PREVIOUS YEAR QUESTION PAPER
2018

Section- A

1. If $a*b$ denotes the larger of 'a' and 'b' and if $a \circ b = (a*b)+3$, then write the value of $(5)\circ(10)$, where $*$ and $\circ$ are binary operations.

Sol: $(5)\circ(10) = (5*10)+3 = 10+3 = 13$

2. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60^0 and their scalar product is $\frac{9}{2}$

Sol: Given :

$|\overline{a}| = |\overline{b}|$ and $\theta = 60^o$ and $\overline{a}\cdot\overline{b} = \frac{9}{2}$

$\therefore \overline{a}\cdot\overline{b} = |\overline{a}||\overline{b}|\cos\theta$

$\frac{9}{2} = |\overline{a}|\,|\overline{a}|\cos 60^o$

$\frac{9}{2} = |\overline{a}|^2 \times \frac{1}{2}$

$|\overline{a}|^2 = 9$

$|\overline{a}| = 3 = |\overline{b}|$

3. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew symmetric, find the values of 'a' and 'b'.

Sol: $\because$ A is skew symmetric matrix

$a_{12} = -a_{21} \Rightarrow a = -2$

and $a_{31} = -a_{13} \Rightarrow b = 3$

4. Find the value of $\tan^{-1}\sqrt{3}-\cot^{-1}\left(-\sqrt{3}\right)$

Sol: $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)=k\left(\text{say}\right)$

as $\cot^{-1}\left(-x\right)=\pi-\cot^{-1}x$

$\therefore k=\tan^{-1}\left(\sqrt{3}\right)-\left(\pi-\cot^{-1}\left(\sqrt{3}\right)\right)$

$=\frac{\pi}{3}-\left(\pi-\frac{\pi}{6}\right)$

$=\frac{\pi}{3}-\pi+\frac{\pi}{6}$

$=\frac{\pi}{2}-\pi$

$=-\frac{\pi}{2}$

Section- B

5. The total cost $C\left(x\right)$ associated with the production of x units of an item is given by

$C\left(x\right)=0.005x^3-0.02x^2+30x+5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of total cost at any level of output.

Sol: $C\left(x\right)=0.005x^3-0.02x^2+30x+5000$

$\text{Marginal cost}\left(C_M\right)=\frac{d}{dx}\left(C\left(x\right)\right)=0.005\times 3x^2-0.02\times 2x+30$

$\because x=3$

$C_M=0.005\times 3\times 9-0.02\times 2\times 3+30$

$=0.135-0.12+30$

$=30.135-0.12$

$=30.015$

6. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to *x*.

Sol: Let $y=\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$\therefore y = \tan^{-1}\left(\frac{2\cos^2 x/2}{2\sin x/2 \cos x/2}\right)$$

$$= \tan^{-1}\left(\cot x/2\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)$$

$$\therefore y = \frac{\pi}{2}-\frac{x}{2}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right) = -\frac{1}{2}$$

7. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ compute A^{-1} and show that $2A^{-} = 9I - A$.

Sol: $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$$|A| = 14 - 12 = 2$$

$$\therefore A_{11} = 7 \qquad A_{12} = 4 \qquad A_{31} = 3 \qquad A_{22} = 2$$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{22} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\text{L.H.S.} = 2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\text{R.H.S.} = 9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

L.H.S. =R.H.S.

8. Prove that : $3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right)$, $X \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Sol: When $-\frac{1}{2} \le x \le \frac{1}{2}$

We have,

$$-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \Rightarrow -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$$

Also, $-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore \ \sin 3\theta = 3x - 4x^3$

$\Rightarrow 3\theta = \sin^{-1}\left(3x - 4x^3\right)$

$\Rightarrow 3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right)$

9. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Sol: $S = \{(1,1),(1,2).....(6,6)\}$

$\therefore n(s) = 36$

A = Red die resulted in a number less than 4.

$= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),$

$(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)\}$

$n(A) = {}^{18}C_1 = 18$

B = sum of number is 8

$B = \{(4,4),(6,2),(2,6),(5,3),(3,5)\}$

$n(B) = {}^{5}C_1 = 5$

$A \cap B = \{(5,3),(6,2)\}$

$n(A \cap B) = {}^{2}C_1 = 2$

$\therefore P\left(\frac{B}{A}\right)$ = Probability of sum of number 8 when Red die resulted in a number less than

$4 = \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)} = \frac{2}{18} = \frac{1}{9}$

10. If θ is the angle between two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$ find $\sin\theta$.

Sol: $\bar{a} = \hat{i} - 2j + 3\hat{k},\ \bar{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = (4)\hat{i} + 8\hat{j} + 4\hat{k}$$

$$|\bar{a} \times \bar{b}| = \sqrt{(4)^2 + (8)^2 + (4)^2} = \sqrt{16 + 64 + 16} = \sqrt{96}$$

$$= 4\sqrt{6}$$

$$|\bar{a}| = \sqrt{(1)+(4)+9} = \sqrt{14}$$

$$|\bar{b}| = \sqrt{9+4+1} = \sqrt{14}$$

$$\sin\theta = \left|\frac{\sqrt{96}}{\sqrt{14}\times\sqrt{14}}\right| = \frac{4\sqrt{16}}{14} = \frac{2\sqrt{6}}{7}$$

11. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitary constants.

Sol: $y = ae^{bx}\times e^{5}$

$y = ae^{bx}\times e^{5}$

$y = \alpha e^{bx}$ where $e^{5}a = \alpha$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \alpha b e^{bx}$$

$$\therefore \frac{dy}{dx} = by$$

$$\frac{\frac{dy}{dx}}{y} = b$$

Again differentiate w.r.t. 'x'

$$\frac{y\frac{d^2y}{dx^2} - \frac{dy}{dx}\times\frac{dy}{dx}}{y^2} = 0$$

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$$

12. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

Sol: $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

$$I = \int \frac{1 - 2\sin^2 x + 2\sin^2 x}{\cos^2 x}dx$$

$$I = \int \sec^2 x \, dx$$

$$I = \tan x + C$$

Section- C

13. If $y=\sin(\sin x)$, prove that $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$

Sol: $\mathrm{y}=\sin(\sin \mathrm{x})$ .....(1)

$$\frac{dy}{dx}=\cos(\sin x)\times\cos x \Rightarrow \frac{\frac{dy}{dx}}{\cos x}=\cos(\sin x).....(2)$$

$$\frac{d^2y}{dx^2}=-\cos(\sin x)\times\sin x-\cos x\ \sin(\sin x)\cos x \qquad ....(3)$$

Put (1) and (2) in (3)

$$\frac{d^2y}{dx^2}=-\left(\frac{\frac{dy}{dx}}{\cos x}\right)\times\sin x-y\cos^2 x$$

$$\frac{d^2y}{dx^2}=-\frac{dy}{dx}\tan x-y\cos^2 x$$

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+y\cos^2 x=0$$

14. Find the particular solution of the differential equation $e^x\tan y dx+\left(2-e^x\right)\sec^2 y dy=0$, given that $y=\frac{\pi}{4}$ when $x=0$

Sol: $e^x\tan y\,dx+(2-e^x)\sec^2 y\,dy=0$

$$e^x\tan y\,dx=(e^x-2)\sec^2 y\,dy$$

$$\int\frac{e^x dx}{e^x-2}=\int\frac{\sec^2 y\,dy}{\tan y}$$

$$\ln\left|e^x-2\right|=\ln\left|\tan y\right|+\ln C$$

$$\ln\left|e^x-2\right|=\ln(C\tan y)$$

$$e^x-2=C\tan y$$

Given: $x=0,\ y=\frac{\pi}{4}$

$$e^o-2=C\tan\left(\frac{\pi}{4}\right)$$

$$e^{o}-2=C\tan\left(\frac{\pi}{4}\right)$$

$$1-2=C\times 1 \Rightarrow C=-1$$

$$\therefore e^{x}-2=-\tan y$$

$$e^{x}-2+\tan y=0$$

(OR)

Find the particular solution of the differential equation $\frac{dy}{dx}+2y\tan x=\sin x$, given that $y=0$ when $x=\frac{\pi}{3}$

Sol: $$\frac{dy}{dx}+(2\tan x)y=\sin x$$

$$\frac{dy}{dx}+py=Q$$

$P=2\tan x$ and $Q=\sin x$

$$I.F=e^{\int Pdx}=e^{2\int \tan x\,dx}=e^{2\ln\sec x}$$

$$=e^{\ln\sec^2 x}=\sec^2 x$$

Soln. $y(I.F)=\int Q(I.F)dx$

$$y\cdot\sec^2 x=\int \sin x\times\sec^2 x\,dx$$

$$y\sec^2 x=\int \tan x\sec x\,dx$$

$$y\sec^2 x=\sec x+C$$

Given $y=0$ $\quad x=\frac{\pi}{3}$

$$\sec\frac{\pi}{3}+C=0$$

$$C=-\sec\frac{\pi}{3}=-2$$

$$\therefore y\sec^2 x=\sec x-2$$

$$y\sec^2 x-\sec x+2=0$$

15. Find the shortest distance between the lines.

$\vec{r}=\left(4\hat{i}-\hat{j}\right)+\lambda\left(\hat{i}+2\hat{j}-3\hat{k}\right)$ and $\vec{r}=\left(\hat{i}-\hat{j}+2\hat{k}\right)+\mu\left(2\hat{i}+4\hat{j}-5\hat{k}\right)$

Sol: $\bar{r}=\left(4\hat{i}-\hat{j}\right)+\lambda\left(\hat{i}+2\hat{j}-3\hat{k}\right)=\bar{a}+\lambda\bar{b}$ (say)

$\bar{r}=\left(\hat{i}-\hat{j}+2\hat{k}\right)+\mu\left(2\hat{i}+4\hat{j}-5\hat{k}\right)=\bar{c}+\mu\bar{d}$ (say)

$\therefore \bar{c}-\bar{a}=\left(\hat{i}-\hat{j}+2\hat{k}\right)-\left(4\hat{i}-\hat{j}\right)=-3\hat{i}+0\hat{j}+2\hat{k}$

$$\therefore \bar{b}\times\bar{d}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix}=2\hat{i}-\hat{j}+0\hat{k}$$

$|\bar{b}\times\bar{d}|=\sqrt{4+1}=\sqrt{5}$

$$\text{Shortest distance}=\left|\frac{(\bar{c}-\bar{a})\cdot(\bar{b}\times\bar{d})}{|\bar{b}\times\bar{d}|}\right|$$

$$=\left|\frac{-6}{\sqrt{5}}\right|=\frac{6}{\sqrt{5}}\text{ units}$$

16. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X.

Sol: X can take values as 2,3,4,5 such that

$P\left(X=2\right)$ = probability that the larger of two number 2.

= prob. of getting 1 in first selection and 2 in second selection getting 2 in first selection and 1 in second selection.

$$\therefore P\left(X=2\right)=\frac{1}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{4}=\frac{2}{20}$$

similarly,

$$\therefore P\left(X=3\right)=\frac{2}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{2}{4}=\frac{4}{20}$$

$$\therefore P\left(X=4\right)=\frac{3}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{3}{4}=\frac{6}{20}$$

$$\therefore P\left(X=5\right)=\frac{4}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{4}{4}=\frac{8}{20}$$

X	2	3	4	5
$P(X)$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{6}{20}$	$\frac{8}{20}$

$$E(X)=2\times\frac{2}{20}+3\times\frac{4}{20}+4\times\frac{6}{20}+5\times\frac{8}{20}$$

$$=\frac{80}{20}=4$$

$$E(X^2)=4\times\frac{2}{20}+9\times\frac{4}{20}+16\times\frac{6}{20}+25\times\frac{8}{20}$$

$$=\frac{340}{20}=17$$

$$V(X)=E(X^2)-(E(X))^2$$

$$=17-16$$
$$=1$$

17. Using propeties of determinants, prove that

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}=9(3xyz+xy+yz+zx)$$

Sol:

$$L.H.S.=\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$$

$C_1\to C_1-C_2$; $C_3\to C_3-C_2$

$$=\begin{vmatrix} 0 & 1 & 3x \\ 3y & 1 & 0 \\ -3z & 1+3z & -3z \end{vmatrix}$$

$$=(3\times3)\begin{vmatrix} 0 & 1 & x \\ y & 1 & 0 \\ -z & 1+3z & -z \end{vmatrix}$$

$$=9\left[-1(-yz-0)+x(y+3zy+z)\right]$$

$$=9(yz+xy+3xyz+xz)$$

$= 9(3xyz + xy + yz + zx) = R.H.S.$

Hence proved.

18. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$

Sol: $\because\ P(x_1, y_1) \equiv (2, y_1)$ lies on $16x^2 + 9y^2 = 145$

$16(2)^2 + 9y_1^2 = 145$

$9y_1^2 = 145 - 64$

$9y_1^2 = 81$

$y_1^2 = 9$

$y_1 = \pm 3$

But $y_1 > 0 \quad \therefore\ y_1 = 3$

$\therefore\ P \equiv (2,3)$

$16x^2 + 9y^2 = 145 \qquad ...(i)$

$$32x + 18y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-32x}{18y} = \frac{-16x}{9y}$$

Slope of tangent $= m_{(2,3)} = \dfrac{-16\times 2}{9\times 3} = \dfrac{-32}{27}$

Slope of normal $= m'_{(2,3)} = \dfrac{27}{32}$

Equation of tangent is,

$$(y-3) = \frac{-32}{27}(x-2)$$

$27y - 81 = -32x + 64$

$32x + 27y - 145 = 0$

Equation of normal is,

$$(y-3) = \frac{27}{32}(x-2)$$

$32y - 96 = 27x - 54$

$27x - 32y - 54 + 96 = 0$

$27x - 32y + 42 = 0$

(OR)

Find the intervals in which the function $f(x)=\frac{x^4}{4}-x^3-5x^2+24x+12$ is

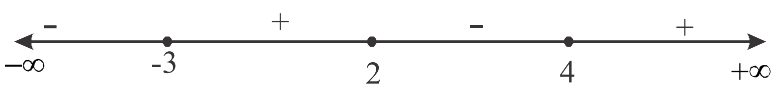
(a) strictly increasing, (b) strictly decreasing.

Sol: $f'(x)=x^3-3x^2-10x+24$

$f'(x)=(x+3)(x-2)(x-4)$

$f(x)$ is strictly increasing

if $f'(x)>0$



$\therefore x\in(-3,2)\cup(4,\infty)$

$f(x)$ is strictly decreasing if $f'(x)<0$

$\therefore x\in(-\infty,-3)\cup(2,4)$

19. Find : $\int\frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Sol: Let $I=\int\frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Let $\sin x=t$

$\cos dx=dt$

$\therefore I=\int\frac{2}{(1-t)(1+t^2)}dt$

Consider

$$\frac{2}{(1-t)(1+t^2)}=\frac{A}{1-t}+\frac{Bt+C}{t^2+1}$$

$$=\frac{A(t^2+1)+(Bt+C)(1-t)}{(1-t)(t^2+1)}$$

$\therefore 2=At^2+A+Bt+C-Bt^2-Ct$

$=(A-B)t^2+(B-C)t+(A+C)$

$\therefore A-B=0,\ B-C=0\quad A+C=2$

$A=1,\ B=1,\ C=1$

$$\therefore I=\int\left(\frac{1}{1-t}+\frac{2t}{2(t^2+1)}+\frac{1}{t^2+1}\right)dt$$

$$=-\log|1-t|+\frac{1}{2}\log\left|t^2+1\right|+\tan^{-1}(t)+C$$

$$=\frac{1}{2}\log\left|\frac{t^2+1}{(1-t)^2}\right|+\tan^{-1}(t)+C$$

$$=\frac{1}{2}\log\left|\frac{\sin^2 x+1}{(1-\sin x)^2}\right|+\tan^{-1}(\sin x)+C$$

20. Suppose a girl throws a die. If she gets 1 or 2 she tosses a coin three times and notes the number of tails. If she gets 3,4,5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3,4,5 or 6 with the ride ?

Sol: Let A be the event that girl will get 1 or 2

$$P(A)=\frac{2}{6}=\frac{1}{3}$$

Let B be the event that girl will get 3, 4, 5 or 6

$$P(B)=\frac{4}{6}=\frac{2}{3}$$

$P\left(T/A\right)=$ Probability of exactly one till given she will get 1 or 2 $=\frac{3}{8}$

$P\left(T/B\right)=$ Probability of exactly one till given she will get 3, 4, 5 or 6 $=\frac{1}{2}$

$$P\left(B/T\right)=\frac{P(B)\times P\left(T/B\right)}{P(A)\times P\left(T/A\right)+P(B)\times P\left(T/B\right)}$$

$$=\frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{2}}$$

$$=\frac{\frac{1}{3}}{\frac{1}{8}+\frac{1}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{11}{8\times 3}}$$

$$= \frac{8}{11}$$

21. Let $\vec{a} = 4\hat{i}+5\hat{j}-\hat{k},\ \vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\cdot\vec{a}=21$

Sol: Since $\overline{d}$ is perpendicular to both $\overline{c}$ and $\overline{b}$, therefore, if is parallel to $\overline{c}\times\overline{b}$

$\therefore \overline{d} = \lambda(\overline{c}\times\overline{b})$

$$= \lambda\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 1 & -4 & 5 \end{vmatrix}$$

$$= \lambda\left\{(5-4)\hat{i}-(15+1)\hat{j}+(-12-1)\hat{k}\right\}$$

$$= \lambda\left\{\hat{i}-16\hat{j}-13\hat{k}\right\}$$

Given that

$$\overline{d}\cdot\overline{a}=21$$

$$\lambda\left\{\hat{i}-16\hat{j}-13\hat{k}\right\}\cdot\left(4\hat{i}+5\hat{j}-\hat{k}\right)=21$$

$$\lambda(4-80+13)=21$$

$$\lambda=\frac{21}{-63}=-\frac{1}{3}$$

$$\therefore \overline{d}=-\frac{1}{3}\left(\hat{i}-16\hat{j}-13\hat{k}\right)$$

$$=\left(-\frac{1}{3}\right)\hat{i}+\left(\frac{16}{3}\right)\hat{j}+\left(\frac{13}{3}\right)\hat{k}$$

22. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width . If the cost is to be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question ?

Sol: Let the length, width and height of the open tank be x, x and y units respectively. Then, its volume is x^2y and the total suface area is x^2+4xy.

It is given that the tank can hold a given quantity of water. This means that its volume is constant. Let it be V. Then,

$V = x^2y$

The cost of the material will be least if the total surface area is least. Let S denote the total surface area. Then,

$S = x^2+4xy$

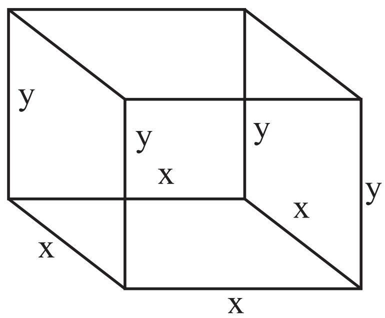
We have to minimize S object to the condition that the volume V is constant.

Now,

$$S = x^2+4xy$$

$$\Rightarrow \quad S = x^2+\frac{4V}{x}$$

$$\Rightarrow \quad \frac{dS}{dx} = 2x-\frac{4V}{x^2} \text{ and } \frac{d^2S}{dx^2} = 2+\frac{8V}{x^3}$$



The critical numbers of S are given by $\frac{dS}{dx} = 0.$

Now, $\frac{dS}{dx} = 0$

$$\Rightarrow \quad 2x-\frac{4V}{x^2} = 0$$

$$\Rightarrow \quad 2x^3-4V = 0$$

$$\Rightarrow \quad 2x^3 = 4x^2y$$

$$\Rightarrow \quad x = 2y$$

Clearly, $\frac{d^2S}{dx^2} = 2+\frac{8V}{x^3} > 0$ for all x.

Hence, S is minimum when x = 2y i.e. the depth (height) of the tank is half of its width.

Comment : Base is directly proportional to height.

23. If $\left(x^2+y^2\right)^2 = xy$, find $\frac{dy}{dx}$.

Sol: Given :

$$\left(x^2+y^2\right)^2 = xy$$

$$x^4+y^4+2x^2y^2 = xy$$

diff. w.r.t. x.

$$4x^3+4y^3\frac{dy}{dx}+2\left(2x^2y\frac{dy}{dx}+2xy^2\right)=\left(x\frac{dy}{dx}+y\right)$$

$$4y^3\frac{dy}{dx}+4x^2y\frac{dy}{dx}-x\frac{dy}{dx}=y-4x^3-4xy^2$$

$$\frac{dy}{dx}\left(4y^3+4x^2y-x\right)=y-4x^3-4xy^2$$

$$\frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4y^3+4x^2y-x}$$

(OR)

If $x=a(2\theta-\sin 2\theta)$ and $y=a(1-\cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$.

Sol: $y=a[1-\cos 2\theta]$, $\quad \frac{dy}{d\theta}=a(0-2\sin 2\theta)$

$$\therefore \frac{dy}{d\theta}=-2a\sin 2\theta$$

$x=a(2\theta-\sin 2\theta)$, $\quad \frac{dx}{d\theta}=a(2-2\cos 2\theta)$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{-2a\sin 2\theta}{2a[1-\cos 2\theta]}\quad\left(\because \frac{dx}{d\theta}\neq 0\right)$$

$$\Rightarrow \frac{-2\sin\theta\ \cos\theta}{2\sin^2\theta}=-\cot\theta$$

$$\therefore \frac{dy}{dx}=-\cot\left(\frac{\pi}{3}\right)=-\frac{1}{\sqrt{3}}$$

Section- D

24. Evaluate: $\int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\sin 2x}dx$

Sol: Let $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\sin 2x}dx$

Here, we express the denominator in terms sin x - cos x which is integration of numerator.

Clearly, $(\sin x-\cos x)^2=\sin^2 x+\cos^2 x-2\sin x\cos x=1-\sin 2x$

$\Rightarrow \sin 2x=1-(\sin x-\cos x)^2$

$$\therefore\ I=\int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\left\{1-(\sin x-\cos x)^2\right\}}dx$$

$$\Rightarrow I=\int_0^{\pi/4}\frac{\sin x+\cos x}{25-9(\sin x-\cos x)^2}dx$$

Let $\sin x-\cos x=t.$ Then, $d(\sin x-\cos x)=dt\Rightarrow(\cos x+\sin x)dx=dt.$

Also, $x=0\Rightarrow t=\sin 0-\cos 0=-1$ and $x=\frac{\pi}{4}\Rightarrow t=\sin\frac{\pi}{4}-\cos\frac{\pi}{4}=0$

$$\therefore\ I=\int_{-1}^{0}\frac{dt}{25-9t^2}=\frac{1}{9}\int_{-1}^{0}\frac{dt}{\frac{25}{9}-t^2}=\frac{1}{9}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{3}\right)^2-t^2}$$

$$\Rightarrow\ I=\frac{1}{9}\times\frac{1}{2(5/3)}\left[\log\left|\frac{5/3+t}{5/3-t}\right|\right]_{-1}^{0}$$

$$\Rightarrow I=\frac{1}{30}\left[\log 1-\log\left(\frac{2/3}{8/3}\right)\right]=\frac{1}{30}\left[\log 1-\log\left(\frac{1}{4}\right)\right]=\frac{1}{30}\left[\log 1+\log 4\right]=\frac{1}{30}\log 4=\frac{1}{15}\log 2$$

(OR)

Evaluate : $\int_1^3\left(x^2+3x+e^x\right)dx$

Sol: $I=\int_1^3(x^2+3x+e^x)dx=\int_a^b f(x)dx$ (say)

when $f(x)=x^2+3x+e^x$; $a=1,\ b=3$

$$h=\frac{b-a}{n}=\frac{3-1}{n}=\frac{2}{n}$$

$$f(a+rh)=f(1+rh)=(1+rh)^2+3(1+rh)+e^{1+rh}$$

$$=4+5rh+r^2h^2+e\times e^{rh}$$

$$=r^2h^2+5rh+4+e\times e^{rh}$$

$$\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{r=1}^{n}h\,f(a+rh)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} h\,(r^2h^2 + 5rh + 4 + e\times e^{rh})$$

$$= \lim_{n\to\infty} \left(\sum_{r=1}^{n} r^2h^3 + 5\sum_{r=1}^{n} rh^2 + \sum_{r=1}^{n} 4h + e\sum_{r=1}^{n} e^{rh} \times h \right)$$

$$= \lim_{n\to\infty} \left(\frac{8}{n^3}\times\frac{n(n+1)(2n+1)}{6} + 5\times\frac{4}{n^2}\times\frac{n(n+1)}{2} + 4\times\frac{2}{n}\times n + e\left(e^h\left(\frac{e^{nh}-1}{e^h-1}\right)\right)\cdot h \right)$$

$$= \lim_{n\to\infty} \left[\left(\frac{8}{6}\times\frac{n}{n}\times\left(\frac{n}{n}+\frac{1}{n}\right)\times\left(\frac{2n}{n}+\frac{1}{n}\right) + \frac{20}{2}\times\frac{n}{n}\times\left(\frac{n}{n}+\frac{1}{n}\right) + 8 + \frac{e^{n+1}(e^2-1)}{\left(\frac{e^h-1}{h}\right)} \right) \right]$$

as $n\to\infty \;\therefore\; h\to 0$

$$\int_a^b f(x)dx = \frac{4}{3}\times 1\times \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\times \lim_{n\to\infty}\left(2+\frac{1}{n}\right) + \lim_{n\to\infty} 10\times 1\times\left(1+\frac{1}{n}\right) + 8 + \lim_{h\to 0}\frac{e^{h+1}(e^2-1)}{\left(\frac{e^h-1}{h}\right)}$$

$$= \frac{4}{3}\times 1\times 2 + 10\times 1\times 1 + 8 + \frac{e(e^2-1)}{1}$$

$$= \frac{8}{3} + 10 + 8 + e^3 - e$$

$$= \frac{8}{3} + 18 + e^3 - e$$

$$= \frac{62}{3} + e^3 - e$$

25. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand - operated. It takes 4 minutes on the automatic and 6 minutes on the hand operated machines to manufacture a packet of screws 'B'. Each machine is availble for at most 4 hours on any day. The manufacturer can sell a packet of screws 'A' at a profit of 70 paise and screws 'B' at a profit of Rs. 1. Assuming that he can sell all the screws he manufactrures, how many packets of each type should the factory owner produce in a day in order to maximize his profit ? Formulate the above LPP and solve it graphically and find the maximum profit.

Sol: Let the factory manufactures x screws of type A and y screws of type B on each day.

$\therefore\; x \ge 0, y \ge 0$

Given that

	Screw A	Screw B	Availibility
Automatic machine	4	6	$4\times 60=240$ minutes
Hand operate machine	6	3	$4\times 60=240$ minutes
Profit	70 paise	1 rupee	

The constraints are

$4x+6y\le 240$

$6x+3y\le 240$

Total profit

$z=0.70x+1y$

$\therefore$ $L.P.P.$ is

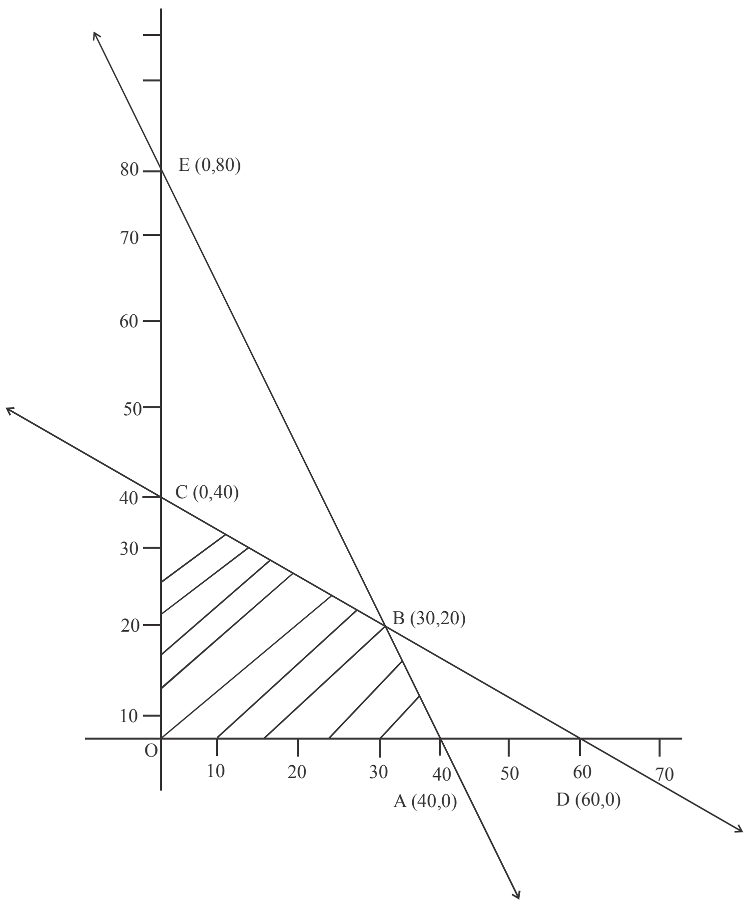
maximise $z=0.7x+y$

subject to ,

$2x+3y\le 120$

$2x+y\le 80$

$x\ge 0, y\ge 0$



$\therefore$ common feasible region is $OCBAO$

Correct point	$Z=0.7x+y$
$A(40,0)$	$Z(A)=28$
$B(30,20)$	$Z(B)=41$ maximum
$C(0,40)$	$Z(C)=40$
$O(0,0)$	$Z(O)=0$

The maximum value of 'Z' is 41 at $(30,20)$. Thus the factory showed produce 30 packages at screw A and 20 packages of screw B to get the maximum profit of Rs.41

26. Let $A=\{x\in Z:0\le x\le 12\}$ show that $R=\{(a,b):a,b\in A,|a-b|\}$ is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2].

Sol: We have,

$R=\{(a,b):|a-b| \text{ is a multiple of } 4\}$, where $a,b\in A=\{x\in Z:0\le x\le 12\}=\{0,1,2,...,12\}$.

We observe the following properties of relation R.

Reflexivity : For any $a\in A$, we have

$|a-a|=0$, which is a multiple of 4.

$\Rightarrow (a,a)\in R$

Thus, $(a,a)\in R$ for all $a\in A$.

So, R is reflexive.

Symmetry : Let $(a,b)\in R$. Then,

$(a,b)\in R$

$\Rightarrow \quad |a-b|$ is a multiple of 4

$\Rightarrow \quad |a-b|=4\lambda$ for some $\lambda\in N$

$\Rightarrow \quad |b-a|=4\lambda$ for some $\lambda\in N$ $\quad [\because |a-b|=|b-a|]$

$\Rightarrow \quad (b,a)\in R$

So, R is symmetric.

Transitivity : Let $(a,b)\in R$ and $(b,c)\in R$. Then,

$(a,b)\in R$ and $(b,c)\in R$

$\Rightarrow \quad |a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4

$\Rightarrow \quad |a-b|=4\lambda$ and $|b-c|=4\mu$ for some $\lambda,\mu\in N$

$\Rightarrow \quad a-b=\pm 4\lambda$ and $b-c=\pm 4\mu$

$\Rightarrow \quad a-c=\pm 4\lambda\pm 4\mu$

$\Rightarrow \quad a - c$ is a multiple of 4

$\Rightarrow \quad |a - c|$ is a multiple of 4

$\Rightarrow \quad (a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

So, R is transitive.

Hence, R is an equivalence relation.

Let x be an element of A such that $(x, 1) \in R$. Then,

$|x - 1|$ is a multiple of 4

$\Rightarrow \quad |x - 1| = 0, 4, 8, 12$

$\Rightarrow \quad x - 1 = 0, 4, 8, 12$

$\Rightarrow \quad x = 1, 5, 9 \qquad [\because 13 \notin A]$

Hence, the set of all elements of A which are related to 1 is {1, 5, 9} i.e. [1] = [1, 5, 9].

& $[2] = [2, 6, 10]$

(OR)

Show that the function $f : R \to R$ defined by $f(x) = \frac{x}{x^2 + 1}, \forall x \in R$ is neither one – one nor onto.

Also, if $g : R \to R$ is defined as $g(x) = 2x - 1$ find $fog(x)$

Sol: $f : R \to R,\ f(x) = \frac{x}{x^2 + 1},\ \forall x \in R$

$f(x_1) = \frac{x_1}{x_2^2 + 1}$

$f(x_1) = f(x_2)$

$\frac{x_1}{x_1^2 + 1} = \frac{x_2}{x_2^2 + 1}$

$x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$

$x_1 x_2^2 - x_2 x_1^2 + x_1 - x_2 = 0$

$x_1 x_2 (x_2 - x_1) - 1(x_2 - x_1) = 0$

$(x_1 x_2 - 1)(x_2 - x_1) = 0$

$x_1 x_2 = 1$ or $x_1 = x_2$

$\therefore f(x)$ is not one-one

also $y=\dfrac{x}{x^2+1}$

$x^2y-x+y=0$

$\Delta\geq 0$ if x is real

$\therefore B^2-4AC\geq 0$

$(-1)^2-4\times y\times y\geq 0$

$1-4y^2\geq 0$

$(1-2y)(1+2y)\geq 0$

$(2y-1)(2y+1)\leq 0$

$\therefore -\dfrac{1}{2}\leq y\leq\dfrac{1}{2}$

Codomain $\in R$

But range $\in\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

$\therefore$ Function is not onto

$f(x)=\dfrac{x}{x^2+1}$ as $f:R\to R$

$g(x)=2x-1$ as $g:R\to R$

$(fog)(x)=f(g(x))=\dfrac{g(x)}{(g(x)^2+1)}$

$$=\frac{2x-1}{(2x-1)^2+1}$$

$$=\frac{2x-1}{4x^2-4x+1+1}$$

$$=\frac{2x-1}{4x^2-4x+2}$$

27. Using integration, find the area of the region in the first quadrant enclosed by the x – axis, the line $y=x$ and the circle $x^2+y^2=32$

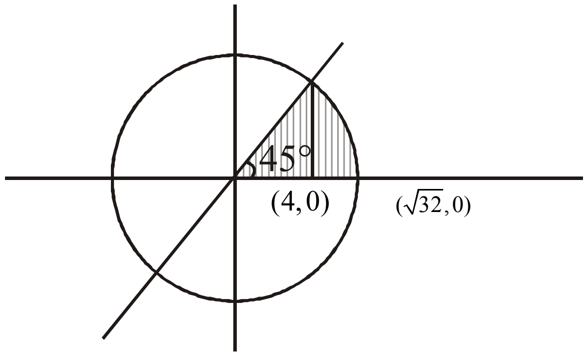
Sol: Put $y=x$ in $x^2+y^2=32$

$\therefore x^2+x^2=32$

$2x^2 = 32$

$x^2 = 16$

$x = 4$



$$A = \int_0^4 y_{\text{line}} dx + \int_4^{\sqrt{32}} y_{\text{circle}} dx$$

$$A = \int_0^4 x dx + \int_4^{\sqrt{32}} (\sqrt{32 - x^2}) dx$$

$$= \left(\frac{x^2}{2}\right)_0^4 + \int_4^{\sqrt{32}} \sqrt{(\sqrt{32})^2 - x^2} dx$$

$$= (8) + \left(\frac{x}{2}\sqrt{32 - x^2} + \frac{32}{2}\sin^{-1}\left(\frac{x}{\sqrt{32}}\right)\right)^{\sqrt{32}}$$

$$= (8) + \left(0 + 16 \times \frac{\pi}{2} - \left(2\sqrt{16} + 16\sin^{-1}\left(\frac{4}{\sqrt{32}}\right)\right)\right)$$

$$= 8 + 8\pi - 8 - 16\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 8\pi - 16 \times \frac{\pi}{4} = 8\pi - 4\pi = 4\pi \text{ sq units}$$

28. If $A = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$, find A^{-1}. Use it solve the system of equations.

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

Sol: $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$\therefore |A| = 2(-4+4) + 3(-6+4) + 5(3-2) = 0 - 6 + 5 = -1 \neq 0$

Now, $A_{11} = 0,\ A_{12} = 2, A_{12} = 1$

$A_{21} = -1, A_{22} = -9, A_{23} = -5$

$A_{31} = 2, A_{32} = 23, A_{33} = 13$

$$\therefore\ A^{-1} = \frac{1}{|A|}(\text{adj}A) = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \qquad ....(1)$$

Now, the given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ x \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

The solution of the systemof equations is given by $X = A^{-1}B$,

$X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \qquad [\text{Using}(1)]$$

$$= \begin{bmatrix} 0-5+6 \\ -22-45+39 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1, y = 2,$ and $z = 3$

(OR)

Using elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

Sol: $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$$

$|A| = 1(-25+28) - 2(-10+14) + 3(-8+10)$

$= 3 - 2(4) + 3(2) = 9 - 8 = 1 \neq 0$

A^{-1} exists.

$A \cdot A^{-1} = I$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_1$; $R_3 \Rightarrow R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - R_3; R_2 \Rightarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$I \cdot A^{-1} = A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

29. Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$ and the plane $\vec{r} \cdot \left(\hat{i} - \hat{j} + \hat{k}\right) = 5$

Sol: Cartesian equation of line and plane,

$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$: (Line)

$x - y + z - 5 = 0$: (Plane)

Let Q (α, β, γ) be point of intersection of line and plane which will satisfy both equation.

$\frac{\alpha - 2}{3} = \frac{\beta + 1}{4} = \frac{\gamma - 2}{2} = \lambda$ (say)

$\alpha = 3\lambda + 2, \beta = 4\lambda - 1, \gamma = 2\lambda + 2$

also $\alpha - \beta + \gamma - 5 = 0$

$3\lambda + 2 - 4\lambda + 1 + 2\lambda + 2 - 5 = 0$

$\lambda = 0$

$\therefore \ \alpha = 2, \beta = -1, \gamma = 2 \Rightarrow Q \equiv (2, -1, 2)$

$$\ell(PQ) = \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$$

$$= \sqrt{9 + 16 + 144}$$

$$= \sqrt{169}$$

= 13 units