $\angle DBC = \angle BCD$

$\Rightarrow$ BD = DC [sides opposite to equal angles are equal]

$\Rightarrow$ D lies on the perpendicular bisector of BC.

Hence, angle bisector of $\angle A$ and perpendicular bisector of BC intersect on the circumcircle of ΔABC **Proved.**