Previous Year Question Paper

2020

- Please check that this question paper contains **15** printed pages.
- Code number given on the right hand side of the question paper should be written on the title page of the answer-book by the candidate.
- Please check that this question paper contains **36** questions.
- Please write down the Serial Number of the question before attempting it.
- 15 minutes of time has been allotted to read this question paper. The question paper will be distributed at 10.15 a.m. From 10.15 a.m. to 10.30 a.m., the students will read the question paper only and will not write any answer on the answer script during this period.

MATHEMATICS

Time Allowed: **3** hours Maximum Marks: **80**

General Instructions:

Read the following instructions very carefully and strictly follow them:

(i) This question paper comprises **four** Sections A, B, C and D. This question paper carries **36** questions. **All** questions are compulsory.

(ii) **Section A** – Questions no. **1** to **20** comprises of **20** questions of **1** mark each.

(iii) **Section B** – Questions no. **21** to **26** comprises of **6** questions of **2** marks each.

(iv) **Section C** – Questions no. **27** to **32** comprises of **6** questions of **4** marks each.

(v) **Section D** – Questions no. **33** to **36** comprises of **4** questions of **6** marks each.

(vi) There is no overall choice in the question paper. However, an internal choice has been provided in 3 questions of one mark, 2 questions of two marks, 2 questions of four marks and 2 questions of six marks. Only one of the choices in such questions have to be attempted.

(vii) In addition to this, separate instructions are given with each section and question, wherever necessary.

(viii) Use of calculators is **not** permitted.

SECTION - A

Question numbers 1 to 20 carry 1 mark each.

Question numbers 1 to 10 are multiple choice type questions. Select the correct option.

1. The area of a triangle formed by vertices O, A and B, where $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{OB}=-3\hat{i}-2\hat{j}+\hat{k}$ is **1 Mark**

(A) $3\sqrt{5}$ sq. units

(B) $5\sqrt{5}$ sq. units

(C) $6\sqrt{5}$ sq. units

(D) 4 sq. units

Ans: Given, $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{OB}=-3\hat{i}-2\hat{j}+\hat{k}$.

We know, are of a triangle if it's vectors are given is, $\frac{1}{2}\left|\vec{A}\times\vec{B}\right|$.

Therefore, here, $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k}$

And, $\overrightarrow{OB}=-3\hat{i}-2\hat{j}+\hat{k}$

Therefore, $\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ -3 & -2 & 1\end{vmatrix}$

$$\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\left|\hat{i}\{(2)(1)-(-2)(3)\}-\hat{j}\{(1)(1)-(-3)(3)\}+\hat{k}\{(1)(-2)-(-3)(2)\}\right|$$

$\Rightarrow \left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\times\left|\hat{i}\{2+6\}-\hat{j}\{1+9\}+\hat{k}\{-2+6\}\right|$

$\Rightarrow \left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\times\left|8\hat{i}-10\hat{j}+4\hat{k}\right|$

$\Rightarrow \left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\sqrt{64+100+16}$

$\Rightarrow \left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\sqrt{180}$

$\Rightarrow \left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\times 6\sqrt{5}$

$\Rightarrow \left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=3\sqrt{5}$

Therefore, the area is $3\sqrt{5}$ sq. units.

Thus, the correct answer is A.

2. If $\cos\left(\sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x\right)=0$, then x is equal to **1 Mark**

(A) $\frac{1}{\sqrt{5}}$

(B) $-\frac{2}{\sqrt{5}}$

(C) $\frac{2}{\sqrt{5}}$

(D) 1

Ans: Given, $\cos\left(\sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x\right)=0$

Taking $\cos^{-1}$ on both sides, we get,

$\Rightarrow \sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x=\cos^{-1}(0)$

$\Rightarrow \sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x=\cos^{-1}x=\frac{\pi}{2}$

We know, $\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$, hence,

$\Rightarrow \sin^{-1}\frac{2}{\sqrt{5}}+\frac{\pi}{2}-\sin^{-1}x=\frac{\pi}{2}$

$\Rightarrow \sin^{-1}x=\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$\Rightarrow x=\frac{2}{\sqrt{5}}$

Thus, the correct answer is C.

3. The interval in which the function f given by $f(x)=x^2e^{-x}$ is strictly increasing, is **1 Mark**

(A) $(-\infty,\infty)$

(B) $(-\infty,0)$

(C) $(2,\infty)$

(D) (0,2)

Ans: Given, $f(x)=x^2e^{-x}$

Now, differentiating both sides with respect to x, we get,

$\Rightarrow f'(x)=2xe^{-x}-x^2e^{-x}$

$\Rightarrow f'(x)=xe^{-x}(2-x)$

For, the function to be increasing,

$f'(x)>0$

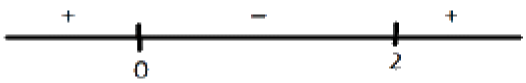
$\Rightarrow xe^{-x}(2-x)>0$

$\Rightarrow x(2-x)>0$

$\Rightarrow x(2-x)<0$

[Since, e^{-x} can never be zero]

Using, the method of intervals, we get,



Since, $x(x-2)<0$, we will take the negative region.

Therefore, $x \in (0,2)$.

The correct option is D.

4. The function $f(x)=\frac{x-1}{x(x^2-1)}$ is discontinuous at **1 Mark**

(A) exactly one point

(B) exactly two points

(C) exactly three points

(D) no point

Ans: Given, $f(x)=\frac{x-1}{x(x^2-1)}$

We can write the function as,

$$\Rightarrow f(x)=\frac{x-1}{x(x-1)(x-1)}$$

Here, the function is discontinuous if,

$x(x-1)(x-1)=0$

x=0 or x-1=0 or x+1=0

x=0 or x=1 or x=-1

Therefore, the function is discontinuous exactly at three points.

The correct option is C.

5. The function $f:R\to[-1,1]$ defined by $f(x)=\cos x$ is **1 Mark**

(A) both one-one and onto

(B) not one-one, but onto

(C) one-one, but not onto

(D) neither one-one, nor onto

Ans: Given, $f:R\to[-1,1]$ defined by $f(x)=\cos x$.

Let, $f(x_1)=f(x_2)$

$\Rightarrow \cos x_1=\cos x_2$

$\Rightarrow x_1=2n\pi\pm x_2, n\in\mathbb{Z}$

Therefore, the above equations have infinitely many solutions.

Hence, it is not a one-one function.

Also, range of cos x is [-1, 1], which is a subset of co-domain R.

Hence, the function is also not onto.

Therefore, the function is neither one-one nor onto.

Thus, the correct option is D.

6. The coordinates of the foot of the perpendicular drawn from the point (2, -3, 4) on the y-axis is **1 Mark**

(A) (2, 3, 4)

(B) (-2, -3, -4)

(C) (0, -3, 0)

(D) (2, 0, 4)

Ans: Given point is P(2, -3, 4).
Any point on y-axis is given by Q(0, k, 0), where k is any real number
So direction ratio of PQ are -2,-3,-k,4.
We know direction ratio of y-axis is given by 0,1,0.
Now since $PQ\perp$ y-axis

$\Rightarrow$ (0)(2)+(1)(-3-k)+(0)(4)=0

$\Rightarrow k=-3$

Hence, coordinate of foot of perpendicular is Q(0, -3, 0).

Thus, the correct option is C.

7. The relation R in the set {1,2,3} given by R={(1,2),(2,1),(1,1)} is 1 Mark

(A) symmetric and transitive, but not reflexive

(B) reflexive and symmetric, but not transitive

(C) symmetric, but neither reflexive nor transitive

(D) an equivalence relation

Ans: The relation is not reflexive because (2,2),(3,3) are not present.

It is symmetric because, $(1,2) \in R$ and also $(2,1) \in R$, which satisfies the condition for a relation to be symmetric perfectly.

And, also, it is transitive because, $(1,2) \in R$, $(2,1) \in R$ and also $(1,1) \in R$, which satisfies the condition for a relation to be transitive perfectly.

Hence, the relation is symmetric and transitive but not reflexive.

Thus, the correct option is A.

8. The angle between the vectors $\hat{i}-\hat{j}$ and $\hat{j}-\hat{k}$ is 1 Mark

(A) $-\frac{\pi}{3}$

(B) 0

(C) $\frac{\pi}{3}$

(D) $\frac{2\pi}{3}$

Ans: Given vectors are

$\vec{a}=\hat{i}-\hat{j}$

$\vec{b}=\hat{i}-\hat{k}$

So, $\vec{a}.\vec{b}=\left(\hat{i}-\hat{j}\right).\left(\hat{i}-\hat{k}\right)$

$\vec{a}.\vec{b}=(1\times0)+(-1\times1)+(0\times(-1))$

$\Rightarrow \vec{a}.\vec{b}=-1$

Also, $|\vec{a}|=\sqrt{1^2+(-1)^2+0}=\sqrt{2}$

$|\vec{b}|=\sqrt{0+1^2}=\sqrt{2}$

We also know, $\vec{a}.\vec{b}=|\vec{a}|.|\vec{b}|.\cos\theta$

$\Rightarrow$ -1=$\sqrt{2}.\sqrt{2}.\cos\theta$

$\Rightarrow \cos\theta=\frac{-1}{2}$

$\Rightarrow \theta=\frac{2\pi}{3}$

The angle between the vectors is $\frac{2\pi}{3}$.

Thus, the correct option is D.

9. If A is a non-singular square matrix of order 3 such that A2=3A , then value of |A| is **1 Mark**

(A) -3

(B) 3

(C) 9

(D) 27

Ans: Given, A2=3A

Taking determinant on both sides,

$\Rightarrow |A^2|=|3A|$

$\Rightarrow |A^2|=3^3|A|$

$\Rightarrow |A^2|=27|A|$

$\Rightarrow |A|=27$

Therefore, the correct option is D.

10. If $\left|\vec{a}\right| = 4$ and $-3 \le \lambda \le 2$, then $\left|\lambda\vec{a}\right|$ lies in **1 Mark**

(A) [0,12]

(B) [2,3]

(C) [8,12]

(D) [-12,8]

Ans: The maximum value of λ is **2.**

So, $\left|\lambda\vec{a}\right| = \left|\lambda\right|.\left|\vec{a}\right|$

$\Rightarrow \left|\lambda\vec{a}\right| = 2.4 = 8$

The minimum value of λ is -3**.**

So, $\left|\lambda\vec{a}\right| = \left|\lambda\right|.\left|\vec{a}\right|$

$\Rightarrow \left|\lambda\vec{a}\right| = \left|-3\right|.4$

$\Rightarrow \left|\lambda\vec{a}\right| = 3.4 = 12$

So, there are no value of λ which is negative.

For, λ =0, we get,

$\left|\lambda\vec{a}\right| = \left|\lambda\right|.\left|\vec{a}\right|$

$\Rightarrow \left|\lambda\vec{a}\right| = 0.4 = 0$

Therefore, the smallest value of $\left|\lambda\vec{a}\right|$ is 0 .

Therefore, $\left|\lambda\vec{a}\right|$ lies in [0, 12].

Thus, the correct option is A**.**

Fill in the blanks in question numbers 11 to 15.

11. If the radius of the circle is increasing at the rate of 0.5 cm/s, then the rate of increase of its circumference is ______________ . **1 Mark**

Ans: Let r be the radius and C the circumference of the circle.

Then, $C=2\pi r$

It is given that $\frac{dr}{dt}=0.5\ \text{cm/s}$

Now, $C=2\pi r$

Differentiating both sides w.r.t t, we get,

$\Rightarrow \frac{dr}{dt}=2\pi.\frac{dr}{dt}$

$\Rightarrow \frac{dC}{dt}=2\pi.05$

$\Rightarrow \frac{dC}{dt}=\pi\ \text{cm/s}.$

Therefore, the rate in increase of the circumference is π cm/s.

12. If $\begin{vmatrix} 2x & -9 \\ -2 & x \end{vmatrix}=\begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix}$, then value of x is _____________ . **1 Mark**

Ans: $\begin{vmatrix} 2x & -9 \\ -2 & x \end{vmatrix}=\begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix}$

(2x)(x)-(-9)(-2)=(-4)(-2)-(8)(1)

2x2-18=8-8

2x2-18=0

2x2-18=0

2x2=18

x2=9

$x=\pm 3$

Therefore, the value of x is ± 3.

13. The corner points of the feasible region of an LPP are (0,0), (0,8), (2,7), (5,4) and (6,0). The maximum profit P=3x+2y occurs at the point ______.

1 Mark

Ans: $P_{(0,0)}=3(0)+2=0$

$P_{(0,8)}=3(0)+2(8)=16$

$P_{(2,7)}=3(2)+2(7)=20$

$P_{(5,4)}=3(5)+2(4)=23$

$P_{(6,0)}=3(6)+2(0)=18$

The maximum value is at (5,4).

14. The range of the principal value branch of the function $y\text{-}\sec^{-1}x$ is ____________ .

1 Mark

Ans: We know, $\sec^{-1}x \in [0,\pi]-\left\{\frac{\pi}{2}\right\}$.

Therefore, the range of the principal value branch of the function $y\text{-}\sec^{-1}x$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$.

Or

The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is ___________ . **1 Mark**

Ans: $\cos^{-1}\left(-\frac{1}{2}\right)$

We know, $\cos^{-1}\frac{2\pi}{3}=-\frac{1}{2}$

$=\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

$=\frac{2\pi}{3}$

Thus, the principle value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $=\frac{2\pi}{3}$.

15. The distance between parallel planes 2x+y-2z-6=0 and 4x+2y-4z is ____________ units. **1 Mark**

Ans: Given, 2x+y-2z-6=0 --------(1)

4x+2y-4z -------(2)

Multiplying (1) by 2, we get,

4x+2y-4z-12=0

4x+2y-4z=12

Therefore, we can write,

c1=12, c2=0

And, a=4, b=2, c=-4.

Therefore, the distance between the parallel lines is,

$$\left|\frac{c_2-c_1}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{0-12}{\sqrt{(4)^2+(2)^2+(-4)^2}}\right|$$

$$=\left|\frac{12}{\sqrt{16+4+16}}\right|$$

$$=\left|\frac{12}{\sqrt{36}}\right|$$

$$=\left|\frac{12}{6}\right|$$

=2 units

Or

If P(1,0,-3) is the foot of the perpendicular from the origin to the plane, then the Cartesian equation of the plane is ______________ . **1 Mark**

Ans: The given foot of the perpendicular is P(1,0,-3).

The direction coefficients of the perpendicular are (1-0,0-0,-3,-0)

=(1,0,-3) .

Therefore, the equation of the plane is

$$a\left(x-x_1\right)+b\left(y-y_1\right)+c\left(z-z_1\right)=0$$

$$\Rightarrow 1\left(x-1\right)+b\left(y-0\right)-3\left(z-\left(-3\right)\right)=0$$

$$\Rightarrow x-1-3\left(z+3\right)=0$$

$$\Rightarrow x-1-3z-9=0$$

$$\Rightarrow x-3z-10=0$$

Therefore, the equation of the plane is $x-3z-10=0$.

Question numbers 16 to 20 are very short answer type questions.

16. Evaluate : $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x\,dx$ **1 Mark**

Ans: Here, $f\left(x\right)=x\cos^2 x$

Now, $f\left(-x\right)=\left(-x\right)\cos^2\left(-x\right)$

$$\Rightarrow f\left(-x\right)=-x\cos^2 x = -f\left(-x\right)$$

Therefore, it is an odd function.

So, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x\,dx=0$.

17. Find the coordinates of the point where the line $\frac{x-1}{3}=\frac{y+4}{7}=\frac{z+4}{2}$ cuts the xy-plane. **1 Mark**

Ans: If the line $\frac{x-1}{3}=\frac{y+4}{7}=\frac{z+4}{2}$ cuts the XY plane.

Then, z=0.

So, let coordinates of point be (x,y,0).

Now, $\frac{x-1}{3}=\frac{y+4}{7}=\frac{z+4}{2}=k$

Thus, x=3k+1, y=7k-4, z=2k-4

Since, z=0

2k-4=0

2k=4

k=2

Now, x=3(2)+1=7

y=7(2)-4=10

Therefore, the point is (7,10,0).

18. Find the value of k, so that the function $f(x)=\begin{cases} kx^2+5 & \text{if } x\le 1 \\ 2 & \text{if } x>1 \end{cases}$ is continuous at x=1 . **1 Mark**

Ans: For, x>1, the value of f(x), such that,

$f(x)_{x\to 1^+}=2$

For, $x\le 1$, the value of f(x), such that,

$f(x)_{x\to 1^+}=k(1)^2+5$

$f(x)_{x\to 1^+}=k+5$

For the function to be continuous

$f(x)_{x\to 1^+}=f(x)_{x\to 1^-}$

2=k+5

k=-3

19. Find the integrating factor of the differential equation

$\mathbf{x\frac{dy}{dx}=2x^2+y}$

1 Mark

Ans: Given, $x\frac{dy}{dx}=2x^2+y$

Dividing both sides by x, we get,

$$\frac{dy}{dx}=2x+\frac{y}{x}$$

$$\frac{dy}{dx}-\frac{y}{x}=2x$$

Therefore, $P(x)=\frac{1}{x}$

Thus, integrating factor, IF = $e^{\int=\frac{1}{x}dx}$

$$=e^{-\int=\frac{1}{x}dx}$$

$$=e^{-\log x}$$

$$=e^{\log x^{-1}}$$

$$=x^{-1}$$

$$=\frac{1}{x}$$

Thus, the integrating factor is $\frac{1}{x}$.

20. Differentiate $=\sec^2\left(x^2\right)$ with respect to x^2. 1 Mark

Ans: We need to find

$$\frac{d\left(\sec^2\left(x^2\right)\right)}{dx^2}$$

Let x2=t

So, $\frac{d\left(\sec^2(t)\right)}{dt}$

$=2.\sec t.(\sec t)'$

$=2.\sec t.\sec t.\tan t$

$=2.\sec^2 t.\tan t$

Putting t=x2

$=2.\sec^2 x^2.\tan x^2$

Or

If $y=f\left(x^2\right)$ and $f'(x)=e^{\sqrt{x}}$, then find $\frac{dy}{dx}$. **1 Mark**

Ans: Given, $y=f\left(x^2\right)$

Differentiating both sides w.r.t x, we get,

$=\frac{dy}{dx}y=f'\left(x^2\right).2x$

Also, given, $f'(x)=e^{\sqrt{x}}$.

$f'\left(x^2\right)=e^{\sqrt{x^2}}=e^x$

$\therefore \frac{dy}{dx}=2xe^x$.

SECTION - B

Question numbers 21 to 26 carry 2 marks each.

21. Find a vector r equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units. **2 Marks**

Ans: We have $|\vec{r}|=3\sqrt{3}$

Since, $\vec{r}$ is equally inclined to the three axes, direction cosines of the unit vector $\vec{r}$ will be same.

i.e., l=m=n

Now, we know that,

$l^2+m^2+n^2=1$

$\Rightarrow l^2+l^2+l^2=1$

$\Rightarrow 3l^2=1$

$\Rightarrow l^2=\frac{1}{3}$

$\Rightarrow l=\pm\frac{1}{\sqrt{3}}$

So, $\hat{r}=\pm\frac{1}{\sqrt{3}}\hat{i}\pm\frac{1}{\sqrt{3}}\hat{j}\pm\frac{1}{\sqrt{3}}\hat{k}$

$\therefore \vec{r}=|\vec{r}|.\hat{r}$

$=2\sqrt{3}\left[\pm\frac{1}{\sqrt{3}}\hat{i}\pm\frac{1}{\sqrt{3}}\hat{j}\pm\frac{1}{\sqrt{3}}\hat{k}\right]$

$=\pm 2\left[\hat{i}+\hat{j}+\hat{k}\right]$

Or

Find the angle between unit vectors a and b so that $\sqrt{3}\vec{a}\text{-b}$ is also a unit vector. **2 Marks**

Ans: a and b are unit vectors and $\sqrt{3}\vec{a}\text{-b}$ is also unit vector

To find: Angle between a and b

Suppose angle between a and b is θ.

$\vec{a}.\vec{b}=|\vec{a}|.|\vec{b}|.\cos\theta$ (Dot product of two vectors)

$\vec{a}.\vec{b}=\cos\theta$

As $\vec{a}$ and $\vec{b}$ are unit vector so, $|\vec{a}|=|\vec{b}|=1$.

$\sqrt{3}\vec{a}\text{-b}$ is also unit vector i.e. $|\sqrt{3}\vec{a}\text{-}\vec{b}|=1$

Squaring both sides, we get,

$\left(\sqrt{3}\vec{a}\text{-}\vec{b}\right)^2=1$

$$\left(\sqrt{3}\right)^2\left|\vec{a}\right|^2+\left|\vec{b}\right|^2 -2.\sqrt{3}.\left|\vec{a}.\vec{b}\right|=1$$

$$\Rightarrow 3.1+1-2.\sqrt{3}.\cos\theta=1$$

[Since, $\vec{a}.\vec{b}=\cos\theta$]

$$\Rightarrow 4\text{-}2\sqrt{3}.\cos\theta=1$$

$$\Rightarrow 2\sqrt{3}.\cos\theta=3$$

$$\Rightarrow \cos\theta=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta=\frac{\pi}{6}$$

Therefore, the angle between the two unit vectors is $\frac{\pi}{6}$.

22. If $A=\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$ and $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find scalar k so that A2+I=kA. 2 Marks

Ans: $A=\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$, $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A2+I=kA

A2=A×A

$A^2 = A\times A$

$$\Rightarrow A^2=\begin{bmatrix} (-3)(-3)+(2)(1) & (-3)(2)+(2)(1) \\ 1(-3)+1(-1) & 1(2)+(-1)(-1) \end{bmatrix}$$

$$\Rightarrow A^2=\begin{bmatrix} 9+2 & -6-2 \\ -3-1 & 2+1 \end{bmatrix}$$

$$\Rightarrow A^2=\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

A2+I=kA

$$\Rightarrow \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = k \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 11+1 & -8+0 \\ -4+0 & 3+1 \end{bmatrix} = \begin{bmatrix} -3k & 2k \\ 1k & -1k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12 & -8 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} -3k & 2k \\ 1k & -1k \end{bmatrix}$$

Therefore, comparing the terms on both sides, we get,

k=-4.

23. If $f(x)=\sqrt{\frac{\sec x-1}{\sec x+1}}$, find $f'\left(\frac{\pi}{3}\right)$. **2 Marks**

Ans: Given, $f(x)=\sqrt{\frac{\sec x-1}{\sec x+1}}$

Using, $\sec x=\frac{1}{\cos x}$.

$$f(x)=\sqrt{\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}}$$

$$f(x)=\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$f(x)=\sqrt{\frac{2\sin^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}$$

Since, $1-\cos x=2\sin^2\frac{x}{2}$ and $1+\cos x=2\cos^2\frac{x}{2}$. So, we get,

$$f(x)=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\tan\frac{x}{2}$$

Differentiating both sides w.r.t x, we get,

$$f'(x)=\frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

$$f'(x)=\sec^2\frac{x}{2}.\frac{1}{2}$$

$$f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$$

Therefore, $f'\left(\frac{\pi}{3}\right)=\frac{1}{2}\sec^2\left(\frac{\pi}{2.3}\right)$

$$\Rightarrow f'\left(\frac{\pi}{3}\right)=\frac{1}{2}\sec^2\left(\frac{\pi}{6}\right)$$

$$\Rightarrow f'\left(\frac{\pi}{3}\right)=\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2$$

$$\Rightarrow f'\left(\frac{\pi}{3}\right)=\frac{2}{3}$$

Or

Find $f'(x)$ if $f(x)=(\tan x)^{\tan x}$. **2 Marks**

Ans: Given, $f(x)=(\tan x)^{\tan x}$

Let, $f(x)=(\tan x)^{\tan x}=y$

Taking log on both sides, we get,

$$\log y=\log\left(\tan x^{\tan x}\right)$$

$$\Rightarrow \log y=\tan x\log(\tan x)$$

Differentiating both sides w.r.t x , we get,

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\tan x.\frac{1}{\tan x}.\sec^2x+\log(\tan x).\sec^2x$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\sec^2x+\sec^2x\log(\tan x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[\sec^2 x+\sec^2 x\log\left(\tan x\right)\right]$$

$$\Rightarrow \frac{dy}{dx}=y\tan x^{\tan x}\sec^2 x\left[1+\log\left(\tan x\right)\right]$$

Since, $y=\tan x^{\tan x}$.

Therefore, $f'\left(x\right)=y\tan x^{\tan x}\sec^2 x\left[1+\log\left(\tan x\right)\right]$.

24. Find the value of integral: $\int\frac{\tan^3 x}{\cos^3 x}dx$ **2 Marks**

Ans: $I=\int\frac{\tan^3 x}{\cos^3 x}dx$

$$\Rightarrow I=\int\frac{\sin^3 x}{\cos^3 x.\cos^3 x}dx$$

$$\Rightarrow I=\int\frac{\sin^3 x.\sin x}{\cos^6 x}dx$$

$$\Rightarrow I=\int\frac{\left(1-\cos^2 x\right)\sin x}{\cos^6 x}dx$$

$$\Rightarrow I=\int\frac{\sin x}{\cos^6 x}dx-\int\frac{\sin x}{\cos^4 x}dx$$

Now, let, $\cos x=t$

$-\sin x dx=dt$

$\sin x dx=-dt$

Therefore, $I=\int-\frac{1}{t^6}dt+\int\frac{1}{t^4}dt$

$$\Rightarrow I=-\int\frac{1}{t^6}dt+\int\frac{1}{t^4}dt$$

$$\Rightarrow I=-\left[\frac{t^{-6+1}}{-6+1}\right]+\left[\frac{t^{-4+1}}{-4+1}\right]+c$$

$\Rightarrow I=-\left[\frac{t^{-5}}{-5}\right]+\left[\frac{t^{-3}}{-3}\right]+c$

$\Rightarrow I=\frac{1}{5t^5}-\frac{1}{3t^3}+c$

Substituting t=cosx, we get,

$\Rightarrow I=\frac{1}{5\cos^5 x}-\frac{1}{3\cos^3 x}+c$

Therefore, $\int\frac{\tan^3 x}{\cos^3 x}dx=\frac{1}{5\cos^5 x}-\frac{1}{3\cos^3 x}+c$.

25. Show that the plane x-5y-2z=1 contains the line $\frac{x-5}{3}=y=2-z$. 2 Marks

Ans: Given:

Plane: x-5y-2z=1

In vector form, we can write the equation of plane as,

$\vec{r}.\left(\hat{i}-5\hat{j}-2\hat{k}\right)=1$

Direction ratio of the plane $\vec{P}=\left(\hat{i}-5\hat{j}-2\hat{k}\right)$

Line: $\frac{x-5}{3}=y=2-z$

In vector form, we can write the equation of line as,

$\vec{r}=\left(5\hat{i}+2\hat{k}\right)+\lambda\left(3\hat{i}+\hat{j}-\hat{k}\right)$

Direction ratio of the plane

$\vec{L}=\left(3\hat{i}+\hat{j}-\hat{k}\right)$

Now, $\vec{p}.\vec{L}=\left(\hat{i}-5\hat{j}-2\hat{k}\right).\left(3\hat{i}+\hat{j}-\hat{k}\right)$

$\vec{p}.\vec{L}=(1)(3)+(-5)(1)+(-2)(-1)$

$\Rightarrow \vec{p}.\vec{L}=3-5+2=0$

Hence this given plane contain the given line.

26. A fair dice is thrown two times. Find the probability distribution of the number of sixes. Also determine the mean of the number of sixes. 2 Marks

Ans: The dice is thrown twice.

Therefore, the sample space is

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Therefore, no. of sample with 0 sixes =25

No. of sample with 1 sixes =10

no. of sample with 2 sixes =1

X	0 Sixes	1 Six	2 Sixes
p(X)	25	10	1

Now, Mean $=\sum X.P(X)$

$=0\times\frac{25}{36}+1\times\frac{10}{36}+2\times\frac{1}{36}$

$=\frac{10}{36}+\frac{2}{36}$

$=\frac{12}{36}$

$=\frac{1}{3}$

SECTION - C

Question numbers 27 to 32 carry 4 marks each.

27. Solve the following differential equation:

$\left(1-e^{\frac{y}{x}}\right)dy+e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx=0\left(x\neq 0\right).$ **4 Marks**

Ans: Given, $\left(1-e^{\frac{y}{x}}\right)dy+e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx=0$

$$\Rightarrow\left(1-e^{\frac{y}{x}}\right)dy=-e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx$$

$$\Rightarrow\frac{dy}{dx}=\frac{-e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{\left(1-e^{\frac{y}{x}}\right)}---(1)$$

Now, let, $\frac{dy}{dx}=F\left(x,y\right)$

$$\therefore\frac{dy}{dx}=F\left(x,y\right)=\frac{-e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{\left(1-e^{\frac{y}{x}}\right)}$$

Now, $F\left(\lambda x,\lambda y\right)=\frac{-e^{\frac{\lambda y}{\lambda x}}\left(1-\frac{\lambda y}{\lambda x}\right)}{\left(1-e^{\frac{\lambda y}{\lambda x}}\right)}$

$$F(\lambda x,\lambda y)=\frac{-e^{\frac{\lambda y}{\lambda x}}\left(1-\frac{\lambda y}{\lambda x}\right)}{\left(1-e^{\frac{\lambda y}{\lambda x}}\right)}=F(x,y)$$

So, $F(\lambda x,\lambda y)=F(x,y)=\lambda^{o}F(x,y)$

Thus, F(x, y) is a homogeneous function.

Therefore, the given differential equation is a homogeneous differential equation.

Now, let, $y=xv$

$$\Rightarrow \frac{dy}{dx}=v+x.\frac{dv}{dx}$$

Now, substituting these values in (1), we get,

$$\Rightarrow \frac{dy}{dx}=\frac{-e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{\left(1-e^{\frac{y}{x}}\right)}$$

$$\Rightarrow v+x.\frac{dy}{dx}=\frac{-e^{v}(1-v)}{(1+e^{v})}$$

$$\Rightarrow x.\frac{dy}{dx}=\frac{-e^{v}(1-v)}{(1+e^{v})}-v$$

$$\Rightarrow x.\frac{dy}{dx}=\frac{-e^{v}+ve^{v}}{(1+e^{v})}-v$$

$$x.\frac{dy}{dx}=\frac{-e^{v}+ve^{v}-v(1+e^{v})}{(1+e^{v})}-v$$

$$\Rightarrow x.\frac{dy}{dx}=\frac{-e^{v}+ve^{v}-v-ve^{v}}{\left(1+e^{v}\right)}-v$$

$$\Rightarrow x.\frac{dy}{dx}=\frac{-e^{v}-v}{\left(1+e^{v}\right)}$$

Now, by method of substitution of differential equation, we get,

$$\Rightarrow \frac{1+e^{v}}{v+e^{v}}dv=-\frac{dx}{x}$$

Now, integrating both sides,

$$\Rightarrow \int\frac{1+e^{v}}{v+e^{v}}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{1+e^{v}}{v+e^{v}}dv=-\log|x|+\log c$$

Now, putting $v+e^{v}=t$

$$\Rightarrow \left(1+e^{v}\right)dv=dt$$

Thus, our equation becomes,

$$\Rightarrow \int\frac{dt}{t}=-\log|x|+\log c$$

$$\Rightarrow \log|t|=-\log|x|+\log c$$

Putting back, $t=v+e^{v}$, we get,

$$\Rightarrow \log\left|v+e^{v}\right|=-\log|x|+\log c$$

$$\Rightarrow \log\left|v+e^{v}\right|+\log|x|=\log c$$

$$\Rightarrow \log\left|\left(v+e^{v}\right).|x|\right|=\log c$$

$\Rightarrow \log\left|\left(v+e^{v}\right).x\right|=\log c$

$\Rightarrow \log\left|vx+e^{v}x\right|=\log c$

$\Rightarrow vx+e^{v}x=c$

Putting back, $\Rightarrow y=vx \Rightarrow v=\frac{y}{x}$, we get,

$\Rightarrow y+e^{\frac{y}{x}}x=c$

28. A cottage industry manufactures pedestal lamps and wooden shades. Both the products require machine time as well as craftsman time in the making. The number of hour(s) required for producing 1 unit of each and the corresponding profit is given in the following table:

Item	Machine Time	Craftsman Time	Profit(in ₹)
Pedestal lamp	**1.5 hours**	**3 hours**	**30**
Wooden shades	**3hours**	**1 hours**	**20**

In a day, the factory has availability of not more than 42 hours of machine time and 24 hours of craftsman time. Assuming that all items manufactured are sold, how should the manufacturer schedule his daily production in order to maximise the profit? Formulate it as an LPP and solve it graphically. **4 Marks**

Ans: Let number of pedestal lamps =x

Number of wooden shades =y

Maximize Profit: P=30x+20y

According to the question:

$1.5x+3y\leq 42$

$\Rightarrow \frac{x}{28}+\frac{y}{14}\leq 1$

Therefore, the intercepts will be (28, 0), (0, 14).

$3x+y \le 24$

$\Rightarrow \frac{x}{8}+\frac{y}{24} \le 1$

Therefore, the intercepts will be $(8,0),(0,24)$.

$x \ge 0$, $y \ge 0$

Check profit at Corner points

At C(0,0),

P30(0)+20(0)=0

At B(0. 14),

P=30(0)+20(14)=280

At F(4,12),

P=30(4)+20(12)=360 [Max]

At D(8,0),

P=30(8)+20(0)=240

Maximum profit = Rs 360 at (number of pedestal lamps) x=4 and (Number of wooden shades) y=12.

29. Evaluate the value of integral: $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx$ **4 Marks**

Ans: Given, $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)dx$

$= \int_0^{\frac{\pi}{2}} \sin 2x \cos x \tan^{-1}(\sin x)dx$

Let, sin x= t

Differentiating both sides w.r.t x

Cosx dx=dt ,

x	0	$\frac{\pi}{2}$
t = sinx	Sin 0 =0	$\sin\left(\frac{\pi}{2}\right)=1$

Substituting x and dx, we get,

$=\int_0^1 2t \tan^{-1}(t)dt$

$=2\int_0^1 t \tan^{-1}(t)dt$

Now, using integration by parts, with function 1 as tan-1t and function 2 as t, we get,

$=2\left[\tan^{-1}t\int tdt-\int\left\{\frac{d\left(\tan^{-1}t\right)}{dt}\int tdt\right\}dt\right]$

$=2\left[\tan^{-1}t\int t\left\{\frac{t^2}{2}\right\}-\int\left\{\left(\frac{1}{t^2+1}\right)\left(\frac{t^2}{2}\right)\right\}dt\right]$

$=2\left[\frac{t^2}{2}\tan^{-1}t-\frac{1}{2}\int\left(\frac{t^2}{t^2+1}\right)dt\right]$

$=t^2\tan^{-1}t-\int\left(\frac{t^2}{t^2+1}\right)td$

Let $I_1=\int\frac{t^2}{t^2+1}dt$.

$$\Rightarrow I_1 = \int \frac{t^2 + 1 - 1}{t^2 + 1} dt$$

$$\Rightarrow I_1 = \int dt - \int \frac{1}{1+t^2} dt$$

$$\Rightarrow I_1 = t - \tan^{-1}$$

Thus, our equation becomes,

$$= t^2 \tan^{-1} t \left[t - \tan^{-1} t \right]$$

$$= t^2 \tan^{-1} t - t + \tan^{-1} t$$

Now, $2\int_0^1 t \tan^{=1}(t) dt = \left[t^2 \tan^{-1} t - t + \tan^{-1} t \right]_0^1$

$$2\int_0^1 t \tan^{=1}(t) dt = \left[1^2 \tan^{-1} 1 - 1 - + \tan^{-1} 1 \right] - \left[0 - 0 + \tan^{-1} 0 \right]$$

$$\Rightarrow 2\int_0^1 t \tan^{=1}(t) dt = \left[\frac{\pi}{4} - 1 + \frac{\pi}{4} \right] - 0$$

$$\Rightarrow 2\int_0^1 t \tan^{=1}(t) dt = \frac{\pi}{2} - 1$$

Therefore, $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx = \frac{\pi}{2} - 1.$

30. Check whether the relation R in the set N of natural numbers given by R={(a,b):a is a divisor of b} is reflexive, symmetric or transitive. Also determine whether R is an equivalence relation. 4 Marks

Ans: Reflexivity:

Let there be a natural number n ,

We know that n divides n, which implies nRn.

So, Every natural number is related to itself in relation R.

Thus, relation R is reflexive .

Transitivity:

Let there be three natural numbers a,b,c and let aRb, bRc

aRb implies a divides a and bRc implies b divides c, which as combined implies that a divides c i.e. aRc.

So, Relation R is also transitive.

Symmetry:

Let there be two natural numbers a,b and let aRb,

aRb implies a divides b but it can't be assured that b necessarily divides a.

For ex, 2R4 as 2 divides 4 but 4 does not divide 2 .

Thus Relation R is not symmetric.

Hence, the relation is not an equivalence relation.

Or

Prove that $\mathbf{\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\sin^{-1}\frac{4}{5}}$**.** **4 Marks**

Ans: To Prove, $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\sin^{-1}\frac{4}{5}$

LHS= $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}$

$$=\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\tan^{-1}\left(\frac{\frac{17}{36}}{\frac{34}{36}}\right)$$

$$=\tan^{-1}\left(\frac{1}{2}\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{2.\frac{1}{2}}{1+\frac{1}{4}}\right)$$

$$\left[\because \tan^{-1}x=\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right]$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{1}{\frac{5}{4}}\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$$

$$\text{RHS}= =\frac{1}{2}\sin^{-1}\frac{4}{5}$$

Hence, LHS = RHS.

31. Find the equation of the plane passing through the points (1,0,-2), (3,-1,0) and perpendicular to the plane 2x-y+z=8 . Also find the distance of the plane thus obtained from the origin. 4 Marks

Ans: Given points, P(1,0,-2,), Q(3,-1,0)

Given plane, 2x-y+z=8.

Normal vector of given plane, $\vec{n}_1=2\hat{i}-\hat{j}+\hat{k}$

Now, $\overrightarrow{PQ}=2\hat{i}-\hat{j}+2\hat{k}$

Normal vector of required plane,

$\vec{n}_2=\left(\vec{n}_1\times\overrightarrow{PQ}\right)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 2 & -1 & 2\end{vmatrix}$$

$=-1.\hat{i}-2\hat{j}+0\hat{k}$

$=-\hat{i}-2\hat{j}$

Required equation of plane,

-1(x-1)+(-2)(y-0)+0(z-2)=0

-x+1-2y+0=0

x+2y=1

Therefore, the required equation of the plane is, x+2y=1.

Now, distance from origin (0,0,0) is,

$$=\left|\frac{a_1.a+b_1.b+c_1.c+d}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{0.1+0.2+0.0+(-1)}{\sqrt{1^2 2^2+0^2}}\right|$$

$$=\left|\frac{-1}{\sqrt{1+4}}\right|$$

$= \frac{1}{\sqrt{5}}$

Therefore, distance from origin is $\frac{1}{\sqrt{5}}$ units.

32. If $\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}$, prove that $\frac{dy}{dx}=\frac{x+y}{x-y}$. **4 Marks**

Ans: Given, $\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\log\left(x^2+y^2\right)$

Now, differentiating both sides w.r.t x,

$$\Rightarrow \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right)=\frac{1}{2}.\frac{1}{x^2+y^2}.\frac{d}{dx}\left(x^2+y^2\right)$$

$$\Rightarrow \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{x\frac{d}{dx}-y}{x^2}\right)=\frac{1}{2}.\frac{1}{x^2+y^2}.\frac{d}{dx}\left(2x+2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{x^2}{x^2+y^2}.\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)=\frac{1}{2}.\frac{2\left(x+y\frac{dy}{dx}\right)}{x^2+y^2}$$

$$\Rightarrow x+y\frac{dy}{dx}=x\frac{dy}{dx}-y$$

$$\Rightarrow x\frac{dy}{dx}-y\frac{dy}{dx}=x+y$$

$$\Rightarrow \frac{dy}{dx}(x-y)=(x+y)$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y}{x-y}$$

Hence, proved.

Or

If $y=e^{acos^{=1}x}$, -1<x<1, then show that

$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-ay^2=0$. **4 Marks**

Ans: Given curve is $y=e^{acos^{=1}x}$

Differentiating given curve,

$$y'=e^{acos^{-1}x}.\frac{d}{dx}\left(acos^{-1}x\right)$$

$$y'=e^{acos^{-1}x}.\frac{(-a)}{\sqrt{1-x^2}}$$

$$\Rightarrow y'=\frac{-ay}{\sqrt{1-x^2}}--(1)$$

$$\left[\therefore y=e^{acos^{-1}x}\right]$$

On differentiating above equation again w.r.t x, we get

$$\Rightarrow y^n=\frac{-a\left(-ae^{acos^{-1}x}+\frac{x.e^{acos^{-1}x}}{\sqrt{1-x^2}}\right)}{\left(1-x^2\right)}$$

$$\Rightarrow \left(1-x^2\right)y^n=-a\left(-ae^{acos^{-1}x}+\frac{x.e^{acos^{-1}x}}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \left(1-x^2\right)y^n = a^2 e^{a\cos^{-1}x} - \frac{a.e^{a\cos^{-1}x}}{\sqrt{1-x^2}}$$

$$\Rightarrow \left(1-x^2\right)y^n = a^2 y + xy'$$

[from (1)]

$$\Rightarrow \left(1-x^2\right)y^n - xy' - a^2 y = 0$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0$$

Hence, proved.

SECTION - D

Question numbers 33 to 36 carry 6 marks each.

33. Amongst all open (from the top) right circular cylindrical boxes of volume 125π cm3, find the dimensions of the box which has the least surface area. 6 Marks

Ans: Given that the volume of the right circular cylindrical box $V=125\pi\ cm^3$.

Let the radius of the cylinder be h and the height be equal to h.

Volume, $V=\pi r^2 h$

$$\Rightarrow 125\pi = \pi r^2 h$$

$$\Rightarrow r^2 h = 125$$

$$\Rightarrow h = \frac{125}{r^2}$$

Surface area of the box, $S=\pi rh+\pi r^2$

$$S=\pi r\left(\frac{125}{r^2}\right)+\pi r^2$$

$S=\left(\frac{250}{r}\right)+\pi r^2$

Differentiating S w.r.t r to find the point of minima,

$\Rightarrow \frac{dS}{dr}=\frac{-250\pi}{r^2}+2\pi r$

Therefore, for the point of minima,

$\Rightarrow \frac{dS}{dr}=0$

$\Rightarrow \frac{-250\pi}{r^2}+2\pi r=0$

$\Rightarrow 2\pi r=\frac{250\pi}{r^2}$

$\Rightarrow r^3=125$

$\Rightarrow r = 5cm$

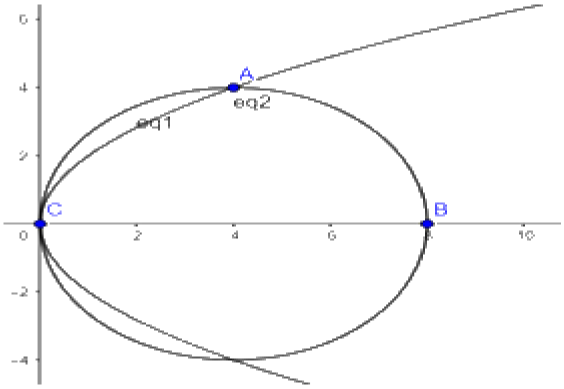
[Only positive value will be considered, as length can't have negative value]

Now, $h=\frac{125}{r^2}=\frac{125}{25}=5\,cm$

The dimension of the cylindrical box is radius, r=5 cm and height, h=5cm.

34. Using integration, find the area lying above x-axis and included between the circle $x^2+y^2=8x$ and inside the parabola $y^2=4x$. **6 Marks**

Ans:



The given equations are

$x^2+y^2=8x$ -------(1)

$y^2=4x$ -----(2)

From (1),

$x^2-8x+y^2=0$

$\Rightarrow x^2-2.5x+16+y^2=16$

$\Rightarrow (x-4)^2+y^2=(4)^2$ ------------(3)

Therefore, the equation (1) is a circle with centre (4,0) and has a radius 4.

Also, $y^2=4x$ is a parabola with vertex at origin and the axis along the x-axis opening in the positive direction.

To find the intersection points of the curves, we solve both the equation.

$\therefore x^2+4x=8x$

$x^2-4x=0$

X(x-4)=0

x=0 and x=4

When, $x=4, y=\pm 4$.

But since, it is given above the x-axis.

So, y=4.

Therefore, area, $A=\int_0^4 |y_2-y_1| dx$

$$=\int_0^4 (y_2-y_1) dx$$

$[\therefore y_2>y_1]$

$$=\int_0^4 \left[\sqrt{\left(16-(x-4)^2\right)}-2\sqrt{x}\right] dx$$

[from (2) and (3)]

$$=\int_0^4 \left[\sqrt{\left(16-(x-4)^2\right)}\right] dx-\int_0^4 -2\sqrt{x} dx$$

$$\left[\frac{(x-4)}{2}\sqrt{16-(x-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_0^4-\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$\left[\frac{(x-4)}{2}\sqrt{16-(x-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_0^4-\frac{4}{3}\left[x^{\frac{3}{2}}\right]_0^4$$

$$=\left[\left(\frac{(4-4)}{2}\sqrt{16-(4-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{4-4}{4}\right)\right)-\left(\frac{(0-4)}{2}\sqrt{16-(0-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{0-4}{2}\right)\right)\right]+\frac{4}{3}\left[3^{\frac{3}{2}}-0^{\frac{3}{2}}\right]$$

$$=\left[\left(\frac{(0)}{2}\sqrt{16-0}+8\sin^{-1}(0)\right)\left(-2\sqrt{16-16}+8\sin^{-1}(-1)\right)\right]-\frac{4}{3}\left[4^{\frac{3}{2}}\right]$$

$$=\left[(0+0)-\left(0-8\sin^{-1}(1)\right)\right]-\frac{4}{3}\left[4^{\frac{3}{2}}\right]$$

$=8\frac{\pi}{2}-\frac{4}{3}.2^3$

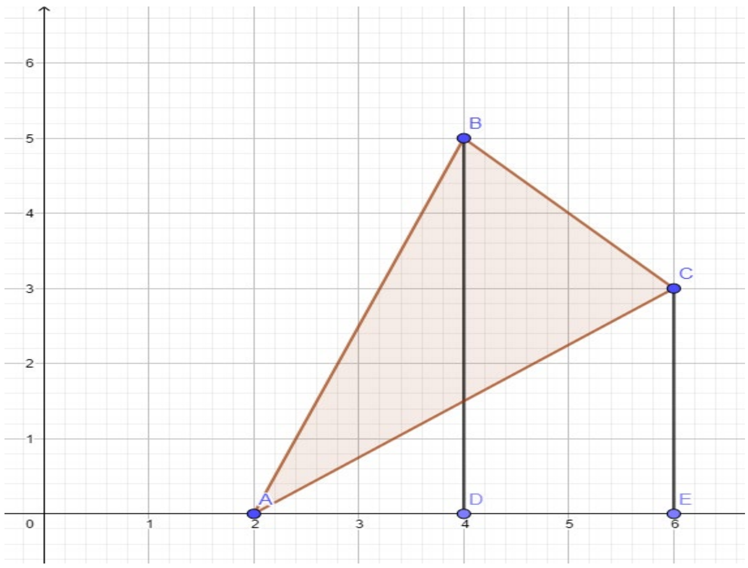
$=4\pi-\frac{32}{3}$

Hence, the required area of the region is $4\pi-\frac{32}{3}$ sq. units.

Or

Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are A(2,0), B(4,5) and C(6,3). 6 Marks

Ans: The vertices of Δ ABC are A(2,0), B(4,5) and C(6,3).



Equation of line segment AB is

$y-0=\left(\frac{5-0}{4-2}\right)(x-2)$

$\Rightarrow y=\left(\frac{5}{2}\right)(x-2)---(1)$

Equation of line segment BC is

$$y-5=\left(\frac{3-5}{6-4}\right)(x-4)$$

$$\Rightarrow y-5=\left(\frac{-2}{2}\right)(x-4)$$

$$\Rightarrow y-5=-x+4$$

$$\Rightarrow y=-x+4-------(2)$$

Equation of line segment CA is

$$\Rightarrow y-3=\left(\frac{0-3}{2-6}\right)(x-6)$$

$$\Rightarrow y-3=\left(\frac{-3}{4}\right)(x-6)$$

$$\Rightarrow 4(y-3)=3(x-6)$$

$$\Rightarrow 4y-12=3x-18$$

$$\Rightarrow 4y=3x-6$$

$$\Rightarrow y=\frac{3}{4}(x-2)---(3)$$

Area(Δ ABC)=Area(ABDA)+Area(BDECB)-Area(AECA)

$$=\frac{5}{2}\int_2^4 (x-2)\,dx+\int_4^6 (-x+9)\,dx-\int_2^6 \frac{3}{4}(x-2)dx$$

$$=\frac{5}{2}\int_2^4 (x-2)\,dx+\int_4^6 (-x+9)\,dx-\frac{3}{4}\int_2^6 (x-2)dx$$

$$=\frac{5}{2}\left[\frac{x^2}{2}-2x\right]_2^4+\left[\frac{-x^2}{2}+9x\right]_4^6-\frac{3}{4}\left[\frac{x^2}{2}-2x\right]_2^6$$

$=\frac{5}{2}[8-8-2+4]+[-18+54+8-36]-\frac{3}{4}[18-12-2+4]$

$=\frac{5}{2}[2]+[8]-\frac{3}{4}[8]$

$= 5+8-6$

$= 7$ sq. units

Therefore, the area of the triangle is 7 sq. units.

35. If $A=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$, find A^{-1} and use it to solve the following system of equations:

5x-y+4z=5

2x+3y+5z=2

5x-2y+6z=-1 **6 Marks**

Ans: Given, $A=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$.

Now, $|A|=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$

$\Rightarrow |A|=5(18+10)+1(12-25)+4(-4-15)$

$\Rightarrow |A|=140-13-76$

$\Rightarrow |A|=51$

Now, we have to find the cofactor matrix.

$=\left[A_{ij}\right]_{3\times 3}$, where, $A_{ij}=(-1)^{i+j}M_{ji}$

$$A_{11}=(-1)^{1+1}M_{11}=\begin{vmatrix}3 & 5\\ -2 & 6\end{vmatrix}=18+10=82$$

$$A_{12}=(-1)^{1+2}M_{12}=\begin{vmatrix}2 & 5\\ 5 & 6\end{vmatrix}=-(12-25)=13$$

$$A_{13}=(-1)^{1+3}M_{13}=\begin{vmatrix}2 & 3\\ 5 & -2\end{vmatrix}=-4-15=-19$$

$$A_{21}=(-1)^{2+1}M_{21}=\begin{vmatrix}-1 & 4\\ -2 & 6\end{vmatrix}=-(6+8)=-2$$

$$A_{22}=(-1)^{2+2}M_{22}=\begin{vmatrix}5 & 4\\ 5 & 6\end{vmatrix}=30-20=10$$

$$A_{23}=(-1)^{2+3}M_{23}=\begin{vmatrix}5 & -1\\ 5 & -2\end{vmatrix}=-(-10+5)=5$$

$$A_{31}=(-1)^{3+1}M_{31}=\begin{vmatrix}5 & 4\\ 2 & 5\end{vmatrix}=-5-12=-17$$

$$A_{32}=(-1)^{3+2}M_{32}=-\begin{vmatrix}5 & 4\\ 2 & 5\end{vmatrix}=-(25-8)=-17$$

$$A_{33}=(-1)^{3+3}M_{33}=\begin{vmatrix}5 & -1\\ 2 & 3\end{vmatrix}=15+2=17$$

Therefore, the cofactor matrix is,

$$\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}=\begin{bmatrix}28 & 13 & -19\\ -2 & 10 & 5\\ -17 & -17 & 17\end{bmatrix}$$

$$\therefore \text{adj}A=\begin{bmatrix}28 & 13 & -19\\ -2 & 10 & 5\\ -17 & -17 & 17\end{bmatrix}$$

$\therefore A^{-1}=\frac{1}{|A|}.\text{adj}A$

$$=\frac{1}{51}\begin{bmatrix} 28 & 13 & -19 \\ -2 & 10 & 5 \\ -17 & -17 & 17 \end{bmatrix}$$

Now, given set of equations is,

5x-y+4z=5

2x+3y+5z=2

5x-2y+6z=-1

The equations can be written in matrix form as,

$$\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

This is of the form AX=B, where

$$A=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix},X=\begin{bmatrix} x \\ y \\ z \end{bmatrix},B=\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

Now, multiplying AX=B by A-1, we get,

A-1(AX)=A-1B

A-1(AX)=A-1B

IX=A-1B

Now, substituting the values, we get,

$$\Rightarrow X=\frac{1}{51}\begin{bmatrix} 28 & 13 & -19 \\ -2 & 10 & 5 \\ -17 & -17 & 17 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow X=\frac{1}{51}\begin{bmatrix}140-4+17\\65+20+17\\-95+10-17\end{bmatrix}$$

$$\Rightarrow X=\frac{1}{51}\begin{bmatrix}153\\102\\-102\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\2\\-2\end{bmatrix}$$

Therefore, by equality of matrices.

x=3, y=2, z=-2

This is the required solution.

Or

If x,y,z are different and $\begin{vmatrix}x & x^2 & 1+x^3\\y & y^2 & 1+y^3\\z & z^2 & 1+z^3\end{vmatrix}=0$**, then using properties of determinants show that 1+xyz=0.** **6 Marks**

Ans: Given, $\begin{vmatrix}x & x^2 & 1+x^3\\y & y^2 & 1+y^3\\z & z^2 & 1+z^3\end{vmatrix}=0$.

Let, $\Delta=\begin{vmatrix}x & x^2 & 1+x^3\\y & y^2 & 1+y^3\\z & z^2 & 1+z^3\end{vmatrix}$

Now, expanding elements of C3 into two determinants,

$$=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

Taking x,y,z common from R1,R2,R3 in 2nd determinant,

$$=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Replacing $C_3 \leftrightarrow C_2$ in 1st determinant,

$$=(-1)\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Replacing $C_1 \leftrightarrow C_2$ in 1st determinant,

$$=(-1)(-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz)$$

Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$=\begin{vmatrix} 1 & x & x^2 \\ 1-1 & y-x & y^2-x^2 \\ 1-1 & z-x & z^2-x^2 \end{vmatrix}(1+xyz)$$

$$=\begin{vmatrix} 1 & x & x^2 \\ 1-1 & y-x & (y-x)(y+x) \\ 1-1 & z-x & (z-x)(z+x) \end{vmatrix}(1+xyz)$$

Taking common factor (y-x) from R2 and (z-x) from R3,

$$=\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & (y+x) \\ 0 & 1 & (z+x) \end{vmatrix}(1+xyz)(y-x)(z-x)$$

Expanding determinant through C1, we get,

$$\left[1\{(z+x)-(y-x)\}\right](1+xyz)(y-x)(z-x)$$

$$=[z-y](1+xyz)(y-x)(z-x)$$

$$=(1+xyz)(y-x)(z-x)(z-y)$$

Given, x,y,z are different.

Therefore, $(x-y) \neq (z-x) \neq (z-y) \neq 0$

Given, $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$(1+xyz)(y-x)(z-x)(z-y)=0$$

Since, $(x-y) \neq (z-x) \neq (z-y) \neq 0$

Therefore, $1+xyz=0$

Hence, proved.

36. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn randomly one-by-one without replacement and

are found to be both kings. Find the probability of the lost card being a king.
6 Marks

Ans: Let E1 be the event that the card is a king.

And, E2 be the event that the card is not a king.

Let A denote the lost card.

Out of 52 cards 4 are king and 48 are non-king.

Probability that the card is a king, $P(E_1)=\frac{4}{52}=\frac{1}{13}$

Probability that the card is not a king, $P(E_2)=\frac{48}{52}=\frac{12}{13}$

Two cards can be drawn out of 4 king in ${}^4C_2$ ways and 2 kings can be drawn out of 51 cards in ${}^{51}C_2$ ways.

Probability of getting two kings out of the remaining cards if the lost card is a king,

$$P\left(\frac{A}{E_1}\right)=\frac{{}^3C_2}{{}^{51}C_1}$$

$$\Rightarrow P\left(\frac{A}{E_1}\right)=\frac{3}{\frac{51.50}{2}}$$

$$\Rightarrow P\left(\frac{A}{E_1}\right)=\frac{1}{425}$$

Probability of getting two kings out of the remaining cards if the lost card is not a king,

$$\Rightarrow P\left(\frac{A}{E_2}\right)=\frac{{}^4C_2}{{}^{51}C_2}$$

$$\Rightarrow P\left(\frac{A}{E_2}\right)=\frac{\frac{4.3}{2}}{\frac{51.50}{2}}$$

$$\Rightarrow P\left(\frac{A}{E_2}\right)=\frac{2}{425}$$

Therefore, probability of getting two cards when on lost card is king, is,

$$P\left(\frac{E_1}{A}\right)=\frac{\frac{1}{13}.\frac{1}{425}}{\frac{1}{13}.\frac{1}{425}+\frac{12}{13}.\frac{2}{425}}$$

$$P\left(\frac{E_1}{A}\right)=\frac{\frac{1}{13}.\frac{1}{425}}{\frac{1}{13}.\frac{1}{425}+\frac{12}{13}.\frac{2}{425}}$$

$$P\left(\frac{E_1}{A}\right)=\frac{1}{25}$$

Probability that the lost card is a king is $\frac{1}{25}$.