PREVIOUS YEAR QUESTION PAPER 2019

General Instructions:

(i) ***All*** *questions are compulsory.*
(ii) *This question paper contains* ***29*** *questions divided into four sections A, B, C and D. Section A comprises of 4 questions of* ***one mark*** *each, Section B comprises of* ***8*** *questions of* ***two marks*** *each, Section C comprises of 11 questions of* ***four marks*** *each and Section D comprises of* ***6*** *questions of* ***six marks*** *each.*
(iii) *All questions in Section* ***A*** *are to be answered in one word, one sentence or as per the exact requirement of the question.*
(iv) *There is no overall choice. However, internal choice has been provided in 1 question of Section* ***A****, 3 questions of Section* ***B****, 3 questions of Section* ***C*** *and 3 questions of Section* ***D****. You have to attempt only* ***one*** *of the alternatives in all such questions.*
(v) *Use of calculators is not permitted. You may ask logarithmic tables, if required.*

SECTION-A

1. If A is a square matrix of order 3 with $|A| = 4$ then write the value of $|-2A|$

Solution. Since, order of the matrix, $n = 3$

$|A| = 4$

$|-2A| = (-2)^n |A|$

$|-2A| = (-2)^3 \times 4$

$|-2A| = -32$

Therefore, the value of $|-2A|$ is -32

2. If $y = \sin^{-1}x + \cos^{-1}x$, find $\frac{dy}{dx}$

Solution.

$$y = \sin^{-1} x + \cos^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x + \cos^{-1} x\right)$$

$$= \frac{d}{dx}\left(\sin^{-1} x\right) + \frac{d}{dx}\left(\cos^{-1} x\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$= 0$$

Therefore, $\frac{dy}{dx} = 0$

3. Write the order and degree of the differential equation $\left(\frac{d^4y}{dx^4}\right)^2 = \left[x + \left(\frac{dy}{dx}\right)^2\right]^3$

Solution. Since,

$$\left(\frac{d^4 y}{dx^4}\right)^2 = \left[x + \left(\frac{dy}{dx}\right)^2\right]^3$$

$$\left(\frac{d^4 y}{dx^4}\right)^2 = x^3 + \left(\frac{dy}{dx}\right)^6 + 3x^2\left(\frac{dy}{dx}\right)^2 + 3x\left(\frac{dy}{dx}\right)^4$$

The highest power raised to $\frac{d^4 y}{dx^4}$ is 2 and degree of the differential equation is 2

4. If the line has the direction ratios -18,12,-4, then what are its direction cosines?

OR

Find the Cartesian equation of the line which passes through the point (-2,4,-5) is parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$

Solution.

The direction ratios of the lines are $-18, 12, -4$

Direction cosines of the lines are $-\frac{18}{\sqrt{18^2+12^2+4^2}}, \frac{12}{\sqrt{18^2+12^2+4^2}}, -\frac{4}{\sqrt{18^2+12^2+4^2}}$

Hence, direction cosine of line are $-\frac{9}{11}, \frac{6}{11}, -\frac{2}{11}$

OR

The cartesion equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$ is $\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$

SECTION - B

5. If * is defined on the set R of all real number by * : a*b = $\sqrt{a^2+b^2}$ find the identity element if exist in R with respect to *

Solution. As per the question

Let b be the identity element then

$a*b = b*a = a$

$a*b = \sqrt{(a)^2+(b)^2} = a$

$\Rightarrow (a)^2 + (b)^2 = (a)^2$

$\Rightarrow b = 0$

Similarly,

$b*a = \sqrt{(b)^2+(a)^2} = a$

$\Rightarrow (b)^2 + (a)^2 = (a)^2$

$\Rightarrow b = 0$

Therefore, 0 is the identity element

6. If A = $\begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and kA = $\begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$ then find the value of k,a and b

Solution. Given,

$$kA=\begin{bmatrix}0 & 3a\\ 2b & 24\end{bmatrix}\ (i)$$

$$A=\begin{bmatrix}0 & 2\\ 3 & -4\end{bmatrix},\text{implies } kA=\begin{bmatrix}0 & 2k\\ 3k & -4k\end{bmatrix}\ (ii)$$

$$\begin{bmatrix}0 & 2k\\ 3k & -4k\end{bmatrix}=\begin{bmatrix}0 & 3\text{a}\\ 2\text{b} & 24\end{bmatrix}$$

$-4\text{k}=24 \Rightarrow \text{k}=-6$

$3a=2k \Rightarrow a=-4$

$2b=3k \Rightarrow b=-9$

7. Find $\int \frac{\sin x-\cos x}{\sqrt{1+\sin 2x}}dx, 0<x<\pi/2$

Solution. According to question,

$$let\ I=\int \frac{\sin x-\cos x}{\sqrt{1+\sin 2x}}dx, 0<x<\frac{\pi}{2}$$

$$I=\int \frac{\sin x-\cos x}{\sqrt{\sin^2 x+\cos^2 x+2\sin x\cdot\cos x}}dx$$

$$=\int \frac{\sin x-\cos x}{\sqrt{(\sin x+\cos x)^2}}dx$$

$$=\int \frac{\sin x-\cos x}{\sin x+\cos x}dx$$

$let \sin x+\cos x=t$

$\Rightarrow (\cos x-\sin x)dx=dt$

$$I=\int \frac{-1}{t}dt$$

$$=-\ln t+C$$

$$=\ln\left(\frac{1}{t}\right)+C$$

$$\Rightarrow I=\ln\left(\frac{1}{\sin x+\cos x}\right)+C$$

8. Find $\int \frac{\sin(x-a)}{\sin(x+a)}dx$

OR

Find $\int (\log x)^2\, dx$

Solution

$$Let\ I = \int \frac{\sin(x-a)}{\sin(x+a)} dx$$

$$\Rightarrow I = \int \frac{\sin[(x+a)-2a]}{\sin(x+a)} dx$$

$$= \int \frac{\sin(x+a)\cdot\cos(2a) - \cos(x+a)\cdot\sin(2a)}{\sin(x+a)} dx$$

$$= \int \cos(2a)\,dx - \int \cot(x+a)\cdot\sin(2a)\,dx$$

$$= x\cdot\cos(2a) - \log|\sin(x+a)|\cdot\sin(2a) + C$$

OR

$$Let\ I = \int (\log x)^2\, dx$$

$$\Rightarrow I = \int 1\cdot(\log x)^2\, dx$$

$$\Rightarrow I = x\cdot(\log x)^2 - \int \frac{2x\log x}{x} dx$$

$$\Rightarrow I = x\cdot(\log x)^2 - I_1 + c_1 \qquad .....(\text{i})$$

$$I_1 = \int 2\cdot\log x dx$$

$$\Rightarrow I_1 = 2x\cdot\log x - 2\int \frac{x}{x} dx$$

$$\Rightarrow I_1 = 2x\cdot\log x - 2x + c_2 \qquad .....(ii)$$

$$I = x\cdot(\log x)^2 - 2x\cdot\log x + 2x + c_1 - c_2$$

$$I = x\cdot(\log x)^2 - 2x\cdot\log x + 2x + C \qquad (where\, C = c_1 - c_2)$$

9. From the differential equation representing the family of curves $y^2 = m(a^2 - x^2)$ by eliminating the arbitrary constant m and a

Solution

The equation $y^2 = m\left(a^2 - x^2\right)$ where m and a are arbitrary constants

$$y^2 = m\left(a^2 - x^2\right) \qquad .....(\text{i})$$

$$2y\frac{dy}{dx} = -2mx \qquad .....(ii)$$

$$\Rightarrow -2m = 2\frac{y}{x}\frac{dy}{dx}$$

$$2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = -2m \qquad .....(iii)$$

$$2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 2\frac{y}{x}\frac{dy}{dx}$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \left(\frac{y}{x}\right)\frac{dy}{dx} = 0$$

therefore the required differential equation is $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \left(\frac{y}{x}\right)\frac{dy}{dx} = 0$

10. Find the unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, where $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

OR

Show that the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\hat{i} - 3\hat{j} + 5\hat{k}$ are coplanner

Solution

$\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ *and* $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

let $\vec{n}$ be the vector perpendicular to $\vec{a}$ and $\vec{b}$

$$\vec{n} = \vec{a} \times \vec{b}$$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = 19\hat{j} + 19\hat{k}$$

$$\hat{n} = \frac{19\hat{j} + 19\hat{k}}{\sqrt{19^2 + 19^2}} = \frac{1}{\sqrt{2}}\left(\hat{j} + \hat{k}\right)$$

OR

let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$

$\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

$$= 1(15-12) + 2(-10+4) + 3(6-3)$$

$$= 3 - 12 + 9$$

$$= 0$$

therefore, $\vec{a}, \vec{b}, \vec{c}$ are coplanar

11. Mother, father and son line up at random for a family photo. If A and B are two events given by A = Son on one end, B = Father in the middle, find P(B/A).

Solution

If mother (M), father (F), and son (S) line up for the family picture, then the sample space will be

S = {MFS, MSF, FMS, FSM, SMF, SFM} = A= {MFS, FMS, SMF, SFM}

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$P(B) = \frac{2}{6} = \frac{1}{3}$$

$$P(A) = \frac{4}{6} = \frac{2}{3}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

12. Let X be a random variable which assumes values x1, x2, x3, x4 such that 2P(X = x1) = 3P(X = x2) = P(X = x3) = 5P(X = x4). Find the probability distribution of X.

OR

A coin is tossed 5 times. Find the probability of getting (i) at least 4 heads, and (ii) at most 4 heads.

Solution

$Let\ P(X = x_3) = x$

$P(X = x_1) = \frac{x}{2}$

$P(X = x_2) = \frac{x}{3}$

$P(X = x_4) = \frac{x}{5}$

$\sum_{i=1}^{4} P(x_i) = 1$

$P(x_1) + P(x_2) + P(x_3) + P(x_4) = 1$

$\frac{x}{2} + \frac{x}{3} + x + \frac{x}{5} = 1$

$x = \frac{30}{61}$

$P(X = x_1) = \frac{15}{61}; P(X = x_2) = \frac{10}{61}; P(X = x_3) = \frac{30}{61}; P(X = x_4) = \frac{6}{61}$

So, the probability distribution function will be

X	1	2	3	4
$P(X = x_i)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

OR

Total number of probability of tossing a coin 5 times is 32

(i) Probability of getting atleast 4 heads

$P(X=4)+P(X=5)$

${}^5C_4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 + {}^5C_5\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5$

$= {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5$

$=\frac{6}{32}=\frac{3}{16}$

(ii) probability of getting at most 4 head

$P(X=1)+P(X=2)+P(X=3)+P(X=4)$

${}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5$

$=\left(\frac{1}{2}\right)^5[5+10+10+5]$

$=\frac{15}{16}$

SECTION – C

14. If $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right), x>0$ **then find the value of x and hence find the value of** $\sec^{-1}\left(\frac{2}{x}\right)$

Solution

$$\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right),x>0$$

$$\Rightarrow \tan^{-1}x-\tan^{-1}\left(\frac{1}{x}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \left[\because\ \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right),\ x>0\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{x-\frac{1}{x}}{1+x\cdot\frac{1}{x}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \frac{x^2-1}{2x}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3}x^2-2x-\sqrt{3}=0$$

$$\Rightarrow \sqrt{3}x^2-3x+x-\sqrt{3}=0$$

$$\Rightarrow \sqrt{3}x\left(x-\sqrt{3}\right)+1\left(x-\sqrt{3}\right)=0$$

$$\Rightarrow \left(x-\sqrt{3}\right)\left(\sqrt{3}x+1\right)=0$$

$$\Rightarrow x=-\frac{1}{\sqrt{3}},\sqrt{3}$$

$$\because\ x>0,\ x=\sqrt{3}$$

$$\Rightarrow \sec^{-1}\left(\frac{2}{x}\right)=\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{2}{x}\right)=\sec^{-1}\left(\sec\frac{\pi}{6}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{2}{x}\right)=\frac{\pi}{6}$$

15. Using properties of determinant prove that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}=4abc$$

Solution

$$Let\,\Delta=\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$R_1 \to R_1 - R_2 - R_3$

$$\Delta=\begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expending R_1

$$\Delta=0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix}-(-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix}+(-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$=2c\left(ab+b^2-bc\right)-2b\left(bc-c^2-ac\right)$$

$$=2abc+2cb^2-2bc^2-2b^2c+2bc^2+2abc$$

$$=4abc$$

16. If $\left(\sin x\right)^y = x+y,\ find\ \dfrac{dy}{dx}$

Solution

$$\left(\sin x\right)^y = x+y$$

$$\log\left(\sin x\right)^y = \log\left(x+y\right)$$

$$\Rightarrow y\log\left(\sin x\right) = \log\left(x+y\right) \qquad .....(\text{i})$$

$$\log\left(\sin x\right)\cdot\frac{dy}{dx}+y\cdot\frac{d}{dx}\left[\log\left(\sin x\right)\right]=\frac{d}{dx}\left[\log\left(x+y\right)\right]$$

$$\Rightarrow \log\left(\sin x\right)\cdot\frac{dy}{dx}+y\cdot\frac{\cos x}{\sin x}=\frac{1}{\left(x+y\right)}\cdot\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left[\log\left(\sin x\right)-\frac{1}{\left(x+y\right)}\right]=\frac{1}{\left(x+y\right)}-y\cdot\cot x$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-\left(xy+y^2\right)\cdot\cot x}{\left(x+y\right)\cdot\log\left(\sin x\right)-1}$$

17. If $y=\left(\sec^{-1}x\right)^2, x>0\, show that\, x^2\left(x^2-1\right)\dfrac{d^2y}{dx^2}+\left(2x^3-x\right)\dfrac{dy}{dx}-2=0$

Solution

$y=\left(\sec^{-1}x\right)^2, x>0$

$\Rightarrow \frac{dy}{dx}=2\sec^{-1}x\cdot\frac{d\left(\sec^{-1}x\right)}{dx}$

$\Rightarrow \frac{dy}{dx}=2\sec^{-1}x\cdot\frac{1}{x\sqrt{x^2-1}}$ .....(i)

$$\Rightarrow \frac{d^2y}{dx^2}=2\left[\frac{1}{x^2\left(x^2-1\right)}\right]+2\sec^{-1}x\left[\frac{-\sqrt{x^2-1}-x\left(\frac{2x}{2\sqrt{x^2-1}}\right)}{x^2\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2}=2\left[\frac{1}{x^2\left(x^2-1\right)}\right]+2\sec^{-1}x\cdot\frac{1}{x\sqrt{x^2-1}}\left[\frac{x\left(1-2x^2\right)}{x^2\left(x^2-1\right)}\right] \quad .....(ii)$$

$$\frac{d^2y}{dx^2}=2\left[\frac{1}{x^2\left(x^2-1\right)}\right]+\frac{dy}{dx}\left[\frac{x\left(1-2x^2\right)}{x^2\left(x^2-1\right)}\right]$$

$$\Rightarrow x^2\left(x^2-1\right)\frac{d^2y}{dx^2}+\left(2x^3-x\right)\cdot\frac{dy}{dx}-2=0$$

18. Find the equation of a tangent and the normal to the curve $y=\frac{(x-7)}{(x-2)(x-3)}$ at the point where it cuts the x-axis

Solution

Equation of the curve is

$$y=\frac{(x-7)}{(x-2)(x-3)}$$

put y=0 in the above equation we get x=7

$$\frac{dy}{dx}=\frac{(x-2)\cdot(x-3)-(x-7)\cdot(2x-5)}{(x-2)^2\cdot(x-3)^2}$$

The slope of the tangent at point $(7,0)$ *is*

$$m_t=\left.\frac{dy}{dx}\right|_{(7,0)}=\frac{20}{400}=\frac{1}{20}$$

$$(y-0)=\frac{1}{20}(x-7)\Rightarrow x-20y-7=0$$

$$m_t\cdot m_n=-1$$

$$\Rightarrow m_n=\frac{-1}{1/20}=-20$$

Equation of the normal is

$$(y-0)=-20(x-7)\Rightarrow 20x+y-140=0$$

19. Find $\int\frac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}dx$

Solution

$$\int \frac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}dx$$

$$\Rightarrow I=\int \frac{2\sin x\cdot\cos x}{(\sin^2 x+1)(\sin^2 x+3)}dx$$

$let \sin^2 x+3=t \Rightarrow 2\sin x\cdot\cos x dx=dt$

Therefore,

$$I=\int \frac{dt}{(t-2)t}$$

$$\Rightarrow I=\frac{1}{2}\int\left(\frac{1}{t-2}-\frac{1}{t}\right)dt$$

$$\Rightarrow I=\frac{1}{2}\left[\ln(t-2)-\ln t\right]+c$$

$$\Rightarrow I=\frac{1}{2}\ln\left(\frac{t-2}{t}\right)+c$$

$$\Rightarrow I=\ln\sqrt{\frac{t-2}{t}}+c$$

$$\Rightarrow I=\ln\sqrt{\frac{\sin^2 x+1}{\sin^2 x+3}}+c$$

20. Prove that $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$ and hence evaluate $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$

Solution

$let\, a+b-x=t$

$\Rightarrow dx=-dt$

$when\, x=a, t=b\, and\; x=b, t=a$

$$\int_a^b f(x)\,dx=-\int_b^a f(a+b-t)\,dt$$

$$=\int_a^b f(a+b-t)\,dt \qquad \left[\because \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx\right]$$

$$=\int_a^b f(a+b-x)\,dx \qquad \left[\because \int_a^b f(x)\,dx=\int_a^b f(t)\,dt\right]$$

$$let\; I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{1+\sqrt{\tan x}}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}}$$

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}\,dx}{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}} \qquad .....(iii)$$

$$2I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}dx=[x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

$$\Rightarrow I=\frac{\pi}{12}$$

21. Show that $\left(1+x^2\right)dy+2xydx=\cot x dx$

Solution

$\left(1+x^2\right)dy+2xydx=\cot xdx$

$\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\cot x}{1+x^2}$

The Linear differential equation is

IF=$e^{\int pdx}=e^{\int \frac{2x}{1+x^2}}=1+x^2$

the general solution is

$y\left(1+x^2\right)=\int\left[\frac{\cot x}{1+x^2}\left(1+x^2\right)\right]dx+C$

$=y\left(1+x^2\right)=\log\left[\sin x\right]+C$

22. let $\vec{a},\vec{b},\vec{c}$ be the three vectors such that $|\vec{a}|=1, |\vec{b}|=2, |\vec{c}|=3.$ If the projection of $\vec{a}$ and $\vec{b}$ is equal to the projection of $\vec{c}$ along $\vec{a}$ and $\vec{b},\vec{c}$ are perpendicular to each other then find $\left|3\vec{a}-2\vec{b}+2\vec{c}\right|$

Solution

$|\vec{a}|=1, |\vec{b}|=2, |\vec{c}|=3$

the projection of $\vec{b}$ along $\vec{a}=\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}$

the projection of $\vec{c}$ along $\vec{a}=\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|}$

$\Rightarrow\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}=\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|}$

$\Rightarrow\vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{a}$ .....(i)

$\left(3\vec{a}-2\vec{b}+2\vec{c}\right)\cdot\left(3\vec{a}-2\vec{b}+2\vec{c}\right)=9|\vec{a}|^2-6\vec{a}\cdot\vec{b}+6\vec{a}\cdot\vec{c}-6\vec{b}\cdot\vec{a}+4|\vec{b}|^2-4\vec{b}\cdot\vec{c}+6\vec{c}\cdot\vec{a}-4\vec{c}\cdot\vec{b}+4|\vec{c}|^2$

$\left|3\vec{a}-2\vec{b}+2\vec{c}\right|^2=9|\vec{a}|^2+4|\vec{b}|^2+4|\vec{c}|^2-12\vec{a}\cdot\vec{b}+12\vec{a}\cdot\vec{c}-8\vec{b}\cdot\vec{c}$

$\left|3\vec{a}-2\vec{b}+2\vec{c}\right|^2=9|\vec{a}|^2+4|\vec{b}|^2+4|\vec{c}|^2$

$\Rightarrow\left|3\vec{a}-2\vec{b}+2\vec{c}\right|^2=9\times1+4\times4+4\times9=61$

$\Rightarrow\left|3\vec{a}-2\vec{b}+2\vec{c}\right|=\sqrt{61}$

SECTION – D

23. Find the value of λ for which the following lines are perpendicular to each other

$$\frac{x-5}{5\lambda+2}=\frac{2-y}{5}=\frac{1-z}{-1};\frac{x}{1}=\frac{y+\frac{1}{2}}{2\lambda}=\frac{z-1}{3}$$

hence, find whether the lines intersect or not

Solution

$$\frac{x-5}{5\lambda+2}=\frac{y-2}{-5}=\frac{z-1}{1} \qquad ...(1)$$

and

$$\frac{x}{1}=\frac{y+\frac{1}{2}}{2\lambda}=\frac{z-1}{3} \qquad ...(2)$$

$a_1=5\lambda+2,\ b_1=-5,\ c_1=1$ *and*

$a_2=1,\ b_2=2\lambda,\ c_2=3$

$a_1a_2+b_1b_2+c_1c_2=0$

$(5\lambda+2)-5(2\lambda)+1(3)=0$

$-5\lambda+5=0$

$\Rightarrow \lambda=-1$

24. If $\mathbf{A}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 1 & -2 & 1\end{bmatrix}$**, find** $\mathbf{A^{-1}}$

hence, solve the following system of equations

$\mathbf{x+y+z=6}$

$\mathbf{y+3z=11}$

$\mathbf{x-2y+z=0}$

Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

Cofactors

$A_{11} = 7, A_{12} = 3, A_{13} = -1$

$A_{21} = -3, A_{22} = 0, A_{23} = 3$

$A_{31} = 2, A_{32} = -3, A_{33} = 1$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$Adj(A) = \begin{bmatrix} 7 & 3 & -1 \\ -3 & 0 & 3 \\ 2 & -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

$|A| = 9$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

For system of equations

$AX = B$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}$$

$x = 1,\ y = 2,\ z = 3$

25. Show that the height of a cylinder, which is open at the top, having a given surface area and greatest volume, is equal to the radius of its base.

Solution

$Let\ R\ be\ the\ radius$

$H\ be\ the\ height$

$V\ be\ the\ volume$

$S\ be\ the\ total\ surface\ area$

$V = \pi R^2 H$

$S = \pi R^2 + 2\pi RH$

$\Rightarrow H = \dfrac{S - \pi R^2}{2\pi R}$

$Substituting\ value\ of\ H\ in\ V$

$V = \dfrac{1}{2}\left(SR - \pi R^3\right)$

$\dfrac{dV}{dR} = \dfrac{1}{2}\left(S - 3\pi R^2\right)$

$\dfrac{dV}{dR} = 0$

$\Rightarrow \dfrac{1}{2}\left(S - 3\pi R^2\right) = 0$

$R = \sqrt{\dfrac{S}{3\pi}}$

$$\begin{aligned}\frac{d^2V}{dR^2} &= \frac{1}{2}\left(0 - 6\pi R\right)\\ &= -3\pi R\end{aligned}$$

$V\ is\ greatest\ when\ R = \sqrt{\dfrac{S}{3\pi}}$

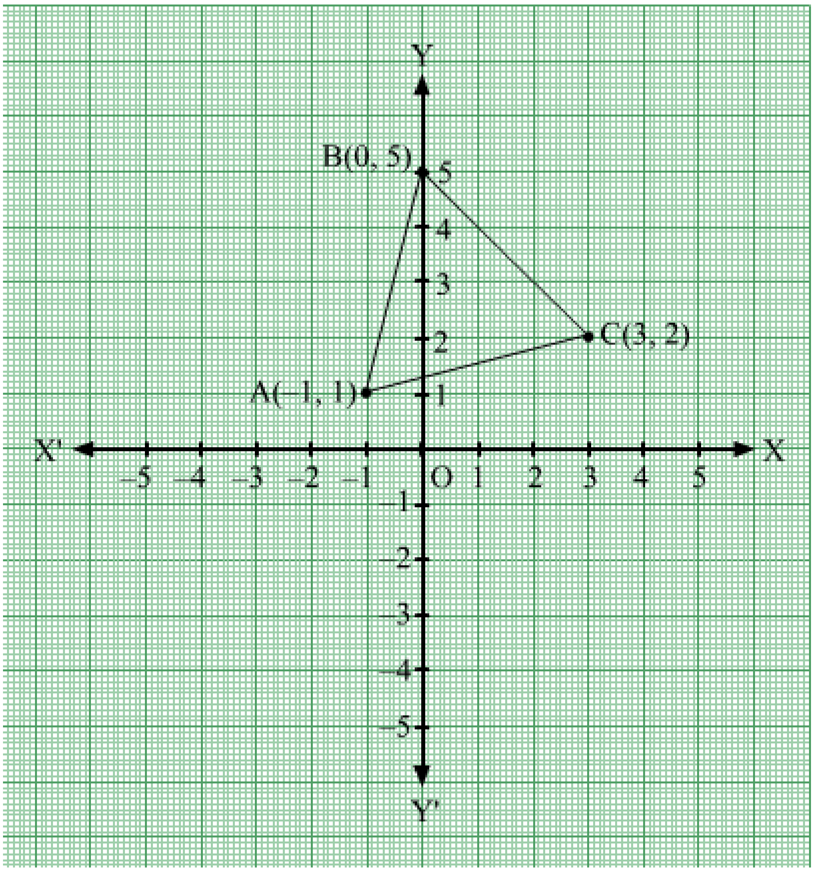
$$H = \frac{S - \pi \times \dfrac{S}{3\pi}}{2\pi\sqrt{\dfrac{S}{3\pi}}}$$

$$H = \frac{\dfrac{2S}{3}}{2\sqrt{\dfrac{\pi S}{3}}}$$

$$H = \sqrt{\frac{S}{3\pi}}$$

26. Find the area of the triangle whose vertices are (-1, 1), (0, 5) and (3, 2), using integration.

Solution



$Let\ A(-1,1),\ B(0,5)\ and\ C(3,2)$

The equation of line AB is

$$y-1=\frac{5-1}{0+1}(x+1)$$

$$y=4x+5$$

The equation of line BC is

$$y-5=\frac{2-5}{3-0}(x-0)$$

$$y=-x+5$$

The equation of line CA is

$$y-2=\frac{1-2}{-1-3}(x-3)$$

$$y=\frac{x}{4}+\frac{5}{4}$$

Required area $=$ *Area of* ΔABC

The equation of line CA is

$$y-2=\frac{1-2}{-1-3}(x-3)$$

$$y=\frac{x}{4}+\frac{5}{4}$$

27. Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=1$and $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$and parallel to x-axis. Hence, find the distance of the plane from x-axis.

Solution

$\vec{a}=2\hat{i}+5\hat{j}-3\hat{k},\vec{b}=-2\hat{i}-3\hat{j}+5\hat{k},\vec{c}=5\hat{i}+3\hat{j}-3\hat{k}$

$(\vec{r}-\vec{a})\cdot\left[\left(\vec{b}-\vec{a}\right)\times(\vec{c}-\vec{a})\right]=0$

$\Rightarrow\left[\vec{r}-\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\right]\cdot\left[\left(-4\hat{i}-8\hat{j}+8\hat{k}\right)\times\left(3\hat{i}-2\hat{j}\right)\right]=0$

$\Rightarrow\left[\vec{r}-\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\right]\cdot\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=0$

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}=0$$

$\Rightarrow(x-2)(16)-(y-5)(-24)+(z+3)(32)=0$

$\Rightarrow 2x+3y+4z=7$

$2(2\lambda+3)+3(2\lambda+1)+4(3\lambda+5)=7$

$\Rightarrow 22\lambda=-22$

$\Rightarrow\lambda=-1$

Therefore, point of intersection is $(1,-1,2)$

28. There are two boxes I and II. Box I contains 3 red and 6 Black balls. Box II contains 5 red and black balls. One of the two boxes, box I and box II is selected at random and a ball is drawn at random. The ball drawn is found to be red. If the probability that this red ball comes out from box II is ' a find the value of n

Solution

$E_1 = selecting\ \ box\ I$

$E_2 = selecting\ \ box\ II$

$A = getting\ \ a\ \ red\ \ ball\ \ from\ selected\ \ box$

$P(E_1) = \frac{1}{2}, P(E_1) = \frac{1}{2}$

$P\left(\frac{A}{E_1}\right) = \frac{3}{9} = \frac{1}{3}$

$P\left(\frac{A}{E_2}\right) = \frac{5}{n+5}$

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$\frac{3}{5} = \frac{\frac{1}{2} \times \frac{5}{n+5}}{\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{5}{n+5}}$$

$$\frac{3}{5} = \frac{15}{n+20}$$

$(n+20)3 = 75$

$3n = 15$

$n = 5$