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Theory of Equations

Topic 1 Quadratic Equations

Objective Questions I (Only one correct option)

1. If α and β are the roots of the quadratic equation, $x^2 + x\sin\theta - 2\sin\theta = 0, \theta \in \left(0, \frac{\pi}{2}\right)$, then $\frac{\alpha^{12}+\beta^{12}}{(\alpha^{-12}+\beta^{-12})(\alpha-\beta)^{24}}$ is equal to **(2019 Main, 10 April I)**

(a) $\frac{2^{12}}{(\sin\theta + 8)^{12}}$ (b) $\frac{2^{6}}{(\sin\theta + 8)^{12}}$

(c) $\frac{2^{12}}{(\sin\theta - 4)^{12}}$ (d) $\frac{2^{12}}{(\sin\theta - 8)^{6}}$

2. Let $p, q \in \mathbf{R}$. If $2 - \sqrt{3}$ is a root of the quadratic equation, $x^2 + px + q = 0$, then **(2019 Main, 9 April I)**

(a) $q^2 - 4p - 16 = 0$

(b) $p^2 - 4q - 12 = 0$

(c) $p^2 - 4q + 12 = 0$

(d) $q^2 + 4p + 14 = 0$

3. If m is chosen in the quadratic equation $(m^2+1)x^2 - 3x + (m^2+1)^2 = 0$ such that the sum of its roots is greatest, then the absolute difference of the cubes of its roots is **(2019 Main, 9 April II)**

(a) $10\sqrt{5}$ (b) $8\sqrt{5}$

(c) $8\sqrt{3}$ (d) $4\sqrt{3}$

4. If α and β are the roots of the equation $x^2 - 2x + 2 = 0$, then the least value of n for which $\left(\frac{\alpha}{\beta}\right)^n = 1$ is

(a) 2 (b) 5 **(2019 Main, 8 April I)**

(c) 4 (d) 3

5. The number of integral values of m for which the equation $(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$, has no real root is **(2019 Main, 8 April II)**

(a) 3 (b) infinitely many

(c) 1 (d) 2

6. The number of integral values of m for which the quadratic expression, $(1 + 2m)\,x^2 - 2(1 + 3m)\,x + 4(1 + m)$, $x \in R$, is always positive, is **(2019 Main, 12 Jan II)**

(a) 6 (b) 8 (c) 7 (d) 3

7. If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2 + m(m-4)x + 2 = 0$, then the least value of m for which $\lambda + \frac{1}{\lambda} = 1$, is **(2019 Main, 12 Jan I)**

(a) $-2 + \sqrt{2}$ (b) $4 - 2\sqrt{3}$

(c) $4 - 3\sqrt{2}$ (d) $2 - \sqrt{3}$

8. If one real root of the quadratic equation $81x^2 + kx + 256 = 0$ is cube of the other root, then a value of k is **(2019 Main, 11 Jan I)**

(a) 100 (b) 144

(c) −81 (d) −300

9. If $5, 5r, 5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to **(2019 Main, 10 Jan I)**

(a) $\frac{5}{4}$ (b) $\frac{7}{4}$ (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

10. The value of λ such that sum of the squares of the roots of the quadratic equation, $x^2 + (3 - \lambda)x + 2 = \lambda$ has the least value is **(2019 Main, 10 Jan II)**

(a) $\frac{4}{9}$ (b) 1 (c) $\frac{15}{8}$ (d) 2

11. The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2 - 11x + \alpha = 0$ are rational numbers is **(2019 Main, 9 Jan II)**

(a) 5 (b) 2

(c) 4 (d) 3

12. Let α and β be two roots of the equation $x^2 + 2x + 2 = 0$, then $\alpha^{15} + \beta^{15}$ is equal to **(2019 Main, 9 Jan I)**

(a) 256 t(b) 512

(c) −256 (d) −512

13. Let $S = \{x \in \mathbf{R} : x \geq 0 \text{ and } 2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0$. Then, S **(2018 Main)**

(a) is an empty set

(b) contains exactly one element

(c) contains exactly two elements

(d) contains exactly four elements

14. If $\alpha, \beta \in \mathbf{C}$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to **(2018 Main)**

(a) −1 (b) 0

(c) 1 (d) 2

15. For a positive integer n, if the quadratic equation, $x(x+1)+(x+1)(x+2)+\ldots+(x+\overline{n-1})(x+n)=10n$ has two consecutive integral solutions, then n is equal to **(2017 Main)**

(a) 12 (b) 9 (c) 10 (d) 11

16. The sum of all real values of x satisfying the equation $(x^2-5x+5)^{x^2+4x-60}=1$ is **(2016 Main)**

(a) 3 (b) – 4 (c) 6 (d) 5

17. Let $-\frac{\pi}{6}<\theta<-\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2-2x\sec\theta+1=0$, and α_2 and β_2 are the roots of the equation $x^2+2x\tan\theta-1=0$. If $\alpha_1>\beta_1$ and $\alpha_2>\beta_2$, then $\alpha_1+\beta_2$ equals **(2016 Adv.)**

(a) $2(\sec\theta-\tan\theta)$ (b) $2\sec\theta$
(c) $-2\tan\theta$ (d) 0

18. Let α and β be the roots of equation $x^2-6x-2=0$. If $a_n=\alpha^n-\beta^n$, for $n\geq 1$, then the value of $\frac{a_{10}-2a_8}{2a_9}$ is

(a) 6 (b) – 6 **(2015 Main)**
(c) 3 (d) – 3

19. In the quadratic equation $p(x)=0$ with real coefficients has purely imaginary roots. Then, the equation $p[p(x)]=0$ has **(2014 Adv.)**

(a) only purely imaginary roots
(b) all real roots
(c) two real and two purely imaginary roots
(d) neither real nor purely imaginary roots

20. Let α and β be the roots of equation $px^2+qx+r=0$, $p\neq 0$. If p, q and r are in AP and $\frac{1}{\alpha}+\frac{1}{\beta}=4$, then the value of $|\alpha-\beta|$ is **(2014 Main)**

(a) $\frac{\sqrt{61}}{9}$ (b) $\frac{2\sqrt{17}}{9}$ (c) $\frac{\sqrt{34}}{9}$ (d) $\frac{2\sqrt{13}}{9}$

21. Let α and β be the roots of $x^2-6x-2=0$, with $\alpha>\beta$. If $a_n=\alpha^n-\beta^n$ for $n\geq 1$, then the value of $\frac{a_{10}-2a_8}{2a_9}$ is

(a) 1 (b) 2 (c) 3 (d) 4 **(2011)**

22. Let p and q be real numbers such that $p\neq 0$, $p^3\neq q$ and $p^3\neq -q$. If α and β are non-zero complex numbers satisfying $\alpha+\beta=-p$ and $\alpha^3+\beta^3=q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is **(2010)**

(a) $(p^3+q)x^2-(p^3+2q)x+(p^3+q)=0$
(b) $(p^3+q)x^2-(p^3-2q)x+(p^3+q)=0$
(c) $(p^3-q)x^2-(5p^3-2q)x+(p^3-q)=0$
(d) $(p^3-q)x^2-(5p^3+2q)x+(p^3-q)=0$

23. Let α, β be the roots of the equation $x^2-px+r=0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2-qx+r=0$. Then, the value of r is **(2007, 3M)**

(a) $\frac{2}{9}(p-q)(2q-p)$ (b) $\frac{2}{9}(q-p)(2p-q)$
(c) $\frac{2}{9}(q-2p)(2q-p)$ (d) $\frac{2}{9}(2p-q)(2q-p)$

24. If a, b, c are the sides of a triangle ABC such that $x^2-2(a+b+c)x+3\lambda(ab+bc+ca)=0$ has real roots, then **(2006, 3M)**

(a) $\lambda<\frac{4}{3}$ (b) $\lambda>\frac{5}{3}$ (c) $\lambda\in\left(\frac{4}{3},\frac{5}{3}\right)$ (d) $\lambda\in\left(\frac{1}{3},\frac{5}{3}\right)$

25. If one root is square of the other root of the equation $x^2+px+q=0$, then the relation between p and q is

(a) $p^3-q(3p-1)+q^2=0$ **(2004, 1M)**
(b) $p^3-q(3p+1)+q^2=0$
(c) $p^3+q(3p-1)+q^2=0$
(d) $p^3+q(3p+1)+q^2=0$

26. The set of all real numbers x for which $x^2-|x+2|+x>0$ is **(2002, 1M)**

(a) $(-\infty,-2)\cup(2,\infty)$ (b) $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$
(c) $(-\infty,-1)\cup(1,\infty)$ (d) $(\sqrt{2},\infty)$

27. The number of solutions of $\log_4(x-1)=\log_2(x-3)$ is

(a) 3 (b) 1 **(2001, 2M)**
(c) 2 (d) 0

28. For the equation $3x^2+px+3=0$, $p>0$, if one of the root is square of the other, then p is equal to **(2000, 1M)**

(a) 1/3 (b) 1 (c) 3 (d) 2/3

29. If α and β $(\alpha<\beta)$ are the roots of the equation $x^2+bx+c=0$, where $c<0<b$, then **(2000, 1M)**

(a) $0<\alpha<\beta$ (b) $\alpha<0<\beta<|\alpha|$
(c) $\alpha<\beta<0$ (d) $\alpha<0<|\alpha|<\beta$

30. The equation $\sqrt{x+1}-\sqrt{x-1}=\sqrt{4x-1}$ has **(1997C, 2M)**

(a) no solution
(b) one solution
(c) two solutions
(d) more than two solutions

31. The equation $x^{\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}}=\sqrt{2}$ has **(1989; 2M)**

(a) atleast one real solution
(b) exactly three real solutions
(c) exactly one irrational solution
(d) complex roots

32. If α and β are the roots of $x^2+px+q=0$ and α^4, β^4 are the roots of $x^2-rx+s=0$, then the equation $x^2-4qx+2q^2-r=0$ has always **(1989, 2M)**

(a) two real roots
(b) two positive roots
(c) two negative roots
(d) one positive and one negative root

33. The equation $x-\frac{2}{x-1}=1-\frac{2}{x-1}$ has **(1984, 2M)**

(a) no root (b) one root
(c) two equal roots (d) infinitely many roots

34. For real x, the function $\frac{(x-a)(x-b)}{(x-c)}$ will assume all real values provided **(1984, 3M)**

(a) $a>b>c$ (b) $a<b<c$
(c) $a>c<b$ (d) $a\leq c\leq b$

35. The number of real solutions of the equation $|x|^2-3|x|+2=0$ is **(1982, 1M)**

(a) 4 (b) 1 (c) 3 (d) 2

36. Both the roots of the equation

$(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)=0$

are always **(1980, 1M)**

(a) positive
(b) negative
(c) real
(d) None of the above

37. Let $a>0, b>0$ and $c>0$. Then, both the roots of the equation $ax^2+bx+c=0$ **(1979, 1M)**

(a) are real and negative
(b) have negative real parts
(c) have positive real parts
(d) None of the above

Assertion and Reason

For the following question, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

38. Let a, b, c, p, q be the real numbers. Suppose α, β are the roots of the equation $x^2+2px+q=0$.

and $\alpha, \dfrac{1}{\beta}$ are the roots of the equation $ax^2+2bx+c=0$,

where $\beta^2 \notin \{-1, 0, 1\}$.

Statement I $(p^2-q)(b^2-ac) \geq 0$

Statement II $b \notin pa$ or $c \notin qa$. **(2008, 3M)**

Fill in the Blanks

39. The sum of all the real roots of the equation $|x-2|^2+|x-2|-2=0$ is...... . **(1997, 2M)**

40. If the products of the roots of the equation $x^2-3kx+2e^{2\log k}-1=0$ is 7, then the roots are real for $k=\ldots$. **(1984, 2M)**

41. If $2+i\sqrt{3}$ is a root of the equation $x^2+px+q=0$, where p and q are real, then $(p, q)=(\ldots,\ldots)$. **(1982, 2M)**

42. The coefficient of x^{99} in the polynomial $(x-1)(x-2)\ldots(x-100)$ is.... **(1982, 2M)**

True/False

43. If $P(x)=ax^2+bx+c$ and $Q(x)=-ax^2+bx+c$, where $ac \neq 0$, then $P(x)\,Q(x)$ has atleast two real roots. **(1985, 1M)**

44. The equation $2x^2+3x+1=0$ has an irrational root. **(1983, 1M)**

Analytical & Descriptive Questions

45. If $x^2-10ax-11b=0$ have roots c and d. $x^2-10cx-11d=0$ have roots a and b, then find $a+b+c+d$. **(2006, 6M)**

46. If α, β are the roots of $ax^2+bx+c=0, (a \neq 0)$ and $\alpha+\delta, \beta+\delta$ are the roots of $Ax^2+Bx+C=0, (A \neq 0)$ for some constant δ, then prove that

$$\frac{b^2-4ac}{a^2}=\frac{B^2-4AC}{A^2}$$ **(2000, 4M)**

47. Let $f(x)=Ax^2+Bx+C$ where, A, B, C are real numbers. prove that if $f(x)$ is an integer whenever x is an integer, then the numbers $2A, A+B$ and C are all integers. Conversely, prove that if the numbers $2A, A+B$ and C are all integers, then $f(x)$ is an integer whenever x is an integer. **(1998, 3M)**

48. Find the set of all solutions of the equation

$$2^{|y|}-|2^{y-1}-1|=2^{y-1}+1$$ **(1997 C, 3M)**

49. Find the set of all x for which

$$\frac{2x}{2x^2+5x+2}>\frac{1}{x+1}$$ **(1987, 3M)**

50. Solve $|x^2+4x+3|+2x+5=0$ **(1987, 5M)**

51. For $a \leq 0$, determine all real roots of the equation

$$x^2-2a|x-a|-3a^2=0$$ **(1986, 5M)**

52. Solve for x : $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$ **(1985, 5M)**

53. If one root of the quadratic equation $ax^2+bx+c=0$ is equal to the nth power of the other, then show that

$$(ac^n)^{\frac{1}{n+1}}+(a^nc)^{\frac{1}{n+1}}+b=0$$ **(1983, 2M)**

54. If α and β are the roots of $x^2+px+q=0$ and γ, δ are the roots of $x^2+rx+s=0$, then evaluate $(\alpha-\gamma)(\beta-\gamma)(\alpha-\delta)(\beta-\delta)$ in terms of p, q, r and s. **(1979, 2M)**

55. Solve $2\log_x a+\log_{ax} a+3\log_b a=0$,

where $a>0, b=a^2x$ **(1978 , 3M)**

56. If α and β are the roots of the equation $x^2+px+1=0$; γ, δ are the roots of $x^2+qx+1=0$, then $q^2-p^2=(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$ **(1978, 2M)**

Passage Type Questions

Let p, q be integers and let α, β be the roots of the equation, $x^2-x-1=0$ where $\alpha \neq \beta$. For $n=0, 1, 2, \ldots\ldots$, let $a_n=p\alpha^n+q\beta^n$.

FACT : If a and b are rational numbers and $a+b\sqrt{5}=0$, then $a=0=b$. **(2017 Adv.)**

57. $a_{12}=$

(a) $a_{11}+2a_{10}$ (b) $2a_{11}+a_{10}$
(c) $a_{11}-a_{10}$ (d) $a_{11}+a_{10}$

58. If $a_4=28$, then $p+2q=$

(a) 14 (b) 7 (c) 21 (d) 12

Topic 2 Common Roots

Objective Questions I (Only one correct option)

1 If α, β and γ are three consecutive terms of a non-constant GP such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then, $\alpha(\beta + \gamma)$ is equal to **(2019 Main, 12 April II)**

(a) 0
(b) $\alpha\beta$
(c) $\alpha\gamma$
(d) $\beta\gamma$

2. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$ have a common root, then $a : b : c$ is **(2013 Main)**

(a) $1 : 2 : 3$ (b) $3 : 2 : 1$
(c) $1 : 3 : 2$ (d) $3 : 1 : 2$

3. A value of b for which the equations $x^2 + bx - 1 = 0$, $x^2 + x + b = 0$ have one root in common is **(2011)**

(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$ (c) $i\sqrt{5}$ (d) $\sqrt{2}$

Fill in the Blanks

4. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, then the numerical value of $a + b$ is… . **(1986, 2M)**

True/False

5. If $x - r$ is a factor of the polynomial $f(x) = a_n x^n$, $+ a_{n-1} x^{n-1} + \ldots + a_0$ repeated m times $(1 < m \leq n)$, then r is a root of $f'(x) = 0$ repeated m times. **(1983, 1M)**

Topic 3 Transformation of Roots

Objective Question I (Only one correct option)

1. Let α, β be the roots of the equation, $(x - a)(x - b) = c, c \neq 0$. Then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

(a) a, c (b) b, c (c) a, b (d) $a + c, b + c$ **(1992, 2M)**

Analytical & Descriptive Question

2. Let a, b and c be real number s with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^3 x^2 + abcx + c^3 = 0$ in terms of α, β. **(2001, 4M)**

Topic 4 Graph of Quadratic Expression

Objective Questions I (Only one correct option)

1. Let $P(4, -4)$ and $Q(9, 6)$ be two points on the parabola, $y^2 = 4x$ and let X be any point on the arc POQ of this parabola, where O is the vertex of this parabola, such that the area of ΔPXQ is maximum. Then, this maximum area (in sq units) is **(2019 Main, 12 Jan I)**

(a) $\frac{125}{2}$ (b) $\frac{75}{2}$
(c) $\frac{625}{4}$ (d) $\frac{125}{4}$

2. Consider the quadratic equation, $(c - 5)x^2 - 2cx + (c - 4) = 0, c \neq 5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval (0, 2) and its other root lies in the interval (2, 3). Then, the number of elements in S is **(2019 Main, 10 Jan I)**

(a) 11 (b) 10
(c) 12 (d) 18

3. If both the roots of the quadratic equation $x^2 - mx + 4 = 0$ are real and distinct and they lie in the interval [1, 5] then m lies in the interval **(2019 Main, 9 Jan II)**

(a) (4, 5) (b) (–5, – 4)
(c) (5, 6) (d) (3, 4)

4. If $a \in R$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where, $[x]$ denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval **(2014 Main)**

(a) $(-1, 0) \cup (0, 1)$ (b) $(1, 2)$
(c) $(-2, -1)$ (d) $(-\infty, -2) \cup (2, \infty)$

5 For all 'x', $x^2 + 2ax + (10 - 3a) > 0$, then the interval in which 'a' lies is **(2004, 1M)**

(a) $a < -5$ (b) $-5 < a < 2$ (c) $a > 5$ (d) $2 < a < 5$

6. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has **(2000, 1M)**

(a) both roots in (a, b)
(b) both roots in $(-\infty, a)$
(c) both roots in $(b, +\infty)$
(d) one root in $(-\infty, a)$ and the other in (b, ∞)

7. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then **(1999, 2M)**

(a) $a < 2$ (b) $2 \leq a \leq 3$ (c) $3 < a \leq 4$ (d) $a > 4$

8. Let $f(x)$ be a quadratic expression which is positive for all real values of x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x **(1990, 2M)**

(a) $g(x) < 0$ (b) $g(x) > 0$ (c) $g(x) = 0$ (d) $g(x) \geq 0$

Analytical & Descriptive Questions

9. If $x^2 + (a-b)x + (1-a-b) = 0$ where $a, b \in R$, then find the values of a for which equation has unequal real roots for all values of b. **(2003, 4M)**

10. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$ **(1995, 5M)**

11. Find all real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 2x - 4 \le 0$. **(1983, 2M)**

Integer Answer Type Question

12. The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values atleast 4, is …… . **(2009)**

Topic 5 Some Special Forms

Objective Questions I (Only one correct option)

1. The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is **(2019 Main, 10 April II)**

(a) 1 (b) 3
(c) 4 (d) 2

2. All the pairs (x, y) that satisfy the inequality $2^{\sqrt{\sin^2 x - 2\sin x + 5}} \cdot \frac{1}{4^{\sin^2 y}} \le 1$ also satisfy the equation **(2019 Main, 10 April I)**

(a) $2|\sin x| = 3\sin y$ (b) $\sin x = |\sin y|$
(c) $\sin x = 2\sin y$ (d) $2\sin x = \sin y$

3. The sum of the solutions of the equation $|\sqrt{x} - 2| + \sqrt{x}(\sqrt{x} - 4) + 2 = 0$ $(x > 0)$ is equal to **(2019 Main, 8 April I)**

(a) 9 (b) 12
(c) 4 (d) 10

4. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in $[0, 1]$ **(2013 Main)**

(a) lies between 1 and 2 (b) lies between 2 and 3
(c) lies between -1 and 0 (d) does not exist

5. Let a, b, c be real numbers, $a \ne 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is the root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies **(1989, 2M)**

(a) $\gamma = \frac{\alpha + \beta}{2}$ (b) $\gamma = \alpha + \frac{\beta}{2}$
(c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$

6. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has **(1983, 1M)**

(a) at least one root in (0, 1)
(b) one root in (2, 3) and the other in $(-2, -1)$
(c) imaginary roots
(d) None of the above

7. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is **(1982, 2M)**

(a) $-4 < x \le 0$ (b) $0 < x < 1$
(c) $-100 < x < 100$ (d) $-\infty < x < \infty$

8. Let a, b, c be non-zero real numbers such that

$$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)dx$$ **(1981, 2M)**

Then, the quadratic equation $ax^2 + bx + c = 0$ has

(a) no root in (0, 2) (b) atleast one root in (1, 2)
(c) a double root in (0, 2) (d) two imaginary roots

Objective Questions II
(One or more than one correct option)

9. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following interval(s) is/are a subset of S? **(2015 Adv.)**

(a) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (b) $\left(-\frac{1}{\sqrt{5}}, 0\right)$ (c) $\left(0, \frac{1}{\sqrt{5}}\right)$ (d) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

10. Let $a \in R$ and $f : R \to R$ be given by $f(x) = x^5 - 5x + a$. Then,

(a) $f(x)$ has three real roots, if $a > 4$
(b) $f(x)$ has only one real root, if $a > 4$
(c) $f(x)$ has three real roots, if $a < -4$
(d) $f(x)$ has three real roots, if $-4 < a < 4$

Passage Based Problems

Read the following passage and answer the questions.

Passage I

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$. **(2010)**

11. The real numbers s lies in the interval

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$ (c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

12. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

(a) $\left(\frac{3}{4}, 3\right)$ (b) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (c) $(9, 10)$ (d) $\left(0, \frac{21}{64}\right)$

13. The function $f'(x)$ is

(a) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$
(b) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$
(c) increasing in $(-t, t)$
(d) decreasing in $(-t, t)$

Passage II

If a continuous function f defined on the real line R, assumes positive and negative values in R, then the equation $f(x) = 0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum values is negative, then the equation $f(x) = 0$ has a root in R. Consider $f(x) = ke^x - x$ for all real x where k is real constant. **(2007, 4M)**

14. The line $y = x$ meets $y = ke^x$ for $k \leq 0$ at

(a) no point (b) one point
(c) two points (d) more than two points

15. The positive value of k for which $ke^x - x = 0$ has only one root is

(a) $\frac{1}{e}$ (b) 1 (c) e (d) $\log_e 2$

16. For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots, is

(a) $\left(0, \frac{1}{e}\right)$ (b) $\left(\frac{1}{e}, 1\right)$
(c) $\left(\frac{1}{e}, \infty\right)$ (d) (0, 1)

True/False

17. If $a < b < c < d$, then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real and distinct. **(1984, 1M)**

Analytical & Descriptive Question

18. Let $-1 \leq p < 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval [1/2, 1] and identify it. **(2001, 4M)**

Answers

Topic 1

1. (a) **2.** (b) **3.** (b) **4.** (c)
5. (a) **6.** (c) **7.** (c) **8.** (d)
9. (b) **10.** (d) **11.** (d) **12.** (c)
13. (c) **14.** (c) **15.** (d) **16.** (a)
17. (c) **18.** (c) **19.** (d) **20.** (d)
21. (c) **22.** (b) **23.** (d) **24.** (a)
25. (a) **26.** (b) **27.** (b) **28.** (c)
29. (b) **30.** (a) **31.** (b) **32.** (a)
33. (a) **34.** (d) **35.** (a) **36.** (c)
37. (b) **38.** (b) **39.** 4 **40.** $k = 2$
41. $(-4, 7)$ **42.** -5050 **43.** True **44.** False
45. 1210 **48.** $y \in \{-1\} \cup [1, \infty)$
49. $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$ **50.** -4 and $(-1 - \sqrt{3})$
51. $x = \{a(1 - \sqrt{2}), a(\sqrt{6} - 1)\}$ **52.** $\pm 2, \pm\sqrt{2}$
54. $(q - s)^2 - rqp - rsp + sp^2 + qr^2$ **55.** $x = a^{-1/2}$ or $a^{-4/3}$
56. $q^2 - p^2$ **57.** (d) **58.** (d)

Topic 2

1. (d) **2.** (a) **3.** (b) **4.** (-1)
5. False

Topic 3

1. (c) **2.** $x = \alpha^2\beta, \alpha\beta^2$

Topic 4

1. (d) **2.** (a) **3.** (a) **4.** (a)
5. (b) **6.** (d) **7.** (a) **8.** (b)
9. $a > 1$ **11.** $x \in [1 - \sqrt{5}, 1) \cup [1 + \sqrt{5}, 2)$ **12.** $k = 2$

Topic 5

1. (a) **2.** (b) **3.** (d) **4.** (d)
5. (d) **6.** (a) **7.** (d) **8.** (b)
9. (a,d) **10.** (b, d) **11.** (c) **12.** (a)
13. (b) **14.** (b) **15.** (a) **16.** (a)
17. True **18.** $x = \cos\left[\frac{1}{3}\cos^{-1} p\right]$

Hints & Solutions

Topic 1 Quadratic Equations

1. Given quadratic equation is

$$x^2 + x\sin\theta - 2\sin\theta = 0, \theta \in \left(0, \frac{\pi}{2}\right)$$

and its roots are α and β.
So, sum of roots $= \alpha + \beta = -\sin\theta$
and product of roots $= \alpha\beta = -2\sin\theta$

$$\Rightarrow \quad \alpha\beta = 2(\alpha + \beta) \quad \ldots(i)$$

Now, the given expression is $\dfrac{\alpha^{12} + \beta^{12}}{(\alpha^{-12} + \beta^{-12})(\alpha - \beta)^{24}}$

$$= \frac{\alpha^{12} + \beta^{12}}{\left(\frac{1}{\alpha^{12}} + \frac{1}{\beta^{12}}\right)(\alpha - \beta)^{24}} = \frac{\alpha^{12} + \beta^{12}}{\left(\frac{\beta^{12} + \alpha^{12}}{\alpha^{12}\beta^{12}}\right)(\alpha - \beta)^{24}}$$

$$= \left[\frac{\alpha\beta}{(\alpha - \beta)^2}\right]^{12} = \left(\frac{\alpha\beta}{(\alpha + \beta)^2 - 4\alpha\beta}\right)^{12}$$

$$= \left[\frac{2(\alpha + \beta)}{(\alpha + \beta)^2 - 8(\alpha + \beta)}\right]^{12} \quad \text{[from Eq. (i)]}$$

$$= \left(\frac{2}{(\alpha + \beta) - 8}\right)^{12} = \left(\frac{2}{-\sin\theta - 8}\right)^{12} \quad [\because \alpha + \beta = -\sin\theta]$$

$$= \frac{2^{12}}{(\sin\theta + 8)^{12}}$$

2. Given quadratic equation is $x^2 + px + q = 0$, where $p, q \in \mathbf{R}$ having one root $2 - \sqrt{3}$, then other root is $2 + \sqrt{3}$ (conjugate of $2 - \sqrt{3}$) [$\because$ irrational roots of a quadratic equation always occurs in pairs]
So, sum of roots $= -p = 4 \Rightarrow p = -4$
and product of roots $= q = 4 - 3 \Rightarrow q = 1$
Now, from options $p^2 - 4q - 12 = 16 - 4 - 12 = 0$

3. Given quadratic equation is

$$(m^2 + 1)x^2 - 3x + (m^2 + 1)^2 = 0 \quad \ldots(i)$$

Let the roots of quadratic Eq. (i) are α and β, so $\alpha + \beta = \dfrac{3}{m^2 + 1}$ and $\alpha\beta = m^2 + 1$

According to the question, the sum of roots is greatest and it is possible only when "$(m^2 + 1)$ is minimum" and "minimum value of $m^2 + 1 = 1$, when $m = 0$".
$\therefore \alpha + \beta = 3$ and $\alpha\beta = 1$, as $m = 0$
Now, the absolute difference of the cubes of roots

$$= |\alpha^3 - \beta^3|$$
$$= |\alpha - \beta||\alpha^2 + \beta^2 + \alpha\beta|$$
$$= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}\,|(\alpha + \beta)^2 - \alpha\beta|$$
$$= \sqrt{9 - 4}\,|9 - 1| = 8\sqrt{5}$$

4. Given, α and β are the roots of the quadratic equation,

$$x^2 - 2x + 2 = 0$$
$$\Rightarrow \quad (x - 1)^2 + 1 = 0$$
$$\Rightarrow \quad (x - 1)^2 = -1$$
$$\Rightarrow \quad x - 1 = \pm i \quad [\text{where } i = \sqrt{-1}]$$
$$\Rightarrow \quad x = (1 + i) \text{ or } (1 - i)$$

Clearly, if $\alpha = 1 + i$, then $\beta = 1 - i$

According to the question $\left(\dfrac{\alpha}{\beta}\right)^n = 1$

$$\Rightarrow \quad \left(\frac{1 + i}{1 - i}\right)^n = 1$$

$$\Rightarrow \quad \left(\frac{(1 + i)(1 + i)}{(1 - i)(1 + i)}\right)^n = 1 \quad \text{[by rationalization]}$$

$$\Rightarrow \quad \left(\frac{1 + i^2 + 2i}{1 - i^2}\right)^n = 1 \Rightarrow \left(\frac{2i}{2}\right)^n = 1 \Rightarrow i^n = 1$$

So, minimum value of n is 4. $[\because i^4 = 1]$

5. **Key Idea**
(i) First convert the given equation in quadratic equation.
(ii) Use, Discriminant, $D = b^2 - 4ac < 0$

Given quadratic equation is

$$(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0 \quad \ldots(i)$$

Now, discriminant

$$D = [-2(1 + 3m)]^2 - 4(1 + m^2)(1 + 8m)$$
$$= 4[(1 + 3m)^2 - (1 + m^2)(1 + 8m)]$$
$$= 4[1 + 9m^2 + 6m - (1 + 8m + m^2 + 8m^3)]$$
$$= 4[-8m^3 + 8m^2 - 2m]$$
$$= -8m(4m^2 - 4m + 1) = -8m(2m - 1)^2$$

According to the question there is no solution of the quadratic Eq. (i), then

$$D < 0$$
$$\therefore \quad -8m(2m - 1)^2 < 0 \quad \Rightarrow \quad m > 0$$

So, there are infinitely many values of 'm' for which, there is no solution of the given quadratic equation.

6. The quadratic expression
$ax^2 + bx + c$, $x \in R$ is always positive,
if $a > 0$ and $D < 0$.
So, the quadratic expression
$(1 + 2m)x^2 - 2(1 + 3m)x + 4(1 + m)$, $x \in R$ will be always positive, if $1 + 2m > 0$ $\quad \ldots(i)$
and $D = 4(1 + 3m)^2 - 4(2m + 1)4(1 + m) < 0$ $\quad \ldots(ii)$
From inequality Eq. (i), we get

$$m > -\frac{1}{2} \quad \ldots(iii)$$

From inequality Eq. (ii), we get

$$1 + 9m^2 + 6m - 4(2m^2 + 3m + 1) < 0$$
$$\Rightarrow \quad m^2 - 6m - 3 < 0$$
$$\Rightarrow \quad [m - (3 + \sqrt{12})][m - (3 - \sqrt{12})] < 0$$
$$\left[\because m^2 - 6m - 3 = 0 \Rightarrow m = \frac{6 \pm \sqrt{36 + 12}}{2} = 3 \pm \sqrt{12}\right]$$

$\Rightarrow \quad 3-\sqrt{12} < m < 3+\sqrt{12}$ ...(iv)

From inequalities Eqs. (iii) and (iv), the integral values of m are $0, 1, 2, 3, 4, 5, 6$

Hence, the number of integral values of m is 7.

7. Let the given quadratic equation in x,
$3m^2x^2 + m(m-4)x + 2 = 0$, $m \neq 0$ have roots α and β, then

$$\alpha+\beta = -\frac{m(m-4)}{3m^2} \text{ and } \alpha\beta = \frac{2}{3m^2}$$

Also, let $\dfrac{\alpha}{\beta} = \lambda$

Then, $\lambda + \dfrac{1}{\lambda} = 1 \Rightarrow \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = 1$ (given)

$\Rightarrow \quad \alpha^2 + \beta^2 = \alpha\beta$

$\Rightarrow \quad (\alpha+\beta)^2 = 3\alpha\beta$

$\Rightarrow \quad \dfrac{m^2(m-4)^2}{9m^4} = 3\dfrac{2}{3m^2}$

$\Rightarrow \quad (m-4)^2 = 18$ $[\because m \neq 0]$

$\Rightarrow \quad m-4 = \pm 3\sqrt{2}$

$\Rightarrow \quad m = 4 \pm 3\sqrt{2}$

The least value of $m = 4 - 3\sqrt{2}$

8. Given quadratic equation is

$$81x^2 + kx + 256 = 0$$

Let one root be α, then other is α^3.

Now, $\alpha + \alpha^3 = -\dfrac{k}{81}$ and $\alpha \cdot \alpha^3 = \dfrac{256}{81}$

[$\because$ for $ax^2+bx+c=0$, sum of roots $= -\dfrac{b}{a}$ and product of roots $= \dfrac{c}{a}$]

$\Rightarrow \quad \alpha^4 = \left(\dfrac{4}{3}\right)^4 \Rightarrow \alpha = \pm\dfrac{4}{3}$

$\therefore \quad k = -81(\alpha + \alpha^3)$

$= -81\alpha(1+\alpha^2)$

$= -81\left(\pm\dfrac{4}{3}\right)\left(1+\dfrac{16}{9}\right) = \pm 300$

9. Let $a = 5$, $b = 5r$ and $c = 5r^2$

We know that, in a triangle sum of 2 sides is always greater than the third side.

$\therefore \quad a+b>c;\ b+c>a$ and $c+a>b$

Now, $\quad a+b>c$

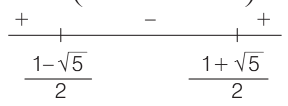
$\Rightarrow 5+5r>5r^2 \quad \Rightarrow 5r^2-5r-5<0$

$\Rightarrow r^2-r-1<0$

$\Rightarrow \left[r-\left(\dfrac{1-\sqrt{5}}{2}\right)\right]\left[r-\left(\dfrac{1+\sqrt{5}}{2}\right)\right]<0$

[$\because$ roots of $ax^2+bx+c=0$ are given by $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ and $r^2-r-1=0$ $\Rightarrow r = \dfrac{1\pm\sqrt{1+4}}{2} = \dfrac{1\pm\sqrt{5}}{2}$]

$\Rightarrow \quad r \in \left(\dfrac{1-\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$ ...(i)



Similarly, $\quad b+c>a$

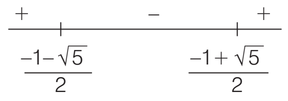
$\Rightarrow \quad 5r+5r^2>5$

$\Rightarrow \quad r^2+r-1>0$

$\left[r-\left(\dfrac{-1-\sqrt{5}}{2}\right)\right]\left[r-\left(\dfrac{-1+\sqrt{5}}{2}\right)\right]>0$

$\left[\because r^2+r-1=0 \Rightarrow r = \dfrac{-1\pm\sqrt{1+4}}{2} = \dfrac{-1\pm\sqrt{5}}{2}\right]$

$\Rightarrow \quad r \in \left(-\infty, \dfrac{-1-\sqrt{5}}{2}\right) \cup \left(\dfrac{-1+\sqrt{5}}{2}, \infty\right)$ ...(ii)



and $\quad c+a>b$

$\Rightarrow \quad 5r^2+5>5r$

$\Rightarrow \quad r^2-r+1>0$

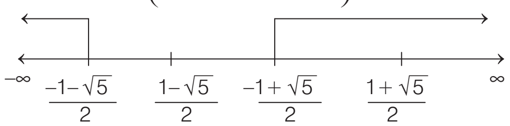
$\Rightarrow r^2 - 2\cdot\dfrac{1}{2}r + \left(\dfrac{1}{2}\right)^2 + 1 - \left(\dfrac{1}{2}\right)^2 > 0$

$\Rightarrow \quad \left(r-\dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$

$\Rightarrow \quad r \in R$ ... (iii)

From Eqs. (i), (ii) and (iii), we get

$$r \in \left(\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$



and $\dfrac{7}{4}$ is the only value that does not satisfy.

10. Given quadratic equation is

$$x^2 + (3-\lambda)x + 2 = \lambda$$

$$x^2 + (3-\lambda)x + (2-\lambda) = 0 \quad \text{... (i)}$$

Let Eq. (i) has roots α and β, then $\alpha+\beta = \lambda-3$ and $\alpha\beta = 2-\lambda$

[$\because$ For $ax^2+bx+c=0$, sum of roots $= -\dfrac{b}{a}$ and product of roots $= \dfrac{c}{a}$]

Now, $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (\lambda-3)^2 - 2(2-\lambda)$

$= \lambda^2 - 6\lambda + 9 - 4 + 2\lambda$

$= \lambda^2 - 4\lambda + 5 = (\lambda^2 - 4\lambda + 4) + 1$

$= (\lambda-2)^2 + 1$

Clearly, $\alpha^2+\beta^2$ will be least when $\lambda = 2$.

11. For the roots of quadratic equation $ax^2 + bx + c = 0$ to be rational $D = (b^2 - 4ac)$ should be perfect square.

In the equation $6x^2 - 11x + \alpha = 0$

$$a = 6,\ b = -11 \text{ and } c = \alpha$$

$\therefore$ For roots to be rational

$D = (-11)^2 - 4(6)(\alpha)$ should be a perfect square.

$\Rightarrow D(\alpha) = 121 - 24\alpha$ should be a perfect square

Now,

$D(1) = 121 - 24 = 97$ is not a perfect square.

$D(2) = 121 - 24 \times 2 = 73$ is not a perfect square.

$D(3) = 121 - 24 \times 3 = 49$ is a perfect square.

$D(4) = 121 - 24 \times 4 = 25$ is a perfect square.

$D(5) = 121 - 24 \times 5 = 1$ is a perfect square.

and for $\alpha \geq 6$, $D(\alpha) < 0$, hence imaginary roots.

$\therefore$ For 3 values of α ($\alpha = 3, 4, 5$), the roots are rational.

12. We have, $x^2 + 2x + 2 = 0$

$$\Rightarrow x = \frac{-2 \pm \sqrt{4-8}}{2} \quad \left[\because \text{roots of } ax^2 + bx + c = 0 \text{ are given by } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$\Rightarrow \quad x = -1 \pm i$$

Let $\alpha = -1 + i$ and $\beta = -1 - i$.

Then, $\alpha^{15} + \beta^{15} = (-1+i)^{15} + (-1-i)^{15}$

$$= -[(1-i)^{15} + (1+i)^{15}]$$

$$= -\left[\left\{\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right\}^{15} + \left\{\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right\}^{15}\right]$$

$$= -\left[\left\{\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right\}^{15} + \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^{15}\right]$$

$$= -(\sqrt{2})^{15}\left[\left(\cos\frac{15\pi}{4} - i\sin\frac{15\pi}{4}\right) + \left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right)\right]$$

[using De' Moivre's theorem $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta,\ n \in Z$]

$$= -(\sqrt{2})^{15}\left[2\cos\frac{15\pi}{4}\right] = -(\sqrt{2})^{15}\left[2 \times \frac{1}{\sqrt{2}}\right]$$

$$\left[\because \cos\frac{15\pi}{4} = \cos\left(4\pi - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

$$= -(\sqrt{2})^{16} = -2^8 = -256.$$

Alternate Method

$$\alpha^{15} + \beta^{15} = (-1+i)^{15} + (-1-i)^{15}$$

$$= -[(1-i)^{15} + (1+i)^{15}]$$

$$= -\left[\frac{(1-i)^{16}}{1-i} + \frac{(1+i)^{16}}{1+i}\right]$$

$$= -\left[\frac{[(1-i)^2]^8}{1-i} + \frac{[(1+i)^2]^8}{1+i}\right]$$

$$= -\left[\frac{[1+i^2-2i]^8}{1-i} + \frac{[1+i^2+2i]^8}{1+i}\right]$$

$$= -\left[\frac{(-2i)^8}{1-i} + \frac{(2i)^8}{1+i}\right]$$

$$= -2^8\left[\frac{1}{1-i} + \frac{1}{1+i}\right] \quad [\because i^{4n} = 1,\ n \in Z]$$

$$= -256\left[\frac{2}{1-(i)^2}\right] = -256\left[\frac{2}{2}\right] = -256$$

13. We have, $2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0$

Let $\sqrt{x} - 3 = y$

$\Rightarrow \quad \sqrt{x} = y + 3$

$\therefore \quad 2|y| + (y+3)(y-3) + 6 = 0$

$\Rightarrow \quad 2|y| + y^2 - 3 = 0$

$\Rightarrow \quad |y|^2 + 2|y| - 3 = 0$

$\Rightarrow \quad (|y| + 3)(|y| - 1) = 0$

$\Rightarrow \quad |y| \neq -3 \Rightarrow |y| = 1$

$\Rightarrow \quad y = \pm 1 \quad \Rightarrow \quad \sqrt{x} - 3 = \pm 1$

$\Rightarrow \quad \sqrt{x} = 4, 2 \quad \Rightarrow \quad x = 16, 4$

14. We have, α, β are the roots of $x^2 - x + 1 = 0$

$\because$ Roots of $x^2 - x + 1 = 0$ are $-\omega, -\omega^2$

$\therefore$ Let $\alpha = -\omega$ and $\beta = -\omega^2$

$\Rightarrow \alpha^{101} + \beta^{107} = (-\omega)^{101} + (-\omega^2)^{107} = -(\omega^{101} + \omega^{214})$

$= -(\omega^2 + \omega) \quad [\because \omega^3 = 1]$

$= -(-1) \quad [\because 1 + \omega + \omega^2 = 0]$

$= 1$

15. Given quadratic equation is

$x(x+1) + (x+1)(x+2) + \ldots + (x + \overline{n-1})(x+n) = 10n$

$\Rightarrow (x^2 + x^2 + \ldots + x^2) + [(1 + 3 + 5 + \ldots + (2n-1)]x + [(1\cdot 2 + 2\cdot 3 + \ldots + (n-1)n] = 10n$

$$\Rightarrow \quad nx^2 + n^2x + \frac{n(n^2-1)}{3} - 10n = 0$$

$$\Rightarrow \quad x^2 + nx + \frac{n^2-1}{3} - 10 = 0$$

$$\Rightarrow \quad 3x^2 + 3nx + n^2 - 31 = 0$$

Let α and β be the roots.

Since, α and β are consecutive.

$\therefore \quad |\alpha - \beta| = 1 \quad \Rightarrow \quad (\alpha - \beta)^2 = 1$

Again, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$\Rightarrow \quad 1 = \left(\frac{-3n}{3}\right)^2 - 4\left(\frac{n^2 - 31}{3}\right)$$

$$\Rightarrow \quad 1 = n^2 - \frac{4}{3}(n^2 - 31) \quad \Rightarrow \quad 3 = 3n^2 - 4n^2 + 124$$

$\Rightarrow \quad n^2 = 121 \quad \Rightarrow \quad n = \pm 11$

$\therefore \quad n = 11 \quad [\because n > 0]$

16. Given, $(x^2 - 5x + 5)^{x^2 + 4x - 60} = 1$

Clearly, this is possible when

I. $x^2 + 4x - 60 = 0$ and $x^2 - 5x + 5 \neq 0$

or

II. $x^2 - 5x + 5 = 1$

or

III. $x^2-5x+5=-1$ and $x^2+4x-60=$ Even integer.

***Case* I** When $x^2+4x-60=0$

$\Rightarrow \quad x^2+10x-6x-60=0$

$\Rightarrow \quad x(x+10)-6(x+10)=0$

$\Rightarrow \quad (x+10)(x-6)=0$

$\Rightarrow \quad x=-10$ or $x=6$

Note that, for these two values of x, $x^2-5x+5\neq 0$

***Case* II** When $x^2-5x+5=1$

$\Rightarrow \quad x^2-5x+4=0$

$\Rightarrow \quad x^2-4x-x+4=0$

$\Rightarrow \quad x(x-4)-1(x-4)=0$

$\Rightarrow \quad (x-4)(x-1)=0 \Rightarrow x=4$ or $x=1$

***Case* III** When $x^2-5x+5=-1$

$\Rightarrow \quad x^2-5x+6=0$

$\Rightarrow \quad x^2-2x-3x+6=0$

$\Rightarrow \quad x(x-2)-3(x-2)=0$

$\Rightarrow \quad (x-2)(x-3)=0$

$\Rightarrow \quad x=2$ or $x=3$

Now, when $x=2$, $x^2+4x-60=4+8-60=-48$, which is an even integer.

When $x=3$, $x^2+4x-60=9+12-60=-39$, which is not an even integer.

Thus, in this case, we get $x=2$.

Hence, the sum of all real values of

$$x=-10+6+4+1+2=3$$

17. Here, $x^2-2x\sec\theta+1=0$ has roots α_1 and β_1.

$$\therefore \quad \alpha_1,\ \beta_1=\frac{2\sec\theta\pm\sqrt{4\sec^2\theta-4}}{2\times 1}$$

$$=\frac{2\sec\theta\pm 2|\tan\theta|}{2}$$

Since, $\theta\in\left(-\frac{\pi}{6},-\frac{\pi}{12}\right)$,

i.e. $\theta\in$ IV quadrant $=\dfrac{2\sec\theta\mp 2\tan\theta}{2}$

$\therefore \quad \alpha_1=\sec\theta-\tan\theta$ and $\beta_1=\sec\theta+\tan\theta$ [as $\alpha_1>\beta_1$]

and $x^2+2x\tan\theta-1=0$ has roots α_2 and β_2.

i.e. $\quad \alpha_2,\ \beta_2=\dfrac{-2\tan\theta\pm\sqrt{4\tan^2\theta+4}}{2}$

$\therefore \quad \alpha_2=-\tan\theta+\sec\theta$

and $\quad \beta_2=-\tan\theta-\sec\theta$ [as $\alpha_2>\beta_2$]

Thus, $\quad \alpha_1+\beta_2=-2\tan\theta$

18. Given, α and β are the roots of the equation $x^2-6x-2=0$.

$\therefore \quad a_n=\alpha^n-\beta^n$ for $n\geq 1$

$a_{10}=\alpha^{10}-\beta^{10}$

$a_8=\alpha^8-\beta^8$

$a_9=\alpha^9-\beta^9$

Now, consider

$$\frac{a_{10}-2a_8}{2a_9}=\frac{\alpha^{10}-\beta^{10}-2(\alpha^8-\beta^8)}{2(\alpha^9-\alpha^9)}$$

$$=\frac{\alpha^8(\alpha^2-2)-\beta^8(\beta^2-2)}{2(\alpha^9-\beta^9)}$$

$$=\frac{\alpha^8\cdot 6\alpha-\beta^8 6\beta}{2(\alpha^9-\beta^9)}=\frac{6\alpha^9-6\beta^9}{2(\alpha^9-6\beta^9)}=\frac{6}{2}=3$$

$\left[\begin{array}{l}\because \alpha \text{ and } \beta \text{ are the roots of} \\ \quad x^2-6x-2=0 \text{ or } x^2=6x+2 \\ \Rightarrow \quad \alpha^2=6\alpha+2 \ \Rightarrow\ \alpha^2-2=6\alpha \\ \text{and} \quad \beta^2=6\beta+2 \ \Rightarrow\ \beta^2-2=6\beta\end{array}\right]$

Alternate Solution

Since, α and β are the roots of the equation

$$x^2-6x-2=0.$$

or $\quad x^2=6x+2$

$\therefore \quad \alpha^2=6\alpha+2$

$\Rightarrow \quad \alpha^{10}=6\alpha^9+2\alpha^8 \quad$ ...(i)

Similarly, $\quad \beta^{10}=6\beta^9+2\beta^8 \quad$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$\alpha^{10}-\beta^{10}=6(\alpha^9-\beta^9)+2(\alpha^8-\beta^8) \quad (\because a_n=\alpha^n-\beta^n)$

$\Rightarrow \quad a_{10}=6a_9+2a_8$

$\Rightarrow \quad a_{10}-2a_8=6a_9 \Rightarrow \dfrac{a_{10}-2a_8}{2a_9}=3$

19. If quadratic equation has purely imaginary roots, then coefficient of x must be equal to zero.

Let $p(x)=ax^2+b$ with a, b of same sign and $a,b\in R$.

Then, $\quad p[p(x)]=a(ax^2+b)^2+b$

$p(x)$ has imaginary roots say ix.

Then, also $ax^2+b\in R$ and $(ax^2+b)^2>0$

$\therefore \quad a(ax^2+b)^2+b\neq 0,\ \forall\ x$

Thus, $\quad p[p(x)]\neq 0,\ \forall\ x$

20. **PLAN** If $ax^2+bx+c=0$ has roots α and β, then $\alpha+\beta=-b/a$ and $\alpha\beta=\dfrac{c}{a}$. Find the values of $\alpha+\beta$ and $\alpha\beta$ and then put in $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$ to get required value.

Given, α and β are roots of $px^2+qx+r=0$, $p\neq 0$.

$\therefore \quad \alpha+\beta=\dfrac{-q}{p},\quad \alpha\beta=\dfrac{r}{p} \quad$ ...(i)

Since, p, q and r are in AP.

$\therefore \quad 2q=p+r \quad$ ...(ii)

Also, $\quad \dfrac{1}{\alpha}+\dfrac{1}{\beta}=4 \Rightarrow \dfrac{\alpha+\beta}{\alpha\beta}=4$

$\Rightarrow \quad \alpha+\beta=4\alpha\beta \Rightarrow \dfrac{-q}{p}=\dfrac{4r}{p} \quad$ [from Eq. (i)]

$\Rightarrow \quad q=-4r$

On putting the value of q in Eq. (ii), we get

$\Rightarrow \quad 2(-4r)=p+r \Rightarrow p=-9r$

Now, $\alpha+\beta=\frac{-q}{p}=\frac{4r}{p}=\frac{4r}{-9r}=-\frac{4}{9}$

and $\alpha\beta=\frac{r}{p}=\frac{r}{-9r}=\frac{1}{-9}$

$\therefore$ $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=\frac{16}{81}+\frac{4}{9}=\frac{16+36}{81}$

$\Rightarrow$ $(\alpha-\beta)^2=\frac{52}{81}$

$\Rightarrow$ $|\alpha-\beta|=\frac{2}{9}\sqrt{13}$

21. $\frac{a_{10}-2a_8}{2a_9}=\frac{(\alpha^{10}-\beta^{10})-2(\alpha^8-\beta^8)}{2(\alpha^9-\beta^9)}$

$=\frac{\alpha^8(\alpha^2-2)-\beta^8(\beta^2-2)}{2(\alpha^9-\beta^9)}$

$\because \alpha$ is root of $x^2-6x-2=0 \Rightarrow \alpha^2-2=6\alpha$

[and β is root of $x^2-6x-2=0 \Rightarrow \beta^2-2=6\beta$]

$=\frac{\alpha^8(6\alpha)-\beta^8(6\beta)}{2(\alpha^9-\beta^9)}=\frac{6(\alpha^9-\beta^9)}{2(\alpha^9-\beta^9)}=3$

22. Sum of roots $=\frac{\alpha^2+\beta^2}{\alpha\beta}$ and product $=1$

Given, $\alpha+\beta=-p$ and $\alpha^3+\beta^3=q$

$\Rightarrow$ $(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)=q$

$\therefore$ $\alpha^2+\beta^2-\alpha\beta=\frac{-q}{p}$ ...(i)

and $(\alpha+\beta)^2=p^2$

$\Rightarrow$ $\alpha^2+\beta^2+2\alpha\beta=p^2$ ...(ii)

From Eqs. (i) and (ii), we get

$\alpha^2+\beta^2=\frac{p^3-2q}{3p}$ and $\alpha\beta=\frac{p^3+q}{3p}$

$\therefore$ Required equation is, $x^2-\frac{(p^3-2q)x}{(p^3+q)}+1=0$

$\Rightarrow (p^3+q)x^2-(p^3-2q)x+(p^3+q)=0$

23. The equation $x^2-px+r=0$ has roots α, β and the equation $x^2-qx+r=0$ has roots $\frac{\alpha}{2}$, 2β.

$\Rightarrow$ $r=\alpha\beta$ and $\alpha+\beta=p$,

and $\frac{\alpha}{2}+2\beta=q \Rightarrow \beta=\frac{2q-p}{3}$ and $\alpha=\frac{2(2p-q)}{3}$

$\Rightarrow$ $\alpha\beta=r=\frac{2}{9}(2q-p)(2p-q)$

24. Since, roots are real, therefore $D\ge 0$

$\Rightarrow$ $4(a+b+c)^2-12\lambda(ab+bc+ca)\ge 0$

$\Rightarrow$ $(a+b+c)^2\ge 3\lambda(ab+bc+ca)$

$\Rightarrow$ $a^2+b^2+c^2\ge(ab+bc+ca)(3\lambda-2)$

$\Rightarrow$ $3\lambda-2\le\frac{a^2+b^2+c^2}{ab+bc+ca}$ ...(i)

Also, $\cos A=\frac{b^2+c^2-a^2}{2bc}<1$

$\Rightarrow$ $b^2+c^2-a^2<2bc$

Similarly, $c^2+a^2-b^2<2ca$

and $a^2+b^2-c^2<2ab$

$\Rightarrow$ $a^2+b^2+c^2<2(ab+bc+ca)$

$\Rightarrow$ $\frac{a^2+b^2+c^2}{ab+bc+ca}<2$ ...(ii)

From Eqs. (i) and (ii), we get

$3\lambda-2<2 \Rightarrow \lambda<\frac{4}{3}$

25. Let the roots of $x^2+px+q=0$ be α and α^2.

$\Rightarrow$ $\alpha+\alpha^2=-p$ and $\alpha^3=q$

$\Rightarrow$ $\alpha(\alpha+1)=-p$

$\Rightarrow$ $\alpha^3\{\alpha^3+1+3\alpha(\alpha+1)\}=-p^3$ [cubing both sides]

$\Rightarrow$ $q(q+1-3p)=-p^3$

$\Rightarrow$ $p^3-(3p-1)q+q^2=0$

26. Given, $x^2-|x+2|+x>0$ ...(i)

***Case* I** When $x+2\ge 0$

$\therefore$ $x^2-x-2+x>0 \Rightarrow x^2-2>0$

$\Rightarrow$ $x<-\sqrt{2}$ or $x>\sqrt{2}$

$\Rightarrow$ $x\in[-2,-\sqrt{2})\cup(\sqrt{2},\infty)$ ...(ii)

***Case* II** When $x+2<0$

$\therefore$ $x^2+x+2+x>0$

$\Rightarrow$ $x^2+2x+2>0$

$\Rightarrow$ $(x+1)^2+1>0$

which is true for all x.

$\therefore$ $x\le -2$ or $x\in(-\infty,-2)$ ...(iii)

From Eqs. (ii) and (iii), we get

$x\in(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$

27. Given, $\log_4(x-1)=\log_2(x-3)=\log_{4^{1/2}}(x-3)$

$\Rightarrow$ $\log_4(x-1)=2\log_4(x-3)$

$\Rightarrow$ $\log_4(x-1)=\log_4(x-3)^2$

$\Rightarrow$ $(x-3)^2=x-1$

$\Rightarrow$ $x^2+9-6x=x-1$

$\Rightarrow$ $x^2-7x+10=0$

$\Rightarrow$ $(x-2)(x-5)=0$

$\Rightarrow$ $x=2$, or $x=5$

$\Rightarrow$ $x=5$ [$\because x=2$ makes $\log(x-3)$ undefined].

Hence, one solution exists.

28. Let α,α^2 be the roots of $3x^2+px+3=0$

Now, $S=\alpha+\alpha^2=-p/3$,

$P=\alpha^3=1$

$\Rightarrow$ $\alpha=1,\omega,\omega^2$

Now, $\alpha+\alpha^2=-p/3$

$\Rightarrow$ $\omega+\omega^2=-p/3$

$\Rightarrow \quad -1 = -p/3$

$\Rightarrow \quad p = 3$

29. Given, $c < 0 < b$

Since, $\alpha + \beta = -b$ ...(i)

and $\alpha\beta = c$ ...(ii)

From Eq. (ii), $c < 0 \Rightarrow \alpha\beta < 0$

$\Rightarrow$ Either α is –ve, β is –ve or α is + ve, β is – ve.

From Eq. (i), $b > 0 \Rightarrow -b < 0 \Rightarrow \alpha + \beta < 0 \Rightarrow$ the sum is negative.

$\Rightarrow$ Modulus of negative quantity is > modulus of positive quantity but $\alpha < \beta$ is given.

Therefore, it is clear that α is negative and β is positive and modulus of α is greater than modulus of β

$\Rightarrow \quad \alpha < 0 < \beta < |\alpha|$

NOTE This question is not on the theory of interval in which root lie, which appears looking at first sight. It is new type and first time asked in the paper. It is important for future. The actual type is interval in which parameter lie.

30. Since, $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$

$\Rightarrow \quad (x+1) + (x-1) - 2\sqrt{x^2-1} = 4x-1$

$\Rightarrow \quad 1 - 2x = 2\sqrt{x^2-1} \Rightarrow 1 + 4x^2 - 4x = 4x^2 - 4$

$\Rightarrow \quad 4x = 5 \Rightarrow x = \frac{5}{4}$

But it does not satisfy the given equation.

Hence, no solution exists.

31. Given, $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$

$\Rightarrow \quad \frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4} = \log_x \sqrt{2}$

$\Rightarrow \quad \frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4} = \frac{1}{2\log_2 x}$

$\Rightarrow \quad 3(\log_2 x)^3 + 4(\log_2 x)^2 - 5(\log_2 x) - 2 = 0$

Put $\log_2 x = y$

$\therefore \quad 3y^3 + 4y^2 - 5y - 2 = 0$

$\Rightarrow \quad (y-1)(y+2)(3y+1) = 0$

$\Rightarrow \quad y = 1, -2, -1/3$

$\Rightarrow \quad \log_2 x = 1, -2, -1/3$

$\Rightarrow \quad x = 2, \frac{1}{2^{1/3}}, \frac{1}{4}$

32. Since, α, β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$.

$\Rightarrow \alpha + \beta = -p; \alpha\beta = q; \alpha^4 + \beta^4 = r$ and $\alpha^4\beta^4 = s$

Let roots of $x^2 - 4qx + (2q^2 - r) = 0$ be α' and β'

Now, $\alpha'\beta' = (2q^2 - r) = 2(\alpha\beta)^2 - (\alpha^4 + \beta^4)$

$= -(\alpha^4 + \beta^4 - 2\alpha^2\beta^2) = -(\alpha^2 - \beta^2)^2 < 0$

$\Rightarrow$ Roots are real and of opposite sign.

33. Given, $x - \frac{2}{x-1} = 1 - \frac{2}{x-1} \Rightarrow x = 1$

But at $x = 1$, the given equation is not defined.

Hence, no solution exist.

34. Let $y = \frac{x^2 - (a+b)x + ab}{x - c}$

$\Rightarrow \quad yx - cy = x^2 - (a+b)x + ab$

$\Rightarrow x^2 - (a+b+y)x + (ab+cy) = 0$

For real roots, $D \geq 0$

$\Rightarrow \quad (a+b+y)^2 - 4(ab+cy) \geq 0$

$\Rightarrow (a+b)^2 + y^2 + 2(a+b)y - 4ab - 4cy \geq 0$

$\Rightarrow \quad y^2 + 2(a+b-2c)y + (a-b)^2 \geq 0$

which is true for all real values of y.

$\therefore \quad D \leq 0$

$4(a+b-2c)^2 - 4(a-b)^2 \leq 0$

$\Rightarrow 4(a+b-2c+a-b)(a+b-2c-a+b) \leq 0$

$\Rightarrow \quad (2a-2c)(2b-2c) \leq 0$

$\Rightarrow \quad (a-c)(b-c) \leq 0$

$\Rightarrow \quad (c-a)(c-b) \leq 0$

$\Rightarrow c$ must lie between a and b

i.e. $a \leq c \leq b$ or $b \leq c \leq a$

35. Since, $|x|^2 - 3|x| + 2 = 0$

$\Rightarrow \quad (|x|-1)(|x|-2) = 0$

$\Rightarrow \quad |x| = 1, 2$

$\therefore \quad x = 1, -1, 2, -2$

Hence, four real solutions exist.

36. $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$

$\Rightarrow \quad 3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$

Now, discriminant $= 4(a+b+c)^2 - 12(ab+bc+ca)$

$= 4\{a^2 + b^2 + c^2 - ab - bc - ca\}$

$= 2\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$

which is always positive.

Hence, both roots are real.

37. Since, $a, b, c > 0$ and $ax^2 + bx + c = 0$

$\Rightarrow \quad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

***Case* I** When $b^2 - 4ac > 0$

$\Rightarrow \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$

and $\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ both roots, are negative.

***Case* II** When $b^2 - 4ac = 0$

$\Rightarrow x = \frac{-b}{2a}$, i.e. both roots are equal and negative

***Case* III** When $b^2 - 4ac < 0$

$\Rightarrow \quad x = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$

have negative real part.

$\therefore$ From above discussion, both roots have negative real parts.

38. Given, $x^2+2px+q=0$

$\therefore \quad \alpha+\beta=-2p$ ... (i)

$\alpha\beta=q$ ... (ii)

And $ax^2+2bx+c=0$

$\therefore \quad \alpha+\frac{1}{\beta}=-\frac{2b}{a}$ ... (iii)

and $\frac{\alpha}{\beta}=\frac{c}{a}$ ... (iv)

Now, $(p^2-q)(b^2-ac)$

$$=\left[\left(\frac{\alpha+\beta}{-2}\right)^2-\alpha\beta\right]\left[\left(\frac{\alpha+\frac{1}{\beta}}{2}\right)^2-\frac{\alpha}{\beta}\right]a^2$$

$$=\frac{(\alpha-\beta)^2}{16}\left(\alpha-\frac{1}{\beta}\right)^2\cdot a^2\geq 0$$

$\therefore$ Statement I is true.

Again, now $pa=-\left(\frac{\alpha+\beta}{2}\right)a=-\frac{a}{2}(\alpha+\beta)$

and $b=-\frac{a}{2}\left(\alpha+\frac{1}{\beta}\right)$

Since, $pa\neq b \Rightarrow \alpha+\frac{1}{\beta}\neq\alpha+\beta$

$\Rightarrow \quad \beta^2\neq 1,\ \beta\neq\{-1,0,1\}$, which is correct.

Similarly, if $c\neq qa$

$\Rightarrow \quad a\frac{\alpha}{\beta}\neq a\alpha\beta \Rightarrow \alpha\left(\beta-\frac{1}{\beta}\right)\neq 0$

$\Rightarrow \quad \alpha\neq 0$ and $\beta-\frac{1}{\beta}\neq 0$

$\Rightarrow \quad \beta\neq\{-1,0,1\}$

Statement II is true.

Both Statement I and Statement II are true. But Statement II does not explain Statement I.

39. Given, $|x-2|^2+|x-2|-2=0$

***Case* I** When $x\geq 2$

$\Rightarrow \quad (x-2)^2+(x-2)-2=0$

$\Rightarrow \quad x^2+4-4x+x-2-2=0$

$\Rightarrow \quad x^2-3x=0$

$\Rightarrow \quad x(x-3)=0$

$\Rightarrow \quad x=0,3$ [0 is rejected]

$\Rightarrow \quad x=3$ ...(i)

***Case* II** When $x<2$

$\Rightarrow \quad \{-(x-2)\}^2-(x-2)-2=0$

$\Rightarrow \quad (x-2)^2-x+2-2=0$

$\Rightarrow \quad x^2+4-4x-x=0$

$\Rightarrow \quad x^2-4x-(x-4)=0$

$\Rightarrow \quad x(x-4)-1(x-4)=0$

$\Rightarrow \quad (x-1)(x-4)=0$

$\Rightarrow \quad x=1,4$ [4 is rejected]

$\Rightarrow \quad x=1$ ...(ii)

Hence, the sum of the roots is $3+1=4$.

Alternate Solution

Given, $|x-2|^2+|x-2|-2=0$

$\Rightarrow (|x-2|+2)(|x-2|-1)=0$

$\therefore \quad |x-2|=-2,1$ [neglecting –2]

$\Rightarrow \quad |x-2|=1 \Rightarrow x=3,1$

$\Rightarrow$ Sum of the roots $=4$

40. Since, $x^2-3kx+2e^{2\log k}-1=0$ has product of roots 7.

$\Rightarrow \quad 2e^{2\log k}-1=7$

$\Rightarrow \quad e^{2\log_e k}=4$

$\Rightarrow \quad k^2=4$

$\Rightarrow \quad k=2$ [neglecting –2]

41. If $2+i\sqrt{3}$ is one of the root of $x^2+px+q=0$. Then, other root is $2-i\sqrt{3}$.

$\Rightarrow \quad -p=2+i\sqrt{3}+2-i\sqrt{3}=4$

and $q=(2+i\sqrt{3})(2-i\sqrt{3})=7$

$\Rightarrow \quad (p,q)=(-4,7)$

42. The coefficient of x^{99} in $(x-1)(x-2)\dots(x-100)$

$$=-(1+2+3+\dots+100)$$
$$=-\frac{100}{2}(1+100)=-50(101)=-5050$$

43. $P(x)\cdot Q(x)=(ax^2+bx+c)(-ax^2+bx+c)$

Now, $D_1=b^2-4ac$ and $D_2=b^2+4ac$

Clearly, $D_1+D_2=2b^2\geq 0$

$\therefore$ Atleast one of D_1 and D_2 is (+ ve). Hence, atleast two real roots.

Hence, statement is true.

44. Given, $2x^2+3x+1=0$

Here, $D=(3)^2-4\cdot 2\cdot 1=1$ which is a perfect square.

$\therefore$ Roots are rational.

Hence, statement is false.

45. Here, $a+b=10c$ and $c+d=10a$

$\Rightarrow \quad (a-c)+(b-d)=10(c-a)$

$\Rightarrow \quad (b-d)=11(c-a)$ ...(i)

Since, 'c' is the root of $x^2-10ax-11b=0$

$\Rightarrow \quad c^2-10ac-11b=0$ ...(ii)

Similarly, 'a' is the root of

$x^2-10cx-11d=0$

$\Rightarrow \quad a^2-10ca-11d=0$ ...(iii)

On subtracting Eq. (iv) from Eq. (ii), we get

$(c^2-a^2)=11(b-d)$ ...(iv)

$\therefore \quad (c+a)(c-a)=11\times 11(c-a)$ [from Eq. (i)]

$\Rightarrow \quad c+a=121$

$\therefore \quad a+b+c+d=10c+10a$

$=10(c+a)=1210$

46. Since, $\alpha + \beta = -\dfrac{b}{a}, \alpha\beta = \dfrac{c}{a}$

and $\quad \alpha + \delta + \beta + \delta = -\dfrac{B}{A}, (\alpha + \delta)(\beta + \delta) = \dfrac{C}{A}$

Now, $\quad \alpha - \beta = (\alpha + \delta) - (\beta + \delta)$

$\Rightarrow \quad (\alpha - \beta)^2 = [(\alpha + \delta) - (\beta + \delta)]^2$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = (\overline{\alpha + \delta} + \overline{\beta + \delta})^2 - 4(\alpha + \delta)(\beta + \delta)$

$\Rightarrow \quad \left(-\dfrac{b}{a}\right)^2 - \dfrac{4c}{a} = \left(-\dfrac{B}{A}\right)^2 - \dfrac{4C}{A}$

$\Rightarrow \quad \dfrac{b^2}{a^2} - \dfrac{4c}{a} = \dfrac{B^2}{A^2} - \dfrac{4C}{A}$

$\Rightarrow \quad \dfrac{b^2 - 4ac}{a^2} = \dfrac{B^2 - 4AC}{A^2}$

47. Suppose $f(x) = Ax^2 + Bx + C$ is an integer, whenever x is an integer.

$\therefore \quad f(0), f(1), f(-1)$ are integers.

$\Rightarrow \quad C, A + B + C, A - B + C$ are integers.

$\Rightarrow \quad C, A + B, A - B$ are integers.

$\Rightarrow \quad C, A + B, (A + B) - (A - B) = 2A$ are integers.

Conversely, suppose $2A$, $A + B$ and C are integers.

Let n be any integer. We have,

$f(n) = An^2 + Bn + C = 2A\left[\dfrac{n(n-1)}{2}\right] + (A + B)n + C$

Since, n is an integer, $n(n-1)/2$ is an integer. Also, $2A$, $A + B$ and C are integers.

We get $f(n)$ is an integer for all integer n.

48. Given, $2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1$

***Case* I** When $y \in (-\infty, 0]$

$\therefore \quad 2^{-y} + (2^{y-1} - 1) = 2^{y-1} + 1$

$\Rightarrow \quad 2^{-y} = 2$

$\Rightarrow \quad y = -1 \in (-\infty, 0] \quad \ldots(i)$

***Case* II** When $y \in (0, 1]$

$\therefore \quad 2^y + (2^{y-1} - 1) = 2^{y-1} + 1$

$\Rightarrow \quad 2^y = 2$

$\Rightarrow \quad y = 1 \in (0, 1] \quad \ldots(ii)$

***Case* III** When $y \in (1, \infty)$

$\therefore \quad 2^y - 2^{y-1} + 1 = 2^{y-1} + 1$

$\Rightarrow \quad 2^y - 2 \cdot 2^{y-1} = 0$

$\Rightarrow \quad 2^y - 2^y = 0$ true for all $y > 1 \quad \ldots(iii)$

From Eqs. (i), (ii) and (iii), we get

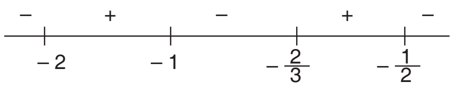
$y \in \{-1\} \cup [1, \infty)$.

49. Given, $\quad \dfrac{2x}{2x^2 + 5x + 2} > \dfrac{1}{x + 1}$

$\Rightarrow \quad \dfrac{2x}{(2x + 1)(x + 2)} - \dfrac{1}{(x + 1)} > 0$

$\Rightarrow \quad \dfrac{2x(x + 1) - (2x + 1)(x + 2)}{(2x + 1)(x + 2)(x + 1)} > 0$

$\Rightarrow \quad \dfrac{-(3x + 2)}{(2x + 1)(x + 1)(x + 2)} > 0$; using number line rule



$\therefore \quad x \in (-2, -1) \cup \left(-\dfrac{2}{3}, -\dfrac{1}{2}\right)$

50. Given, $|x^2 + 4x + 3| + 2x + 5 = 0$

***Case* I** $\quad x^2 + 4x + 3 > 0 \Rightarrow (x < -3$ or $x > -1)$

$\therefore \quad x^2 + 4x + 3 + 2x + 5 = 0$

$\Rightarrow \quad x^2 + 6x + 8 = 0 \Rightarrow (x + 4)(x + 2) = 0$

$\Rightarrow \quad x = -4, -2 \quad$ [but $x < -3$ or $x > -1$]

$\therefore \quad x = -4$ is the only solution. $\quad \ldots(i)$

***Case* II** $\quad x^2 + 4x + 3 < 0 \Rightarrow (-3 < x < -1)$

$\therefore \quad -x^2 - 4x - 3 + 2x + 5 = 0$

$\Rightarrow \quad x^2 + 2x - 2 = 0 \Rightarrow (x + 1)^2 = 3$

$\Rightarrow \quad |x + 1| = \sqrt{3}$

$\Rightarrow \quad x = -1 - \sqrt{3}, -1 + \sqrt{3} \quad$ [but $x \in (-3, -1)$]

$\therefore \quad x = -1 - \sqrt{3}$ is the only solution. $\quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$x = -4$ and $(-1 - \sqrt{3})$ are the only solutions.

51. Here, $\quad a \le 0$

Given, $\quad x^2 - 2a|x - a| - 3a^2 = 0$

***Case* I** When $x \ge a$

$\Rightarrow \quad x^2 - 2a(x - a) - 3a^2 = 0$

$\Rightarrow \quad x^2 - 2ax - a^2 = 0$

$\Rightarrow \quad x = a \pm \sqrt{2}a$

[as $a(1 + \sqrt{2}) < a$ and $a(1 - \sqrt{2}) > a$]

$\therefore$ Neglecting $x = a(1 + \sqrt{2})$ as $x \ge a$

$\Rightarrow \quad x = a(1 - \sqrt{2}) \quad \ldots(i)$

***Case* II** When $x < a \Rightarrow x^2 + 2a(x - a) - 3a^2 = 0$

$\Rightarrow \quad x^2 + 2ax - 5a^2 = 0 \Rightarrow x = -a \pm \sqrt{6}a$

[as $a(\sqrt{6} - 1) < a$ and $a(-1 - \sqrt{6}) > a$]

$\therefore$ Neglecting $x = a(-1 - \sqrt{6}) \Rightarrow x = a(\sqrt{6} - 1) \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$x = \{a(1 - \sqrt{2}), a(\sqrt{6} - 1)\}$

52. Given, $(5 + 2\sqrt{6})^{x^2 - 3} + (5 - 2\sqrt{6})^{x^2 - 3} = 10 \quad \ldots(i)$

Put $y = (5 + 2\sqrt{6})^{x^2 - 3} \Rightarrow (5 - 2\sqrt{6})^{x^2 - 3} = \dfrac{1}{y}$

From Eq. (i), $y + \dfrac{1}{y} = 10$

$\Rightarrow \quad y^2 - 10y + 1 = 0 \Rightarrow y = 5 \pm 2\sqrt{6}$

$\Rightarrow \quad (5 + 2\sqrt{6})^{x^2 - 3} = 5 + 2\sqrt{6}$

or $\quad (5 + 2\sqrt{6})^{x^2 - 3} = 5 - 2\sqrt{6}$

$\Rightarrow \quad x^2 - 3 = 1 \quad$ or $\quad x^2 - 3 = -1$

$\Rightarrow \quad x = \pm 2$ or $x = \pm\sqrt{2}$

$\Rightarrow \quad x = \pm 2, \pm\sqrt{2}$

53. Let α, β are roots of $ax^2 + bx + c = 0$

Given, $\alpha = \beta^n$

$\Rightarrow \alpha\beta = \dfrac{c}{a} \Rightarrow \beta^{n+1} = \dfrac{c}{a}$

$\Rightarrow \beta = \left(\dfrac{c}{a}\right)^{1/(n+1)}$

It must satisfy $ax^2 + bx + c = 0$

i.e. $a\left(\dfrac{c}{a}\right)^{2/(n+1)} + b\left(\dfrac{c}{a}\right)^{1/(n+1)} + c = 0$

$\Rightarrow \dfrac{a \cdot c^{2/(n+1)}}{a^{2/(n+1)}} + \dfrac{b \cdot c^{1/(n+1)}}{a^{1/(n+1)}} + c = 0$

$\Rightarrow \dfrac{c^{1/(n+1)}}{a^{1/(n+1)}}\left\{\dfrac{a \cdot c^{1/(n+1)}}{a^{1/(n+1)}} + b + \dfrac{c \cdot a^{1/(n+1)}}{c^{1/(n+1)}}\right\} = 0$

$\Rightarrow a^{n/(n+1)}c^{1/(n+1)} + b + c^{n/(n+1)}a^{1/(n+1)} = 0$

$\Rightarrow (a^n c)^{1/(n+1)} + (c^n a)^{1/(n+1)} + b = 0$

54. Since, α, β are the roots of $x^2 + px + q = 0$

and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \alpha + \beta = -p, \ \alpha\beta = q$

and $\gamma + \delta = -r, \gamma\delta = s$

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - (\gamma + \delta)\alpha + \gamma\delta][\beta^2 - (\gamma + \delta)\beta + \gamma\delta]$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$

$= (\alpha\beta)^2 + r(\alpha + \beta)\alpha\beta + s(\alpha^2 + \beta^2) + \alpha\beta r^2 + rs(\alpha + \beta) + s^2$

$= q^2 - rqp + s(p^2 - 2q) + qr^2 - rsp + s^2$

$= (q - s)^2 - rqp - rsp + sp^2 + qr^2$

55. The given equation can be rewritten as

$\dfrac{2}{\log_a x} + \dfrac{1}{\log_a ax} + \dfrac{3}{\log_a a^2x} = 0$ $[\because b = a^2x, \text{ given}]$

$\Rightarrow \dfrac{2}{\log_a x} + \dfrac{1}{1 + \log_a x} + \dfrac{3}{2 + \log_a x} = 0$

$\Rightarrow \dfrac{2}{t} + \dfrac{1}{1 + t} + \dfrac{3}{2 + t} = 0$, where $t = \log_a x$

$\Rightarrow 2(1 + t)(2 + t) + 3t(1 + t) + t(2 + t) = 0$

$\Rightarrow 6t^2 + 11t + 4 = 0$

$\Rightarrow (2t + 1)(3t + 4) = 0$

$\Rightarrow t = -\dfrac{1}{2}$ or $-\dfrac{4}{3}$

$\therefore \log_a x = -\dfrac{1}{2}$ or $\log_a x = -\dfrac{4}{3}$

$\Rightarrow x = a^{-1/2}$

or $x = a^{-4/3}$

56. Since, $\alpha + \beta = -p, \alpha\beta = 1$ and $\gamma + \delta = -q, \gamma\delta = 1$

Now, $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$

$= \{\alpha\beta - \gamma(\alpha + \beta) + \gamma^2\}\{\alpha\beta + \delta(\alpha + \beta) + \delta^2\}$

$= \{1 - \gamma(-p) + \gamma^2\}\{1 + \delta(-p) + \delta^2\}$

$= (1 + \gamma^2 + \gamma p)(1 - \delta p + \delta^2) = (-q\gamma + \gamma p)(-\delta p - \delta q)$

$[\because \gamma^2 + q\gamma + 1 = 0 \text{ and } \delta^2 + q\delta + 1 = 0]$

$= (q^2 - p^2)(\gamma\delta) = q^2 - p^2$ $[\because \gamma\delta = 1]$

57. $\alpha^2 = \alpha + 1$

$\beta^2 = \beta + 1$

$a_n = p\alpha^n + q\beta^n$

$= p(\alpha^{n-1} + \alpha^{n-2}) + q(\beta^{n-1} + \beta^{n-2})$

$= a_{n-1} + a_{n-2}$

$\therefore a_{12} = a_{11} + a_{10}$

58. $\alpha = \dfrac{1 + \sqrt{5}}{2}, \beta = \dfrac{1 - \sqrt{5}}{2}$

$a_4 = a_3 + a_2$

$= 2a_2 + a_1$

$= 3a_1 + 2a_0$

$28 = p(3\alpha + 2) + q(3\beta + 2)$

$28 = (p + q)\left(\dfrac{3}{2} + 2\right) + (p - q)\left(\dfrac{3\sqrt{5}}{2}\right)$

$\therefore p - q = 0$

and $(p + q) \times \dfrac{7}{2} = 28$

$\Rightarrow p + q = 8$

$\Rightarrow p = q = 4$

$\therefore p + 2q = 12$

Topic 2 Common Roots

1 Given α, β and γ are three consecutive terms of a non-constant GP.

Let $\alpha = \alpha, \beta = \alpha r, \gamma = \alpha r^2, \{r \neq 0, 1\}$

and given quadratic equation is

$$\alpha x^2 + 2\beta x + \gamma = 0 \quad \text{...(i)}$$

On putting the values of α, β, γ in Eq. (i), we get

$\alpha x^2 + 2\alpha r x + \alpha r^2 = 0$

$\Rightarrow x^2 + 2rx + r^2 = 0$

$\Rightarrow (x + r)^2 = 0$

$\Rightarrow x = -r$

$\because$ The quadratic equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, so $x = -r$ must be root of equation $x^2 + x - 1 = 0$, so

$$r^2 - r - 1 = 0 \quad \text{...(ii)}$$

Now, $\alpha(\beta + \gamma) = \alpha(\alpha r + \alpha r^2)$

$= \alpha^2(r + r^2)$

From the options,

$\beta\gamma = \alpha r \cdot \alpha r^2 = \alpha^2 r^3 = \alpha^2(r + r^2)$

$[\because r^2 - r - 1 = 0 \Rightarrow r^3 = r + r^2]$

$\therefore \alpha(\beta + \gamma) = \beta\gamma$

2. Given equations are $x^2 + 2x + 3 = 0$ ...(i)

and $ax^2 + bx + c = 0$ ...(ii)

Since, Eq. (i) has imaginary roots, so Eq. (ii) will also have both roots same as Eq. (i).

Thus, $\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$

Hence, $a : b : c$ is $1 : 2 : 3$.

3. If $a_1x^2 + b_1x + c_1 = 0$

and $a_2x^2 + b_2x + c_2 = 0$

have a common real root, then

$$\Rightarrow (a_1c_2 - a_2c_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$$

$$\therefore \left.\begin{matrix} x^2 + bx - 1 = 0 \\ x^2 + x + b = 0 \end{matrix}\right\} \text{have a common root.}$$

$$\Rightarrow (1 + b)^2 = (b^2 + 1)(1 - b)$$

$$\Rightarrow b^2 + 2b + 1 = b^2 - b^3 + 1 - b$$

$$\Rightarrow b^3 + 3b = 0$$

$$\therefore b(b^2 + 3) = 0$$

$$\Rightarrow b = 0, \pm\sqrt{3}\,i$$

4. Given equations are $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have common root

On subtracting above equations, we get

$$(a - b)x + (b - a) = 0$$

$$\Rightarrow x = 1$$

$\therefore x = 1$ is the common root.

$$\Rightarrow 1 + a + b = 0$$

$$\Rightarrow a + b = -1$$

5. Since, $(x - r)$ is a factor of the polynomial

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$$

Then, $x = r$ is root of $f'(x) = 0$ repeated $(m - 1)$ times.

Hence, statement is false.

Topic 3 Transformation of Roots

1. Given, α, β are the roots of $(x - a)(x - b) - c = 0$

$$\Rightarrow (x - a)(x - b) - c = (x - \alpha)(x - \beta)$$

$$\Rightarrow (x - a)(x - b) = (x - \alpha)(x - \beta) + c$$

$\Rightarrow a, b$ are the roots of equation $(x - \alpha)(x - \beta) + c = 0$

2. Since, $ax^2 + bx + c = 0$ has roots α and β.

$$\Rightarrow \alpha + \beta = -b/a$$

and $$\alpha\beta = c/a$$

Now, $a^3x^2 + abcx + c^3 = 0$ ...(i)

On dividing the equation by c^2, we get

$$\frac{a^3}{c^2}x^2 + \frac{abcx}{c^2} + \frac{c^3}{c^2} = 0$$

$$\Rightarrow a\left(\frac{ax}{c}\right)^2 + b\left(\frac{ax}{c}\right) + c = 0$$

$\Rightarrow \frac{ax}{c} = \alpha, \beta$ are the roots

$\Rightarrow x = \frac{c}{a}\alpha, \frac{c}{a}\beta$ are the roots

$\Rightarrow x = \alpha\beta\alpha, \alpha\beta\beta$ are the roots

$\Rightarrow x = \alpha^2\beta, \alpha\beta^2$ are the roots

Divide the Eq. (i) by a^3, we get

$$x^2 + \frac{b}{a}\cdot\frac{c}{a}\cdot x + \left(\frac{c}{a}\right)^3 = 0$$

$$\Rightarrow x^2 - (\alpha + \beta)\cdot(\alpha\beta)x + (\alpha\beta)^3 = 0$$

$$\Rightarrow x^2 - \alpha^2\beta x - \alpha\beta^2 x + (\alpha\beta)^3 = 0$$

$$\Rightarrow x(x - \alpha^2\beta) - \alpha\beta^2(x - \alpha^2\beta) = 0$$

$$\Rightarrow (x - \alpha^2\beta)(x - \alpha\beta^2) = 0$$

$\Rightarrow x = \alpha^2\beta, \alpha\beta^2$ which is the required answer.

Alternate Solution

Since, $a^3x^2 + abcx + c^3 = 0$

$$\Rightarrow x = \frac{-abc \pm \sqrt{(abc)^2 - 4\cdot a^3\cdot c^3}}{2a^3}$$

$$\Rightarrow x = \frac{-(b/a)(c/a) \pm \sqrt{(b/a)^2(c/a)^2 - 4(c/a)^3}}{2}$$

$$\Rightarrow x = \frac{(\alpha + \beta)(\alpha\beta) \pm \sqrt{(\alpha + \beta)^2(\alpha\beta)^2 - 4(\alpha\beta)^3}}{2}$$

$$\Rightarrow x = \frac{(\alpha + \beta)(\alpha\beta) \pm \alpha\beta\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}}{2}$$

$$\Rightarrow x = \alpha\beta\left[\frac{(\alpha + \beta) \pm \sqrt{(\alpha - \beta)^2}}{2}\right]$$

$$\Rightarrow x = \alpha\beta\left[\frac{(\alpha + \beta) \pm (\alpha - \beta)}{2}\right]$$

$$\Rightarrow x = \alpha\beta\left[\frac{\alpha + \beta + \alpha - \beta}{2}, \frac{\alpha + \beta - \alpha + \beta}{2}\right]$$

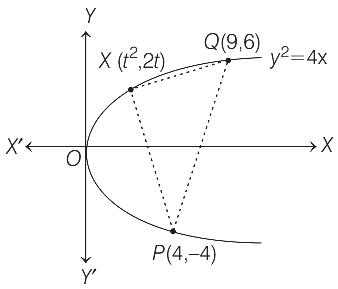
$$\Rightarrow x = \alpha\beta\left[\frac{2\alpha}{2}, \frac{2\beta}{2}\right]$$

$\Rightarrow x = \alpha^2\beta, \alpha\beta^2$ which is the required answer.

Topic 4 Graph of Quadratic Expression

1. Given parabola is $y^2 = 4x$,

Since, X lies on the parabola, so let the coordinates of X be $(t^2, 2t)$. Thus, the coordinates of the vertices of the triangle PXQ are $P(4, -4), X(t^2, 2t)$ and $Q(9, 6)$.



$$\therefore \text{Area of } \Delta PXQ = \frac{1}{2}\left\|\begin{matrix} 4 & -4 & 1 \\ t^2 & 2t & 1 \\ 9 & 6 & 1 \end{matrix}\right\|$$

$= \frac{1}{2} \left| [4(2t - 6) + 4(t^2 - 9) + 1(6t^2 - 18t] \right|$

$= \frac{1}{2} | [8t - 24 + 4t^2 - 36 + 6t^2 - 18t] |$

$= | 5t^2 - 5t - 30 | = | 5(t + 2)\,(t - 3) |$

Now, as X is any point on the arc POQ of the parabola, therefore ordinate of point X, $2t \in (-4, 6) \Rightarrow t \in (-2, 3)$.

$\therefore$ Area of $\Delta PXQ = -5(t + 2)\,(t - 3) = -5t^2 + 5t + 30$

[$\because | x - a | = -(x - a)$, if $x < a$]

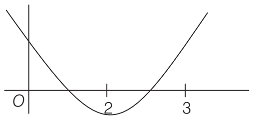
The maximum area (in square units)

$$= -\left[\frac{25 - 4(-5)\,(30)}{4(-5)}\right] = \frac{125}{4}$$

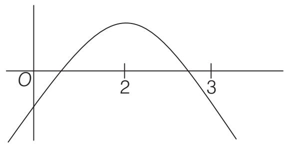
[$\because$ Maximum value of quadratic expression $ax^2 + bx + c$, when $a < 0$ is $-\frac{D}{4a}$]

2. Let $f(x) = (c - 5)x^2 - 2\,cx + (c - 4) = 0$.

Then, according to problem, the graph of $y = f(x)$ will be either of the two ways, shown below.



Or



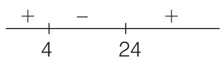
In both cases $f(0) \cdot f(2) < 0$ and $f(2) f(3) < 0$

Now, consider $\quad f(0) f(2) < 0$

$\Rightarrow \quad (c - 4)\,[4\,(c - 5) - 4c + (c - 4)] < 0$

$\Rightarrow \quad (c - 4)\,(c - 24) < 0$

$\Rightarrow \quad c \in (4, 24) \quad \ldots \text{(i)}$

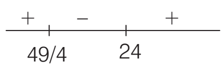


Similarly, $f(2) \cdot f(3) < 0$

$\Rightarrow [4\,(c - 5) - 4c + (c - 4)]$

$[9(c - 5) - 6c + (c - 4)] < 0$

$\Rightarrow \quad (c - 24)\,(4c - 49) < 0$



$\Rightarrow \quad c \in \left(\frac{49}{4}, 24\right) \quad \ldots \text{(ii)}$

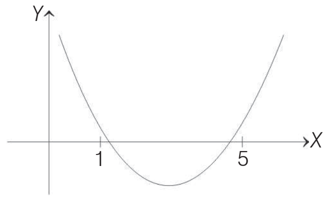
From Eqs. (i) and (ii), we get

$$c \in \left(\frac{49}{4}, 24\right)$$

$\therefore$ Integral values of c are 13, 14, ……, 23.

Thus, 11 integral values of c are possible.

3. According to given information, we have the following graph



Now, the following conditions should satisfy

(i) $D > 0 \Rightarrow b^2 - 4ac > 0$

$\Rightarrow \quad m^2 - 4 \times 1 \times 4 > 0$

$\Rightarrow \quad m^2 - 16 > 0$

$\Rightarrow \quad (m - 4)\,(m + 4) > 0$

$\Rightarrow \quad m \in (-\infty, -4) \cup (4, \infty)$

(ii) The vertex of the parabola should lie between $x = 1$ and $x = 5$

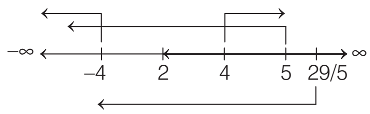
$\therefore \quad -\frac{b}{2a} \in (1, 5) \Rightarrow 1 < \frac{m}{2} < 5 \Rightarrow m \in (2, 10)$

(iii) $f(1) > 0 \Rightarrow 1 - m + 4 > 0$

$\Rightarrow m < 5 \Rightarrow m \in (-\infty, 5)$

(iv) $f(5) > 0 \Rightarrow 25 - 5m + 4 > 0 \Rightarrow 5m < 29 \Rightarrow m \in \left(-\infty, \frac{29}{5}\right)$

From the values of m obtained in (i), (ii), (iii) and (iv), we get $m \in (4, 5)$.



4. Put $t = x - [x] = \{X\}$, which is a fractional part function and lie between $0 \le \{X\} < 1$ and then solve it.

Given, $a \in R$ and equation is

$$-3\{x - [x]\}^2 + 2\{x - [x]\} + a^2 = 0$$

Let $t = x - [x]$, then equation is

$$-3t^2 + 2\,t + a^2 = 0$$

$\Rightarrow \quad t = \frac{1 \pm \sqrt{1 + 3a^2}}{3}$

$\because \quad t = x - [x] = \{X\} \quad$ [fractional part]

$\therefore \quad 0 \le t \le 1$

$$0 \le \frac{1 \pm \sqrt{1 + 3a^2}}{3} \le 1$$

Taking positive sign, we get

$$0 \le \frac{1 + \sqrt{1 + 3a^2}}{3} < 1 \quad [\because \{x\} > 0]$$

$\Rightarrow \quad \sqrt{1 + 3a^2} < 2 \;\Rightarrow\; 1 + 3a^2 < 4$

$\Rightarrow \quad a^2 - 1 < 0 \;\Rightarrow\; (a + 1)\,(a - 1) < 0$

$\therefore \quad a \in (-1, 1)$, for no integer solution of a, we consider $(-1, 0) \cup (0, 1)$

5. As we know, $ax^2 + bx + c > 0$ for all $x \in R$, iff $a > 0$ and $D < 0$.

Given equation is $x^2 + 2ax + (10 - 3a) > 0, \forall x \in R$

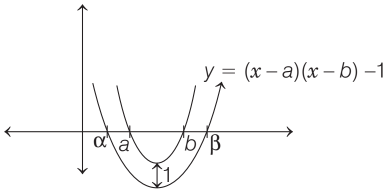
Now, $D < 0$

$\Rightarrow \quad 4a^2 - 4(10 - 3a) < 0$

$\Rightarrow \quad 4(a^2 + 3a - 10) < 0$

$\Rightarrow \quad (a + 5)(a - 2) < 0 \Rightarrow a \in (-5, 2)$

6.



From graph, it is clear that one of the roots of $(x - a)(x - b) - 1 = 0$ lies in $(-\infty, a)$ and other lies in (b, ∞).

7. Let $f(x) = x^2 - 2ax + a^2 + a - 3$

Since, both root are less than 3.

$\Rightarrow \quad \alpha < 3, \beta < 3$

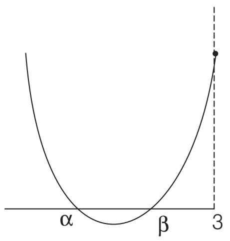
$\Rightarrow \quad$ Sum, $S = \alpha + \beta < 6$

$\Rightarrow \quad \dfrac{\alpha + \beta}{2} < 3$

$\Rightarrow \quad \dfrac{2a}{2} < 3$

$\Rightarrow \quad a < 3 \quad \ldots(i)$

Again, product, $P = \alpha\beta$



$\Rightarrow \quad P < 9 \quad \Rightarrow \quad \alpha\beta < 9$

$\Rightarrow \quad a^2 + a - 3 < 9$

$\Rightarrow \quad a^2 + a - 12 < 0$

$\Rightarrow \quad (a - 3)(a + 4) < 0$

$\Rightarrow \quad -4 < a < 3 \quad \ldots(ii)$

Again, $D = B^2 - 4AC \geq 0$

$\Rightarrow \quad (-2a)^2 - 4 \cdot 1\,(a^2 + a - 3) \geq 0$

$\Rightarrow \quad 4a^2 - 4a^2 - 4a + 12 \geq 0$

$\Rightarrow \quad -4a + 12 \geq 0 \quad \Rightarrow \quad a \leq 3 \quad \ldots(iii)$

Again, $a\, f(3) > 0$

$\Rightarrow \quad 1\,[(3)^2 - 2a\,(3) + a^2 + a - 3] > 0$

$\Rightarrow \quad 9 - 6a + a^2 + a - 3 > 0$

$\Rightarrow \quad a^2 - 5a + 6 > 0$

$\Rightarrow \quad (a - 2)(a - 3) > 0$

$\therefore \quad a \in (-\infty, 2) \cup (3, \infty) \quad \ldots(iv)$

From Eqs. (i), (ii), (iii) and (iv), we get

$a \in (-4, 2)$.

NOTE There is correction in answer $a < 2$ should be $-4 < a < 2$.

8. Let $f(x) = ax^2 + bx + c > 0, \forall x \in R$

$\Rightarrow \quad a > 0$

and $\quad b^2 - 4ac < 0 \quad \ldots(i)$

$\therefore \quad g(x) = f(x) + f'(x) + f''(x)$

$\Rightarrow \quad g(x) = ax^2 + bx + c + 2ax + b + 2a$

$\Rightarrow \quad g(x) = ax^2 + x\,(b + 2a) + (c + b + 2a)$

whose discriminant

$= (b + 2a)^2 - 4a\,(c + b + 2a)$

$= b^2 + 4a^2 + 4ab - 4ac - 4ab - 8a^2$

$= b^2 - 4a^2 - 4ac = (b^2 - 4ac) - 4a^2 < 0 \quad$ [from Eq. (i)]

$\therefore\; g(x) > 0\; \forall\, x$, as $a > 0$ and discriminant < 0.

Thus, $g(x) > 0, \forall\, x \in R$.

9. Given,

$x^2 + (a - b)\,x + (1 - a - b) = 0$ has real and unequal roots.

$\Rightarrow \quad D > 0$

$\Rightarrow \quad (a - b)^2 - 4(1)(1 - a - b) > 0$

$\Rightarrow \quad a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$

Now, to find the values of 'a' for which equation has unequal real roots for all values of b.

i.e. Above equation is true for all b.

or $b^2 + b(4 - 2a) + (a^2 + 4a - 4) > 0$, is true for all b.

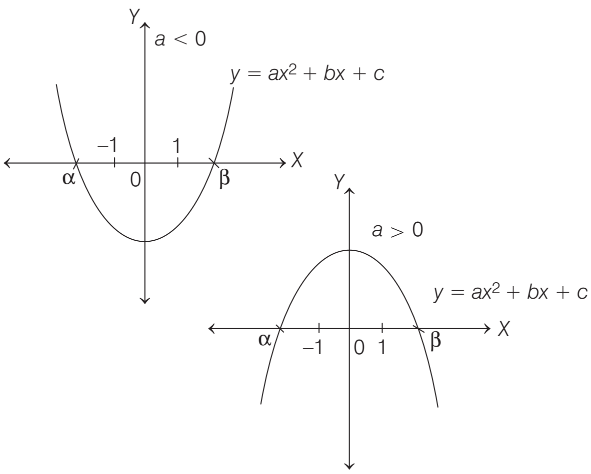
$\therefore$ Discriminant, $D < 0$

$\Rightarrow \quad (4 - 2a)^2 - 4\,(a^2 + 4a - 4) < 0$

$\Rightarrow \quad 16 - 16a + 4a^2 - 4a^2 - 16a + 16 < 0$

$\Rightarrow \quad -32a + 32 < 0 \quad \Rightarrow \quad a > 1$

10.



From figure, it is clear that, if $a > 0$, then $f(-1) < 0$ and $f(1) < 0$ and if $a < 0$, $f(-1) > 0$ and $f(1) > 0$. In both cases, $af(-1) < 0$ and $af(1) < 0$.

$\Rightarrow \quad a\,(a - b + c) < 0 \quad$ and $\quad a\,(a + b + c) < 0$

On dividing by a^2, we get

$1 - \dfrac{b}{a} + \dfrac{c}{a} < 0 \quad$ and $\quad 1 + \dfrac{b}{a} + \dfrac{c}{a} < 0$

On combining both, we get

$1 \pm \dfrac{b}{a} + \dfrac{c}{a} < 0$

$\Rightarrow \quad 1 + \left|\dfrac{b}{a}\right| + \dfrac{c}{a} < 0$

11. Since, $x^2-3x+2>0$ and $x^2-2x-4\le 0$

$\Rightarrow\quad (x-1)(x-2)>0 \quad$ and $\quad x^2-2x+1\le 5$

$\Rightarrow\quad (x<1$ or $x>2)\quad$ and $\quad (1-\sqrt{5}\le x\le 1+\sqrt{5})$

$\therefore\quad x\in[1-\sqrt{5},\ 1)\cup[1+\sqrt{5},2)$

12. (i) Given, $x^2-8kx+16(k^2-k+1)=0$

Now, $D=64\{k^2-(k^2-k+1)\}=64(k-1)>0$

$k>1$

(ii) $-\dfrac{b}{2a}>4 \quad\Rightarrow\quad \dfrac{8k}{2}>4 \quad\Rightarrow\quad k>1$

(iii) $f(4)\ge 0$

$\Rightarrow\quad 16-32k+16(k^2-k+1)\ge 0$

$\Rightarrow\quad k^2-3k+2\ge 0$

$\Rightarrow\quad (k-2)(k-1)\ge 0$

$\Rightarrow\quad k\le 1 \quad$ or $\quad k\ge 2$

Hence, $\quad k=2$

Topic 5 Some Special Forms

1. Given equation $5+|2^x-1|=2^x(2^x-2)$

***Case* I**

If $2^x-1\ge 0\Rightarrow x\ge 0$,

then $5+2^x-1=2^x(2^x-2)$

Put $2^x=t$, then

$5+t-1=t^2-2t \quad\Rightarrow\quad t^2-3t-4=0$

$\Rightarrow t^2-4t+t-4=0 \quad\Rightarrow\quad t(t-4)+1(t-4)=0$

$\Rightarrow\quad t=4$ or $-1 \quad\Rightarrow\quad t=4 \qquad (\because t=2^x>0)$

$\Rightarrow\quad 2^x=4\Rightarrow x=2>0$

$\Rightarrow x=2$ is the solution.

***Case* II**

If $2^x-1<0\Rightarrow x<0$,

then $5+1-2^x=2^x(2^x-2)$

Put $2^x=y$, then $6-y=y^2-2y$

$\Rightarrow\quad y^2-y-6=0 \quad\Rightarrow\quad y^2-3y+2y-6=0$

$\Rightarrow\ (y+2)(y-3)=0\ \Rightarrow\ y=3$ or -2

$\Rightarrow\quad y=3$ (as $y=2^x>0$) $\Rightarrow 2^x=3$

$\Rightarrow\ x=\log_2 3>0$

So, $x=\log_2 3$ is not a solution.

Therefore, number of real roots is one.

2. Given, inequality is

$$2^{\sqrt{\sin^2 x-2\sin x+5}}\cdot\frac{1}{4^{\sin^2 y}}\le 1$$

$\Rightarrow 2^{\sqrt{(\sin x-1)^2+4}}\cdot 2^{-2\sin^2 y}\le 1$

$\Rightarrow 2^{\sqrt{(\sin x-1)^2+4}}\le 2^{2\sin^2 y}$

$\Rightarrow \sqrt{(\sin x-1)^2+4}\le 2\sin^2 y$

[if $a>1$ and $a^m\le a^n\Rightarrow m\le n$]

$\because$ Range of $\sqrt{(\sin x-1)^2+4}$ is $[2,2\sqrt{2}]$

and range of $2\sin^2 y$ is $[0,2]$.

$\therefore$ The above inequality holds, iff

$\sqrt{(\sin x-1)^2+4}\ =2=2\sin^2 y$

$\Rightarrow \sin x=1$ and $\sin^2 y=1$

$\Rightarrow \sin x=|\sin y|$ [from the options]

3. **Key Idea** Reduce the given equation into quadratic equation.

Given equation is

$|\sqrt{x}-2|+\sqrt{x}(\sqrt{x}-4)+2=0$

$\Rightarrow |\sqrt{x}-2|+x-4\sqrt{x}+4=2$

$\Rightarrow |\sqrt{x}-2|+(\sqrt{x}-2)^2=2$

$\Rightarrow (|\sqrt{x}-2|)^2+|\sqrt{x}-2|-2=0$

Let $|\sqrt{x}-2|=y$, then above equation reduced to

$y^2+y-2=0 \quad\Rightarrow y^2+2y-y-2=0$

$\Rightarrow\ y(y+2)-1(y+2)=0\ \Rightarrow\ (y+2)(y-1)=0$

$\Rightarrow\quad y=1,-2$

$\therefore\quad y=1 \qquad [\because y=|\sqrt{x}-2|\ge 0]$

$\Rightarrow\quad |\sqrt{x}-2|=1$

$\Rightarrow\quad \sqrt{x}-2=\pm 1$

$\Rightarrow\quad \sqrt{x}=3$ or 1

$\Rightarrow\quad x=9$ or 1

$\therefore$ Sum of roots $=9+1=10$

4. Let $f(x)=2x^3+3x+k$

On differentiating w.r.t. x, we get

$f'(x)=6x^2+3>0,\ \forall\ x\in R$

$\Rightarrow f(x)$ is strictly increasing function.

$\Rightarrow f(x)=0$ has only one real root, so two roots are not possible.

5. Since, α is a root of $a^2x^2+bx+c=0$

$\Rightarrow\quad a^2\alpha^2+b\alpha+c=0 \quad$ ... (i)

and β is a root of $\quad a^2x^2-bx-c=0$

$\Rightarrow\quad a^2\beta^2-b\beta-c=0 \quad$ ... (ii)

Let $\quad f(x)=a^2x^2+2bx+2c$

$\therefore\quad f(\alpha)=a^2\alpha^2+2b\alpha+2c$

$=a^2\alpha^2-2a^2\alpha^2=-a^2\alpha^2$

[from Eq. (i)]

and $\quad f(\beta)=\alpha^2\beta^2+2b\beta+2c$

$=a^2\beta^2+2a^2\beta^2=3a^2\beta^2$ [from Eq. (ii)]

$\Rightarrow\quad f(\alpha)\,f(\beta)<0$

$f(x)$ must have a root lying in the open interval (α,β).

$\therefore\quad \alpha<\gamma<\beta$

6. Let $\quad f(x)=ax^3+bx^2+cx+d \quad$ ...(i)

$\therefore\quad f(0)=d\quad$ and $\quad f(1)=a+b+c+d=d$

$[\because a+b+c=0]$

$\therefore\quad f(0)=f(1)$

f is continuous in the closed interval $[0,1]$ and f is derivable in the open interval $(0,1)$.

Also, $\quad f(0)=f(1)$.

$\therefore$ By Rolle's theorem, $f'(\alpha)=0$ for $0<\alpha<1$

Now, $f'(x) = 3ax^2 + 2bx + c$

$\Rightarrow$ $f'(\alpha) = 3a\alpha^2 + 2b\alpha + c = 0$

$\therefore$ Eq. (i) has exist atleast one root in the interval (0, 1).

Thus, $f'(x)$ must have root in the interval (0, 1) or $3ax^2 + 2bx + c = 0$ has root $\in (0, 1)$.

7. Given, $x^{12} - x^9 + x^4 - x + 1 > 0$

Here, three cases arises:

***Case* I** When $x \le 0 \Rightarrow x^{12} > 0, -x^9 > 0, x^4 > 0, -x > 0$

$\therefore \quad x^{12} - x^9 + x^4 - x + 1 > 0, \forall x \le 0 \quad$ ...(i)

***Case* II** When $0 < x \le 1$

$x^9 < x^4$ and $x < 1 \Rightarrow -x^9 + x^4 > 0$ and $1 - x > 0$

$\therefore \quad x^{12} - x^9 + x^4 - x + 1 > 0, \forall 0 < x \le 1 \quad$ ...(ii)

***Case* III** When $x > 1 \Rightarrow x^{12} > x^9$ and $x^4 > x$

$\therefore \quad x^{12} - x^9 + x^4 - x + 1 > 0, \forall x > 1 \quad$ ...(iii)

From Eqs. (i), (ii) and (iii), the above equation holds for all $x \in R$.

8. Consider,

$$f(x) = \int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\, dx$$

Obviously, $f(x)$ is continuous and differentiable in the interval [1, 2].

Also, $f(1) = f(2)$ [given]

$\therefore$ By Rolle's theorem, there exist atleast one point $k \in (1, 2)$, such that $f'(k) = 0$.

Now, $f'(x) = (1 + \cos^8 x)(ax^2 + bx + c)$

$f'(k) = 0$

$\Rightarrow \quad (1 + \cos^8 k)(ak^2 + bk + c) = 0$

$\Rightarrow \quad ak^2 + bk + c = 0 \quad$ [as $(1 + \cos^8 k) \ne 0$]

$\therefore \quad x = k$ is root of $ax^2 + bx + c = 0$,

where $k \in (1, 2)$

9. Given, x_1 and x_2 are roots of $\alpha x^2 - x + \alpha = 0$.

$\therefore \quad x_1 + x_2 = \dfrac{1}{\alpha}$ and $x_1 x_2 = 1$

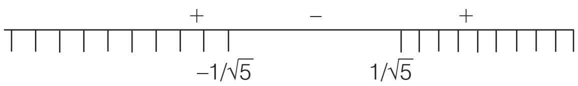
Also, $|x_1 - x_2| < 1$

$\Rightarrow \quad |x_1 - x_2|^2 < 1 \Rightarrow (x_1 - x_2)^2 < 1$

or $\quad (x_1 + x_2)^2 - 4x_1 x_2 < 1$

$\Rightarrow \quad \dfrac{1}{\alpha^2} - 4 < 1$ or $\dfrac{1}{\alpha^2} < 5$

$\Rightarrow \quad 5\alpha^2 - 1 > 0$ or $(\sqrt{5}\alpha - 1)(\sqrt{5}\alpha + 1) > 0$



$\therefore \quad \alpha \in \left(-\infty, -\dfrac{1}{\sqrt{5}}\right) \cup \left(\dfrac{1}{\sqrt{5}}, \infty\right) \quad$ ...(i)

Also, $D > 0$

$\Rightarrow \quad 1 - 4\alpha^2 > 0$ or $\alpha \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$\alpha \in \left(-\frac{1}{2}, \frac{-1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$$

10. **PLAN**

(i) Concepts of curve tracing are used in this question.

(ii) Number of roots are taken out from the curve traced.

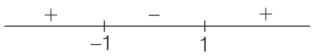
Let $y = x^5 - 5x$

(i) As $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$

(ii) Also, at $x = 0, y = 0$, thus the curve passes through the origin.

(iii) $\dfrac{dy}{dx} = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2 - 1)(x^2 + 1)$

$= 5(x - 1)(x + 1)(x^2 + 1)$



Now, $\dfrac{dy}{dx} > 0$ in $(-\infty, -1) \cup (1, \infty)$, thus $f(x)$ is increasing in these intervals.

Also, $\dfrac{dy}{dx} < 0$ in $(-1, 1)$, thus decreasing in $(-1, 1)$.

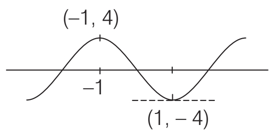
(iv) Also, at $x = -1$, dy/dx changes its sign from + ve to –ve.

$\therefore x = -1$ is point of local maxima.

Similarly, $x = 1$ is point of local minima.

Local maximum value, $y = (-1)^5 - 5(-1) = 4$

Local minimum value, $y = (1)^5 - 5(1) = -4$



Now, let $y = -a$

As evident from the graph, if $-a \in (-4, 4)$

i.e. $a \in (-4, +4)$

Then, $f(x)$ has three real roots and if $-a > 4$ or $-a < -4$, then $f(x)$ has one real root.

i.e. for $a < -4$ or $a > 4$, $f(x)$ has one real root.

11. Given, $f(x) = 4x^3 + 3x^2 + 2x + 1$

$f'(x) = 2(6x^2 + 3x + 1)$

$\Rightarrow \quad D = 9 - 24 < 0$

Hence, $f(x) = 0$ has only one real root.

$$f\left(-\frac{1}{2}\right) = 1 - 1 + \frac{3}{4} - \frac{4}{8} > 0$$

$$f\left(-\frac{3}{4}\right) = 1 - \frac{6}{4} + \frac{27}{16} - \frac{108}{64}$$

$$= \frac{64 - 96 + 108 - 108}{64} < 0$$

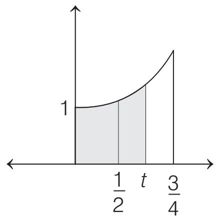
$f(x)$ changes its sign in $\left(-\dfrac{3}{4}, -\dfrac{1}{2}\right)$

Hence, $f(x) = 0$ has a root in $\left(-\dfrac{3}{4}, -\dfrac{1}{2}\right)$.

12. $\int_0^{1/2} f(x)\,dx < \int_0^{t} f(x)\,dx < \int_0^{3/4} f(x)\,dx$

Now, $\int f(x)\,dx = \int (1 + 2x + 3x^2 + 4x^3)\,dx$

$= x + x^2 + x^3 + x^4$



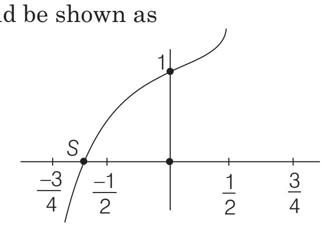
$\Rightarrow \quad \int_0^{1/2} f(x)\,dx = \frac{15}{16} > \frac{3}{4}, \quad \int_0^{3/4} f(x)\,dx = \frac{530}{256} < 3$

13. As, $f''(x) = 2(12x + 3)$

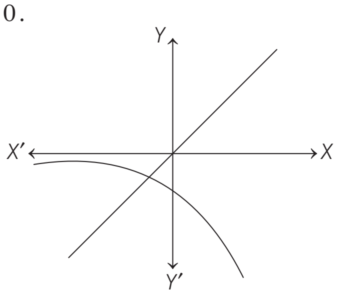
$f'(x) > 0$, when $x > -\frac{1}{4}$ and

$f'(x) < 0$, when $x < -\frac{1}{4}$.

$\therefore$ It could be shown as



14. Let $y = x$ intersect the curve $y = ke^x$ at exactly one point when $k \le 0$.



15. Let $\quad f(x) = ke^x - x$

$f'(x) = ke^x - 1 = 0$

$\Rightarrow \quad x = -\ln k$

$f''(x) = ke^x$

$\therefore \quad [f''(x)]_{x=-\ln k} = 1 > 0$

Hence, $\quad f(-\ln k) = 1 + \ln k$

For one root of given equation

$1 + \ln k = 0$

$\Rightarrow \quad k = \frac{1}{e}$

16. For two distinct roots, $1 + \ln k < 0 \ (k > 0)$

$\ln k < -1 \Rightarrow \quad k < \frac{1}{e}$

Hence, $\quad k \in \left(0, \frac{1}{e}\right)$

17. Let $f(x) = (x-a)(x-c) + 2(x-b)(x-d)$

Here, $\quad f(a) = +\text{ ve}$

$f(b) = -\text{ ve}$

$f(c) = -\text{ ve}$

$f(d) = +\text{ ve}$

$\therefore$ There exists two real and distinct roots one in the interval (a, b) and other in (c, d).

Hence, statement is true.

18. Let $f(x) = 4x^3 - 3x - p$ $\quad$ ...(i)

Now, $\quad f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right) - p = \frac{4}{8} - \frac{3}{2} - p$

$= -(1 + p)$

$f(1) = 4(1)^3 - 3(1) - p = 1 - p$

$\Rightarrow \quad f\left(\frac{1}{2}\right) \cdot f(1) = -(1+p)(1-p)$

$= (p+1)(p-1) = p^2 - 1$

Which is ≤ 0, $\forall \ p \in [-1, 1]$.

$\therefore$ $f(x)$ has atleast one root in $\left[\frac{1}{2}, 1\right]$.

Now, $f'(x) = 12x^2 - 3 = 3(2x-1)(2x+1)$

$= \frac{3}{4}\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) > 0$ in $\left[\frac{1}{2}, 1\right]$

$\Rightarrow$ $f(x)$ is an increasing function in $[1/2, 1]$

Therefore, $f(x)$ has exactly one root in $[1/2, 1]$ for any $p \in [-1, 1]$.

Now, let $x = \cos\theta$

$\therefore \quad x \in \left[\frac{1}{2}, 1\right] \Rightarrow \theta \in \left[0, \frac{\pi}{3}\right]$

From Eq. (i),

$4\cos^3\theta - 3\cos\theta = p \Rightarrow \cos 3\theta = p$

$\Rightarrow \quad 3\theta = \cos^{-1} p$

$\Rightarrow \quad \theta = \frac{1}{3}\cos^{-1} p$

$\Rightarrow \quad \cos\theta = \cos\left(\frac{1}{3}\cos^{-1} p\right)$

$\Rightarrow \quad x = \cos\left(\frac{1}{3}\cos^{-1} p\right)$