3

Sequences and Series

Topic 1 Arithmetic Progression (AP)

Objective Questions I (Only one correct option)

1. If $a_1, a_2, a_3, \ldots, a_n$ are in AP and $a_1 + a_4 + a_7 + \ldots + a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to **(2019 Main, 10 April I)**

(a) 64 (b) 76
(c) 98 (d) 38

2. If 19th term of a non-zero AP is zero, then its (49th term) : (29th term) is **(2019 Main, 11 Jan II)**

(a) 1 : 3 (b) 4 : 1
(c) 2 : 1 (d) 3 : 1

3. For any three positive real numbers a, b and c, if $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$, then **(2017 Main)**

(a) b, c and a are in GP
(b) b, c and a are in AP
(c) a, b and c are in AP
(d) a, b and c are in GP

4. If T_r is the rth term of an AP, for $r = 1, 2, 3, \ldots$. If for some positive integers m and n, we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals **(1998, 2M)**

(a) $\frac{1}{mn}$ (b) $\frac{1}{m} + \frac{1}{n}$ (c) 1 (d) 0

Analytical and Descriptive Question

5. If $a_1, a_2, \ldots, a_n$ are in arithmetic progression, where $a_i > 0, \forall i$, then show that

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$

(1982, 2M)

True/False

6. $n_1, n_2, \ldots, n_p$ are p positive integers, whose sum is an even number, then the number of odd integers among them is odd. **(1985, 1M)**

Integer Answer Type Question

7. The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side? **(2017 Adv.)**

Topic 2 Sum of *n* Terms of an AP

Objective Questions I (Only one correct option)

1. If $a_1, a_2, a_3, \ldots$ are in AP such that $a_1 + a_7 + a_{16} = 40$, then the sum of the first 15 terms of this AP is **(2019 Main, 12 April II)**

(a) 200 (b) 280 (c) 120 (d) 150

2. Let S_n denote the sum of the first n terms of an AP. If $S_4 = 16$ and $S_6 = -48$, then S_{10} is equal to **(2019 Main, 12 April I)**

(a) -260 (b) -410 (c) -320 (d) -380

3. For $x \in R$, let $[x]$ denote the greatest integer $\leq x$, then the sum of the series

$$\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \ldots + \left[-\frac{1}{3} - \frac{99}{100}\right] \text{ is}$$

(2019 Main, 12 April I)

(a) -153 (b) -133
(c) -131 (d) -135

4. If the sum and product of the first three terms in an AP are 33 and 1155, respectively, then a value of its 11th term is **(2019 Main, 9 April II)**

(a) 25 (b) -36
(c) -25 (d) -35

5. Let the sum of the first n terms of a non-constant AP $a_1, a_2, a_3 \ldots$ be $50n + \frac{n(n-7)}{2}A$, where A is a constant. If d is the common difference of this AP, then the ordered pair (d, a_{50}) is equal to **(2019 Main, 9 April I)**

(a) $(A, 50 + 46A)$ (b) $(50, 50 + 45A)$
(c) $(50, 50 + 46A)$ (d) $(A, 50 + 45A)$

6. The sum of all two digit positive numbers which when divided by 7 yield 2 or 5 as remainder is **(2019 Main, 10 Jan I)**

(a) 1256 (b) 1465 (c) 1356 (d) 1365

7. Let $a_1, a_2, \ldots\ldots a_{30}$ be an AP, $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{(2i-1)}$. If $a_5 = 27$ and $S - 2T = 75$, then a_{10} is equal to **(2019 Main, 9 Jan I)**

(a) 42 (b) 57
(c) 52 (d) 47

8. Let $b_i > 1$ for $i = 1, 2, \ldots, 101$. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are in AP with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in AP, such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \ldots + b_{51}$ and $s = a_1 + a_2 + \ldots + a_{51}$, then **(2016 Adv.)**

(a) $s > t$ and $a_{101} > b_{101}$ (b) $s > t$ and $a_{101} < b_{101}$
(c) $s < t$ and $a_{101} > b_{101}$ (d) $s < t$ and $a_{101} < b_{101}$

9. If the sum of first n terms of an AP is cn^2, then the sum of squares of these n terms is **(2009)**

(a) $\frac{n(4n^2-1)c^2}{6}$ (b) $\frac{n(4n^2+1)c^2}{3}$
(c) $\frac{n(4n^2-1)c^2}{3}$ (d) $\frac{n(4n^2+1)c^2}{6}$

10. If the sum of the first $2n$ terms of the AP series 2,5,8,..., is equal to the sum of the first n terms of the AP series 57, 59, 61,..., then n equals **(2001, 1M)**

(a) 10 (b) 12
(c) 11 (d) 13

Objective Question II

(One or more than one correct option)

11. If $S_n = \sum_{2}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then, S_n can take value(s) **(2013 Adv.)**

(a) 1056 (b) 1088
(c) 1120 (d) 1332

Passage Based Problems

Read the following passage and answer the questions.

Passage

Let V_r denotes the sum of the first r terms of an arithmetic progression (AP) whose first term is r and the common difference is $(2r-1)$. Let $T_r = V_{r+1} - V_r$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots$ **(2007, 8M)**

12. The sum $V_1 + V_2 + \ldots + V_n$ is

(a) $\frac{1}{12}n(n+1)(3n^2-n+1)$ (b) $\frac{1}{12}n(n+1)(3n^2+n+2)$
(c) $\frac{1}{2}n(2n^2-n+1)$ (d) $\frac{1}{3}(2n^3-2n+3)$

13. T_r is always

(a) an odd number (b) an even number
(c) a prime number (d) a composite number

14. Which one of the following is a correct statement ?

(a) $Q_1, Q_2, Q_3, \ldots$ are in an AP with common difference 5
(b) $Q_1, Q_2, Q_3, \ldots$ are in an AP with common difference 6
(c) $Q_1, Q_2, Q_3, \ldots$ are in an AP with common difference 11
(d) $Q_1 = Q_2 = Q_3 = \ldots$

Fill in the Blanks

15. Let p and q be the roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then $A = \ldots$ and $B = \ldots$. **(1997, 2M)**

16. The sum of the first n terms of the series $1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 + 2\cdot 6^2 + \ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is .... . **(1988, 2M)**

17. The sum of integers from 1 to 100 that are divisible by 2 or 5 is ...... **(1984, 2M)**

Analytical & Descriptive Questions

18. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that resulting sum is the square of an integer **(2000, 4M)**

19. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP. Find the intervals in which β and γ lie. **(1996, 3M)**

20. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon. **(1980, 3M)**

Integer Answer Type Questions

21. Suppose that all the terms of an arithmetic progression are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this AP is **(2015 Adv.)**

22. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20$ is equal to **(2013 Adv.)**

23. Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is equal to ...... **(2011)**

24. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$. If $\frac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \ldots + a_{11}}{11}$ is...... **(2010)**

Topic 3 Geometric Progression (GP)

Objective Questions I (Only one correct option)

1. Let a, b and c be in GP with common ratio r, where $a \neq 0$ and $0 < r \leq \frac{1}{2}$. If $3a$, $7b$ and $15c$ are the first three terms of an AP, then the 4th term of this AP is **(2019 Main, 10 April II)**

(a) $5a$ (b) $\frac{2}{3}a$ (c) a (d) $\frac{7}{3}a$

2. If three distinct numbers a, b and c are in GP and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then which one of the following statements is correct? **(2019 Main, 8 April II)**

(a) d, e and f are in GP (b) $\frac{d}{a}, \frac{e}{b}$ and $\frac{f}{c}$ are in AP

(c) d, e and f are in AP (d) $\frac{d}{a}, \frac{e}{b}$ and $\frac{f}{c}$ are in GP

3. The product of three consecutive terms of a GP is 512. If 4 is added to each of the first and the second of these terms, the three terms now form an AP. Then, the sum of the original three terms of the given GP is **(2019 Main, 12 Jan I)**

(a) 36 (b) 28 (c) 32 (d) 24

4. Let $a_1, a_2, \ldots, a_{10}$ be a GP. If $\frac{a_3}{a_1} = 25$, then $\frac{a_9}{a_5}$ equals **(2019 Main, 11 Jan I)**

(a) 5^3 (b) $2(5^2)$ (c) $4(5^2)$ (d) 5^4

5. Let a, b and c be the 7th, 11th and 13th terms respectively of a non-constant AP. If these are also the three consecutive terms of a GP, then $\frac{a}{c}$ is equal to **(2019 Main, 9 Jan II)**

(a) 2 (b) $\frac{7}{13}$ (c) 4 (d) $\frac{1}{2}$

6. If a, b and c be three distinct real numbers in GP and $a + b + c = xb$, then x cannot be **(2019 Main, 9 Jan I)**

(a) 4 (b) 2 (c) -2 (d) -3

7. If the 2nd, 5th and 9th terms of a non-constant AP are in GP, then the common ratio of this GP is **(2016 Main)**

(a) $\frac{8}{5}$ (b) $\frac{4}{3}$ (c) 1 (d) $\frac{7}{4}$

8. Let $f(x) = ax^2 + bx + c$, $a \neq 0$ and $\Delta = b^2 - 4ac$. If $\alpha + \beta$, $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ are in GP, then **(2005, 1M)**

(a) $\Delta \neq 0$ (b) $b\Delta = 0$ (c) $c\Delta = 0$ (d) $bc \neq 0$

9. Let a, b, c be in an AP and a^2, b^2, c^2 be in GP. If $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is **(2002, 1M)**

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$ (c) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ (d) $\frac{1}{2} - \frac{1}{\sqrt{2}}$

10. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in GP, then the integer values of p and q respectively are **(2001, 1M)**

(a) $-2, -32$ (b) $-2, 3$ (c) $-6, 3$ (d) $-6, -32$

11. If a, b, c, d and p are distinct real numbers such that

$$(a^2 + b^2 + c^2)\, p^2 - 2\,(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \leq 0,$$ then a, b, c, d

(a) are in AP (b) are in GP **(1987, 2M)**

(c) are in HP (d) satisfy $ab = cd$

12. If a, b, c are in GP, then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in **(1985, 2M)**

(a) AP (b) GP

(c) HP (d) None of these

13. The third term of a geometric progression is 4. The product of the first five terms is **(1982, 2M)**

(a) 4^3 (b) 4^5

(c) 4^4 (d) None of these

Analytical & Descriptive Questions

14. Find three numbers a, b, c between 2 and 18 such that (i) their sum is 25. (ii) the numbers $2, a, b$ are consecutive terms of an AP. (iii) the numbers b, c, 18 are consecutive terms of a GP. **(1983, 2M)**

15. Does there exist a geometric progression containing 27,8 and 12 as three of its term? If it exists, then how many such progressions are possible? **(1982, 2M)**

16. If the mth, nth and pth terms of an AP and GP are equal and are x, y, z, then prove that $x^{y-z} \cdot y^{z-x} \cdot z^{x-y} = 1$. **(1979, 3M)**

Topic 4 Sum of *n* Terms & Infinite Terms of a GP

Objective Questions I (Only one correct option)

1. The sum $\sum_{k=1}^{20} k \frac{1}{2^k}$ is equal to **(2019 Main, 8 April II)**

(a) $2 - \frac{11}{2^{19}}$ (b) $1 - \frac{11}{2^{20}}$

(c) $2 - \frac{3}{2^{17}}$ (d) $2 - \frac{21}{2^{20}}$

2. Let $S_n = 1 + q + q^2 + \ldots + q^n$ and

$$T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n,$$ where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2 \cdot S_1 + \ldots + {}^{101}C_{101} \cdot S_{100} = \alpha T_{100}$, then α is equal to **(2019 Main, 11 Jan II)**

(a) 2^{100} (b) 202

(c) 200 (d) 2^{99}

3. The sum of an infinite geometric series with positive terms is 3 and the sum of the cubes of its terms is $\frac{27}{19}$. Then, the common ratio of this series is **(2019 Main, 11 Jan I)**

(a) $\frac{4}{9}$ (b) $\frac{2}{3}$ (c) $\frac{2}{9}$ (d) $\frac{1}{3}$

4. Three positive numbers form an increasing GP. If the middle term in this GP is doubled, then new numbers are in AP. Then, the common ratio of the GP is **(2014 Main)**

(a) $\sqrt{2}+\sqrt{3}$ (b) $3+\sqrt{2}$
(c) $2-\sqrt{3}$ (d) $2+\sqrt{3}$

5. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9 = k(10)^9$, then k is equal to **(2014 Main)**

(a) $\frac{121}{10}$ (b) $\frac{441}{100}$ (c) 100 (d) 110

6. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777,… , is **(2013 Main)**

(a) $\frac{7}{81}(179-10^{-20})$ (b) $\frac{7}{9}(99-10^{-20})$
(c) $\frac{7}{81}(179+10^{-20})$ (d) $\frac{7}{9}(99+10^{-20})$

7. An infinite GP has first term x and sum 5, then x belongs to **(2004, 1M)**

(a) $x<-10$ (b) $-10<x<0$ (c) $0<x<10$ (d) $x>10$

8. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then **(2000, 2M)**

(a) $a=4/7,\ r=3/7$ (b) $a=2,\ r=3/8$
(c) $a=3/2,\ r=1/2$ (d) $a=3,\ r=1/4$

9. Sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$ is equal to **(1988, 2M)**

(a) 2^n-n-1 (b) $1-2^{-n}$ (c) $n+2^{-n}-1$ (d) 2^n+1

Topic 5 Harmonic Progression (HP)

Objective Questions I (Only one correct option)

1. If $a_1, a_2, a_3, \ldots$ are in a harmonic progression with $a_1=5$ and $a_{20}=25$. Then, the least positive integer n for which $a_n<0$, is **(2012)**

(a) 22 (b) 23 (c) 24 (d) 25

2. If the positive numbers a, b, c, d are in AP. Then, abc, abd, acd, bcd are **(2001, 1M)**

(a) not in AP/GP/HP (b) in AP
(c) in GP (d) in HP

3. Let $a_1, a_2, \ldots, a_{10}$ be in AP and h_1, h_2, equal to….., h_{10} be in HP. If $a_1=h_1=2$ and $a_{10}=h_{10}=3$, then a_4h_7 is **(1999, 2M)**

(a) 2 (b) 3 (c) 5 (d) 6

4. If $x>1, y>1, z>1$ are in GP, then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in **(1998, 2M)**

(a) AP (b) HP (c) GP (d) None of these

Objective Question II

(One or more than one correct option)

10. Let $S_1, S_2, \ldots$ be squares such that for each $n \ge 1$ the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq cm? **(1999, 3M)**

(a) 7 (b) 8 (c) 9 (d) 10

Analytical & Descriptive Questions

11. Let $A_n = \left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$, $B_n = 1 - A_n$. Find a least odd natural number n_0, so that $B_n > A_n, \forall\ n \ge n_0$. **(2006, 6M)**

12. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series, whose first terms are 1, 2, 3,..., n and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$ respectively, then find the values of $S_1^2 + S_2^2 + S_3^2 + \ldots + S_{2n-1}^2$. **(1991, 4M)**

13. The sum of the squares of three distinct real numbers, which are in GP, is S^2. If their sum is aS, then show that

$$a^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$$

(1986, 5M)

Integer Answer Type Questions

14. Let S_k, where $k = 1, 2, , \ldots, 100$, denotes the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then, the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k|$ is …… **(2010)**

Assertion and Reason

For the following question, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows:

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

5. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in GP. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

Statement I The numbers b_1, b_2, b_3, b_4 are neither in AP nor in GP.

Statement II The numbers b_1, b_2, b_3, b_4 are in HP. **(2008, 3M)**

Fill in the Blank

6. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$' are in HP. Then $\cos x \cdot \sec\left(\frac{y}{2}\right) = \dots$. **(1997C, 2M)**

Analytical & Descriptive Questions

7. If a, b, c are in AP, a^2, b^2, c^2 are in HP, then prove that either $a = b = c$ or $a, b, -\frac{c}{2}$ form a GP. **(2003, 4M)**

8. Let a and b be positive real numbers. If a, A_1, A_2, b are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, B are in harmonic progression, then show that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}$$ **(2002, 5M)**

9. (i) The value of $x + y + z$ is 15. If a, x, y, z, b are in AP while the value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is $\frac{5}{3}$. If a, x, y, z, b are in HP, then find a and b.

(ii) If x, y, z are in HP, then show that $\log(x+z) + \log(x+z-2y) = 2\log(x-z)$. **(1978, 3M)**

Topic 6 Relation between AM, GM, HM and Some Special Series

Objective Questions I (Only one correct option)

1. The sum of series $1 + \frac{1^3+2^3}{1+2} + \frac{1^3+2^3+3^3}{1+2+3} + \dots + \frac{1^3+2^3+3^3+\dots+15^3}{1+2+3+\dots+15} - \frac{1}{2}(1+2+3+\dots+15)$ is equal to **(2019 Main, 10 April II)**

(a) 620 (b) 660 (c) 1240 (d) 1860

2. The sum of series $\frac{3\times 1^3}{1^2} + \frac{5\times(1^3+2^3)}{1^2+2^2} + \frac{7\times(1^3+2^3+3^3)}{1^2+2^2+3^2} + \dots\dots\dots +$ upto 10th term, is **(2019 Main, 10 April I)**

(a) 680 (b) 600 (c) 660 (d) 620

3. The sum of the series $1 + 2\times 3 + 3\times 5 + 4\times 7 + \dots$ upto 11th term is **(2019 Main, 9 April II)**

(a) 915 (b) 946 (c) 916 (d) 945

4. If the sum of the first 15 terms of the series $\left(\frac{3}{4}\right)^3 + \left(1\frac{1}{2}\right)^3 + \left(2\frac{1}{4}\right)^3 + 3^3 + \left(3\frac{3}{4}\right)^3 + \dots$ is equal to $225k$, then k is equal to **(2019 Main, 12 Jan II)**

(a) 108 (b) 27 (c) 54 (d) 9

5. Let $S_k = \frac{1+2+3+\dots+k}{k}$. If $S_1^2 + S_2^2 + \dots + S_{10}^2 = \frac{5}{12}A$, then A is equal to **(2019 Main, 12 Jan I)**

(a) 156 (b) 301 (c) 283 (d) 303

6. Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$ is **(2019 Main, 11 Jan II)**

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{1}{4}$ (d) $\frac{m+n}{6mn}$

7. The sum of the following series $1 + 6 + \frac{9(1^2+2^2+3^2)}{7} + \frac{12(1^2+2^2+3^2+4^2)}{9} + \frac{15(1^2+2^2+\dots+5^2)}{11} + \dots$ up to 15 terms is **(2019 Main, 9 Jan II)**

(a) 7510 (b) 7820 (c) 7830 (d) 7520

8. Let $a_1, a_2, a_3, \dots, a_{49}$ be in AP such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \dots + a_{17}^2 = 140$ m, then m is equal to **(2018 Main)**

(a) 66 (b) 68 (c) 34 (d) 33

9. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series

$$1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 + 2\cdot 6^2 + \dots$$

If $B - 2A = 100\lambda$, then λ is equal to **(2018 Main)**

(a) 232 (b) 248 (c) 464 (d) 496

10. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \dots$, is $\frac{16}{5}m$, then m is equal to **(2016 Main)**

(a) 102 (b) 101 (c) 100 (d) 99

11. If m is the AM of two distinct real numbers l and $n (l, n > 1)$ and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals **(2015)**

(a) $4l^2mn$ (b) $4lm^2n$ (c) lmn^2 (d) $l^2m^2n^2$

12. The sum of first 9 terms of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots$ is **(2015)**

(a) 71 (b) 96 (c) 142 (d) 192

13. If $\alpha \in \left(0, \frac{\pi}{2}\right)$, then $\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to **(2003, 2M)**

(a) $2\tan\alpha$ (b) 1 (c) 2 (d) $\sec^2\alpha$

14. If $a_1, a_2, \dots, a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + \dots + a_{n-1} + 2a_n$ is **(2002, 1M)**

(a) $n\,(2c)^{1/n}$ (b) $(n+1)c^{1/n}$
(c) $2nc^{1/n}$ (d) $(n+1)\,(2c)^{1/n}$

15. If a, b, c are positive real numbers such that $a + b + c + d = 2$, then $M = (a+b)(c+d)$ satisfies the relation **(2000, 2M)**

(a) $0 < M \le 1$ (b) $1 \le M \le 2$
(c) $2 \le M \le 3$ (d) $3 \le M \le 4$

16. The harmonic mean of the roots of the equation $(5+\sqrt{2})\,x^2 - (4+\sqrt{5})\,x + 8 + 2\sqrt{5} = 0$ is **(1999, 2M)**

(a) 2 (b) 4 (c) 6 (d) 8

17. The product of n positive numbers is unity, then their sum is **(1991, 2M)**

(a) a positive integer (b) divisible by n
(c) equal to $n + \dfrac{1}{n}$ (d) never less than n

18. If a, b and c are distinct positive numbers, then the expression $(b+c-a)(c+a-b)(a+b-c) - abc$ is **(1991, 2M)**

(a) positive (b) negative
(c) non-positive (d) non-negative

19. If $x_1, x_2, \dots, x_n$ are any real numbers and n is any positive integer, then **(1982, 1M)**

(a) $n\sum_{i=1}^{n} x_i^2 < \left(\sum_{i=1}^{n} x_i\right)^2$ (b) $n\sum_{i=1}^{n} x_i^2 \ge \left(\sum_{i=1}^{n} x_i\right)^2$

(c) $n\sum_{i=1}^{n} x_i^2 \ge n\left(\sum_{i=1}^{n} x_i\right)^2$ (d) None of these

Passage Based Problems

Passage

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \ge 2$, let A_{n-1} and H_{n-1} has arithmetic, geometric and harmonic means as A_n, G_n, H_n, respectively. **(2007, 8M)**

20. Which one of the following statements is correct?

(a) $G_1 > G_2 > G_3 > \dots$
(b) $G_1 < G_2 < G_3 < \dots$
(c) $G_1 = G_2 = G_3 = \dots$
(d) $G_1 < G_3 < G_5 < \dots$ and $G_2 > G_4 > G_6 > \dots$

21. Which of the following statements is correct?

(a) $A_1 > A_2 > A_3 > \dots$
(b) $A_1 < A_2 < A_3 < \dots$
(c) $A_1 > A_3 > A_5 > \dots$ and $A_2 < A_4 < A_6 < \dots$
(d) $A_1 < A_3 < A_5 < \dots$ and $A_2 > A_4 > A_6 > \dots$

22. Which of the following statements is correct ?

(a) $H_1 > H_2 > H_3 > \dots$
(b) $H_1 < H_2 < H_3 < \dots$
(c) $H_1 > H_3 > H_5 > \dots$ and $H_2 < H_4 < H_6 < \dots$
(d) $H_1 < H_3 < H_5 < \dots$ and $H_2 > H_4 > H_6 > \dots$

Objective Question II

(One or more than one correct option)

23. For a positive integer n let

$a(n) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dots + \dfrac{1}{(2^n) - 1}$, then **(1999, 3M)**

(a) $a(100) \le 100$ (b) $a(100) > 100$
(c) $a(200) \le 100$ (d) $a(200) > 100$

24. If the first and the $(2n-1)$th term of an AP, GP and HP are equal and their nth terms are a, b and c respectively, then **(1988, 2M)**

(a) $a = b = c$
(b) $a \ge b \ge c$
(c) $a + c = b$
(d) $ac - b^2 = 0$

Fill in the Blanks

25. If x be is the arithmetic mean and y, z be two geometric means between any two positive numbers, then $\dfrac{y^3 + z^3}{xyz} = \dots$ **(1997C, 2M)**

26. If the harmonic mean and geometric mean of two positive numbers are in the ratio 4 : 5. Then, the two numbers are in the ratio… . **(1992, 2M)**

True/False

27. If x and y are positive real numbers and m, n are any positive integers, then $\dfrac{x^n y^m}{(1+x^{2n})(1+y^{2m})} > \dfrac{1}{4}$. **(1989, 1M)**

28. For $0 < a < x$, the minimum value of function $\log_a x + \log_x a$ is 2.

Analytical & Descriptive Questions

29. If a, b, c are positive real numbers, then prove that

$$\{(1+a)(1+b)(1+c)\}^7 > 7^7 a^4 b^4 c^4$$ **(2004, 4M)**

30. Let $a_1, a_2, \dots$ be positive real numbers in geometric progression. For each n, if A_n, G_n, H_n are respectively, the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, \dots, a_n$. Then, find an expression for the geometric mean of $G_1, G_2, \dots, G_n$ in terms of $A_1, A_2, \dots, A_n, H_1, H_2, \dots, H_n$. **(2001, 5M)**

31. If p is the first of the n arithmetic means between two numbers and q be the first on n harmonic means between the same numbers. Then, show that q does not lie between p and $\left(\dfrac{n+1}{n-1}\right)^2 p$. **(1991, 4M)**

32. If $a > 0$, $b > 0$ and $c > 0$, then prove that

$$(a+b+c)\,\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge 9$$ **(1984, 2m)**

Integer Answer Type Question

33. Let a,b,c be positive integers such that b/a is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is $b+2$, then the value of $\frac{a^2+a-14}{a+1}$ is **(2014 Adv.)**

34. The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a>0$ is ...... **(2011)**

Answers

Topic 1

1. (b)	**2.** (d)	**3.** (b)	**4.** (c)
6. False	**7.** (6)		

Topic 2

1. (a)	**2.** (c)	**3.** (b)	**4.** (c)
5. (a)	**6.** (c)	**7.** (c)	**8.** (b)
9. (c)	**10.** (c)	**11.** (a, d)	**12.** (b)
13. (d)	**14.** (b)	**15.** $(A=-3, B=77)$	
16. $\left[\frac{n^2(n+1)}{2}\right]$	**17.** (3050)		
19. $\beta \in \left[-\infty, \frac{1}{3}\right]$ and $\gamma \in \left[-\frac{1}{27}, \infty\right]$	**20.** (9)	**21.** (9)	
22. (5)	**23.** (9)	**24.** (0)	

Topic 3

1. (c)	**2.** (b)	**3.** (b)	**4.** (d)
5. (c)	**6.** (b)	**7.** (b)	**8.** (c)
9. (d)	**10.** (a)	**11.** (b)	**12.** (a)
13. (b)	**14.** $(a=5)$ $(b=8)$ $(c=12)$	**15.** Yes, infinite	

Topic 4

1. (a)	**2.** (a)	**3.** (b)	
4. (d)	**5.** (c)	**6.** (c)	**7.** (c)
8. (d)	**9.** (c)	**10.** (b, c, d)	
11. (7)	**12.** $\frac{1}{6}(2n)(2n+1)(4n+1)-1$	**14.** (4)	

Topic 5

1. (d)	**2.** (d)	**3.** (d)	**4.** (b)
5. (c)	**6.** $\pm\sqrt{2}$	**9.**(i) $a=1, b=9$	
12. 29			

Topic 6

1. (a)	**2.** (c)	3. (b)	**4.** (b)
5. (d)	**6.** (c)	**7.** (b)	**8.** (c)
9. (b)	**10.** (b)	**11.** (b)	**12.** (b)
13. (a)	**14.** (a)	**15.** (a)	**16.** (b)
17. (d)	**18.** (b)	**19.** (b)	**20.** (c)
21. (a)	**22.** (b)	**23.** (a, d)	**24.** (a, b, d)
25. 2	**26.** 4 : 1	**27.** False	**28.** False
34. (8)			

Hints & Solutions

Topic 1 Arithmetic Progression (AP)

1. **Key Idea** Use nth term of an AP i.e. $a_n = a + (n-1)d$, simplify the given equation and use result.

Given AP is $a_1, a_2, a_3, \dots, a_n$

Let the above AP has common difference 'd', then

$a_1 + a_4 + a_7 + \dots + a_{16}$

$= a_1 + (a_1+3d) + (a_1+6d) + \dots + (a_1+15d)$

$= 6a_1 + (3+6+9+12+15)d$

$\therefore 6a_1 + 45d = 114$ (given)

$\Rightarrow \quad 2a_1 + 15d = 38$ ...(i)

Now, $a_1 + a_6 + a_{11} + a_{16}$

$= a_1 + (a_1+5d) + (a_1+10d) + (a_1+15d)$

$= 4a_1 + 30d = 2(2a_1+15d)$

$= 2 \times 38 = 76$ [from Eq. (i)]

2. Let t_n be the nth term of given AP. Then, we have $t_{19}=0$

$\Rightarrow a + (19-1)d = 0$ $[\because t_n = a + (n-1)d]$

$\Rightarrow \quad a + 18d = 0$ ...(i)

Now, $\frac{t_{49}}{t_{29}} = \frac{a+48d}{a+28d}$

$= \frac{-18d+48d}{-18d+28d}$ [using Eq. (i)]

$= \frac{30d}{10d} = 3:1$

3. We have,

$225a^2 + 9b^2 + 25c^2 - 75ac - 45ab - 15bc = 0$

$\Rightarrow (15a)^2 + (3b)^2 + (5c)^2 - (15a)(5c) - (15a)(3b) - (3b)(5c) = 0$

$\Rightarrow \frac{1}{2}[(15a-3b)^2 + (3b-5c)^2 + (5c-15a)^2] = 0$

$\Rightarrow 15a = 3b, 3b = 5c$ and $5c = 15a$

$\therefore 15a = 3b = 5c$

$\Rightarrow \frac{a}{1} = \frac{b}{5} = \frac{c}{3} = \lambda$ (say)

$\Rightarrow a = \lambda, b = 5\lambda, c = 3\lambda$

$\therefore b, c, a$ are in AP.

4. Let $T_m = a + (m-1)d = \frac{1}{n}$ ...(i)

and $T_n = a + (n-1)d = \frac{1}{m}$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(m-n)d = \frac{1}{n} - \frac{1}{m} = \frac{m-n}{mn}$$

$$\Rightarrow d = \frac{1}{mn}$$

Again, $T_{mn} = a + (mn-1)d = a + (mn - n + n - 1)d$

$= a + (n-1)d + (mn-n)d$

$= T_n + n(m-1)\frac{1}{mn} = \frac{1}{m} + \frac{(m-1)}{m} = 1$

5. Since, $a_1, a_2, \dots, a_n$ are in an AP.

$\therefore (a_2 - a_1) = (a_3 - a_2) = \dots = (a_n - a_{n-1}) = d$

Thus, $\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}}$

$$= \left(\frac{\sqrt{a_2} - \sqrt{a_1}}{d}\right) + \left(\frac{\sqrt{a_3} - \sqrt{a_2}}{d}\right) + \dots + \left(\frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{d}\right)$$

$$= \frac{1}{d}(\sqrt{a_n} - \sqrt{a_1}) = \frac{1}{d}\frac{(a_n - a_1)}{\sqrt{a_n} + \sqrt{a_1}} = \frac{(n-1)}{\sqrt{a_n} + \sqrt{a_1}}$$

6. Since, $n_1, n_2, \dots, n_p$ are p positive integers, whose sum is even and we know that, sum of any two odd integers is even.

$\therefore$ Number of odd integers must be even.

Hence, it is a false statement.

7. Let the sides are $a-d$, a and $a+d$. Then,

$a(a-d) = 48$

and $a^2 - 2ad + d^2 + a^2 = a^2 + 2ad + d^2$

$\Rightarrow a^2 = 4ad$

$\Rightarrow a = 4d$

Thus, $a = 8, d = 2$

Hence, $a - d = 6$

Topic 2 Sum of *n* Terms of an AP

1. Let the common difference of given AP is 'd'.

Since, $a_1 + a_7 + a_{16} = 40$

$\therefore a_1 + a_1 + 6d + a_1 + 15d = 40$ $[\because a_n = a_1 + (n-1)d]$

$\Rightarrow 3a_1 + 21d = 40$ ...(i)

Now, sum of first 15 terms is given by

$$S_{15} = \frac{15}{2}[2a_1 + (15-1)d]$$

$$= \frac{15}{2}[2a_1 + 14d] = 15[a_1 + 7d]$$

From Eq. (i), we have

$$a_1 + 7d = \frac{40}{3}$$

So, $S_{15} = 15 \times \frac{40}{3}$

$= 5 \times 40 = 200$

2. Given S_n denote the sum of the first n terms of an AP.

Let first term and common difference of the AP be 'a' and 'd', respectively.

$\therefore S_4 = 2[2a + 3d] = 16$ (given)

$\left[\because S_n = \frac{n}{2}[2a + (n-1)d]\right]$

$\Rightarrow 2a + 3d = 8$ ... (i)

and $S_6 = 3[2a + 5d] = -48$ [given]

$\Rightarrow 2a + 5d = -16$ ... (ii)

On subtracting Eq. (i) from Eq. (ii), we get

$2d = -24$

$\Rightarrow d = -12$

So, $2a = 44$ [put $d = -12$ in Eq. (i)]

Now, $S_{10} = 5[2a + 9d]$

$= 5[44 + 9(-12)] = 5[44 - 108]$

$= 5 \times (-64) = -320$

3. Given series is

$$\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \dots \dots + \left[-\frac{1}{3} - \frac{99}{100}\right]$$

[where, $[x]$ denotes the greatest integer $\le x$]

Now,

$$\left[-\frac{1}{3}\right], \left[-\frac{1}{3} - \frac{1}{100}\right], \left[-\frac{1}{3} - \frac{2}{100}\right], \dots + \left[-\frac{1}{3} - \frac{66}{100}\right]$$

all the term have value -1

and $\left[-\frac{1}{3} - \frac{67}{100}\right], \left[-\frac{1}{3} - \frac{68}{100}\right], \dots, \left[-\frac{1}{3} - \frac{99}{100}\right]$ all the term have value -2.

So, $\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \dots + \left[-\frac{1}{3} - \frac{66}{100}\right]$

$= -1 - 1 - 1 - 1 \dots$ 67 times.

$= (-1) \times 67 = -67$

and $\left[-\frac{1}{3} - \frac{67}{100}\right] + \left[-\frac{1}{3} - \frac{68}{100}\right] + \dots + \left[-\frac{1}{3} - \frac{99}{100}\right]$

$= -2 - 2 - 2 - 2 \dots$ 33 times

$= (-2) \times 33 = -66$

$\therefore \left[-\frac{1}{3}\right]+\left[-\frac{1}{3}-\frac{1}{100}\right]+\left[-\frac{1}{3}-\frac{2}{100}\right]+\ldots+\left[-\frac{1}{3}-\frac{99}{100}\right]$

$=(-67)+(-66)=-133.$

Alternate Solution

$\because [-x]=-[x]-1$, if $x \notin$ Integer,

and $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx]$,

$n \in N$.

So given series

$$\left[-\frac{1}{3}\right]+\left[-\frac{1}{3}-\frac{1}{100}\right]+\left[-\frac{1}{3}-\frac{2}{100}\right]+\ldots\ldots+\left[\frac{-1}{3}-\frac{99}{100}\right]$$

$$=\left(-\left[\frac{1}{3}\right]-1\right)+\left(-\left[\frac{1}{3}+\frac{1}{100}\right]-1\right)$$

$$+\left(-\left[\frac{1}{3}+\frac{2}{100}\right]-1\right)+\ldots+\left(-\left[\frac{1}{3}+\frac{99}{100}\right]-1\right)$$

$$=(-1)\times 100-\left[\frac{1}{3}\times 100\right]=-100-33=-133.$$

4. Let first three terms of an AP as $a-d, a, a+d$.

So, $3a=33 \Rightarrow a=11$

[given sum of three terms $=33$ and product of terms $=1155$]

$\Rightarrow (11-d)11(11+d)=1155$ [given]

$\Rightarrow 11^2-d^2=105$

$\Rightarrow d^2=121-105=16$

$\Rightarrow d=\pm 4$

So the first three terms of the AP are either 7, 11, 15 or 15, 11, 7.

So, the 11th term is either $7+(10\times 4)=47$ or $15+(10\times(-4))=-25$.

5 **Key Idea** Use the formula of sum of first n terms of AP, i.e $S_n=\frac{n}{2}[2a+(n-1)d]$

Given AP, is

$a_1, a_2, a_3,\ldots$ having sum of first n-terms

$$=\frac{n}{2}[2a_1+(n-1)d]$$

[where, d is the common difference of AP]

$$=50n+\frac{n(n-7)}{2}A \quad \text{(given)}$$

$$\Rightarrow \frac{1}{2}[2a_1+(n-1)d]=50+\frac{n-7}{2}A$$

$$\Rightarrow \frac{1}{2}[2a_1+nd-d]=\left(50-\frac{7}{2}A\right)+\frac{n}{2}A$$

$$\Rightarrow \left(a_1-\frac{d}{2}\right)+\frac{nd}{2}=\left(50-\frac{7}{2}A\right)+\frac{n}{2}A$$

On comparing corresponding term, we get

$d=A$ and $a_1-\frac{d}{2}=50-\frac{7}{2}A$

$$\Rightarrow a_1-\frac{A}{2}=50-\frac{7}{2}A \quad [\because d=A]$$

$$\Rightarrow a_1=50-3A$$

So $a_{50}=a_1+49d$

$=(50-3A)+49A$ $[\because d=A]$

$=50+46A$

Therefore, $(d, a_{50})=(A, 50+46A)$

6. Clearly, the two digit number which leaves remainder 2 when divided by 7 is of the form $N=7k+2$ [by Division Algorithm]

For, $k=2, N=16$
$k=3, N=23$
$\vdots \quad \vdots$
$k=13, N=93$

$\therefore$ 12 such numbers are possible and these numbers forms an AP.

Now, $S=\frac{12}{2}[16+93]=654$ $\left(\because S_n=\frac{n}{2}(a+l)\right)$

Similarly, the two digit number which leaves remainder 5 when divided by 7 is of the form $N=7k+5$

For $k=1, N=12$
$k=2, N=19$
$\vdots$
$k=13, N=96$

$\therefore$ 13 such numbers are possible and these numbers also forms an AP.

Now, $S'=\frac{13}{2}[12+96]=702$ $\left(\because S_n=\frac{n}{2}(a+l)\right)$

Total sum $=S+S'=654+702=1356$

7. We have, $S=a_1+a_2+\ldots+a_{30}$

$=15[2a_1+29d]$ ...(i)

(where d is the common difference) $\left[\because S_n=\frac{n}{2}[2a+(n-1)d]\right]$

and $T=a_1+a_3+\ldots+a_{29}$

$=\frac{15}{2}[2a_1+14\times 2d)]$

($\because$ common difference is $2d$)

$\Rightarrow 2T=15[2a_1+28d]$ ...(ii)

From Eqs. (i) and (ii), we get

$S-2T=15d=75$ $[\because S-2T=75]$

$\Rightarrow \quad d = 5$

Now, $\quad a_{10} = a_5 + 5d$

$= 27 + 25 = 52$

8. If $\log b_1, \log b_2, \ldots, \log b_{101}$ are in AP, with common difference $\log_e 2$, then $b_1, b_2, \ldots, b_{101}$ are in GP, with common ratio 2.

$\therefore b_1 = 2^0 b_1, b_2 = 2^1 b_1, b_3 = 2^2 b_1, \ldots, b_{101} = 2^{100} b_1 \quad \ldots(i)$

Also, $a_1, a_2, \ldots, a_{101}$ are in AP.

Given, $\quad a_1 = b_1$ and $a_{51} = b_{51}$

$\Rightarrow \quad a_1 + 50D = 2^{50} b_1$

$\Rightarrow \quad a_1 + 50D = 2^{50} a_1 \quad [\because a_1 = b_1] \ldots(ii)$

Now, $\quad t = b_1 + b_2 + \ldots + b_{51}$

$\Rightarrow \quad t = b_1 \dfrac{(2^{51} - 1)}{2 - 1} \quad \ldots(iii)$

and $\quad s = a_1 + a_2 + \ldots + a_{51}$

$= \dfrac{51}{2}(2a_1 + 50D) \quad \ldots(iv)$

$\therefore \quad t = a_1(2^{51} - 1) \quad [\because a_1 = b_1]$

or $\quad t = 2^{51} a_1 - a_1 < 2^{51} a_1 \quad \ldots(v)$

and $\quad s = \dfrac{51}{2}[a_1 + (a_1 + 50D)] \quad$ [from Eq. (ii)]

$= \dfrac{51}{2}[a_1 + 2^{50} a_1]$

$= \dfrac{51}{2} a_1 + \dfrac{51}{2} 2^{50} a_1$

$\therefore \quad s > 2^{51} a_1 \quad \ldots(vi)$

From Eqs. (v) and (vi), we get $s > t$

Also, $\quad a_{101} = a_1 + 100D$ and $b_{101} = 2^{100} b_1$

$\therefore \quad a_{101} = a_1 + 100\left(\dfrac{2^{50} a_1 - a_1}{50}\right)$ and $b_{101} = 2^{100} a_1$

$\Rightarrow \quad a_{101} = a_1 + 2^{51} a_1 - 2a_1 = 2^{51} a_1 - a_1$

$\Rightarrow \quad a_{101} < 2^{51} a_1$ and $b_{101} > 2^{51} a_1$

$\Rightarrow \quad b_{101} > a_{101}$

9. Let $S_n = cn^2$

$S_{n-1} = c(n-1)^2 = cn^2 + c - 2cn$

$\therefore \quad T_n = 2cn - c \quad [\because T_n = S_n - S_{n-1}]$

$T_n^2 = (2cn - c)^2 = 4c^2 n^2 + c^2 - 4c^2 n$

$\therefore$ Sum $= \Sigma T_n^2 = \dfrac{4c^2 \cdot n(n+1)(2n+1)}{6} + nc^2 - 2c^2 n(n+1)$

$= \dfrac{2c^2 n(n+1)(2n+1) + 3nc^2 - 6c^2 n(n+1)}{3}$

$= \dfrac{nc^2(4n^2 + 6n + 2 + 3 - 6n - 6)}{3} = \dfrac{nc^2(4n^2 - 1)}{3}$

10. According to given condition,

$S_{2n} = S'_n$

$\Rightarrow \dfrac{2n}{2}[2 \times 2 + (2n-1) \times 3] = \dfrac{n}{2}[2 \times 57 + (n-1) \times 2]$

$\Rightarrow \quad (4 + 6n - 3) = \dfrac{1}{2}(114 + 2n - 2)$

$\Rightarrow \quad 6n + 1 = 57 + n - 1 \quad \Rightarrow \quad 5n = 55$

$\therefore \quad n = 11$

11. PLAN Convert it into differences and use sum of n terms of an AP, i.e. $S_n = \dfrac{n}{2}[2a + (n-1)d]$

Now, $\quad S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} \cdot k^2$

$= -(1)^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 + \ldots$

$= (3^2 - 1^2) + (4^2 - 2^2) + (7^2 - 5^2) + (8^2 - 6^2) + \ldots$

$= 2\{\underbrace{(4 + 6 + 12 + \ldots)}_{n \text{ terms}} + \underbrace{(6 + 14 + 22 + \ldots)}_{n \text{ terms}}\}$

$= 2\left[\dfrac{n}{2}\{2 \times 4 + (n-1)8\} + \dfrac{n}{2}\{2 \times 6 + (n-1)8\}\right]$

$= 2\,[n(4 + 4n - 4) + n(6 + 4n - 4)]$

$= 2\,[4n^2 + 4n^2 + 2n] = 4n(4n + 1)$

Here, $1056 = 32 \times 33, 1088 = 32 \times 34,$

$1120 = 32 \times 35, 1332 = 36 \times 37$

1056 and 1332 are possible answers.

12. Here, $V_r = \dfrac{r}{2}[2r + (r-1)(2r-1)] = \dfrac{1}{2}(2r^3 - r^2 + r)$

$\therefore \Sigma V_r = \dfrac{1}{2}[2\Sigma r^3 - \Sigma r^2 + \Sigma r]$

$= \dfrac{1}{2}\left[2\left\{\dfrac{n(n+1)}{2}\right\}^2 - \dfrac{n(n+1)(2n+1)}{6} + \dfrac{n(n+1)}{2}\right]$

$\Rightarrow \quad = \dfrac{n(n+1)}{12}[3n(n+1) - (2n+1) + 3]$

$= \dfrac{1}{12} n(n+1)(3n^2 + n + 2)$

13. $V_{r+1} - V_r = (r+1)^3 - r^3 - \dfrac{1}{2}[(r+1)^2 - r^2] + \dfrac{1}{2}$

$= 3r^2 + 2r - 1$

$\therefore \quad T_r = 3r^2 + 2r - 1 = (r+1)(3r-1)$

which is a composite number.

14. Since, $T_r = 3r^2 + 2r - 1$

and $T_{r+1} = 3(r+1)^2 + 2(r+1) - 1$

$\therefore \quad Q_r = T_{r+1} - T_r = 3[2r+1] + 2[1]$

$\Rightarrow \quad Q_r = 6r + 5$

$\Rightarrow \quad Q_{r+1} = 6(r+1) + 5$

Common difference $= Q_{r+1} - Q_r = 6$

15. Given, $\; p + q = 2, pq = A$

and $\quad r + s = 18, rs = B$

and it is given that p, q, r, s are in an AP.

Therefore, let $p = a - 3d, q = a - d, r = a + d$

and $\quad s = a + 3d$

Since, $\quad p < q < r < s$

We have, $d > 0$

Now, $2 = p + q = a - 3d + a - d = 2a - 4d$

$\Rightarrow \quad a - 2d = 1 \quad \ldots(i)$

Again, $18 = r + s = a + d + a + 3d$

$18 = 2a + 4d$

$\Rightarrow \quad 9 = a + 2d \quad \ldots(ii)$

On subtracting Eq. (i) from Eq. (ii), we get

$$8 = 4d \Rightarrow d = 2$$

On putting in Eq. (ii), we get $a = 5$

$\therefore \quad p = a - 3d = 5 - 6 = -1$

$q = a - d = 5 - 2 = 3$

$r = a + d = 5 + 2 = 7$

and $s = a + 3d = 5 + 6 = 11$

Therefore, $A = pq = -3$ and $B = rs = 77$

16. Here, $1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 + \ldots$ upto n terms

$$= \frac{n\,(n+1)^2}{2} \quad \text{[when } n \text{ is even]} \ldots \text{(i)}$$

When n is odd, $1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 \ldots + n^2$

$$= \{1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + \ldots + 2\,(n-1)^2\} + n^2$$

$$= \left\{\frac{(n-1)\,(n)^2}{2}\right\} + n^2 \quad \text{[from Eq. (i)]}$$

$$= n^2\left(\frac{n-1}{2} + 1\right) = n^2\,\frac{(n+1)}{2}$$

$\therefore \quad 1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + \ldots$ upto n terms, when n is odd

$$= \frac{n^2\,(n+1)}{2}$$

17. Integers divisible by 2 are {2,4,6,8,10, ...,100}.

Integers divisible by 5 are {5,10,15, ...,100}.

Thus, sum of integers divisible by 2

$$= \frac{50}{2}\,(2 + 100) = 50 \times 51 = 2550$$

Sum of integers divisible by 5

$$= \frac{20}{2}\,(5 + 100) = 10 \times 105 = 1050$$

Sum of integers divisible by 10

$$= \frac{10}{2}\,(10 + 100) = 5 \times 110 = 550$$

$\therefore$ Sum of integers from 1 to 100 divisible by 2 or 5

$$= 2550 + 1050 - 550$$

$$= 2550 + 500 = 3050$$

18. Let four consecutive terms of the AP are $a - 3d$, $a - d$, $a + d$, $a + 3d$, which are integers.

Again, required product

$P = (a - 3d)(a - d)(a + d)(a + 3d) + (2d)^4$

[by given condition]

$$= (a^2 - 9d^2)(a^2 - d^2) + 16d^4$$

$$= a^4 - 10a^2d^2 + 9d^4 + 16d^4 = (a^2 - 5d^2)^2$$

Now, $a^2 - 5d^2 = a^2 - 9d^2 + 4d^2$

$= (a - 3d)(a + 3d) + (2d)^2$

$= I \cdot I + I^2$ [given]

$= I^2 + I^2 = I^2$

$= I$ [where, I is any integer]

Therefore, $P = (I)^2 =$ Integer

19. Since, x_1, x_2, x_3 are in an AP. Let $x_1 = a - d$, $x_2 = a$ and $x_3 = a + d$ and x_1, x_2, x_3 be the roots of $x^3 - x^2 + \beta x + \gamma = 0$

$\therefore \quad \Sigma\alpha = a - d + a + a + d = 1$

$\Rightarrow \quad a = 1/3 \quad \ldots(i)$

$\Sigma\alpha\beta = (a - d)\,a + a\,(a + d) + (a - d)\,(a + d) = \beta \quad \ldots(ii)$

and $\alpha\beta\gamma = (a - d)\,a\,(a + d) = -\gamma \quad \ldots(iii)$

From Eq. (i),

$$3a = 1 \Rightarrow a = 1/3$$

From Eq. (ii), $3a^2 - d^2 = \beta$

$\Rightarrow \quad 3\,(1/3)^2 - d^2 = \beta$ [from Eq. (i)]

$\Rightarrow \quad 1/3 - \beta = d^2$

NOTE In this equation, we have two variables β and γ but we have only one equation. So, at first sight it looks that this equation cannot solve but we know that $d^2 \geq 0$, $\forall d \in R$, then β can be solved. This trick is frequently asked in IIT examples.

$\Rightarrow \quad \frac{1}{3} - \beta \geq 0 \quad [\because d^2 \geq 0]$

$\Rightarrow \quad \beta \leq \frac{1}{3} \Rightarrow \beta \in [-\infty, 1/3]$

From Eq. (iii), $a\,(a^2 - d^2) = -\gamma$

$\Rightarrow \quad \frac{1}{3}\left(\frac{1}{9} - d^2\right) = -\gamma \Rightarrow \frac{1}{27} - \frac{1}{3}\,d^2 = -\gamma$

$\Rightarrow \quad \gamma + \frac{1}{27} = \frac{1}{3}\,d^2 \Rightarrow \gamma + \frac{1}{27} \geq 0$

$\Rightarrow \quad \gamma \geq -1/27$

$\Rightarrow \quad \gamma \in \left[-\frac{1}{27}, \infty\right)$

Hence, $\beta \in (-\infty, 1/3]$ and $\gamma \in [-1/27, \infty)$

20. Since, angles of polygon are in an AP.

$\therefore$ Sum of all angles

$$= (n - 2) \times 180° = \frac{n}{2}\{2\,(120°) + (n - 1)\,5°\}$$

$\Rightarrow \quad 5n^2 - 125n + 720 = 0$

$\Rightarrow \quad n^2 - 25n + 144 = 0$

$\Rightarrow \quad (n - 9)\,(n - 16) = 0$

$\Rightarrow \quad n = 9, 16$

If $n = 9$, then largest angle $= a + 8d = 160°$

Again, if $n = 16$, the n largest angle

$$= a + 15d = 120° + 75 = 195°$$

which is not possible.

[since, any angle of polygon cannot be $> 180°$]

Hence, $n = 9$ [neglecting $n = 16$]

21. Given, $\dfrac{S_7}{S_{11}} = \dfrac{6}{11}$ and $130 < t_7 < 140$

$$\Rightarrow \frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11} \Rightarrow \frac{7(2a+6d)}{(2a+10d)} = 6$$

$\Rightarrow \quad a = 9d \quad \ldots$(i)

Also, $\quad 130 < t_7 < 140$

$\Rightarrow \quad 130 < a + 6d < 140$

$\Rightarrow \quad 130 < 9d + 6d < 140 \quad$ [from Eq. (i)]

$\Rightarrow \quad 130 < 15d < 140$

$\Rightarrow \quad \dfrac{26}{3} < d < \dfrac{28}{3} \quad$ [since, d is a natural number]

$\therefore \quad d = 9$

22. Let number of removed cards be k and $(k+1)$.

$\therefore \quad \dfrac{n(n+1)}{2} - k - (k+1) = 1224$

$\Rightarrow \quad n^2 + n - 4k = 2450 \Rightarrow n^2 + n - 2450 = 4k$

$\Rightarrow \quad (n+50)(n-49) = 4k$

$\therefore \quad n > 49$

Let $\quad n = 50$

$\therefore \quad 100 = 4k$

$\Rightarrow \quad k = 25$

Now $\quad k - 20 = 5$

23. Given, $a_1 = 3, m = 5n$ and $a_1, a_2, \ldots$, is an AP.

$\therefore \quad \dfrac{S_m}{S_n} = \dfrac{S_{5n}}{S_n}$ is independent of n.

$$= \frac{\frac{5n}{2}[2\times 3 + (5n-1)d]}{\frac{n}{2}[2\times 3 + (n-1)d]} = \frac{5\{(6-d)+5n\}}{(6-d)+n},$$

independent of n

If $\quad 6 - d = 0 \quad \Rightarrow \quad d = 6$

$\therefore \quad a_2 = a_1 + d = 3 + 6 = 9$

or If $d = 0$, then $\dfrac{S_m}{S_n}$ is independent of n.

$\therefore \quad a_2 = 9$

24. $a_k = 2a_{k-1} - a_{k-2}$

$\Rightarrow a_1, a_2, \ldots, a_{11}$ are in an AP.

$$\therefore \quad \frac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = \frac{11a^2 + 35 \times 11d^2 + 10ad}{11} = 90$$

$\Rightarrow 225 + 35d^2 + 150d = 90$

$\Rightarrow 35d^2 + 150d + 135 = 0 \Rightarrow d = -3, -\dfrac{9}{7}$

Given, $\quad a_2 < \dfrac{27}{2}$

$\therefore \quad d = -3$ and $d \neq -\dfrac{9}{7}$

$$\Rightarrow \quad \frac{a_1 + a_2 + \ldots + a_{11}}{11} = \frac{11}{2}[30 - 10\times 3] = 0$$

Topic 3 Geometric Progression (GP)

1. **Key Idea** Use n^{th} term of AP i.e., $a_n = a + (n-1)d$, If a, A, b are in AP, then $2A = a + b$ and n^{th} term of G.P. i.e., $a_n = ar^{n-1}$.

It is given that, the terms a, b, c are in GP with common ratio r, where $a \neq 0$ and $0 < r \leq \dfrac{1}{2}$.

So, let, $b = ar$ and $c = ar^2$

Now, the terms $3a$, $7b$ and $15c$ are the first three terms of an AP, then

$2(7b) = 3a + 15c$

$\Rightarrow \quad 14ar = 3a + 15ar^2 \quad$ [as $b = ar$, $c = ar^2$]

$\Rightarrow \quad 14r = 3 + 15r^2 \quad$ [as $a \neq 0$]

$\Rightarrow \quad 15r^2 - 14r + 3 = 0$

$\Rightarrow \quad 15r^2 - 5r - 9r + 3 = 0$

$\Rightarrow 5r(3r-1) - 3(3r-1) = 0$

$\Rightarrow \quad (3r-1)(5r-3) = 0$

$\Rightarrow \quad r = \dfrac{1}{3}$ or $\dfrac{3}{5}$

as, $r \in \left(0, \dfrac{1}{2}\right]$, so $r = \dfrac{1}{3}$

Now, the common difference of AP $= 7b - 3a$

$$= 7ar - 3a = a\left(\frac{7}{3} - 3\right) = -\frac{2a}{3}$$

So, 4^{th} term of AP $= 3a + 3\left(\dfrac{-2a}{3}\right) = a$

2. (b) Given, three distinct numbers a, b and c are in GP.

$\therefore \quad b^2 = ac \quad \ldots$(i)

and the given quadratic equations

$ax^2 + 2bx + c = 0 \quad \ldots$(ii)

$dx^2 + 2ex + f = 0 \quad \ldots$(iii)

For quadratic Eq. (ii),

the discriminant $D = (2b)^2 - 4ac$

$= 4(b^2 - ac) = 0 \quad$ [from Eq. (i)]

$\Rightarrow$ Quadratic Eq. (ii) have equal roots, and it is equal to $x = -\dfrac{b}{a}$, and it is given that quadratic Eqs. (ii) and (iii) have a common root, so

$$d\left(-\frac{b}{a}\right)^2 + 2e\left(-\frac{b}{a}\right) + f = 0$$

$\Rightarrow \quad db^2 - 2eba + a^2f = 0$

$\Rightarrow \quad d(ac) - 2eab + a^2f = 0 \quad [\because b^2 = ac]$

$\Rightarrow \quad dc - 2eb + af = 0 \quad [\because a \neq 0]$

$\Rightarrow \quad 2eb = dc + af$

$\Rightarrow \quad 2\dfrac{e}{b} = \dfrac{dc}{b^2} + \dfrac{af}{b^2}$

[dividing each term by b^2]

$\Rightarrow \quad 2\left(\dfrac{e}{b}\right) = \dfrac{d}{a} + \dfrac{f}{c} \quad [\because b^2 = ac]$

So, $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

Alternate Solution

Given, three distinct numbers a, b and c are in GP. Let $a = a$, $b = ar$, $c = ar^2$ are in GP, which satisfies $ax^2 + 2bx + c = 0$

$\therefore \quad ax^2 + 2(ar)x + ar^2 = 0$

$\Rightarrow \quad x^2 + 2rx + r^2 = 0 \quad [\because a \neq 0]$

$\Rightarrow \quad (x + r)^2 = 0 \Rightarrow x = -r.$

According to the question, $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root.

So, $x = -r$ satisfies $dx^2 + 2ex + f = 0$

$\therefore \quad d(-r)^2 + 2e(-r) + f = 0$

$\Rightarrow \quad dr^2 - 2er + f = 0$

$\Rightarrow \quad d\left(\frac{c}{a}\right) - 2e\left(\frac{c}{b}\right) + f = 0$

$\Rightarrow \quad \frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0$

$\Rightarrow \quad \frac{d}{a} + \frac{f}{c} = \frac{2e}{b} \quad [\because c \neq 0]$

3. Let the three consecutive terms of a GP are $\frac{a}{r}$, a and ar.

Now, according to the question, we have

$\frac{a}{r} \cdot a \cdot ar = 512$

$\Rightarrow \quad a^3 = 512$

$\Rightarrow \quad a = 8 \quad \ldots \text{(i)}$

Also, after adding 4 to first two terms, we get

$\frac{8}{r} + 4, 8 + 4, 8r$ are in AP

$\Rightarrow \quad 2\,(12) = \frac{8}{r} + 4 + 8r$

$\Rightarrow \quad 24 = \frac{8}{r} + 8r + 4 \quad \Rightarrow \quad 20 = 4\left(\frac{2}{r} + 2r\right)$

$\Rightarrow \quad 5 = \frac{2}{r} + 2r \quad \Rightarrow \quad 2r^2 - 5r + 2 = 0$

$\Rightarrow \quad 2r^2 - 4r - r + 2 = 0$

$\Rightarrow 2r(r - 2) - 1(r - 2) = 0$

$\Rightarrow \quad (r - 2)\,(2r - 1) = 0$

$\Rightarrow \quad r = 2, \frac{1}{2}$

Thus, the terms are either 16, 8, 4 or 4, 8, 16. Hence, required sum = 28.

4. Let r be the common ratio of given GP, then we have the following sequence $a_1, a_2 = a_1 r, a_3 = a_1 r^2, \ldots, a_{10} = a_1 r^9$

Now, $\quad a_3 = 25\, a_1$

$\Rightarrow \quad a_1 r^2 = 25\, a_1$

$\Rightarrow \quad r^2 = 25$

Consider, $\frac{a_9}{a_5} = \frac{a_1 r^8}{a_1 r^4} = r^4 = (25)^2 = 5^4$

5. Let A be the Ist term of AP and d be the common difference.

$\therefore \quad$ 7th term $= a = A + 6d$

$[\because n\text{th term} = A + (n - 1)d]$

11th term $= b = A + 10d$

13th term $= c = A + 12d$

$\because a, b, c$ are also in GP

$\therefore \quad b^2 = ac$

$\Rightarrow (A + 10d)^2 = (A + 6d)\,(A + 12d)$

$\Rightarrow A^2 + 20Ad + 100d^2 = A^2 + 18Ad + 72d^2$

$\Rightarrow 2Ad + 28d^2 = 0$

$\Rightarrow 2d(A + 14d) = 0$

$\Rightarrow d = 0$ or $A + 14d = 0$

But $\quad d \neq 0 \quad$ [$\because$ the series is non constant AP]

$\Rightarrow \quad A = -14d$

$\therefore \quad a = A + 6d = -14d + 6d = -8d$

and $\quad c = A + 12d = -14d + 12d = -2d$

$\Rightarrow \quad \frac{a}{c} = \frac{-8d}{-2d} = 4$

6. Let $b = ar$ and $c = ar^2$, where r is the common ratio.

Then, $\quad a + b + c = xb$

$\Rightarrow \quad a + ar + ar^2 = xar$

$\Rightarrow \quad 1 + r + r^2 = xr \quad \ldots \text{(i)} \; [\because a \neq 0]$

$\Rightarrow \quad x = \frac{1 + r + r^2}{r} = 1 + r + \frac{1}{r}$

We know that, $r + \frac{1}{r} \geq 2$ (for $r > 0$)

and $\quad r + \frac{1}{r} \leq -2$ (for $r < 0$) [using AM $\geq$ GM]

$\therefore \quad 1 + r + \frac{1}{r} \geq 3$

or $\quad 1 + r + \frac{1}{r} \leq -1$

$\Rightarrow \quad x \geq 3$ or $x \leq -1$

$\Rightarrow \quad x \in (-\infty, -1] \cup [3, \infty)$

Hence, x cannot be 2.

Alternate Method

From Eq. (i), we have

$1 + r + r^2 = xr$

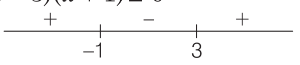
$\Rightarrow \quad r^2 + (1 - x)r + 1 = 0$

For real solution of r, $D \geq 0$.

$\Rightarrow \quad (1 - x)^2 - 4 \geq 0$

$\Rightarrow \quad x^2 - 2x - 3 \geq 0$

$\Rightarrow \quad (x - 3)(x + 1) \geq 0$



$\Rightarrow x \in (-\infty, -1] \cup [3, \infty)$

$\therefore \; x$ cannot be 2.

7. Let a be the first term and d be the common difference. Then, we have $a + d, a + 4d, a + 8d$ in GP,

i.e. $\quad (a + 4d)^2 = (a + d)\,(a + 8d)$

$\Rightarrow \quad a^2 + 16d^2 + 8ad = a^2 + 8ad + ad + 8d^2$

$\Rightarrow \quad 8d^2 = ad$

$\Rightarrow \quad 8d = a \quad [\because d \neq 0]$

Now, common ratio,

$$r=\frac{a+4d}{a+d}=\frac{8d+4d}{8d+d}=\frac{12d}{9d}=\frac{4}{3}$$

8. Since, $(\alpha+\beta), (\alpha^2+\beta^2), (\alpha^3+\beta^3)$ are in GP.

$\Rightarrow \quad (\alpha^2+\beta^2)^2=(\alpha+\beta)(\alpha^3+\beta^3)$

$\Rightarrow \quad \alpha^4+\beta^4+2\alpha^2\beta^2=\alpha^4+\beta^4+\alpha\beta^3+\beta\alpha^3$

$\Rightarrow \quad \alpha\beta(\alpha^2+\beta^2-2\alpha\beta)=0$

$\Rightarrow \quad \alpha\beta(\alpha-\beta)^2=0$

$\Rightarrow \quad \alpha\beta=0 \quad \text{or} \quad \alpha=\beta$

$\Rightarrow \quad \frac{c}{a}=0 \quad \text{or} \quad \Delta=0$

$\Rightarrow \quad c\Delta=0$

9. Since, a, b and c are in an AP.

Let $\quad a=A-D, b=A, c=A+D$

Given, $\quad a+b+c=\frac{3}{2}$

$\Rightarrow \quad (A-D)+A+(A+D)=\frac{3}{2}$

$\Rightarrow \quad 3A=\frac{3}{2} \Rightarrow A=\frac{1}{2}$

$\therefore$ The number are $\frac{1}{2}-D, \frac{1}{2}, \frac{1}{2}+D$.

Also, $\left(\frac{1}{2}-D\right)^2, \frac{1}{4}, \left(\frac{1}{2}+D\right)^2$ are in GP.

$\therefore \quad \left(\frac{1}{4}\right)^2=\left(\frac{1}{2}-D\right)^2\left(\frac{1}{2}+D\right)^2 \Rightarrow \frac{1}{16}=\left(\frac{1}{4}-D^2\right)^2$

$\Rightarrow \quad \frac{1}{4}-D^2=\pm\frac{1}{4} \Rightarrow D^2=\frac{1}{2} \Rightarrow D=\pm\frac{1}{\sqrt{2}}$

$\therefore \quad a=\frac{1}{2}\pm\frac{1}{\sqrt{2}}$

So, out of the given values, $a=\frac{1}{2}-\frac{1}{\sqrt{2}}$ is the right choice.

10. $\left.\begin{array}{r}\alpha+\beta=1\\ \alpha\beta=p\end{array}\right\}$ and $\left.\begin{array}{r}\lambda+\delta=4\\ \lambda\delta=q\end{array}\right\}$

Let r be the common ratio.

Since, α, β, γ and δ are in GP.

Therefore, $\quad \beta=\alpha r, \gamma=\alpha r^2$

and $\quad \delta=\alpha r^3$

Then, $\quad \alpha+\alpha r=1 \Rightarrow \alpha(1+r)=1 \quad$ ...(i)

and $\quad \alpha r^2+\alpha r^3=4 \Rightarrow \alpha r^2(1+r)=4 \quad$ ...(ii)

From Eqs. (i) and (ii), $r^2=4 \Rightarrow r=\pm 2$

Now, $\quad \alpha\cdot\alpha r=p$ and $\alpha r^2\cdot\alpha r^3=q$

On putting $\quad r=-2$, we get

$\alpha=-1, p=-2$ and $q=-32$

Again putting $r=2$, we get $\alpha=1/3$ and $p=-\frac{2}{9}$

Since, q and p are integers.

Therefore, we take $p=-2$ and $q=-32$.

11. Here, $(a^2+b^2+c^2)p^2-2(ab+bc+cd)p+(b^2+c^2+d^2)\le 0$

$\Rightarrow (a^2p^2-2abp+b^2)+(b^2p^2-2bcp+c^2)+(c^2p^2-2cdp+d^2)\le 0$

$\Rightarrow (ap-b)^2+(bp-c)^2+(cp-d)^2\le 0$

[since, sum of squares is never less than zero]

Since, each of the squares is zero.

$\therefore \quad (ap-b)^2=(bp-c)^2=(cp-d)^2=0$

$\Rightarrow \quad p=\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$

$\therefore \quad a, b, c, d$ are in GP.

12. Since, a, b, c are in GP.

$\Rightarrow \quad b^2=ac$

Given, $\quad ax^2+2bx+c=0$

$\Rightarrow \quad ax^2+2\sqrt{ac}\,x+c=0$

$\Rightarrow \quad (\sqrt{a}\,x+\sqrt{c})^2=0 \Rightarrow x=-\sqrt{\frac{c}{a}}$

Since, $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have common root.

$\therefore \quad x=-\sqrt{c/a}$ must satisfy.

$dx^2+2ex+f=0$

$\Rightarrow \quad d\cdot\frac{c}{a}-2e\sqrt{\frac{c}{a}}+f=0 \Rightarrow \frac{d}{a}-\frac{2e}{\sqrt{ac}}+\frac{f}{c}=0$

$\Rightarrow \quad \frac{2e}{b}=\frac{d}{a}+\frac{f}{c} \quad [\because b^2=ac]$

Hence, $\quad \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in an AP.

13. Here, $t_3=4 \Rightarrow ar^2=4$

$\therefore$ Product of first five terms $=a\cdot ar\cdot ar^2\cdot ar^3\cdot ar^4$

$=a^5r^{10}=(ar^2)^5=4^5$

14. If $a, b, c\in(2, 18)$, then

$a+b+c=25 \quad$ ....(i)

Since, 2, a, b are in AP.

$\Rightarrow \quad 2a=b+2 \quad$ .... (ii)

and $b, c, 18$ are in GP.

$\Rightarrow \quad c^2=18b \quad$ ... (iii)

From Eqs. (i), (ii) and (iii),

$\frac{b+2}{2}+b+\sqrt{18b}=25$

$\Rightarrow \quad 3b+2+6\sqrt{2}\sqrt{b}=50$

$\Rightarrow \quad 3b+6\sqrt{2}\sqrt{b}-48=0$

$\Rightarrow \quad b+2\sqrt{2}\sqrt{b}-16=0$

$\Rightarrow \quad b+4\sqrt{2}\sqrt{b}-2\sqrt{2}\sqrt{b}-16=0$

$\Rightarrow \quad \sqrt{b}(\sqrt{b}+4\sqrt{2})-2\sqrt{2}(\sqrt{b}+4\sqrt{2})=0$

$\Rightarrow \quad (\sqrt{b}-2\sqrt{2})(\sqrt{b}+4\sqrt{2})=0$

$\Rightarrow \quad b=8, a=5$

and $\quad c=12$

15. Let 27, 8, 12 be three terms of a GP.

$\Rightarrow \quad t_m = 27, t_n = 8 \quad \text{and} \quad t_p = 12$

$AR^{m-1} = 27, AR^{n-1} = 8$

and $AR^{p-1} = 12$

$\therefore \quad R = \left(\frac{27}{8}\right)^{1/(m-n)} \quad \text{and} \quad R = \left(\frac{8}{12}\right)^{1/(n-p)}$

$\Rightarrow \quad \left(\frac{27}{8}\right)^{1/(m-n)} = \left(\frac{2}{3}\right)^{1/(n-p)}$

$\Rightarrow \quad 3^{3/(m-n)} \cdot 3^{1/(n-p)} = 2^{1/(n-p)} \cdot 2^{3/(m-n)}$

$\Rightarrow \quad \frac{3^{\frac{3}{m-n} + \frac{1}{n-p}}}{2^{\frac{1}{n-p} + \frac{3}{m-n}}} = 1$

$\therefore \quad \frac{3}{m-n} + \frac{1}{n-p} = 0 \text{ and } \frac{1}{n-p} + \frac{3}{m-n} = 0$

$\Rightarrow \quad 3(n-p) = n - m \quad \text{and} \quad 2n = 3p - m$

Hence, there exists infinite GP for which 27, 8 and 12 as three of its terms.

16. Let a, d be the first term and common difference of an AP and b, r be the first term and common ratio of a GP.

Then, $\quad x = a + (m-1)d$ and $x = br^{m-1}$

$y = a + (n-1)d$ and $y = br^{n-1}$

$z = a + (p-1)d$ and $z = br^{p-1}$

Now, $x - y = (m-n)d$, $y - z = (n-p)d$

and $\quad z - x = (p-m)d$

Again now, $x^{y-z} \cdot y^{z-x} \cdot z^{x-y}$

$= [br^{m-1}]^{(n-p)d} \cdot [br^{n-1}]^{(p-m)d} \cdot [br^{p-1}]^{(m-n)d}$

$= b^{[n-p+p-m+m-n]d} \cdot r^{[(m-1)(n-p)+(n-1)(p-m)+(p-1)(m-n)]d}$

$= b^0 \cdot r^0 = 1$

Topic 4 Sum of *n* Terms and Infinite Terms of a GP

1. Let $S = \sum_{k=1}^{20} k\left(\frac{1}{2^k}\right)$

$$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots + \frac{20}{2^{20}} \quad \ldots\text{(i)}$$

On multiplying by $\left(\frac{1}{2}\right)$ both sides, we get

$$\frac{S}{2} = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \ldots + \frac{19}{2^{20}} + \frac{20}{2^{21}} \quad \ldots\text{(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$S - \frac{S}{2} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{20}} - \frac{20}{2^{21}}$$

$$\Rightarrow \quad \frac{S}{2} = \frac{\frac{1}{2}\left(1 - \frac{1}{2^{20}}\right)}{1 - \frac{1}{2}} - \frac{20}{2^{21}}$$

$\left[\because \text{sum of GP} = \frac{a(1-r^n)}{1-r}, r < 1\right]$

$$\frac{S}{2} = 1 - \frac{1}{2^{20}} - \frac{20}{2^{21}} = 1 - \frac{1}{2^{20}} - \frac{10}{2^{20}} = 1 - \frac{11}{2^{20}}$$

$$\Rightarrow \quad S = 2 - \frac{11}{2^{19}}$$

2. (a) We have, $S_n = 1 + q + q^2 + \ldots + q^n$ and

$$T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$$

Also, we have

$${}^{101}C_1 + {}^{101}C_2 S_1 + {}^{101}C_3 S_2 + \ldots + {}^{101}C_{101} S_{100} = \alpha T_{100}$$

$$\Rightarrow {}^{101}C_1 + {}^{101}C_2(1+q) + {}^{101}C_3(1+q+q^2) + \ldots + {}^{101}C_{101}(1+q+q^2+\ldots+q^{100}) = \alpha \cdot T_{100}$$

$$\Rightarrow {}^{101}C_1 + {}^{101}C_2 \frac{(1-q^2)}{1-q} + {}^{101}C_3\left(\frac{1-q^3}{1-q}\right) + {}^{101}C_4\left(\frac{1-q^4}{1-q}\right) + \ldots + {}^{101}C_{101}\left(\frac{1-q^{101}}{1-q}\right)$$

$= \alpha \cdot T_{100} \quad \left[\because \text{for a GP, } S_n = a\left(\frac{1-r^n}{1-r}\right), r \neq 1\right]$

$$\Rightarrow \frac{1}{1-q}[\{{}^{101}C_1 + {}^{101}C_2 + \ldots + {}^{101}C_{101}\} - \{{}^{101}C_1 q + {}^{101}C_2 q^2 + \ldots + {}^{101}C_{101} q^{101}\} = \alpha \cdot T_{100}$$

$$\Rightarrow \frac{1}{(1-q)}[(2^{101} - 1) - ((1+q)^{101} - 1)] = \alpha T_{100}$$

$[\because {}^nC_0 + {}^nC_1 + \ldots + {}^nC_n = 2^n]$

$$\Rightarrow \frac{2^{101} - (q+1)^{101}}{1-q} = \alpha \left[1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^{100}\right]$$

$$\Rightarrow \frac{2^{101} - (q+1)^{101}}{1-q} = \alpha \left[1 \cdot \frac{1 - \left(\frac{q+1}{2}\right)^{101}}{1 - \frac{q+1}{2}}\right]$$

$\left[\because q \neq 1 \Rightarrow q + 1 \neq 2 \Rightarrow \frac{q+1}{2} \neq 1\right]$

$$= \frac{\alpha\,[2^{101} - (q+1)^{101}]}{(1-q) \cdot 2^{100}} \Rightarrow \alpha = 2^{100}$$

3. Let the GP be $a, ar, ar^2, ar^3, \ldots \infty$; where $a > 0$ and $0 < r < 1$.

Then, according the problem, we have

$$3 = \frac{a}{1-r}$$

and $\quad \frac{27}{19} = a^3 + (ar)^3 + (ar^2)^3 + (ar^3)^3 + \ldots$

$$\Rightarrow \quad \frac{27}{19} = \frac{a^3}{1-r^3} \quad \left[\because S_\infty = \frac{a}{1-r}\right]$$

$\Rightarrow \quad \frac{27}{19} = \frac{(3(1-r))^3}{1-r^3} \left[\because 3 = \frac{a}{1-r} \Rightarrow a = 3(1-r)\right]$

$\Rightarrow \quad \frac{27}{19} = \frac{27(1-r)(1+r^2-2r)}{(1-r)(1+r+r^2)}$

$[\because (1-r)^3 = (1-r)(1-r)^2]$

$\Rightarrow \quad r^2 + r + 1 = 19(r^2 - 2r + 1)$

$\Rightarrow 18r^2 - 39r + 18 = 0$

$\Rightarrow \quad 6r^2 - 13r + 6 = 0$

$\Rightarrow (3r-2)(2r-3) = 0$

$\therefore \quad r = \frac{2}{3}$ or $r = \frac{3}{2}$ (reject) $\quad [\because 0 < r < 1]$

4. Let a, ar, ar^2 are in GP, where $(r > 1)$.

On multiplying middle term by 2, we have

$a, 2ar, ar^2$ are in an AP.

$\Rightarrow \quad 4ar = a + ar^2$

$\Rightarrow \quad r^2 - 4r + 1 = 0$

$\Rightarrow \quad r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\Rightarrow \quad r = 2 + \sqrt{3}$ [since, AP is increasing]

5. Given,

$k \cdot 10^9 = 10^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9$

$\Rightarrow \quad k = 1 + 2\left(\frac{11}{10}\right) + 3\left(\frac{11}{10}\right)^2 + \ldots + 10\left(\frac{11}{10}\right)^9 \quad \ldots(i)$

$\left(\frac{11}{10}\right)k = 1\left(\frac{11}{10}\right) + 2\left(\frac{11}{10}\right)^2 + \ldots + 9\left(\frac{11}{10}\right)^9 + 10\left(\frac{11}{10}\right)^{10} \ldots(ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$$k\left(1 - \frac{11}{10}\right) = 1 + \frac{11}{10} + \left(\frac{11}{10}\right)^2 + \ldots + \left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10}$$

$$\Rightarrow k\left(\frac{10-11}{10}\right) = \frac{1\left[\left(\frac{11}{10}\right)^{10} - 1\right]}{\left(\frac{11}{10} - 1\right)} - 10\left(\frac{11}{10}\right)^{10}$$

$\left[\because \text{ In GP, sum of } n \text{ terms} = \frac{a(r^n - 1)}{r-1}, \text{ when } r > 1\right]$

$$\Rightarrow \quad -k = 10\left[10\left(\frac{11}{10}\right)^{10} - 10 - 10\left(\frac{11}{10}\right)^{10}\right]$$

$\therefore \quad k = 100$

6. Let $S = 0.7 + 0.77 + 0.777 + \ldots$

$$= \frac{7}{10} + \frac{77}{10^2} + \frac{777}{10^3} + \ldots \text{ upto 20 terms}$$

$$= 7\left[\frac{1}{10} + \frac{11}{10^2} + \frac{111}{10^3} + \ldots \text{ upto 20 terms}\right]$$

$$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ upto 20 terms}\right]$$

$$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots + \text{upto 20 terms}\right]$$

$$= \frac{7}{9}\left[(1 + 1 + \ldots + \text{ upto 20 terms}) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \text{ upto 20 terms}\right)\right]$$

$$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left\{1 - \left(\frac{1}{10}\right)^{20}\right\}}{1 - \frac{1}{10}}\right]$$

$\left[\because \sum_{i=1}^{20} = 20 \text{ and sum of } n \text{ terms of GP, } S_n = \frac{a(1-r^n)}{1-r} \text{ when } (r < 1)\right]$

$$= \frac{7}{9}\left[20 - \frac{1}{9}\left\{1 - \left(\frac{1}{10}\right)^{20}\right\}\right]$$

$$= \frac{7}{9}\left[\frac{179}{9} + \frac{1}{9}\left(\frac{1}{10}\right)^{20}\right] = \frac{7}{81}[179 + (10)^{-20}]$$

7. We know that, the sum of infinite terms of GP is

$$S_\infty = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ \infty, & |r| \ge 1 \end{cases}$$

$\therefore \quad S_\infty = \frac{x}{1-r} = 5 \quad [|r| < 1]$

or $\quad 1 - r = \frac{x}{5}$

$\Rightarrow \quad r = \frac{5-x}{5}$ exists only when $|r| < 1$.

i.e. $\quad -1 < \frac{5-x}{5} < 1$

or $\quad -10 < -x < 0$

$\Rightarrow \quad 0 < x < 10$

8. Since, sum $= 4$ and second term $= \frac{3}{4}$.

It is given first term a and common ratio r.

$\Rightarrow \quad \frac{a}{1-r} = 4, \; ar = \frac{3}{4}$

$\Rightarrow \quad r = \frac{3}{4a}$

$\Rightarrow \quad \frac{a}{1 - \frac{3}{4a}} = 4$

$\Rightarrow \quad \frac{4a^2}{4a-3} = 4$

$\Rightarrow \quad (a-1)(a-3) = 0$

$\Rightarrow \quad a = 1$ or 3

When $a = 1, r = 3/4$

and when $a = 3, r = 1/4$

9. Sum of the n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\dots$ upto n terms can be written as

$$\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+\left(1-\frac{1}{16}\right)+\dots \text{ upto } n \text{ terms}$$

$$= n-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+n \text{ terms}\right)$$

$$= n-\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = n+2^{-n}-1$$

10. Let a_n denotes the length of side of the square S_n.

We are given, a_n = length of diagonal of S_{n+1}.

$\Rightarrow \quad a_n = \sqrt{2}\, a_{n+1}$

$\Rightarrow \quad a_{n+1} = \frac{a_n}{\sqrt{2}}$

This shows that $a_1, a_2, a_3, \dots$ form a GP with common ratio $1/\sqrt{2}$.

Therefore, $a_n = a_1\left(\frac{1}{\sqrt{2}}\right)^{n-1}$

$\Rightarrow \quad a_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1}$ $\quad [\because a_1 = 10, \text{ given}]$

$\Rightarrow \quad a_n^2 = 100\left(\frac{1}{\sqrt{2}}\right)^{2(n-1)}$

$\Rightarrow \quad \frac{100}{2^{n-1}} \le 1$ $\quad [\because a_n^2 \le 1, \text{ given}]$

$\Rightarrow \quad 100 \le 2^{n-1}$

This is possible for $n \ge 8$.

Hence, (b), (c), (d) are the correct answers.

11. $B_n = 1 - A_n > A_n$

$$\Rightarrow \quad A_n < \frac{1}{2} \Rightarrow \frac{3}{4}\,\frac{\left(1-\left(-\frac{3}{4}\right)^n\right)}{1+\frac{3}{4}} < \frac{1}{2}$$

$$\Rightarrow \quad \left(-\frac{3}{4}\right)^n > -\frac{1}{6}$$

Obviously, it is true for all even values of n.

But for

$$n=1,\ -\frac{3}{4} < -\frac{1}{6}$$

$$n=3, \left(-\frac{3}{4}\right)^3 = -\frac{27}{64} < -\frac{1}{6}$$

$$n=5, \left(-\frac{3}{4}\right)^5 = -\frac{243}{1024} < -\frac{1}{6}$$

and for $n = 7$,

$$\left(-\frac{3}{4}\right)^7 = -\frac{2187}{12288} > -\frac{1}{6}$$

Hence, minimum odd natural number $n_0 = 7$.

12. Consider an infinite GP with first term 1, 2, 3, ..., n and common ratios $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n+1}$.

$$\therefore \quad S_1 = \frac{1}{1-1/2} = 2$$

$$S_2 = \frac{2}{1-1/3} = 3$$

$$\vdots \quad \vdots \quad \vdots$$

$$S_{2n-1} = \frac{2n-1}{1-1/2n} = 2n$$

$$\therefore \quad S_1^2 + S_2^2 + S_3^2 + \dots + S_{2n-1}^2$$

$$= 2^2 + 3^2 + 4^2 + \dots + (2n)^2$$

$$= \frac{1}{6}(2n)(2n+1)(4n+1) - 1$$

13. Let three numbers in GP be a, ar, ar^2.

$\therefore \quad a^2 + a^2r^2 + a^2r^4 = S^2 \quad$ ...(i)

and $\quad a + ar + ar^2 = a\,S \quad$ ...(ii)

On dividing Eq. (i) by Eq. (ii) after squaring it, we get

$$\frac{a^2(1+r^2+r^4)}{a^2(1+r+r^2)^2} = \frac{S^2}{a^2S^2}$$

$$\Rightarrow \quad \frac{(1+r^2)^2 - r^2}{(1+r+r^2)^2} = \frac{1}{a^2}$$

$$\Rightarrow \quad \frac{(1+r^2-r)}{(1+r^2+r)} = \frac{1}{a^2}$$

$$\Rightarrow \quad a^2 = \frac{r+\frac{1}{r}+1}{r+\frac{1}{r}-1}$$

Put $\quad r + \frac{1}{r} = y$

$$\therefore \quad \frac{y+1}{y-1} = a^2$$

$$\Rightarrow \quad y + 1 = a^2y - a^2$$

$$\Rightarrow \quad y = \frac{a^2+1}{a^2-1} \quad \left[\because |y| = \left|r+\frac{1}{r}\right| > 2\right]$$

$$\Rightarrow \quad \left|\frac{a^2+1}{a^2-1}\right| > 2 \quad [\text{where}, (a^2-1) \ne 0]$$

$$\Rightarrow \quad |a^2+1| > 2|a^2-1|$$

$$\Rightarrow \quad (a^2+1)^2 - \{2(a^2-1)\}^2 > 0$$

$$\Rightarrow \{(a^2+1) - 2(a^2-1)\}\{(a^2+1) + 2(a^2-1)\} > 0$$

$$\Rightarrow \quad (-a^2+3)(3a^2-1) > 0$$

$$\therefore \quad \frac{1}{3} < a^2 < 3$$

$$\therefore \quad a^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3) \quad [\because a^2 \ne 1]$$

14. We have, $S_k = \dfrac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \dfrac{1}{(k-1)!}$

Now, $(k^2-3k+1)S_k = \{(k-2)(k-1)-1\}\times\frac{1}{(k-1)!}$

$$= \frac{1}{(k-3)!} - \frac{1}{(k-1)!}$$

$\Rightarrow \sum_{k=1}^{100} |(k^2-3k+1)S_k| = 1+1+2-\left(\frac{1}{99!}+\frac{1}{98!}\right) = 4-\frac{100^2}{100!}$

$\Rightarrow \frac{100^2}{100!} + \sum_{k=1}^{100} |(k^2-3k+1)S_k| = 4$

Topic 5 Harmonic Progression (HP)

1. **PLAN** nth term of HP, $t_n = \frac{1}{a+(n-1)n}$

Here, $a_1 = 5$, $a_{20} = 25$ for HP

$\therefore \quad \frac{1}{a} = 5$ and $\frac{1}{a+19d} = 25$

$\Rightarrow \quad \frac{1}{5} + 19d = \frac{1}{25} \quad \Rightarrow 19d = \frac{1}{25} - \frac{1}{5} = -\frac{4}{25}$

$\therefore \quad d = \frac{-4}{19\times 25}$

Since, $a_n < 0$

$\Rightarrow \quad \frac{1}{5} + (n-1)d < 0$

$\Rightarrow \quad \frac{1}{5} - \frac{4}{19\times 25}(n-1) < 0 \quad \Rightarrow \quad (n-1) > \frac{95}{4}$

$\Rightarrow \quad n > 1 + \frac{95}{4}$ or $n > 24.75$

$\therefore$ Least positive value of $n = 25$

2. Since, a, b, c, d are in AP.

$\Rightarrow \frac{a}{abcd}, \frac{b}{abcd}, \frac{c}{abcd}, \frac{d}{abcd}$ are in AP.

$\Rightarrow \frac{1}{bcd}, \frac{1}{cda}, \frac{1}{abd}, \frac{1}{abc}$ are in AP.

$\Rightarrow$ bcd, cda, abd, abc are in HP.

$\Rightarrow$ abc, abd, cda, bcd are in HP.

3. Since, $a_1, a_2, a_3, \ldots, a_{10}$ are in AP.

Now, $a_{10} = a_1 + 9d$

$\Rightarrow \quad 3 = 2 + 9d$

$\Rightarrow \quad d = 1/9$ and $a_4 = a_1 + 3d$

$\Rightarrow \quad a_4 = 2 + 3(1/9) = 2 + 1/3 = 7/3$

Also, $h_1, h_2, h_3, \ldots, h_{10}$ are in HP.

$\Rightarrow \quad \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \ldots, \frac{1}{h_{10}}$ are in AP.

Given, $h_1 = 2, h_{10} = 3$

$\therefore \quad \frac{1}{h_{10}} = \frac{1}{h_1} + 9d_1 \Rightarrow \frac{1}{3} = \frac{1}{2} + 9d_1$

$\Rightarrow \quad -\frac{1}{6} = 9d_1$

$\Rightarrow \quad d_1 = -\frac{1}{54}$ and $\frac{1}{h_7} = \frac{1}{h_1} + 6d_1$

$\Rightarrow \quad \frac{1}{h_7} = \frac{1}{2} + \frac{6\times 1}{-54}$

$\Rightarrow \quad \frac{1}{h_7} = \frac{1}{2} - \frac{1}{9} \Rightarrow h_7 = \frac{18}{7}$

$\therefore \quad a_4 h_7 = \frac{7}{3}\times\frac{18}{7} = 6$

4. Let the common ratio of the GP be r. Then,

$$y = xr \text{ and } z = xr^2$$

$\Rightarrow \ln y = \ln x + \ln r$ and $\ln z = \ln x + 2\ln r$

Let $A = 1 + \ln x, D = \ln r$

Then, $\frac{1}{1+\ln x} = \frac{1}{A}, \frac{1}{1+\ln y} = \frac{1}{1+\ln x + \ln r} = \frac{1}{A+D}$

and $\frac{1}{1+\ln z} = \frac{1}{1+\ln x + 2\ln r} = \frac{1}{A+2D}$

Therefore, $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in HP.

5. Let $a_1 = 1, a_2 = 2, \Rightarrow a_3 = 4, a_4 = 8$

$\therefore \quad b_1 = 1, b_2 = 3, b_3 = 7, b_4 = 15$

Clearly, b_1, b_2, b_3, b_4 are not in HP.

Hence, Statement II is false.

Statement I is already true.

6. Since, $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in HP.

$\therefore \quad \cos x = \frac{2\cos(x-y)\cos(x+y)}{\cos(x-y)+\cos(x+y)}$

$\Rightarrow \cos x(2\cos x\cdot\cos y) = 2\{\cos^2 x - \sin^2 y\}$

$\Rightarrow \quad \cos^2 x\cdot\cos y = \cos^2 x - \sin^2 y$

$\Rightarrow \quad \cos^2 x(1-\cos y) = \sin^2 y$

$\Rightarrow \quad \cos^2 x\cdot 2\sin^2\frac{y}{2} = 4\sin^2\frac{y}{2}\cdot\cos^2\frac{y}{2}$

$\Rightarrow \quad \cos^2 x\cdot\sec^2\frac{y}{2} = 2$

$\therefore \quad \cos x\cdot\sec\frac{y}{2} = \pm\sqrt{2}$

7. Since, a, b, c are in an AP.

$\therefore 2b = a + c$

and a^2, b^2, c^2 are in HP.

$\Rightarrow \quad b^2 = \frac{2a^2c^2}{a^2+c^2} \Rightarrow \left(\frac{a+c}{2}\right)^2 = \frac{2a^2c^2}{a^2+c^2}$

$\Rightarrow \quad (a^2+c^2)(a^2+c^2+2ac) = 8a^2c^2$

$\Rightarrow \quad (a^2+c^2) + 2ac(a^2+c^2) = 8a^2c^2$

$\Rightarrow (a^2+c^2) + 2ac(a^2+c^2) + a^2c^2 = 9a^2c^2$

$\Rightarrow \quad (a^2+c^2+ac)^2 = 9a^2c^2$

$\Rightarrow \quad a^2+c^2+ac = 3ac$

$\Rightarrow \quad a^2+b^2-2ac = 0$

$\Rightarrow \quad (a-c)^2 = 0 \quad \Rightarrow \quad a = c$

and if $a = c \Rightarrow b = c$ or $a^2+c^2+ac = -3ac$

$\Rightarrow \quad a^2+c^2+2ac = -2ac$

$\Rightarrow \quad (a+c)^2 = -2ac$

$\Rightarrow \quad 4b^2 = -2ac \Rightarrow b^2 = -\frac{ac}{2}$

Hence, $\quad a, b, -\frac{c}{2}$ are in GP.

$\therefore$ Either $a = b = c$ or $a, b, -\frac{c}{2}$ are in GP.

8. Since, a, A_1, A_2, b are in AP.

$\Rightarrow \quad A_1 + A_2 = a + b$

a, G_1, G_2, b are in GP $\Rightarrow G_1G_2 = ab$

and $\quad a, H_1, H_2, b$ are in HP.

$\Rightarrow \quad H_1 = \frac{3ab}{2b+a}, H_2 = \frac{3ab}{b+2a}$

$\therefore \quad \frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$

$\Rightarrow \quad \frac{H_1 + H_2}{H_1H_2} = \frac{A_1 + A_2}{G_1G_2} = \frac{1}{a} + \frac{1}{b} \quad \ldots\text{(i)}$

Now, $\quad \frac{G_1G_2}{H_1H_2} = \frac{ab}{\left(\frac{3ab}{2b+a}\right)\left(\frac{3ab}{b+2a}\right)}$

$= \frac{(2a+b)(a+2b)}{9ab} \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii), we get

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}$$

9. (i) Now, $\quad a + b = (a + x + y + z + b) - (x + y + z)$

$= \frac{5}{2}(a+b) - 15$

[since, a, x, y, z are in AP]

$\therefore \quad \text{Sum} = \frac{5}{2}(a+b) \Rightarrow a + b = 10 \quad \ldots\text{(i)}$

Since, a, x, y, z, b are in HP, then $\frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in AP.

Now, $\quad \frac{1}{a} + \frac{1}{b} = \left(\frac{1}{a} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{b}\right) - \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$

$= \frac{5}{2}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{5}{3}$

$\Rightarrow \quad \frac{a+b}{ab} = \frac{10}{9} \Rightarrow ab = \frac{9 \times 10}{10} \quad$ [from Eq. (i)]

$\Rightarrow \quad ab = 9 \quad \ldots\text{(ii)}$

On solving Eqs. (i) and (ii), we get

$a = 1, \ b = 9$

(ii) LHS $= \log(x+z) + \log(x+z-2y)$

$= \log(x+z) + \log\left[x + z - 2\left(\frac{2xz}{x+z}\right)\right] \quad \left[\because y = \frac{2xz}{x+z}\right]$

$= \log(x+z) + \log\frac{(x-z)^2}{(x+z)}$

$= 2\log(x-z) = \text{RHS}$

Topic 6 Relation between AM, GM, HM and Some Special Series

1. Given series,

$S = 1 + \frac{1^3 + 2^3}{1+2} + \frac{1^3 + 2^3 + 3^3}{1+2+3} + \ldots + \frac{1^3 + 2^3 + 3^3 + \ldots + 15^3}{1+2+3+\ldots+15} - \frac{1}{2}(1+2+3+\ldots+15)$

$= S_1 - S_2$ (let)

where,

$S_1 = 1 + \frac{1^3+2^3}{1+2} + \frac{1^3+2^3+3^3}{1+2+3} + \ldots + \frac{1^3 + 2^3 + 3^3 + \ldots + 15^3}{1+2+3+\ldots+15}$

$= \sum_{n=1}^{15} \frac{1^3 + 2^3 + \ldots + n^3}{1 + 2 + \ldots + n} = \sum_{n=1}^{15} \frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)}{2}}$

$\left[\because \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2 \text{ and } \sum_{r=1}^{n} r = \frac{n(n+1)}{2}\right]$

$= \sum_{n=1}^{15} \frac{n(n+1)}{2} = \frac{1}{2}\sum_{n=1}^{15}(n^2 + n)$

$= \frac{1}{2}\left[\frac{15 \times 16 \times 31}{6} + \frac{15 \times 16}{2}\right]$

$\left[\because \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{1}{2}[(5 \times 8 \times 31) + (15 \times 8)]$

$= (5 \times 4 \times 31) + (15 \times 4)$

$= 620 + 60 = 680$

and $S_2 = \frac{1}{2}(1 + 2 + 3 + \ldots + 15)$

$= \frac{1}{2} \times \frac{15 \times 16}{2} = 60$

Therefore, $S = S_1 - S_2 = 680 - 60 = 620.$

2. Given series is

$\frac{3 \times 1^3}{1^2} + \frac{5 \times (1^3 + 2^3)}{1^2 + 2^2} + \frac{7 \times (1^3 + 2^3 + 3^3)}{1^2 + 2^2 + 3^3} + \ldots$

So, nth term

$T_n = \frac{(3 + (n-1)2)(1^3 + 2^3 + 3^3 \ldots + n^3)}{1^2 + 2^2 + 3^2 + \ldots + n^2}$

$= \frac{(2n+1) \times \left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)(2n+1)}{6}}$

$\left[\because \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2 \text{ and } \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}\right]$

So, $T_n = \frac{3n(n+1)}{2} = \frac{3}{2}(n^2+n)$

Now, sum of the given series upto n terms

$$S_n = \Sigma T_n = \frac{3}{2}[\Sigma n^2 + \Sigma n]$$
$$= \frac{3}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$\therefore$ $$S_{10} = \frac{3}{2}\left[\frac{10\times 11\times 21}{6} + \frac{10\times 11}{2}\right]$$
$$= \frac{3}{2}[(5\times 11\times 7) + (5\times 11)]$$
$$= \frac{3}{2}\times 55(7+1) = \frac{3}{2}\times 55\times 8 = 3\times 55\times 4$$
$$= 12\times 55 = 660$$

3. *(b)* Given series is
$1 + (2\times 3) + (3\times 5) + (4\times 7) + \ldots$upto 11 terms.

Now, the rth term of the series is $a_r = r(2r-1)$

$\therefore$Sum of first 11-terms is

$$S_{11} = \sum_{r=1}^{11} r(2r-1) = \sum_{r=1}^{11}(2r^2 - r) = 2\sum_{r=1}^{11} r^2 - \sum_{r=1}^{11} r$$
$$= 2\,\frac{11\times(11+1)(2\times 11+1)}{6} - \frac{11\times(11+1)}{2}$$
$$\left[\because \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6} \text{ and } \sum_{r=1}^{n} r = \frac{n(n+1)}{2}\right]$$
$$= \left(\frac{11\times 12\times 23}{3}\right) - \left(\frac{11\times 12}{2}\right)$$
$$= (11\times 4\times 23) - (11\times 6) = 11(92-6) = 11\times 86 = 946$$

4. Given series is
$$\left(\frac{3}{4}\right)^3 + \left(1\frac{1}{2}\right)^3 + \left(2\frac{1}{4}\right)^3 + 3^3 + \left(3\frac{3}{4}\right)^3 + \ldots$$
Let $$S = \left(\frac{3}{4}\right)^3 + \left(\frac{6}{4}\right)^3 + \left(\frac{9}{4}\right)^3 + \left(\frac{12}{4}\right)^3 + \left(\frac{15}{4}\right)^3 + \ldots + \text{upto 15 terms}$$
$$= \left(\frac{3}{4}\right)^3 [1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots + 15^3]$$
$$= \left(\frac{3}{4}\right)^3 \left(\frac{15\times 16}{2}\right)^2$$
$$\left[\because 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2, n\in N\right]$$
$$= \frac{27}{64}\times\frac{225\times 256}{4}$$
$$= 27\times 225$$
$\Rightarrow$ $S = 27\times 225 = 225\,k$ [given]

$\Rightarrow$ $k = 27.$

5. Since, $S_k = \frac{1+2+3+\ldots+k}{k}$
$$= \frac{k(k+1)}{2k} = \frac{k+1}{2}$$
So, $$S_k^2 = \left(\frac{k+1}{2}\right)^2 = \frac{1}{4}(k+1)^2 \quad \ldots \text{(i)}$$
Now, $$\frac{5}{12}A = S_1^2 + S_2^2 + S_3^2 + \ldots S_{10}^2 = \sum_{k=1}^{10} S_k^2$$
$$\Rightarrow \frac{5}{12}A = \frac{1}{4}\sum_{k=1}^{10}(k+1)^2 = \frac{1}{4}[2^2 + 3^2 + 4^2 + \ldots 11^2]$$
$$= \frac{1}{4}\left[\frac{11\times(11+1)(2\times 11+1)}{6} - 1^2\right]$$
$$\left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$
$$= \frac{1}{4}\left[\frac{11\times 12\times 23}{6} - 1\right] = \frac{1}{4}[(22\times 23) - 1]$$
$$= \frac{1}{4}[506-1] = \frac{1}{4}[505]$$
$$\Rightarrow \frac{5}{12}A = \frac{505}{4} \Rightarrow A = 303$$

6. Consider, $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$
$$= \frac{1}{(x^m + x^{-m})(y^n + y^{-n})}$$
By using AM $\geq$ GM (because $x, y \in R^+$), we get
$(x^m + x^{-m}) \geq 2$ and $(y^n + y^{-n}) \geq 2$

$\left[\because \text{If } x > 0, \text{ then } x + \frac{1}{x} \geq 2\right]$

$\Rightarrow (x^m + x^{-m})(y^n + y^{-n}) \geq 4$

$$\Rightarrow \frac{1}{(x^m + x^{-m})(y^n + y^{-n})} \leq \frac{1}{4}$$
$\therefore$ Maximum value $= \frac{1}{4}$.

7. General term of the given series is
$$T_r = \frac{3r(1^2 + 2^2 + \ldots + r^2)}{2r+1} = \frac{3r[r(r+1)(2r+1)]}{6(2r+1)}$$
$$= \frac{1}{2}(r^3 + r^2)$$
Now, required sum $= \sum_{r=1}^{15} T_r = \frac{1}{2}\sum_{r=1}^{15}(r^3 + r^2)$
$$= \frac{1}{2}\left\{\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}\right\}_{n=15}$$
$$= \frac{1}{2}\left\{\frac{n(n+1)}{2}\left[\frac{n^2+n}{2} + \frac{2n+1}{3}\right]\right\}_{n=15}$$
$$= \frac{1}{2}\left\{\frac{n(n+1)}{2}\,\frac{(3n^2+7n+2)}{6}\right\}_{n=15}$$
$$= \frac{1}{2}\times\frac{15\times 16}{2}\times\frac{(3\times 225 + 105 + 2)}{6} = 7820$$

8. We have, $a_1, a_2, a_3, \dots a_{49}$ are in AP.

$\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$

Let $a_1 = a$ and $d =$ common difference

$\because \quad a_1 + a_5 + a_9 + \cdots + a_{49} = 416$

$\therefore \quad a + (a + 4d) + (a + 8d) + \dots (a + 48d) = 416$

$\Rightarrow \quad \frac{13}{2}(2a + 48d) = 416$

$\Rightarrow \quad a + 24d = 32 \quad \dots(i)$

Also $\quad , a_9 + a_{43} = 66$

$\therefore \quad a + 8d + a + 42d = 66$

$\Rightarrow \quad 2a + 50d = 66$

$\Rightarrow \quad a + 25d = 33 \quad \dots(ii)$

Solving Eqs. (i) and (ii), we get

$a = 8$ and $d = 1$

Now, $a_1^2 + a_2^2 + a_3^2 + \cdots + a_{17}^2 = 140m$

$8^2 + 9^2 + 10^2 + \dots + 24^2 = 140m$

$\Rightarrow (1^2 + 2^2 + 3^2 + \dots + 24^2) - (1^2 + 2^2 + 3^2 + \dots + 7^2) = 140m$

$\Rightarrow \quad \frac{24 \times 25 \times 49}{6} - \frac{7 \times 8 \times 15}{6} = 140m$

$\Rightarrow \quad \frac{3 \times 7 \times 8 \times 5}{6}(7 \times 5 - 1) = 140m$

$\Rightarrow \quad 7 \times 4 \times 5 \times 34 = 140m$

$\Rightarrow \quad 140 \times 34 = 140m \Rightarrow \quad m = 34$

9. We have,

$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots$

A = sum of first 20 terms

B = sum of first 40 terms

$\therefore A = 1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots + 2 \cdot 20^2$

$A = (1^2 + 2^2 + 3^2 + \dots + 20^2) + (2^2 + 4^2 + 6^2 + \dots + 20^2)$

$A = (1^2 + 2^2 + 3^2 + \dots + 20^2) + 4(1^2 + 2^2 + 3^2 + \dots + 10^2)$

$A = \frac{20 \times 21 \times 41}{6} + \frac{4 \times 10 \times 11 \times 21}{6}$

$A = \frac{20 \times 21}{6}(41 + 22) = \frac{20 \times 41 \times 63}{6}$

Similarly

$B = (1^2 + 2^2 + 3^2 + \dots + 40^2) + 4(1^2 + 2^2 + \dots + 20^2)$

$B = \frac{40 \times 41 \times 81}{6} + \frac{4 \times 20 \times 21 \times 41}{6}$

$B = \frac{40 \times 41}{6}(81 + 42) = \frac{40 \times 41 \times 123}{6}$

Now, $B - 2A = 100\lambda$

$\therefore \quad \frac{40 \times 41 \times 123}{6} - \frac{2 \times 20 \times 21 \times 63}{6} = 100\lambda$

$\Rightarrow \quad \frac{40}{6}(5043 - 1323) = 100\lambda \quad \Rightarrow \frac{40}{6} \times 3720 = 100\lambda$

$\Rightarrow \quad 40 \times 620 = 100\lambda \qquad \Rightarrow \lambda = \frac{40 \times 620}{100} = 248$

10. Let S_{10} be the sum of first ten terms of the series. Then, we have

$$S_{10} = \left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \dots \text{ to 10 terms}$$

$$= \left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + 4^2 + \left(\frac{24}{5}\right)^2 + \dots \text{ to 10 terms}$$

$$= \frac{1}{5^2}(8^2 + 12^2 + 16^2 + 20^2 + 24^2 + \dots \text{ to 10 terms})$$

$$= \frac{4^2}{5^2}(2^2 + 3^2 + 4^2 + 5^2 + \dots \text{ to 10 terms})$$

$$= \frac{4^2}{5^2}(2^2 + 3^2 + 4^2 + 5^2 + \dots + 11^2)$$

$$= \frac{16}{25}((1^2 + 2^2 + \dots + 11^2) - 1^2)$$

$$= \frac{16}{25}\left(\frac{11 \cdot (11 + 1)(2 \cdot 11 + 1)}{6} - 1\right)$$

$$= \frac{16}{25}(506 - 1) = \frac{16}{25} \times 505 \Rightarrow \frac{16}{5}m = \frac{16}{25} \times 505 = 101$$

11. Given, m is the AM of l and n.

$\therefore \quad l + n = 2m \quad \dots (i)$

and G_1, G_2, G_3 are geometric means between l and n.

l, G_1, G_2, G_3, n are in GP.

Let r be the common ratio of this GP.

$\therefore \quad G_1 = lr, G_2 = lr^2, G_3 = lr^3, \quad n - lr^4 \quad \Rightarrow \quad r = \left(\frac{n}{l}\right)^{\frac{1}{4}}$

Now, $G_1^4 + 2G_2^4 + G_3^4 = (lr)^4 + 2(lr^2)^4 + (lr^3)^4$

$= l^4 \times r^4(1 + 2r^4 + r^8) = l^4 \times r^4(r^4 + 1)^2$

$= l^4 \times \frac{n}{l}\left(\frac{n + l}{l}\right)^2 = ln \times 4m^2 = 4lm^2n$

12. **PLAN** Write the nth term of the given series and simplify it to get its lowest form. Then, apply, $S_n = \sum T_n$

Given series is $\frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \dots$

Let T_n be the nth term of the given series.

$$\therefore \quad T_n = \frac{1^3 + 2^3 + 3^3 + \dots + n^3}{1 + 3 + 5 + \dots + \text{upto } n \text{ terms}}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{n^2} = \frac{(n+1)^2}{4}$$

$$S_9 = \sum_{n=1}^{9} \frac{(n+1)^2}{4} = \frac{1}{4}(2^2 + 3^2 + \dots + 10^2) + 1^2 - 1^2]$$

$$= \frac{1}{4}\left[\frac{10(10 + 1)(20 + 1)}{6} - 1\right] = \frac{384}{4} = 96$$

13. Here, $\alpha \in (0, \frac{\pi}{2}) \Rightarrow \tan\alpha > 0$

$$\therefore \quad \frac{\sqrt{x^2 + x} + \frac{\tan^2\alpha}{\sqrt{x^2 + x}}}{2} \geq \sqrt{\sqrt{x^2 + x} \cdot \frac{\tan^2\alpha}{\sqrt{x^2 + x}}}$$

[using AM ≥ GM]

$$\Rightarrow \quad \sqrt{x^2 + x} + \frac{\tan^2\alpha}{\sqrt{x^2 + x}} \geq 2\tan\alpha$$

14. Given, $a_1 a_2 a_3 \dots a_n = c$

$\Rightarrow \quad a_1 a_2 a_3 \dots (a_{n-1})(2a_n) = 2c \qquad \dots(i)$

$\therefore \quad \dfrac{a_1 + a_2 + a_3 + \dots + 2a_n}{n} \ge (a_1 \cdot a_2 \cdot a_3 \dots 2a_n)^{1/n}$ [using AM ≥ GM]

$\Rightarrow \quad a_1 + a_2 + a_3 + \dots + 2a_n \ge n(2c)^{1/n}$ [from Eq. (i)]

$\Rightarrow$ Minimum value of

$$a_1 + a_2 + a_3 + \dots + 2a_n = n(2c)^{1/n}$$

15. Since, AM ≥ GM, then

$$\frac{(a+b)+(c+d)}{2} \ge \sqrt{(a+b)(c+d)} \Rightarrow M \le 1$$

Also, $(a+b)+(c+d) > 0$ [$\because a, b, c, d > 0$]

$\therefore \quad 0 < M \le 1$

16. Let α, β be the roots of given quadratic equation. Then,

$$\alpha + \beta = \frac{4+\sqrt{5}}{5+\sqrt{2}} \text{ and } \alpha\beta = \frac{8+2\sqrt{5}}{5+\sqrt{2}}$$

Let H be the harmonic mean between α and β, then

$$H = \frac{2\alpha\beta}{\alpha+\beta} = \frac{16+4\sqrt{5}}{4+\sqrt{5}} = 4$$

17. Since, product of n positive numbers is unity.

$\Rightarrow \quad x_1 \cdot x_2 \cdot x_3 \dots x_n = 1 \qquad \dots(i)$

Using AM ≥ GM, $\dfrac{x_1 + x_2 + \dots + x_n}{n} \ge (x_1 \cdot x_2 \dots x_n)^{1/n}$

$\Rightarrow \quad x_1 + x_2 + \dots + x_n \ge n\,(1)^{1/n}$ [from Eq. (i)]

Hence, sum of n positive numbers is never less than n.

18. Since, AM > GM

$\therefore \quad \dfrac{(b+c-a)+(c+a-b)}{2} > (b+c-a)(c+a-b)^{1/2}$

$\Rightarrow \quad c > [(b+c-a)(c+a-b)]^{1/2} \qquad \dots(i)$

Similarly $\quad b > [(a+b-c)(b+c-a)]^{1/2} \qquad \dots(ii)$

and $\quad a > [(a+b-c)(c+a-b)]^{1/2} \qquad \dots(iii)$

On multiplying Eqs. (i), (ii) and (iii), we get

$$abc > (a+b-c)(b+c-a)(c+a-b)$$

Hence, $(a+b-c)(b+c-a)(c+a-b) - abc < 0$

19. Since, $x_1, x_2, \dots, x_n$ are positive real numbers.

$\therefore$ Using nth power mean inequality

$$\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} \ge \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^2$$

$$\Rightarrow \quad \frac{n^2}{n}\left(\sum_{i=1}^{n} x_i^2\right) \ge \left(\sum_{i=1}^{n} x_i\right)^2 \Rightarrow n\left(\sum_{i=1}^{n} x_i^2\right) \ge \left(\sum_{i=1}^{n} x_i\right)^2$$

20. Let a and b are two numbers. Then,

$$A_1 = \frac{a+b}{2};\ G_1 = \sqrt{ab};\ H_1 = \frac{2ab}{a+b}$$

$$A_n = \frac{A_{n-1} + H_{n-1}}{2},$$

$$G_n = \sqrt{A_{n-1}H_{n-1}},$$

$$H_n = \frac{2A_{n-1}H_{n-1}}{A_{n-1} + H_{n-1}}$$

Clearly, $G_1 = G_2 = G_3 = \dots = \sqrt{ab}$.

21. A_2 is AM of A_1 and H_1 and $A_1 > H_1$

$\Rightarrow \quad A_1 > A_2 > H_1$

A_3 is AM of A_2 and H_2 and $A_2 > H_2$

$\Rightarrow \quad A_2 > A_3 > H_2$

$\vdots \quad \vdots \quad \vdots$

$\therefore \quad A_1 > A_2 > A_3 > \dots$

22. As above, $A_1 > H_2 > H_1,\ A_2 > H_3 > H_2$

$\therefore \quad H_1 < H_2 < H_3 < \dots$

23. Given, $a(n) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dots + \dfrac{1}{2^n - 1}$

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \dots + \frac{1}{7}\right) + \left(\frac{1}{8} + \dots + \frac{1}{15}\right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^n - 1}\right)$$

$$< 1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4} + \dots + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^{n-1}}\right)$$

$$= 1 + \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + \dots + \frac{2^{n-1}}{2^{n-1}}$$

$$= \underbrace{1 + 1 + 1 + 1 + \dots + 1}_{(n) \text{ times}} = n$$

Thus, $a(100) \le 100$

Therefore, (a) is the answer.

Again, $a(n) = 1 + \dfrac{1}{2} + \left(\dfrac{1}{3} + \dfrac{1}{4}\right) + \left(\dfrac{1}{5} + \dots + \dfrac{1}{8}\right)$

$$+ \dots + \left(\frac{1}{2^{n-1}+1} + \dots + \frac{1}{2^n}\right) - \frac{1}{2^n}$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^n} + \dots + \frac{1}{2^n}\right) - \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \dots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n}$$

$$= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{n \text{ times}} - \frac{1}{2^n} = \left(1 - \frac{1}{2^n}\right) + \frac{n}{2}$$

Therefore, $a(200) > \left(1 - \dfrac{1}{2^{200}}\right) + \dfrac{200}{2} > 100$

Therefore, (d) is also the answer.

24. Since, first and $(2n-1)$th terms are equal.

Let first term be x and $(2n-1)$ th term be y, whose middle term is t_n.

Thus, in arithmetic progression, $t_n = \frac{x+y}{2} = a$

In geometric progression, $t_n = \sqrt{xy} = b$

In harmonic progression, $t_n = \frac{2xy}{x+y} = c$

$\Rightarrow \quad b^2 = ac$ and $a > b > c$ [using AM > GM > HM]

Here, equality holds (i.e. $a = b = c$) only if all terms are same. Hence, options (a), (b) and (d) are correct.

25. Let the two positive numbers be a and b.

$\therefore \quad x = \frac{a+b}{2}$ [since, x is AM between a and b] ... (i)

and $\frac{a}{y} = \frac{y}{z} = \frac{z}{b}$ [since, y, z are GM's between a and b]

$\therefore \quad a = \frac{y^2}{z}$ and $b = \frac{z^2}{y}$

On substituting the values of a and b in Eq. (i), we get

$$2x = \frac{y^2}{z} + \frac{z^2}{y}$$

$$\Rightarrow \quad \frac{y^3 + z^3}{yz} = 2x$$

$$\Rightarrow \quad \frac{y^3 + z^3}{xyz} = 2$$

26. Let the two positive numbers be ka and a, $a > 0$.

Then, $G = \sqrt{ka \cdot a} = \sqrt{k} \cdot a$

and $H = \frac{2(ka)a}{ka + a} = \frac{2ka}{k+1}$

Again, $\frac{H}{G} = \frac{4}{5}$ [given]

$$\Rightarrow \quad \frac{\frac{2ka}{k+1}}{\sqrt{k}\,a} = \frac{4}{5} \quad \Rightarrow \quad \frac{2\sqrt{k}}{k+1} = \frac{4}{5}$$

$$\Rightarrow \quad 5\sqrt{k} = 2k + 2$$

$$\Rightarrow \quad 2k - 5\sqrt{k} + 2 = 0$$

$$\Rightarrow \quad \sqrt{k} = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm 3}{4} = 2, \frac{1}{2}$$

$\Rightarrow \quad k = 4, 1/4.$

Hence, the required ratio is 4 : 1.

27. Using AM $\geq$ GM,

$$\frac{1 + x^{2n}}{2} \geq \sqrt{1 \cdot x^{2n}}$$

$$\Rightarrow \quad \frac{1 + x^{2n}}{2} \geq x^n$$

$$\Rightarrow \quad \frac{x^n}{1 + x^{2n}} \leq \frac{1}{2}$$

$$\therefore \quad \frac{x^n \cdot y^m}{(1 + x^{2n})(1 + y^{2m})} \leq \frac{1}{4}$$

Hence, it is false statement.

28. Since, $\dfrac{\log_a x + \dfrac{1}{\log_a x}}{2} > 1$, using AM > GM

Here, equality holds only when $x = a$ which is not possible. So, $\log_a x + \log_x a$ is greater than 2.

Hence, it is a false statement.

29. Here, $(1 + a)(1 + b)(1 + c)$

$$= 1 + a + b + c + ab + bc + ca + abc \quad \text{...(i)}$$

Since, $\frac{a + b + c + ab + bc + ca + abc}{7} \geq (a^4 b^4 c^4)^{1/7}$

[using AM $\geq$ GM]

$$\Rightarrow \quad a + b + c + ab + bc + ca + abc \geq 7(a^4 b^4 c^4)^{1/7}$$

$$\Rightarrow \quad 1 + a + b + c + ca + abc > 7(a^4 b^4 c^4)^{1/7} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$(1 + a)(1 + b)(1 + c) > 7(a^4 b^4 c^4)^{1/7}$$

or $\{(1 + a)(1 + b)(1 + c)\}^7 > 7^7 (a^4 b^4 c^4)$

30. Let G_m be the geometric mean of $G_1, G_2, \ldots, G_n$.

$$\Rightarrow \quad G_m = (G_1 \cdot G_2 \ldots G_n)^{1/n}$$

$$= [(a_1) \cdot (a_1 \cdot a_1 r)^{1/2} \cdot (a_1 \cdot a_1 r \cdot a_1 r^2)^{1/3} \ldots (a_1 \cdot a_1 r \cdot a_1 r^2 \ldots a_1 r^{n-1})^{1/n}]^{1/n}$$

where, r is the common ratio of GP $a_1, a_2, \ldots, a_n$.

$$= [(a_1 \cdot a_1 \ldots n \text{ times}) \ (r^{1/2} \cdot r^{3/3} \cdot r^{6/4} \ldots r^{\frac{(n-1)n}{2n}})]^{1/n}$$

$$= [a_1^n \cdot r^{\frac{1}{2} + 1 + \frac{3}{2} + \cdots + \frac{n-1}{2}}]^{1/n}$$

$$= a_1 \left[r^{\frac{1}{2}\left[\frac{(n-1)n}{2}\right]} \right]^{1/n} = a_1 \left[r^{\frac{n-1}{4}} \right] \quad \text{...(i)}$$

Now, $A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} = \frac{a_1 (1 - r^n)}{n(1 - r)}$

and
$$H_n = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)}$$

$$= \frac{n}{\frac{1}{a_1}\left(1 + \frac{1}{r} + \ldots + \frac{1}{r^{n-1}}\right)}$$

$$= \frac{a_1 n (1 - r)\, r^{n-1}}{1 - r^n}$$

$$\therefore \quad A_n \cdot H_n = \frac{a_1 (1 - r^n)}{n\,(1 - r)} \times \frac{a_1 n\,(1 - r) r^{n-1}}{(1 - r^n)} = a_1^2 r^{n-1}$$

$$\Rightarrow \quad \prod_{k=1}^{n} A_k H_k = \prod_{k=1}^{n} (a_1^2 r^{n-1})$$

$$= (a_1^2 \cdot a_1^2 \cdot a_1^2 \ldots n \text{ times}) \times r^0 \cdot r^1 \cdot r^2 \ldots r^{n-1}$$

$$= a_1^{2n} \cdot r^{1 + 2 + \ldots + (n-1)}$$

$$= a_1^{2n} r^{\frac{n(n-1)}{2}} = [a_1 r^{\frac{n-1}{4}}]^{2n}$$

$$= [G_m]^{2n} \quad \text{[from Eq. (i)]}$$

$$G_m = \left[\prod_{k=1}^{n} A_k H_k\right]^{1/2n}$$

$$\Rightarrow \quad G_m = (A_1 A_2 \dots A_n H_1 H_2 \dots H_n)^{1/2n}$$

31. Let two numbers be a and b and $A_1, A_2, \dots, A_n$ be n arithmetic means between a and b. Then, $a, A_1, A_2, \dots, A_n, b$ are in AP with common difference

$$d = \frac{b-a}{n+1}$$

$$\therefore \quad p = A_1 = a + d = a + \frac{b-a}{n+1}$$

$$\Rightarrow \quad p = \frac{na+b}{n+1} \quad \dots\text{(i)}$$

Let $H_1, H_2, \dots, H_n$ be n harmonic means between a and b.

$\therefore \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \dots, \frac{1}{H_n}, \frac{1}{b}$ is an AP with common difference, $D = \frac{(a-b)}{(n+1)\,ab}$.

$$\therefore \quad \frac{1}{q} = \frac{1}{a} + D \Rightarrow \frac{1}{q} = \frac{1}{a} + \frac{(a-b)}{(n+1)\,ab}$$

$$\Rightarrow \quad \frac{1}{q} = \frac{nb+a}{(n+1)\,ab}$$

$$\Rightarrow \quad q = \frac{(n+1)\,ab}{nb+a} \quad \dots\text{(ii)}$$

From Eq. (i),

$$b = (n+1)\,p - na.$$

Putting it in Eq. (ii), we get

$$q\{n(n+1)\,p - n^2a + a\} = (n+1)\,a\{(n+1)p - na\}$$

$$\Rightarrow \quad n(n+1)\,a^2 - \{(n+1)^2 p + (n^2-1)q\}a + n(n+1)\,pq = 0$$

$$\Rightarrow na^2 - \{(n+1)\,p + (n-1)q\}\,a + npq = 0$$

Since, a is real, therefore

$$\{(n+1)\,p + (n-1)q\}^2 - 4n^2pq > 0$$

$$\Rightarrow (n+1)^2p^2 + (n-1)^2q^2 + 2\,(n^2-1)\,pq - 4n^2pq > 0$$

$$\Rightarrow \quad (n+1)^2p^2 + (n-1)^2q^2 - 2\,(n^2+1)\,pq > 0$$

$$\Rightarrow \quad q^2 - \frac{2\,(n^2+1)}{(n-1)^2}\,pq + \left(\frac{n+1}{n-1}\right)^2 p^2 > 0$$

$$\Rightarrow \quad q^2 - \left\{1 + \left(\frac{n+1}{n-1}\right)^2\right\}pq + \left(\frac{n+1)}{n-1}\right)^2 p^2 > 0$$

$$\Rightarrow \quad (q-p)\left\{q - \left(\frac{n+1}{n-1}\right)^2 p\right\} > 0$$

$$\Rightarrow \quad q < p \text{ or } q > \left(\frac{n+1}{n-1}\right)^2 p$$

$$\because \quad \left\{\left(\frac{n+1}{n-1}\right)^2 p > p\right\}$$

Hence, q cannot lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

32. Since $a, b, c > 0$

$$\Rightarrow \quad \frac{(a+b+c)}{3} > (abc)^{1/3} \quad \dots\text{(i)}$$

[using AM ≥ GM]

$$\text{Also,} \quad \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} \geq \left(\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\right)^{1/3} \quad \dots\text{(ii)}$$

[using AM ≥ GM]

On multiplying Eqs. (i) and (ii), we get

$$\frac{(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}{9} \geq (abc)^{1/3}\,\frac{1}{(abc)^{1/3}}$$

$$\therefore \quad (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$$

33. **Plan**

(i) If a, b, c are in GP, then they can be taken as a, ar, ar^2 where r, $(r \neq 0)$ is the common ratio.

(ii) Arithmetic mean of $x_1, x_2, \dots, x_n = \frac{x_1 + x_2 + \dots + x_n}{n}$

Let a, b, c be a, ar, ar^2, where $r \in N$

$$\text{Also,} \quad \frac{a+b+c}{3} = b + 2$$

$$\Rightarrow \quad a + ar + ar^2 = 3\,(ar) + 6$$

$$\Rightarrow \quad ar^2 - 2ar + a = 6$$

$$\Rightarrow \quad (r-1)^2 = \frac{6}{a}$$

Since, $6/a$ must be perfect square and $a \in N$.

So, a can be 6 only.

$\Rightarrow \; r - 1 = \pm 1 \;\Rightarrow\; r = 2$

$$\text{and} \quad \frac{a^2 + a - 14}{a+1} = \frac{36 + 6 - 14}{7} = 4$$

34. Using AM ≥ GM,

$$\frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}}{8}$$

$$\geq (a^{-5}\cdot a^{-4}\cdot a^{-3}\cdot a^{-3}\cdot a^{-3}\cdot 1\cdot a^8\cdot a^{10})^{\frac{1}{8}}$$

$$\Rightarrow a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10} \geq 8\cdot 1$$

Hence, minimum value is 8.