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Matrices and Determinants

Topic 1 Types of Matrices, Addition, Subtraction, Multiplication and Transpose of a Matrix

Objective Question I (Only one correct option)

1. If A is a symmetric matrix and B is a skew-symmetric matrix such that $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, then AB is equal to **(2019 Main, 12 April I)**

(a) $\begin{bmatrix} -4 & -2 \\ -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -2 \\ 1 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & 2 \\ 1 & 4 \end{bmatrix}$

2. The total number of matrices $A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}$, $(x, y \in R, x \neq y)$ for which $A^T A = 3I_3$ is **(2019 Main, 9 April II)**

(a) 2 (b) 4 (c) 3 (d) 6

3. Let $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, (\alpha \in R)$ such that $A^{32} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then, a value of α is **(2019 Main, 8 April I)**

(a) $\frac{\pi}{32}$ (b) 0 (c) $\frac{\pi}{64}$ (d) $\frac{\pi}{16}$

4. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}$ and $Q = [q_{ij}]$ be two 3×3 matrices such that $Q - P^5 = I_3$. Then, $\frac{q_{21} + q_{31}}{q_{32}}$ is equal to **(2019 Main, 12 Jan I)**

(a) 10 (b) 135 (c) 9 (d) 15

5. Let $A = \begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix}$. If $AA^T = I_3$, then $|p|$ is **(2019 Main, 11 Jan I)**

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

6. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix, such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals **(2016 Adv.)**

(a) 52 (b) 103

(c) 201 (d) 205

7. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ c & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9\,I$, where, I is 3×3 identity matrix, then the ordered pair (a, b) is equal to **(2015 Main)**

(a) $(2, -1)$ (b) $(-2, 1)$

(c) $(2, 1)$ (d) $(-2, -1)$

8. If $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T Q^{2005} P$ is **(2005, 1M)**

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2005 \\ 2005 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 2005 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then value of α for which $A^2 = B$, is **(2003, 1M)**

(a) 1 (b) -1

(c) 4 (d) no real values

10. If A and B are square matrices of equal degree, then which one is correct among the following? **(1995, 2M)**

(a) $A + B = B + A$

(b) $A + B = A - B$

(c) $A - B = B - A$

(d) $AB = BA$

Objective Question II

One or more than one correct option)

11. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary, 3×3, non-zero, symmetric matrix. Then, which of the following matrices is/are skew-symmetric? **(2015 Adv.)**

(a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$
(c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$

12. For 3×3 matrices M and N, which of the following statement(s) is/are not correct ? **(2013 Adv.)**

(a) N^TMN is symmetric or skew-symmetric, according as M is symmetric or skew-symmetric
(b) $MN - NM$ is symmetric for all symmetric matrices M and N
(c) MN is symmetric for all symmetric matrices M and N
(d) (adj M) (adj N) = adj(MN) for all invertible matrices M and N

13. Let ω be a complex cube root of unity with $\omega\neq0$ and $P=[p_{ij}]$ be an $n\times n$ matrix with $p_{ij}=\omega^{i+j}$. Then, $P^2\neq0$ when n is equal to **(2013 Adv.)**

(a) 57 (b) 55 (c) 58 (d) 56

Passage Based Problems

Passage I

Let a, b and c be three real numbers satisfying

$$[a\ \ b\ \ c]\begin{bmatrix}1 & 9 & 7\\ 8 & 2 & 7\\ 7 & 3 & 7\end{bmatrix}=[0\ \ 0\ \ 0] \qquad \text{...(i)}$$

(2011)

14. If the point $P(a, b, c)$, with reference to Eq. (i), lies on the plane $2x+y+z=1$, then the value of $7a+b+c$ is

(a) 0 (b) 12 (c) 7 (d) 6

15. Let $b=6$, with a and c satisfying Eq. (i). If α and β are the roots of the quadratic equation $ax^2+bx+c=0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is equal to

(a) 6 (b) 7 (c) $\frac{6}{7}$ (d) ∞

16. Let ω be a solution of $x^3-1=0$ with Im $(\omega)>0$. If $a=2$ with b and c satisfying Eq. (i) then the value of $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is

(a) -2 (b) 2
(c) 3 (d) -3

Passage II

Let p be an odd prime number and T_p be the following set of 2×2 matrices

$$T_p=\left\{A=\begin{bmatrix}a & b\\ c & a\end{bmatrix}; a, b, c\in\{0, 1, 2, \dots, p-1\}\right\}$$

(2010)

17. The number of A in T_p such that det (A) is not divisible by p, is

(a) $2p^2$ (b) p^3-5p
(c) p^3-3p (d) p^3-p^2

18. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is

(a) $(p-1)(p^2-p+1)$ (b) $p^3-(p-1)^2$
(c) $(p-1)^2$ (d) $(p-1)(p^2-2)$

19. The number of A in T_p such that A is either symmetric or skew-symmetric or both and det (A) is divisible by p is

(a) $(p-1)^2$ (b) $2(p-1)$
(c) $(p-1)^2+1$ (d) $2p-1$

NOTE The trace of a matrix is the sum of its diagonal entries.

Analytical and Descriptive Questions

20. If matrix $A=\begin{bmatrix}a & b & c\\ b & c & a\\ c & a & b\end{bmatrix}$, where a, b, c are real positive numbers, $abc=1$ and $A^TA=I$, then find the value of $a^3+b^3+c^3$. **(2003, 2M)**

Integer Type Question

21. Let $z=\frac{-1+\sqrt{3}i}{2}$, where $i=\sqrt{-1}$, and $r, s\in\{1, 2, 3\}$. Let $P=\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}$ and I be the identity matrix of order 2. Then, the total number of ordered pairs (r, s) for which $P^2=-I$ is **(2016 Adv.)**

Topic 2 Properties of Determinants

Objective Questions I (Only one correct option)

1. A value of $\theta\in(0, \pi/3)$, for which

$$\begin{vmatrix}1+\cos^2\theta & \sin^2\theta & 4\cos6\theta\\ \cos^2\theta & 1+\sin^2\theta & 4\cos6\theta\\ \cos^2\theta & \sin^2\theta & 1+4\cos6\theta\end{vmatrix}=0, \text{ is}$$

(2019 Main, 12 April II)

(a) $\frac{\pi}{9}$ (b) $\frac{\pi}{18}$ (c) $\frac{7\pi}{24}$ (d) $\frac{7\pi}{36}$

2. The sum of the real roots of the equation $\begin{vmatrix}x & -6 & -1\\ 2 & -3x & x-3\\ -3 & 2x & x+2\end{vmatrix}=0$, is equal to **(2019 Main, 10 April II)**

(a) 0 (b) -4
(c) 6 (d) 1

3. If $\Delta_1 = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

and $\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}, x \neq 0,$

then for all $\theta \in \left(0, \frac{\pi}{2}\right)$ **(2019 Main, 10 April I)**

(a) $\Delta_1 + \Delta_2 = -2(x^3 + x - 1)$
(b) $\Delta_1 - \Delta_2 = -2x^3$
(c) $\Delta_1 + \Delta_2 = -2x^3$
(d) $\Delta_1 - \Delta_2 = x(\cos 2\theta - \cos 4\theta)$

4. If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$, then the inverse of $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is **(2019 Main, 9 April I)**

(a) $\begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 13 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}$

5. Let α and β be the roots of the equation $x^2 + x + 1 = 0$. Then, for $y \neq 0$ in **R**,

$\begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$ is equal to **(2019 Main, 9 April I)**

(a) $y(y^2 - 1)$ (b) $y(y^2 - 3)$ (c) $y^3 - 1$ (d) y^3

6. Let the numbers 2, b, c be in an AP and $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$.

If $\det(A) \in [2, 16]$, then c lies in the interval **(2019 Main, 8 April II)**

(a) $[3, 2 + 2^{3/4}]$ (b) $(2 + 2^{3/4}, 4)$ (c) $[4, 6]$ (d) $[2, 3)$

7. If $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$; then for all $\theta \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$, $\det(A)$ lies in the interval **(2019 Main, 12 Jan II)**

(a) $\left(\frac{3}{2}, 3\right]$ (b) $\left[\frac{5}{2}, 4\right)$ (c) $\left(0, \frac{3}{2}\right]$ (d) $\left(1, \frac{5}{2}\right]$

8. If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= (a + b + c)(x + a + b + c)^2, x \neq 0$ and $a + b + c \neq 0$, then x is equal to **(2019 Main, 11 Jan II)**

(a) $-(a + b + c)$ (b) $-2(a + b + c)$
(c) $2(a + b + c)$ (d) abc

9. Let $a_1, a_2, a_3, \ldots, a_{10}$ be in GP with $a_i > 0$ for $i = 1, 2, \ldots, 10$ and S be the set of pairs (r, k), $r, k \in N$ (the set of natural numbers) for which

$$\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^r a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k & \log_e a_6^r a_7^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k & \log_e a_9^r a_{10}^k \end{vmatrix} = 0$$

Then, the number of elements in S, is **(2019 Main, 10 Jan II)**

(a) 4 (b) 2
(c) 10 (d) infinitely many

10. Let $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}$, where $b > 0$. Then, the minimum value of $\frac{\det(A)}{b}$ is **(2019 Main, 10 Jan II)**

(a) $-\sqrt{3}$ (b) $-2\sqrt{3}$ (c) $2\sqrt{3}$ (d) $\sqrt{3}$

11. Let $d \in R$, and

$A = \begin{bmatrix} -2 & 4+d & (\sin\theta)-2 \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{bmatrix}$, $\theta \in [0, 2\pi]$. If the minimum value of det(A) is 8, then a value of d is **(2019 Main, 10 Jan I)**

(a) -5 (b) -7 (c) $2(\sqrt{2} + 1)$ (d) $2(\sqrt{2} + 2)$

12. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2$, then the ordered pair (A, B) is equal to **(2018 Main)**

(a) $(-4, -5)$ (b) $(-4, 3)$ (c) $(-4, 5)$ (d) $(4, 5)$

13. Let ω be a complex number such that $2\omega + 1 = z$, where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to **(2017 Main)**

(a) $-z$ (b) z (c) -1 (d) 1

14. If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and

$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$= K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then K is equal to **(2014 Main)**

(a) $\alpha\beta$ (b) $\frac{1}{\alpha\beta}$ (c) 1 (d) -1

15. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \leq i, j \leq 3$. If the determinant of P is 2, then the determinant of the matrix Q is **(2012)**

(a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}

16. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then the value of α is **(2004, 1M)**

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 5

17. The number of distinct real roots of **(2001, 1M)**

$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ is

(a) 0 (b) 2
(c) 1 (d) 3

18. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$,

then $f(100)$ is equal to **(1999, 2M)**

(a) 0 (b) 1 (c) 100 (d) -100

19. The parameter on which the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$$

does not depend upon, is **(1997, 2M)**

(a) a (b) p (c) d (d) x

20. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$, if **(1997C, 2M)**

(a) x, y, z are in AP (b) x, y, z are in GP
(c) x, y, z are in HP (d) xy, yz, zx are in AP

21. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1. Let C be the subset of A consisting of all determinants with value -1. Then, **(1981, 2M)**

(a) C is empty
(b) B has as many elements as C
(c) $A = B \cup C$
(d) B has twice as many elements as C

Objective Question II

(One or more than one correct option)

22. Which of the following is(are) NOT the square of a 3×3 matrix with real entries? **(2017 Adv.)**

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

23. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha ?$$

(2015 Adv.)

(a) -4 (b) 9 (c) -9 (d) 4

24. Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then **(2014 Adv.)**

(a) determinant of $(M^2 + MN^2)$ is 0
(b) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)\, U$ is zero matrix
(c) determinant of $(M^2 + MN^2) \geq 1$
(d) for a 3×3 matrix U, if $(M^2 + MN^2)\, U$ equals the zero matrix, then U is the zero matrix

25. The determinant $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$

is equal to zero, then **(1986, 2M)**

(a) a, b, c are in AP
(b) a, b, c are in GP
(c) a, b, c are in HP
(d) $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

Numerical Value

26. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is ....

Fill in the Blanks

27. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is...... . **(1993, 2M)**

28. The value of the determinant $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$ is ... . **(1988, 2M)**

29. Given that $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, the other two roots are... and... . **(1983, 2M)**

30. The solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is... . **(1981, 2M)**

31. Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda + 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$

be an identity in λ, where p, q, r, s and t are constants. Then, the value of t is.... . **(1981, 2M)**

True/False

32. The determinants $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ are not identically equal. **(1983, 1M)**

Analytical and Descriptive Questions

33. If M is a 3×3 matrix, where $M^T M = I$ and $\det(M) = 1$, then prove that $\det(M - I) = 0$ **(2004, 2M)**

34. Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$$ represents a straight line. **(2001, 6M)**

35. Prove that for all values of θ

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$$

(2000, 3M)

36. Suppose, $f(x)$ is a function satisfying the following conditions

(a) $f(0) = 2, f(1) = 1$

(b) f has a minimum value at $x = 5/2$, and

(c) for all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

where a, b are some constants. Determine the constants a, b and the function $f(x)$. **(1998, 3M)**

37. Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$, where a, b and c are respectively the pth, qth and rth terms of a harmonic progression. **(1997C, 2M)**

38. Let $a > 0, d > 0$. Find the value of the determinant

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

39. For all values of A, B, C and P, Q, R, show that **(1994, 4M)**

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

40. For a fixed positive integer n, if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix},$$

then show that $\left[\frac{D}{(n!)^3} - 4\right]$ is divisible by n. **(1992, 4M)**

41. If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

Then, find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$. **(1991, 4M)**

42. Let the three digit numbers $A28$, $3B9$ and $62C$, where A, B and C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant

$\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k. **(1990, 4M)**

43. Let $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

Show that $\sum_{a=1}^{n} \Delta_a = c \in$ constant. **(1989, 5M)**

44. Show that

$$\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

(1985, 3M)

45. If α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x), B(x)$ and $C(x)$ be polynomials of degree 3, 4 and 5 respectively, then show that

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

is divisible by $f(x)$, where prime denotes the derivatives. **(1984, 3M)**

46. Without expanding a determinant at any stage, show that

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$$

where A and B are determinants of order 3 not involving x. **(1982, 5M)**

47. Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative. **(1981, 4M)**

Integer Type Question

48. The total number of distincts $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is **(2016 Adv.)**

Topic 3 Adjoint and Inverse of a Matrix

Objective Questions I (Only one correct option)

1. If $B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$ is the inverse of a 3×3 matrix A, then the sum of all values of α for which $\det(A) + 1 = 0$, is **(2019 Main, 12 April I)**

(a) 0 (b) -1 (c) 1 (d) 2

2. If $A = \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$ then A is **(2019 Main, 9 Jan II)**

(a) invertible only when $t = \pi$
(b) invertible for every $t \in R$
(c) not invertible for any $t \in R$
(d) invertible only when $t = \dfrac{\pi}{2}$

3. Let A and B be two invertible matrices of order 3×3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det(BA^{-1}B^T)$ is equal to **(2019 Main, 11 Jan II)**

(a) 1 (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{16}$ (d) 16

4. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \dfrac{\pi}{12}$, is equal to **(2019 Main, 9 Jan I)**

(a) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

5. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to **(2017 Main)**

(a) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

(c) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$ (d) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

6. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and A adj $A = AA^T$, then $5a + b$ is equal to **(2016 Main)**

(a) -1 (b) 5 (c) 4 (d) 13

7. If A is a 3×3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then BB^T is equal to **(2014 Main)**

(a) $I + B$ (b) I (c) B^{-1} (d) $(B^{-1})^T$

8. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to **(2013 Main)**

(a) 4 (b) 11 (c) 5 (d) 0

9. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **(2012)**

(a) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $PX = X$ (c) $PX = 2X$ (d) $PX = -X$

10. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a, b and c is either ω or ω^2. Then, the number of distinct matrices in the set S is **(2011)**

(a) 2 (b) 6 (c) 4 (d) 8

11. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to

(a) M^2 (b) $-N^2$ (c) $-M^2$ (d) MN **(2011)**

12. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $6A^{-1} = A^2 + cA + dI$, then (c, d) is **(2005, 1M)**

(a) $(-6, 11)$ (b) $(-11, 6)$
(c) $(11, 6)$ (d) $(6, 11)$

Objective Questions II

(One or more than one correct option)

13. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in R$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\dfrac{k}{8}$ and $\det(Q) = \dfrac{k^2}{2}$, then **(2016 Adv.)**

(a) $\alpha = 0, k = 8$ (b) $4\alpha - k + 8 = 0$
(c) $\det(P \text{ adj } (Q)) = 2^9$ (d) $\det(Q \text{ adj } (P)) = 2^{13}$

14. Let M be a 2×2 symmetric matrix with integer entries. Then, M is invertible, if **(2014 Adv.)**

(a) the first column of M is the transpose of the second row of M
(b) the second row of M is the transpose of the first column of M
(c) M is a diagonal matrix with non-zero entries in the main digonal
(d) the product of entries in the main diagonal of M is not the square of an integer

15. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is/are

(a) -2 (b) -1
(c) 1 (d) 2

Integer Answer Type Question

16. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$$

If $\det(\text{adj } A) + \det(\text{adj} B) = 10^6$, then $[k]$ is equal to...... **(2010)**

Topic 4 Solving System of Equations

Objective Questions I (Only one correct option)

1. If $[x]$ denotes the greatest integer $\leq x$, then the system of liner equations $[\sin\theta]x + [-\cos\theta]y = 0$, $[\cot\theta]x + y = 0$ **(2019 Main, 12 April II)**

(a) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and has a unique solution if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$.

(b) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

(c) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and have infinitely many solutions if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(d) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

2. Let λ be a real number for which the system of linear equations

$x + y + z = 6$, $4x + \lambda y - \lambda z = \lambda - 2$ and $3x + 2y - 4z = -5$

has infinitely many solutions. Then λ is a root of the quadratic equation **(2019 Main, 10 April II)**

(a) $\lambda^2 - 3\lambda - 4 = 0$ (b) $\lambda^2 + 3\lambda - 4 = 0$
(c) $\lambda^2 - \lambda - 6 = 0$ (d) $\lambda^2 + \lambda - 6 = 0$

3. If the system of linear equations

$x + y + z = 5$
$x + 2y + 2z = 6$
$x + 3y + \lambda z = \mu, (\lambda, \mu \in R)$, has infinitely many solutions, then the value of $\lambda + \mu$ is **(2019 Main, 10 April I)**

(a) 7 (b) 12
(c) 10 (d) 9

4. If the system of equations $2x + 3y - z = 0$, $x + ky - 2z = 0$ and $2x - y + z = 0$ has a non-trivial solution (x, y, z), then $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k$ is equal to **(2019 Main, 9 April II)**

(a) -4 (b) $\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) $\frac{3}{4}$

5. If the system of linear equations

$x - 2y + kz = 1$, $2x + y + z = 2$, $3x - y - kz = 3$

has a solution (x, y, z), $z \neq 0$, then (x, y) lies on the straight line whose equation is **(2019 Main, 8 April II)**

(a) $3x - 4y - 4 = 0$ (b) $3x - 4y - 1 = 0$
(c) $4x - 3y - 4 = 0$ (d) $4x - 3y - 1 = 0$

6. The greatest value of $c \in R$ for which the system of linear equations $x - cy - cz = 0$, $cx - y + cz = 0$, $cx + cy - z = 0$ has a non-trivial solution, is **(2019 Main, 8 April I)**

(a) -1 (b) $\frac{1}{2}$ (c) 2 (d) 0

7. The set of all values of λ for which the system of linear equations $x - 2y - 2z = \lambda x$, $x + 2y + z = \lambda y$ and $-x - y = \lambda z$ has a non-trivial solution **(2019 Main, 12 Jan II)**

(a) contains exactly two elements.
(b) contains more than two elements.
(c) is a singleton.
(d) is an empty set.

8. An ordered pair (α, β) for which the system of linear equations **(2019 Main, 12 Jan I)**

$(1 + \alpha)x + \beta y + z = 2$
$\alpha x + (1 + \beta)y + z = 3$
$\alpha x + \beta y + 2z = 2$

has a unique solution, is

(a) (2, 4) (b) (– 4, 2) (c) (1, – 3) (d) (–3, 1)

9. If the system of linear equations

$2x + 2y + 3z = a$
$3x - y + 5z = b$
$x - 3y + 2z = c$

where a, b, c are non-zero real numbers, has more than one solution, then **(2019 Main, 11 Jan I)**

(a) $b - c - a = 0$ (b) $a + b + c = 0$
(c) $b - c + a = 0$ (d) $b + c - a = 0$

10. The number of values of $\theta \in (0, \pi)$ for which the system of linear equations

$x + 3y + 7z = 0$,
$-x + 4y + 7z = 0$,
$(\sin 3\theta)x + (\cos 2\theta)y + 2z = 0$

has a non-trivial solution, is **(2019 Main, 10 Jan II)**

(a) two (b) three (c) four (d) one

11. If the system of equations

$x + y + z = 5$ $x + 2y + 3z = 9$
$x + 3y + \alpha z = \beta$

has infinitely many solutions, then $\beta - \alpha$ equals **(2019 Main, 10 Jan I)**

(a) 8 (b) 18 (c) 21 (d) 5

12. If the system of linear equations

$x - 4y + 7z = g$, $3y - 5z = h$, $-2x + 5y - 9z = k$

is consistent, then **(2019 Main, 9 Jan II)**

(a) $2g + h + k = 0$ (b) $g + 2h + k = 0$
(c) $g + h + k = 0$ (d) $g + h + 2k = 0$

13. The system of linear equations

$x + y + z = 2$, $2x + 3y + 2z = 5$
$2x + 3y + (a^2 - 1)z = a + 1$ **(2019 Main, 9 Jan I)**

(a) has infinitely many solutions for $a = 4$
(b) is inconsistent when $a = 4$
(c) has a unique solution for $|a| = \sqrt{3}$
(d) is inconsistent when $|a| = \sqrt{3}$

14. If the system of linear equations

$$x + ky + 3z = 0,\ 3x + ky - 2z = 0$$
$$2x + 4y - 3z = 0$$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to **(2018 Main)**

(a) -10 (b) 10 (c) -30 (d) 30

15. The system of linear equations

$x + \lambda y - z = 0;\ \lambda x - y - z = 0;\ x + y - \lambda z = 0$

has a non-trivial solution for **(2016 Main)**

(a) infinitely many values of λ (b) exactly one value of λ

(c) exactly two values of λ (d) exactly three values of λ

16. The set of all values of λ for which the system of linear equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$ and $-x_1 + 2x_2 = \lambda x_3$ has a non-trivial solution **(2015 Main)**

(a) is an empty set

(b) is a singleton set

(c) contains two elements

(d) contains more than two elements

17. The number of value of k, for which the system of equation

$$(k+1)x + 8y = 4y \quad \Rightarrow \quad kx + (k+3)y = 3k - 1$$

(2013 Main)

has no solution, is

(a) infinite (b) 1 (c) 2 (d) 3

18. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is **(2010)**

(a) 0 (b) $2^9 - 1$ (c) 168 (d) 2

19. Given, $2x - y + 2z = 2$, $x - 2y + z = -4$, $x + y + \lambda z = 4$, then the value of λ such that the given system of equations has no solution, is **(2004, 1M)**

(a) 3 (b) 1 (c) 0 (d) -3

20. The number of values of k for which the system of equations $(k+1)x + 8y = 4k$ and $kx + (k+3)y = 3k - 1$ has infinitely many solutions, is/are **(2002, 1M)**

(a) 0 (b) 1 (c) 2 (d) ∞

21. If the system of equations $x + ay = 0$, $az + y = 0$ and $ax + z = 0$ has infinite solutions, then the value of a is

(a) -1 (b) 1

(c) 0 (d) no real values

22. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution, then possible values of k are **(2000, 2M)**

(a) $-1, 2$ (b) $1, 2$ (c) $0, 1$ (d) $-1, 1$

Assertion and Reason

For the following questions, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows.

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

23. Consider the system of equations $x - 2y + 3z = -1$, $x - 3y + 4z = 1$ and $-x + y - 2z = k$

Statement I The system of equations has no solution for $k \neq 3$ and

Statement II The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 0$. **(2008, 3M)**

Objective Questions II (Only or More Than One)

24. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in R$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

(a) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(b) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(c) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(d) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

Fill in the Blank

25. The system of equations $\lambda x + y + z = 0$, $-x + \lambda y + z = 0$ and $-x - y + \lambda z = 0$ will have a non-zero solution, if real values of λ are given by ... **(1982, 2M)**

Analytical and Descriptive Questions

26. $A = \begin{bmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{bmatrix}$, $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$

If there is a vector matrix X, such that $AX = U$ has infinitely many solutions, then prove that $BX = V$ cannot have a unique solution. If $a\ f\ d \neq 0$. Then, prove that $BX = V$ has no solution. **(2004, 4M)**

27. Let λ and α be real. Find the set of all values of λ for which the system of linear equations

$$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0,$$
$$x + (\cos\alpha)y + (\sin\alpha)z = 0$$

and $$-x + (\sin\alpha)y - (\cos\alpha)z = 0$$

has a non-trivial solution.

For $\lambda = 1$, find all values of α. **(1993, 5M)**

28. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$, respectively. If the system of equations $\alpha_1 y + \alpha_2 z = 0$ and $\beta_1 y + \beta_2 z = 0$ has a non-trivial solution, then prove that $\dfrac{b^2}{q^2} = \dfrac{ac}{pr}$. **(1987, 3M)**

29. Consider the system of linear equations in x, y, z

$(\sin 3\theta)\, x - y + z = 0$, $(\cos 2\theta)\, x + 4y + 3z = 0$ and $2x + 7y + 7z = 0$

Find the values of θ for which this system has non-trivial solution. **(1986, 5M)**

30. Show that the system of equations, $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ has atleast one solution for any real number $\lambda \neq -5$. Find the set of solutions, if $\lambda = -5$. **(1983, 5M)**

31. For what values of m, does the system of equations $3x + my = m$ and $2x - 5y = 20$ has a solution satisfying the conditions $x > 0$, $y > 0$? **(1979, 3M)**

32. For what value of k, does the following system of equations possess a non-trivial solution over the set of rationals

$x + y - 2z = 0$, $2x - 3y + z = 0$, and $x - 5y + 4z = k$

Find all the solutions. **(1979, 3M)**

33. Given, $x = cy + bz$, $y = az + cx$, $z = bx + ay$, where x, y, z are not all zero, prove that $a^2 + b^2 + c^2 + 2abc = 1$. **(1978, 2M)**

Integer Answer Type Question

34. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$ **(2017 Adv.)**

35. Let M be a 3×3 matrix satisfying $M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$,

Then, the sum of the diagonal entries of M is … **(2011)**

Answers

Topic 1

1. (b)	**2.** (b)	**3.** (c)	**4.** (a)
5. (b)	**6.** (b)	**7.** (d)	**8.** (a)
9. (d)	**10.** (a)	**11.** (c, d)	**12.** (c, d)
13. (b, c, d)	**14.** (d)	**15.** (b)	**16.** (a)
17. (d)	**18.** (c)	**19.** (d)	**20.** (4)
21. (1)			

Topic 2

1. (a)	**2.** (a)	**3.** (c)	**4.** (b)
5. (d)	**6.** (c)	**7.** (a)	**8.** (b)
9. (d)	**10.** (c)	**11.** (a)	**12.** (c)
13. (a)	**14.** (c)	**15.** (d)	**16.** (c)
17. (c)	**18.** (a)	**19.** (b)	**20.** (b)
21. (b)	**22.** (a, c)	**23.** (b, c)	**24.** (a, b)
25. (b,d)	**26.** (4)	**27.** (0)	**28.** (0)
29. (2 and 7)	**30.** {−1,2}	**31.** (0)	**32.** False

36. $\left(a = \dfrac{1}{4}, b = -\dfrac{5}{4} \text{ and } f(x) = \dfrac{1}{4}x^2 - \dfrac{5}{4}x + 2\right)$ **37.** (0)

38. $\left(\dfrac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\right)$ **41.** (2)

48. (2)

Topic 3

1. (c)	**2.** (b)	**3.** (c)	**4.** (c)
5. (b)	**6.** (b)	**7.** (b)	**8.** (b)
9. (d)	**10.** (a)	**11.** (c)	**12.** (a)
13. (b,c)	**14.** (c, d)	**15.** (a,d)	**16.** (4)

Topic 4

1. (a)	**2.** (c)	**3.** (c)	**4.** (b)
5. (c)	**6.** (b)	**7.** (c)	**8.** (a)
9. (a)	**10.** (a)	**11.** (a)	**12.** (a)
13. (d)	**14.** (b)	**15.** (d)	**16.** (c)
17. (d)	**18.** (a)	**19.** (b)	**20.** (b)
21. (a)	**22.** (d)	**23.** (a)	**24.** (a,d)

25. $\lambda = 0$ **27.** $-\sqrt{2} \leq \lambda \leq \sqrt{2}, \alpha = n\pi, n\pi + \dfrac{\pi}{4}$

29. $\theta = n\pi, n\pi + (-1)^n \dfrac{\pi}{6}, \ n \in Z$

30. $x = \dfrac{4-5k}{7}, y = \dfrac{13k-9}{7}, z = k$

31. $m < -\dfrac{15}{2}$ or $m > 30$

32. ($k = 0$, the given system has infinitely many solutions)

34. (1) **35.** (9)

Hints & Solutions

Topic 1 Types of Matrices, Addition, Subtraction and Transpose of a Matrix

1. Given matrix A is a symmetric and matrix B is a skew-symmetric.

$\therefore \quad A^T = A$ and $B^T = -B$

Since, $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$ (given)... (i)

On taking transpose both sides, we get

$$(A+B)^T = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}^T$$

$$\Rightarrow \quad A^T + B^T = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} \quad \ldots \text{(ii)}$$

Given, $A^T = A$ and $B^T = -B$

$$\Rightarrow \quad A - B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$$

On solving Eqs. (i) and (ii), we get

$$A = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

So, $AB = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

2. Given matrix

$$A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}, (x, y \in R, x \neq y)$$

for which

$$A^T A = 3I_3$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2x & 2x \\ 2y & y & -y \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Here, two matrices are equal, therefore equating the corresponding elements, we get

$$8x^2 = 3 \text{ and } 6y^2 = 3$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{3}{8}}$$

and $\quad y = \pm\frac{1}{\sqrt{2}}$

$\because$ There are 2 different values of x and y each.

So, 4 matrices are possible such that $A^T A = 3I_3$.

3. Given, matrix $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore \; A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & -\cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha + \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Similarly,

$$A^n = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}, n \in N$$

$$\Rightarrow A^{32} = \begin{bmatrix} \cos(32\alpha) & -\sin(32\alpha) \\ \sin(32\alpha) & \cos(32\alpha) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{(given)}$$

So, $\cos(32\alpha) = 0$ and $\sin(32\alpha) = 1$

$$\Rightarrow \quad 32\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{64}$$

4. Given matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad P = X + I$ (let)

Now, $P^5 = (I + X)^5$

$$= I + {}^5C_1(X) + {}^5C_2(X^2) + {}^5C_3(X^3) + \ldots$$

$[\because I^n = I,\; I \cdot A = A$ and $(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x + \ldots + {}^nC_n x^n]$

Here, $\quad X^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}$

and $\quad X^3 = X^2 \cdot X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow X^4 = X^5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\quad P^5 = I + 5\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} + 10\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 15 & 1 & 0 \\ 135 & 15 & 1 \end{bmatrix}$$

and $\; Q = I + P^5 = \begin{bmatrix} 2 & 0 & 0 \\ 15 & 2 & 0 \\ 135 & 15 & 2 \end{bmatrix} = [q_{ij}]$

$\Rightarrow q_{21} = 15, q_{31} = 135$ and $q_{32} = 15$

Hence, $\dfrac{q_{21} + q_{31}}{q_{32}} = \dfrac{15 + 135}{15} = \dfrac{150}{15} = 10$

5. Given, $AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix} \begin{bmatrix} 0 & p & p \\ 2q & q & -q \\ r & -r & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0+4q^2+r^2 & 0+2q^2-r^2 & 0-2q^2+r^2 \\ 0+2q^2-r^2 & p^2+q^2+r^2 & p^2-q^2-r^2 \\ 0-2q^2+r^2 & p^2-q^2-r^2 & p^2+q^2+r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that, if two matrices are equal, then corresponding elements are also equal, so

$$4q^2 + r^2 = 1 = p^2 + q^2 + r^2, \quad \text{...(i)}$$

$$2q^2 - r^2 = 0 \Rightarrow r^2 = 2q^2 \quad \text{...(ii)}$$

and $p^2 - q^2 - r^2 = 0$ ...(iii)

Using Eqs. (ii) and (iii), we get

$$p^2 = 3q^2 \quad \text{...(iv)}$$

Using Eqs. (ii) and (iv) in Eq. (i), we get

$$4q^2 + 2q^2 = 1$$

$$\Rightarrow 6q^2 = 1$$

$$\Rightarrow 2p^2 = 1 \quad \text{[using Eq. (iv)]}$$

$$p^2 = \frac{1}{2} \Rightarrow |p| = \frac{1}{\sqrt{2}}$$

6. Here, $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$

$$\therefore \quad P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4+4 & 1 & 0 \\ 16+32 & 4+4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4\times 2 & 1 & 0 \\ 16\,(1+2) & 4\times 2 & 1 \end{bmatrix} \quad \text{...(i)}$$

and $$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 4\times 2 & 1 & 0 \\ 16\,(1+2) & 4\times 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4\times 3 & 1 & 0 \\ 16\,(1+2+3) & 4\times 3 & 1 \end{bmatrix} \quad \text{...(ii)}$$

From symmetry,

$$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 4\times 50 & 1 & 0 \\ 16\,(1+2+3+\ldots+50) & 4\times 50 & 1 \end{bmatrix}$$

$$\because \quad P^{50} - Q = I \quad \text{[given]}$$

$$\therefore \begin{bmatrix} 1-q_{11} & -q_{12} & -q_{13} \\ 200-q_{21} & 1-q_{22} & -q_{23} \\ 16\times\frac{50}{2}(51)-q_{31} & 200-q_{32} & 1-q_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 200 - q_{21} = 0, \frac{16\times 50\times 51}{2} - q_{31} = 0,$$

$$200 - q_{32} = 0$$

$$\therefore \quad q_{21} = 200, q_{32} = 200, q_{31} = 20400$$

Thus, $$\frac{q_{31}+q_{32}}{q_{21}} = \frac{20400+200}{200} = \frac{20600}{200} = 103$$

7. Given, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$ and

$$AA^T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}$$

It is given that, $AA^T = 9I$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

On comparing, we get

$$a + 4 + 2b = 0 \Rightarrow a + 2b = -4 \quad \text{...(i)}$$

$$2a + 2 - 2b = 0 \Rightarrow a - b = -1 \quad \text{...(ii)}$$

and $a^2 + 4 + b^2 = 9$ ...(iii)

On solving Eqs. (i) and (ii), we get

$$a = -2, b = -1$$

This satisfies Eq. (iii)

Hence, $(a, b) \equiv (-2, -1)$

8. Now, $$P^TP = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\Rightarrow \quad P^TP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad P^TP = I$$

$$\Rightarrow \quad P^T = P^{-1}$$

Since, $Q = PAP^T$

$$\therefore \quad P^TQ^{2005}P = P^T[(PAP^T)(PAP^T)\ \ldots 2005 \text{ times}]P$$

$$= \underbrace{(P^TP)\,A\,(P^TP)\,A\,(P^TP)\ldots(P^TP)\,A\,(P^TP)}_{2005 \text{ times}}$$

$$= IA^{2005} = A^{2005}$$

$$\therefore \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

… … … … … … … …

… … … … … … … …

$$A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad P^T Q^{2005} P = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

9. Given, $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$\Rightarrow \quad A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix}$

Also, given, $A^2 = B$

$\Rightarrow \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$\Rightarrow \alpha^2 = 1$ and $\alpha + 1 = 5$

Which is not possible at the same time.

$\therefore$ No real values of α exists.

10. If A and B are square matrices of equal degree, then

$$A + B = B + A$$

11. Given, $X^T = -X, Y^T = -Y, Z^T = Z$

(a) Let $P = Y^3 Z^4 - Z^4 Y^3$

Then, $P^T = (Y^3 Z^4)^T - (Z^4 Y^3)^T$

$= (Z^T)^4 (Y^T)^3 - (Y^T)^3 (Z^T)^4$

$= -Z^4 Y^3 + Y^3 Z^4 = P$

$\therefore$ P is symmetric matrix.

(b) Let $P = X^{44} + Y^{44}$

Then, $P^T = (X^T)^{44} + (Y^T)^{44}$

$= X^{44} + Y^{44} = P$

$\therefore$ P is symmetric matrix.

(c) Let $P = X^4 Z^3 - Z^3 X^4$

Then, $P^T = (X^4 Z^3)^T - (Z^3 X^4)^T$

$= (Z^T)^3 (X^T)^4 - (X^T)^4 (Z^T)^3$

$= Z^3 X^4 - X^4 Z^3 = -P$

$\therefore$ P is skew-symmetric matrix.

(d) Let $P = X^{23} + Y^{23}$

Then, $P^T = (X^T)^{23} + (Y^T)^{23} = -X^{23} - Y^{23} = -P$

$\therefore$ P is skew-symmetric matrix.

12. (a) $(N^T M N)^T = N^T M^T (N^T)^T = N^T M^T N$, is symmetric if M is symmetric and skew-symmetric, if M is skew-symmetric.

(b) $(MN - NM)^T = (MN)^T - (NM)^T$

$= NM - MN = -(MN - NM)$

$\therefore$ Skew-symmetric, when M and N are symmetric.

(c) $(MN)^T = N^T M^T = NM \neq MN$

$\therefore$ Not correct.

(d) $(\text{adj } MN) = (\text{adj } N) \cdot (\text{adj } M)$

$\therefore$ Not correct.

13. Here, $P = [p_{ij}]_{n \times n}$ with $p_{ij} = w^{i+j}$

$\therefore$ When $n = 1$

$P = [p_{ij}]_{1 \times 1} = [\omega^2]$

$\Rightarrow \quad P^2 = [\omega^4] \neq 0$

$\therefore$ When $n = 2$

$$P = [p_{ij}]_{2 \times 2} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} \omega^2 & \omega^3 \\ \omega^3 & \omega^4 \end{bmatrix} = \begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix}$$

$P^2 = \begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix}\begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix}$

$\Rightarrow \quad P^2 = \begin{bmatrix} \omega^4 + 1 & \omega^2 + \omega \\ \omega^2 + \omega & 1 + \omega^2 \end{bmatrix} \neq 0$

When $n = 3$

$$P = [p_{ij}]_{3 \times 3} = \begin{bmatrix} \omega^2 & \omega^3 & \omega^4 \\ \omega^3 & \omega^4 & \omega^5 \\ \omega^4 & \omega^5 & \omega^6 \end{bmatrix} = \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix}\begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore \quad P^2 = 0$, when n is a multiple of 3.

$P^2 \neq 0$, when n is not a multiple of 3.

$\Rightarrow \quad n = 57$ is not possible.

$\therefore \quad n = 55, 58, 56$ is possible.

14. As (a, b, c) lies on $2x + y + z = 1 \Rightarrow 2a + b + c = 1$

$\Rightarrow \quad 2a + 6a - 7a = 1$

$\Rightarrow \quad a = 1, b = 6, c = -7$

$\therefore 7a + b + c = 7 + 6 - 7 = 6$

15. If $b = 6 \Rightarrow a = 1$ and $c = -7$

$\therefore ax^2 + bx + c = 0 \Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow (x + 7)(x - 1) = 0$

$\therefore \quad x = 1, -7$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{1} - \frac{1}{7}\right)^n = 1 + \frac{6}{7} + \left(\frac{6}{7}\right)^2 + \cdots + \infty = \frac{1}{1 - \frac{6}{7}}$$

$$= \frac{1}{1/7} = 7$$

16. If $a = 2, b = 12, c = -14$

$$\therefore \quad \frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$$

$$\Rightarrow \quad \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}} = \frac{3}{\omega^2} + 1 + 3\omega^2 = 3\omega + 1 + 3\omega^2$$

$$= 1 + 3(\omega + \omega^2) = 1 - 3 = -2$$

17. The number of matrices for which p does not divide $Tr\ (A) = (p-1)p^2$ of these $(p-1)^2$ are such that p divides $|A|$. The number of matrices for which p divides $Tr\ (A)$ and p does not divides $|A|$ are $(p-1)^2$.

$\therefore \quad$ Required number $= (p-1)p^2 - (p-1)^2 + (p-1)^2$

$= p^3 - p^2$

18. Trace of $A = 2a$, will be divisible by p, iff $a = 0$.

$|A| = a^2 - bc$, for $(a^2 - bc)$ to be divisible by p. There are exactly $(p-1)$ ordered pairs (b, c) for any value of a.

$\therefore$ Required number is $(p-1)^2$.

19. Given, $A=\begin{bmatrix} a & b \\ c & a \end{bmatrix}$, $a, b, c \in \{0, 1, 2, \ldots, p-1\}$

If A is skew-symmetric matrix, then $a=0$, $b=-c$

$\therefore \quad |A| = -b^2$

Thus, P divides $|A|$, only when $b=0$. ...(i)

Again, if A is symmetric matrix, then $b=c$ and $|A| = a^2 - b^2$

Thus, p divides $|A|$, if either p divides $(a-b)$ or p divides $(a+b)$.

p divides $(a-b)$, only when $a=b$,

i.e. $a = b \in \{0, 1, 2, \ldots, (p-1)\}$

i.e. p choices ...(ii)

p divides $(a+b)$.

$\Rightarrow p$ choices, including $a=b=0$ included in Eq. (i).

$\therefore$ Total number of choices are $(p+p-1) = 2p-1$

20. Given, $A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $abc=1$ and $A^T A = I$ ...(i)

Now, $A^T A = I$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow a^2+b^2+c^2 = 1$ and $ab+bc+ca = 0$ ...(ii)

We know, $a^3+b^3+c^3-3abc$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$\Rightarrow a^3+b^3+c^3 = (a+b+c)(1-0)+3$ [from Eqs. (i) and (ii)]

$\therefore a^3+b^3+c^3 = (a+b+c)+3$ ...(iii)

Now, $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

$=1$ ...(iv)

From Eq. (iii), $a^3+b^3+c^3 = 1+3 \Rightarrow a^3+b^3+c^3 = 4$

21. Here, $z = \dfrac{-1+i\sqrt{3}}{2} = \omega$

$\because \quad P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$

$$P^2 = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}\begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{2r}+\omega^{4s} & \omega^{r+2s}[(-1)^r+1] \\ \omega^{r+2s}[(-1)^r+1] & \omega^{4s}+\omega^{2r} \end{bmatrix}$$

Given, $P^2 = -I$

$\therefore \quad \omega^{2r}+\omega^{4s} = -1$ and $\omega^{r+2s}[(-1)^r+1] = 0$

Since, $r \in \{1, 2, 3\}$ and $(-1)^r + 1 = 0$

$\Rightarrow \quad r = \{1, 3\}$

Also, $\omega^{2r}+\omega^{4s} = -1$

If $r=1$, then $\omega^2+\omega^{4s} = -1$

which is only possible, when $s=1$.

As, $\omega^2+\omega^4 = -1$

$\therefore \quad r=1, s=1$

Again, if $r=3$, then

$\omega^6+\omega^{4s} = -1$

$\Rightarrow \quad \omega^{4s} = -2$ [never possible]

$\therefore \quad r \neq 3$

$\Rightarrow (r, s) = (1, 1)$ is the only solution.

Hence, the total number of ordered pairs is 1.

Topic 2 Properties of Determinants

1. Let $\Delta = \begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2$, we get

$$\Delta = \begin{vmatrix} 2 & \sin^2\theta & 4\cos 6\theta \\ 2 & 1+\sin^2\theta & 4\cos 6\theta \\ 1 & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - 2R_3$ and $R_2 \to R_2 - 2R_3$, we get

$$\Delta = \begin{vmatrix} 0 & -\sin^2\theta & -2-4\cos 6\theta \\ 0 & 1-\sin^2\theta & -2-4\cos 6\theta \\ 1 & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$$

On expanding w.r.t. C_1, we get

$\Rightarrow \sin^2\theta\,(2+4\cos 6\theta) + (2+4\cos 6\theta)(1-\sin^2\theta) = 0$

$\Rightarrow 2+4\cos 6\theta = 0 \Rightarrow \cos 6\theta = -\dfrac{1}{2} = \cos\dfrac{2\pi}{3}$

$\Rightarrow \quad 6\theta = \dfrac{2\pi}{3} \Rightarrow \theta = \dfrac{\pi}{9} \quad \left[\because \theta \in \left(0, \dfrac{\pi}{3}\right)\right]$

2. Given equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$

On expansion of determinant along R_1, we get

$x[(-3x)(x+2) - 2x(x-3)] + 6[2(x+2) + 3(x-3)]$

$-1[2(2x) - (-3x)(-3)] = 0$

$\Rightarrow x[-3x^2-6x-2x^2+6x] + 6[2x+4+3x-9]$

$-1[4x-9x] = 0$

$\Rightarrow x(-5x^2) + 6(5x-5) - 1(-5x) = 0$

$\Rightarrow \quad -5x^3+30x-30+5x = 0$

$\Rightarrow \quad 5x^3-35x+30 = 0 \Rightarrow x^3-7x+6 = 0.$

Since all roots are real

$\therefore$ Sum of roots $= -\dfrac{\text{coefficient of } x^2}{\text{coefficient of } x^3} = 0$

3. Given determinants are

$$\Delta_1 = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$= -x^3 + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta - x + x\sin^2\theta$

$= -x^3$

and $\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}, x \neq 0$

$= -x^3$ (similarly as Δ_1)

So, according to options, we get $\Delta_1 + \Delta_2 = -2x^3$

4. Given

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$$

$\because \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix},$

$\begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3+2+1 \\ 0 & 1 \end{bmatrix},$

$\vdots \quad \vdots \quad \vdots$

$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & (n-1)+(n-2)+\ldots+3+2+1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{n(n-1)}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$

Since, both matrices are equal, so equating corresponding element, we get

$\frac{n(n-1)}{2} = 78 \Rightarrow n(n-1) = 156$

$= 13 \times 12 = 13(13-1)$

$\Rightarrow \quad n = 13$

So, $\quad A = \begin{bmatrix} 1 & 13 \\ 0 & 1 \end{bmatrix} = A^{-1} = \begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

$\left[\because \text{if } |A| = 1 \text{ and } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

5. Given, quadratic equation is $x^2 + x + 1 = 0$ having roots α, β.

Then, $\alpha + \beta = -1$ and $\alpha\beta = 1$

Now, given determinant

$$\Delta = \begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} y+1+\alpha+\beta & y+1+\alpha+\beta & y+1+\alpha+\beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

$= \begin{vmatrix} y & y & y \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix} \quad [\because \alpha+\beta = -1]$

On applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} y & 0 & 0 \\ \alpha & y+\beta-\alpha & 1-\alpha \\ \beta & 1-\beta & y+\alpha-\beta \end{vmatrix}$$

$= y[(y+(\beta-\alpha))(y-(\beta-\alpha)) - (1-\alpha)(1-\beta)]$

[expanding along R_1]

$= y[y^2 - (\beta-\alpha)^2 - (1-\alpha-\beta+\alpha\beta)]$

$= y[y^2 - \beta^2 - \alpha^2 + 2\alpha\beta - 1 + (\alpha+\beta) - \alpha\beta]$

$= y[y^2 - (\alpha+\beta)^2 + 2\alpha\beta + 2\alpha\beta - 1 + (\alpha+\beta) - \alpha\beta]$

$= y[y^2 - 1 + 3 - 1 - 1] = y^3 \quad [\because \alpha+\beta = -1 \text{ and } \alpha\beta = 1]$

6. Given, matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$, so

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{vmatrix}$$

On applying, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$,

we get $\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & b-2 & c-2 \\ 4 & b^2-4 & c^2-4 \end{vmatrix}$

$= \begin{vmatrix} b-2 & c-2 \\ b^2-4 & c^2-4 \end{vmatrix}$

$= \begin{vmatrix} b-2 & c-2 \\ (b-2)(b+2) & (c-2)(c+2) \end{vmatrix}$

$= (b-2)(c-2)\begin{vmatrix} 1 & 1 \\ b+2 & c+2 \end{vmatrix}$

[taking common $(b-2)$ from C_1 and $(c-2)$ from C_2]

$= (b-2)(c-2)(c-b)$

Since, 2, b and c are in AP, if assume common difference of AP is d, then

$b = 2 + d$ and $c = 2 + 2d$

So, $\quad |A| = d(2d)d = 2d^3 \in [2, 16] \quad$ [given]

$\Rightarrow \quad d^3 \in [1, 8] \Rightarrow d \in [1, 2]$

$\therefore \quad 2 + 2d \in [2+2, 2+4]$

$= [4, 6] \Rightarrow c \in [4, 6]$

7. Given matrix $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$\Rightarrow \det(A) = |A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$= 1(1+\sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$

$\Rightarrow |A| = 2(1+\sin^2\theta) \quad \ldots$(i)

As we know that, for $\theta \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$

$\sin\theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \sin^2\theta \in \left[0, \frac{1}{2}\right) \Rightarrow 1+\sin^2\theta \in \left[0+1, \frac{1}{2}+1\right)$

$\Rightarrow \quad 1+\sin^2\theta \in \left[1, \frac{3}{2}\right)$

$\Rightarrow 2(1+\sin^2\theta) \in [2,3) \Rightarrow |A| \in [2,3) \subset \left(\frac{3}{2}, 3\right]$

8. Let $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

(taking common $(a+b+c)$ from R_1)

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

Now, expanding along R_1, we get

$\Delta = (a+b+c)\,1.\{(a+b+c)^2 - 0\}$

$= (a+b+c)^3 = (a+b+c)(x+a+b+c)^2$ (given)

$\Rightarrow (x+a+b+c)^2 = (a+b+c)^2$

$\Rightarrow \quad x+a+b+c = \pm(a+b+c)$

$\Rightarrow \quad x = -2(a+b+c) \qquad [\because x \neq 0]$

9. Given, $\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^r a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k & \log_e a_6^r a_7^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k & \log_e a_9^r a_{10}^k \end{vmatrix} = 0$

On applying elementary operations

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k - \log_e a_1^r a_2^k & \log_e a_3^r a_4^k - \log_e a_1^r a_2^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k - \log_e a_4^r a_5^k & \log_e a_6^r a_7^k - \log_e a_4^r a_5^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k - \log_e a_7^r a_8^k & \log_e a_9^r a_{10}^k - \log_e a_7^r a_8^k \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \log_e a_1^r a_2^k & \log_e\left(\frac{a_2^r a_3^k}{a_1^r a_2^k}\right) & \log_e\left(\frac{a_3^r a_4^k}{a_1^r a_2^k}\right) \\ \log_e a_4^r a_5^k & \log_e\left(\frac{a_5^r a_6^k}{a_4^r a_5^k}\right) & \log_e\left(\frac{a_6^r a_7^k}{a_4^r a_5^k}\right) \\ \log_e a_7^r a_8^k & \log_e\left(\frac{a_8^r a_9^k}{a_7^r a_8^k}\right) & \log_e\left(\frac{a_9^r a_{10}^k}{a_7^r a_8^k}\right) \end{vmatrix} = 0$$

$\left[\because \log_e m - \log_e n = \log_e\left(\frac{m}{n}\right)\right]$

[$\because a_1, a_2, a_3 \,\ldots\ldots, a_{10}$ are in GP, therefore put $a_1 = a, a_2 = aR, a_3 = aR^2, \ldots, a_{10} = aR^9$]

$$\Rightarrow \begin{vmatrix} \log_e a^{r+k}R^k & \log_e\left(\frac{a^{r+k}R^{r+2k}}{a^{r+k}R^k}\right) & \log_e\left(\frac{a^{r+k}R^{2r+3k}}{a^{r+k}R^k}\right) \\ \log_e a^{r+k}R^{3r+4k} & \log_e\left(\frac{a^{r+k}R^{4r+5k}}{a^{r+k}R^{3r+4k}}\right) & \log_e\left(\frac{a^{r+k}R^{5r+6k}}{a^{r+k}R^{3r+4k}}\right) \\ \log_e a^{r+k}R^{6r+7k} & \log_e\left(\frac{a^{r+k}R^{7r+8k}}{a^{r+k}R^{6r+7k}}\right) & \log_e\left(\frac{a^{r+k}R^{8r+9k}}{a^{r+k}R^{6r+7k}}\right) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \log_e(a^{r+k}R^k) & \log_e R^{r+k} & \log_e R^{2r+2k} \\ \log_e a^{r+k}R^{3r+4k} & \log_e R^{r+k} & \log_e R^{2r+2k} \\ \log_e a^{r+k}R^{6r+7k} & \log_e R^{r+k} & \log_e R^{2r+2k} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \log_e(a^{r+k}R^k) & \log_e R^{r+k} & 2\log_e R^{r+k} \\ \log_e(a^{r+k}R^{3r+4k}) & \log_e R^{r+k} & 2\log_e R^{r+k} \\ \log_e(a^{r+k}R^{6r+7k}) & \log_e R^{r+k} & 2\log_e R^{r+k} \end{vmatrix} = 0$$

[$\because \log m^n = n \log m$ and here $\log_e R^{2r+2k} = \log_e R^{2(r+k)} = 2\log_e R^{r+k}$]

$\because$ Column C_2 and C_3 are proportional,

So, value of determinant will be zero for any value of $(r, k), r, k \in N$.

$\therefore$ Set 'S' has infinitely many elements.

10. Given matrix, $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}, b > 0$

So, $\det(A) = |A| = \begin{vmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{vmatrix}$

$= 2\,[2(b^2+1) - b^2] - b(2b - b) + 1(b^2 - b^2 - 1)$

$= 2[2b^2 + 2 - b^2] - b^2 - 1$

$= 2b^2 + 4 - b^2 - 1 = b^2 + 3$

$\Rightarrow \quad \frac{\det(A)}{b} = \frac{b^2+3}{b} = b + \frac{3}{b}$

Now, by AM $\geq$ GM, we get

$\frac{b + \frac{3}{b}}{2} \geq \left(b \times \frac{3}{b}\right)^{1/2} \qquad \{\because b > 0\}$

$\Rightarrow \quad b + \frac{3}{b} \geq 2\sqrt{3}$

So, minimum value of $\frac{\det(A)}{b} = 2\sqrt{3}$

11. Given,

$$A = \begin{bmatrix} -2 & 4+d & (\sin\theta) - 2 \\ 1 & (\sin\theta) + 2 & d \\ 5 & (2\sin\theta) - d & (-\sin\theta) + 2 + 2d \end{bmatrix}$$

$$\therefore\ |A| = \begin{vmatrix} -2 & 4+d & (\sin\theta)-2 \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 4+d & (\sin\theta)-2 \\ 1 & (\sin\theta)+2 & d \\ 1 & 0 & 0 \end{vmatrix}$$

$(R_3 \to R_3 - 2R_2 + R_1)$

$= 1\,[(4+d)d - (\sin\theta+2)(\sin\theta-2)]$

(expanding along R_3)

$= (d^2 + 4d - \sin^2\theta + 4)$
$= (d^2 + 4d + 4) - \sin^2\theta$
$= (d+2)^2 - \sin^2\theta$

Note that $|A|$ will be minimum if $\sin^2\theta$ is maximum i.e. if $\sin^2\theta$ takes value 1.

$\because\ |A|_{\min} = 8,$

therefore $(d+2)^2 - 1 = 8$

$\Rightarrow\ (d+2)^2 = 9$

$\Rightarrow\ d+2 = \pm 3$

$\Rightarrow\ d = 1, -5$

12. Given,

$$\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

$\Rightarrow$ Apply $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

Taking common $(5x-4)$ from C_1, we get

$$(5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\therefore (5x-4)\begin{vmatrix} 1 & 2x & 0 \\ 0 & -x-4 & 0 \\ 0 & 0 & -x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

Expanding along C_1, we get

$(5x-4)(x+4)^2 = (A+Bx)(x-A)^2$

Equating, we get, $A = -4$ and $B = 5$

13. Given, $2\omega + 1 = z$

$\Rightarrow\ 2\omega + 1 = \sqrt{-3}$ $\quad [\because z = \sqrt{-3}]$

$\Rightarrow\ \omega = \dfrac{-1+\sqrt{3}i}{2}$

Since, ω is cube root of unity.

$\therefore\ \omega^2 = \dfrac{-1-\sqrt{3}i}{2}$ and $\omega^{3n} = 1$

$$\text{Now, } \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$[\because 1 + \omega + \omega^2 = 0 \text{ and } \omega^7 = (\omega^3)^2 \cdot \omega = \omega]$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 3 & 0 & 0 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$\Rightarrow\ 3(\omega^2 - \omega^4) = 3k$

$\Rightarrow\ (\omega^2 - \omega) = k$

$\therefore\ k = \left(\dfrac{-1-\sqrt{3}i}{2}\right) - \left(\dfrac{-1+\sqrt{3}i}{2}\right) = -\sqrt{3}i = -z$

14. **PLAN** Use the property that, two determinants can be multiplied column-to-row or row-to-column, to write the given determinant as the product of two determinants and then expand.

Given, $f(n) = \alpha^n + \beta^n$, $f(1) = \alpha + \beta$, $f(2) = \alpha^2 + \beta^2$, $f(3) = \alpha^3 + \beta^3$, $f(4) = \alpha^4 + \beta^4$

$$\text{Let } \Delta = \begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1\cdot1+1\cdot1+1\cdot1 & 1\cdot1+1\cdot\alpha+1\cdot\beta & 1\cdot1+1\cdot\alpha^2+1\cdot\beta^2 \\ 1\cdot1+1\cdot\alpha+1\cdot\beta & 1\cdot1+\alpha\cdot\alpha+\beta\cdot\beta & 1\cdot1+\alpha\cdot\alpha^2+\beta\cdot\beta^2 \\ 1\cdot1+1\cdot\alpha^2+1\cdot\beta^2 & 1\cdot1+\alpha^2\cdot\alpha+\beta^2\cdot\beta & 1\cdot1+\alpha^2\cdot\alpha^2+\beta^2\cdot\beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

On expanding, we get $\Delta = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

But given, $\Delta = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

Hence, $K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2 = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

$\therefore\ K = 1$

15. **PLAN** It is a simple question on scalar multiplication, i.e.

$$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Description of Situation Construction of matrix,

i.e. if $a = [a_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Here, $P = [a_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$Q = [b_{ij}]_{3\times3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where, $b_{ij} = 2^{i+j} a_{ij}$

$$\therefore \quad |Q| = \begin{vmatrix} 4a_{11} & 8a_{12} & 16a_{13} \\ 8a_{21} & 16a_{22} & 32a_{23} \\ 16a_{31} & 32a_{32} & 64a_{33} \end{vmatrix}$$

$$= 4\times8\times16 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{vmatrix}$$

$$= 2^9 \times 2 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} \cdot |P| = 2^{12} \cdot 2 = 2^{13}$$

16. We know, $|A^n| = |A|^n$

Since, $|A^3| = 125 \Rightarrow |A|^3 = 125$

$$\Rightarrow \begin{vmatrix} \alpha & 2 \\ 2 & \alpha \end{vmatrix} = 5 \quad \Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha = \pm 3$$

17. Given, $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix}$$

$$= (2\cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\Rightarrow (2\cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0$$

$$\Rightarrow \quad (2\cos x + \sin x)(\sin x - \cos x)^2 = 0$$

$$\Rightarrow \quad 2\cos x + \sin x = 0 \text{ or } \sin x - \cos x = 0$$

$$\Rightarrow \quad 2\cos x = -\sin x \text{ or } \sin x = \cos x$$

$\Rightarrow \cot x = -1/2$ gives no solution in $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$

and $\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$

18. Given,

$$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$$

Applying $C_3 \to C_3 - (C_1 + C_2)$

$$= \begin{vmatrix} 1 & x & 0 \\ 2x & x(x-1) & 0 \\ 3x(x-1) & x(x-1)(x-2) & 0 \end{vmatrix} = 0$$

$$\therefore \quad f(x) = 0 \Rightarrow f(100) = 0$$

19. Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 \to C_1 + C_3$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ \cos(p-d)x + \cos(p+d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x + \sin(p+d)x & \sin px & \sin(p+d)x \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ 2\cos px \cos dx & \cos px & \cos(p+d)x \\ 2\sin px \cos dx & \sin px & \sin(p+d)x \end{vmatrix}$$

Applying $C_1 \to C_1 - 2\cos dx\, C_2$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$$

$$\Rightarrow \Delta = (1+a^2-2a\cos dx)\,[\sin(p+d)x\cos px - \sin px\cos(p+d)x]$$

$$\Rightarrow \Delta = (1+a^2-2a\cos dx)\sin dx$$

which is independent of p.

20. Given, $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$

Applying $C_1 \to C_1 - (pC_2 + C_3)$

$$\Rightarrow \begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2+yp+yp+z) & xp+y & yp+z \end{vmatrix} = 0$$

$$\Rightarrow \quad -(xp^2+2yp+z)(xz-y^2) = 0$$

$$\therefore \quad \text{Either } xp^2+2yp+z = 0 \text{ or } y^2 = xz$$

$\Rightarrow x, y, z$ are in GP.

21. Since, A is the determinant of order 3 with entries 0 or 1 only.

Also, B is the subset of A consisting of all determinants with value 1.

[since, if we interchange any two rows or columns, then among themself sign changes]

Given, C is the subset having determinant with value –1.

$\therefore$ B has as many elements as C.

22. For a matrix to be square of other matrix its determinant should be positive.

(a) and (c) $\to$ Correct

(b) and (d) $\to$ Incorrect

23. Given determinant could be expressed as product of two determinants.

i.e. $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\,\alpha$

$$\Rightarrow \begin{vmatrix} 1+2\alpha+\alpha^2 & 1+4\alpha+4\alpha^2 & 1+6\alpha+9\alpha^2 \\ 4+4\alpha+\alpha^2 & 4+8\alpha+4\alpha^2 & 4+12\alpha+9\alpha^2 \\ 9+6\alpha+\alpha^2 & 9+12\alpha+4\alpha^2 & 9+18\alpha+9\alpha^2 \end{vmatrix} = -648\alpha$$

$$\Rightarrow \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 4 & 2\alpha & \alpha^2 \\ 9 & 3\alpha & \alpha^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$$\Rightarrow \alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$$\Rightarrow -8\alpha^3 = -648\alpha$$

$$\Rightarrow \alpha^3 - 81\alpha = 0 \Rightarrow \alpha(\alpha^2 - 81) = 0$$

$$\therefore \alpha = 0, \pm 9$$

24. **PLAN** (i) If A and B are two non-zero matrices and $AB = BA$, then $(A-B)(A+B) = A^2 - B^2$.

(ii) The determinant of the product of the matrices is equal to product of their individual determinants, i.e. $|AB| = |A||B|$.

Given, $M^2 = N^4 \Rightarrow M^2 - N^4 = 0$

$\Rightarrow (M - N^2)(M + N^2) = 0$ [as $MN = NM$]

Also, $M \neq N^2$

$\Rightarrow M + N^2 = 0$

$\Rightarrow \det(M + N^2) = 0$

Also, $\det(M^2 + MN^2) = (\det M)(\det M + N^2)$

$= (\det M)(0) = 0$

As, $\det(M^2 + MN^2) = 0$

Thus, there exists a non-zero matrix U such that

$$(M^2 + MN^2)U = 0$$

25. Given, $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$

Applying $C_3 \to C_3 - (\alpha C_1 + C_2)$

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(a\alpha^2 + 2b\alpha + c) \end{vmatrix} = 0$$

$\Rightarrow -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$

$\Rightarrow a\alpha^2 + 2b\alpha + c = 0$ or $b^2 = ac$

$\Rightarrow x - \alpha$ is a factor of $ax^2 + 2bx + c$ or a, b, c are in GP.

26. Let $\text{Det}(P) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

Now, maximum value of $\text{Det}(P) = 6$

If $a_1 = 1, a_2 = -1, a_3 = 1, b_2c_3 = b_1c_3 = b_1c_2 = 1$

and $b_3c_2 = b_3c_1 = b_2c_1 = -1$

But it is not possible as

$(b_2c_3)(b_3c_1)(b_1c_2) = -1$ and $(b_1c_3)(b_3c_2)(b_2c_1) = 1$

i.e., $b_1b_2b_3c_1c_2c_3 = 1$ and -1

Similar contradiction occurs when

$a_1 = 1, a_2 = 1, a_3 = 1, b_2c_1 = b_3c_1 = b_1c_2 = 1$

and $b_3c_2 = b_1c_3 = b_1c_2 = -1$

Now, for value to be 5 one of the terms must be zero but that will make 2 terms zero which means answer cannot be 5

Now, $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 4$

Hence, maximum value is 4.

27. Let $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & 1 & \dfrac{\log z}{\log y} \\ \dfrac{\log x}{\log z} & \dfrac{\log y}{\log z} & 1 \end{vmatrix}$$

On dividing and multiplying R_1, R_2, R_3 by $\log x$, $\log y$, $\log z$, respectively.

$$= \frac{1}{\log x \log y \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

28. $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Now, $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \dfrac{1}{abc} \begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$

Applying $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$

$$= \frac{1}{abc} \cdot abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$$

29. Given, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0 \Rightarrow (x+9) \begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Rightarrow (x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix} = 0 \Rightarrow (x+9)(x-2)(x-7)=0$$

$\Rightarrow$ $x = -9, 2, 7$ are the roots.

$\therefore$ Other two roots are 2 and 7.

30. Given, $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

$\Rightarrow$ $1(-10x^2 - 10x) - 4(5x^2 - 5) + 20(2x + 2) = 0$

$\Rightarrow$ $-30x^2 + 30x + 60 = 0$

$\Rightarrow$ $(x-2)(x+1) = 0$

$\Rightarrow$ $x = 2, -1$

Hence, the solution set is $\{-1, 2\}$.

31. Given, $\begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda + 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$

$= p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t$

Thus, the value of t is obtained by putting $\lambda = 0$.

$$\Rightarrow \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = t$$

$\Rightarrow$ $t = 0$

[$\because$ determinants of odd order skew-symmetric matrix is zero]

32. Let $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$

Applying $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$

$$= \frac{1}{abc}\cdot abc\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Hence, statement is false.

33. Since, $M^T M = I$ and $|M| = 1$

$\therefore$ $|M - I| = |IM - M^T M|$ [$\because IM = M$]

$\Rightarrow$ $|M - I| = |(I - M^T)M| = |(I - M)^T||M| = |I - M|$

$= (-1)^3 |M - I|$ [$\because I - M$ is a 3×3 matrix]

$= -|M - I|$

$\Rightarrow$ $2|M - I| = 0$

$\Rightarrow$ $|M - I| = 0$

34. Given, $\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} a^2x - aby - ac & bx + ay & cx + a \\ abx + a^2y & -ax + by - c & cy + b \\ acx + a^2 & cy + b & -ax - by + c \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} (a^2 + b^2 + c^2)x & ay + bx & cx + a \\ (a^2 + b^2 + c^2)y & by - c - ax & b + cy \\ a^2 + b^2 + c^2 & b + cy & c - ax - by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} x & ay + bx & cx + a \\ y & by - c - ax & b + cy \\ 1 & b + cy & c - ax - by \end{vmatrix} = 0$$

[$\because a^2 + b^2 + c^2 = 1$]

Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{ax}\begin{vmatrix} x^2 & axy & ax \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + yR_2 + R_3$

$$\Rightarrow \frac{1}{ax}\begin{vmatrix} x^2 + y^2 + 1 & 0 & 0 \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix} = 0$$

$\Rightarrow$ $\frac{1}{ax}[(x^2 + y^2 + 1)\{(-c - ax)(-ax - by) - b(cy)\}] = 0$

$\Rightarrow$ $\frac{1}{ax}[(x^2 + y^2 + 1)(acx + bcy + a^2x^2 + abxy - bcy)] = 0$

$\Rightarrow$ $\frac{1}{ax}[(x^2 + y^2 + 1)(acx + a^2x^2 + abxy)] = 0$

$\Rightarrow$ $\frac{1}{ax}[ax(x^2 + y^2 + 1)(c + ax + by)] = 0$

$\Rightarrow$ $(x^2 + y^2 + 1)(ax + by + c) = 0$

$\Rightarrow$ $ax + by + c = 0$

which represents a straight line.

35. Let $\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$

Applying $R_2 \to R_2 + R_3$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) + \sin\left(2\theta - \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

Now, $\sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta - \frac{2\pi}{3}\right)$

$$= 2\sin\left(\frac{\theta + \frac{2\pi}{3} + \theta - \frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta + \frac{2\pi}{3} - \theta + \frac{2\pi}{3}}{2}\right)$$

$$= 2\sin\theta\cos\frac{2\pi}{3} = 2\sin\theta\cos\left(\pi - \frac{\pi}{3}\right)$$

$$= -2\sin\theta\cos\frac{\pi}{3} = -\sin\theta$$

and $\cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta - \frac{2\pi}{3}\right)$

$$= 2\cos\left(\frac{\theta + \frac{2\pi}{3} + \theta - \frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta + \frac{2\pi}{3} - \theta + \frac{2\pi}{3}}{2}\right)$$

$$= 2\cos\theta\cos\left(\frac{2\pi}{3}\right) = 2\cos\theta\left(-\frac{1}{2}\right) = -\cos\theta$$

and $\sin\left(2\theta + \frac{4\pi}{3}\right) + \sin\left(2\theta - \frac{4\pi}{3}\right)$

$$= 2\sin\left(\frac{2\theta + \frac{4\pi}{3} + 2\theta - \frac{4\pi}{3}}{2}\right)\cos\left(\frac{2\theta + \frac{4\pi}{3} - 2\theta + \frac{4\pi}{3}}{2}\right)$$

$$= 2\sin 2\theta\cos\frac{4\pi}{3} = 2\sin 2\theta\cos\left(\pi + \frac{\pi}{3}\right)$$

$$= -2\sin 2\theta\cos\frac{\pi}{3} = -\sin 2\theta$$

$$\therefore \Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$$

[since, R_1 and R_2 are proportional]

36. Given, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Applying $R_3 \to R_3 - R_1 - 2R_2$, we get

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix} \quad [C_2 \to C_2 - C_1]$$

$\Rightarrow \quad f'(x) = 2ax + b$

On integrating, we get $f(x) = ax^2 + bx + c$,

where c is an arbitrary constant.

Since, f has maximum at $x = 5/2$.

$\Rightarrow \quad f'(5/2) = 0 \Rightarrow 5a + b = 0$ ...(i)

Also, $f(0) = 2 \Rightarrow c = 2$ and $f(1) = 1$

$\Rightarrow \quad a + b + c = 1$ ...(ii)

On solving Eqs. (i) and (ii) for a, b, we get

$$a = \frac{1}{4}, b = -\frac{5}{4}$$

Thus, $\quad f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

37. Since, a, b, c are pth, qth and rth terms of HP.

$\Rightarrow \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in an AP.

$$\Rightarrow \left.\begin{aligned} \frac{1}{a} &= A + (p-1)D \\ \frac{1}{b} &= A + (q-1)D \\ \frac{1}{c} &= A + (r-1)D \end{aligned}\right\} \quad \text{...(i)}$$

Let $\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ [from Eq. (i)]

$$= abc\begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - (A - D)R_3 - DR_2$

$$= abc\begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

38. Given, $a > 0, d > 0$ and let

$$\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

Taking $\frac{1}{a(a+d)(a+2d)}$ common from R_1,

$\frac{1}{(a+d)(a+2d)(a+3d)}$ from R_2,

$\frac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3

$$\Rightarrow \Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(a+3d) & (a+3d) & (a+d) \\ (a+3d)(a+4d) & (a+4d) & (a+2d) \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta'$$

where, $\Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(a+3d) & (a+3d) & (a+d) \\ (a+3d)(a+4d) & (a+4d) & (a+2d) \end{vmatrix}$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_2$

$$\Rightarrow \Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)2d & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta' = 2d^2 \begin{vmatrix} a+2d & a \\ d & d \end{vmatrix}$$

$$\Delta' = (2d^2)(d)(a+2d-a) = 4d^4$$

$$\therefore \quad \Delta = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

39. Let $\Delta = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \cos A\cos P + \sin A\sin P & \cos(A-Q) & \cos(A-R) \\ \cos B\cos P + \sin B\sin P & \cos(B-Q) & \cos(B-R) \\ \cos C\cos P + \sin C\sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \cos A\cos P & \cos(A-Q) & \cos(A-R) \\ \cos B\cos P & \cos(B-Q) & \cos(B-R) \\ \cos C\cos P & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \begin{vmatrix} \sin A\sin P & \cos(A-Q) & \cos(A-R) \\ \sin B\sin P & \cos(B-Q) & \cos(B-R) \\ \sin C\sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \cos P \begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1\cos Q, C_3 \to C_3 - C_1\cos R$ in first determinant and $C_2 \to C_2 - C_1\sin Q$ and in second determinant

$$\Rightarrow \quad \Delta = \cos P \begin{vmatrix} \cos A & \sin A\sin Q & \sin A\sin R \\ \cos B & \sin B\sin Q & \sin B\sin R \\ \cos C & \sin C\sin Q & \sin C\sin R \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos A\cos Q & \cos A\cos R \\ \sin B & \cos B\cos Q & \cos B\cos R \\ \sin C & \cos C\cos Q & \cos C\cos R \end{vmatrix}$$

$$\Delta = \cos P\sin Q\sin R \begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin B & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix} + \sin P\cos Q\cos R \begin{vmatrix} \sin A & \cos A & \cos A \\ \sin B & \cos B & \cos B \\ \sin C & \cos C & \cos C \end{vmatrix}$$

$$\Delta = 0 + 0 = 0$$

40. Given, $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$

Taking $n!$, $(n+1)!$ and $(n+2)!$ common from R_1, R_2 and R_3, respectively.

$$\therefore \quad D = n!\,(n+1)!\,(n+2)! \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 1 & (n+2) & (n+2)(n+3) \\ 1 & (n+3) & (n+3)(n+4) \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get

$$D = n!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

Expanding along C_1, we get

$$D = (n!)(n+1)!(n+2)![(2n+6) - (2n+4)]$$

$$D = (n!)(n+1)!(n+2)!\,[2]$$

On dividing both side by $(n!)^3$

$$\Rightarrow \quad \frac{D}{(n!)^3} = \frac{(n!)(n!)(n+1)(n!)(n+1)(n+2)2}{(n!)^3}$$

$$\Rightarrow \quad \frac{D}{(n!)^3} = 2(n+1)(n+1)(n+2)$$

$$\Rightarrow \quad \frac{D}{(n!)^3} = 2(n^3 + 4n^2 + 5n + 2) = 2n\,(n^2 + 4n + 5) + 4$$

$$\Rightarrow \frac{D}{(n!)^3} - 4 = 2n\,(n^2 + 4n + 5)$$

which shows that $\left[\frac{D}{(n!)^3} - 4\right]$ is divisible by n.

41. Let $\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix}$$

$$= c\begin{vmatrix} a-p & q-b \\ a-p & 0 \end{vmatrix} + (r-c)\begin{vmatrix} p & b \\ a-p & q-b \end{vmatrix}$$

$$= -c\,(a-p)\,(q-b) + (r-c)\,[p(q-b) - b(a-p)]$$

$$= -c\,(a-p)\,(q-b) + p(r-c)\,(q-b) - b(r-c)(a-p)$$

Since, $\Delta = 0$

$$\Rightarrow -c\,(a-p)\,(q-b) + p(r-c)\,(q-b) - b(r-c)\,(a-p) = 0$$

$$\Rightarrow \quad \frac{c}{r-c} + \frac{p}{p-a} + \frac{b}{q-b} = 0$$

[on dividing both sides by $(a-p)(q-b)(r-c)$]

$$\Rightarrow \quad \frac{p}{p-a} + \frac{b}{q-b} + 1 + \frac{c}{r-c} + 1 = 2$$

$$\Rightarrow \quad \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

42. We know, $A28 = A \times 100 + 2 \times 10 + 8$

$$3B9 = 3 \times 100 + B \times 10 + 9$$

and $\quad 62C = 6 \times 100 + 2 \times 10 + C$

Since, $A28, 3B9$ and $62C$ are divisible by k, therefore there exist positive integers m_1, m_2 and m_3 such that,

$100 \times A + 10 \times 2 + 8 = m_1 k$, $100 \times 3 + 10 \times B + 9 = m_2 k$

and $\quad 100 \times 6 + 10 \times 2 + C = m_3 k \quad$ ... (i)

$\therefore \quad \Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

Applying $R_2 \to 100R_1 + 10R_3 + R_2$

$$\Rightarrow \quad \Delta = \begin{vmatrix} A & 3 & 6 \\ 100A + 2 \times 10 + 8 & 100 \times 3 + 10 \times B + 9 & 100 \times 6 + 10 \times 2 + C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix} \quad \text{[from Eq. (i)]}$$

$$= \begin{vmatrix} A & 3 & 6 \\ m_1k & m_2k & m_3k \\ 2 & B & 2 \end{vmatrix} = k \begin{vmatrix} A & 3 & 6 \\ m_1 & m_2 & m_3 \\ 2 & B & 2 \end{vmatrix}$$

$\therefore \quad \Delta = mk$

Hence, determinant is divisible by k.

43. Given, $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$

$$\therefore \quad \sum_{a=1}^{n} \Delta_a = \begin{vmatrix} \sum_{a=1}^{n} (a-1) & n & 6 \\ \sum_{a=1}^{n} (a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n} (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \frac{n^2(n-1)^2}{4} & 3n^3 & 3n^2 - 3n \end{vmatrix}$$

$$= \frac{n^2(n-1)}{2} \begin{vmatrix} 1 & 1 & 6 \\ \frac{(2n-1)}{3} & 2n & 4n-2 \\ \frac{n(n-1)}{2} & 3n^2 & 3n^2 - 3n \end{vmatrix}$$

$$= \frac{n^3(n-1)}{12} \begin{vmatrix} 1 & 1 & 6 \\ 2n-1 & 6n & 12n-6 \\ n-1 & 6n & 6n-6 \end{vmatrix}$$

Applying $C_3 \to C_3 - 6C_1$

$$= \frac{n^3(n-1)}{12} \begin{vmatrix} 1 & 1 & 0 \\ 2n-1 & 6n & 0 \\ n-1 & 6n & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad \sum_{a=1}^{n} \Delta_a = c$ $\quad$ [$c = 0$, i.e. constant]

44. Let $\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$

Applying $C_3 \to C_3 + C_2$

$$\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

[$\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$]

Applying $C_2 \to C_2 + C_1$

$$\Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

Applying $C_3 \to C_3 + C_2$

$$\Rightarrow \quad \Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

Hence proved.

45. Since, α is repeated root of $f(x) = 0$.

$\therefore \; f(x) = a\,(x - \alpha)^2, a \in$ constant $(\neq 0)$

Let $\quad \phi(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

To show $\phi(x)$ is divisible by $(x - \alpha)^2$, it is sufficient to show that $\phi(\alpha)$ and $\phi'(\alpha) = 0$.

$\therefore \quad \phi(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$= 0$ $\quad$ [$\because R_1$ and R_2 are identical]

Again, $\quad \phi'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$\phi'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$= 0$ $\quad$ [$\because R_1$ and R_3 are identical]

Thus, α is a repeated root of $\phi(x) = 0$.

Hence, $\phi(x)$ is divisible by $f(x)$.

46. Let $\Delta = \begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$

Applying $R_2 \to R_2 - (R_1 + R_3)$, we get

$$\Delta = \begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ -4 & 0 & 0 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + \frac{x^2}{4} R_2$

and $\quad R_3 \to R_3 + \frac{x^2}{4} R_2$, we get

$$\Delta = \begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Applying $R_3 \to R_3 - 2R_1 = \begin{vmatrix} x+0 & x+1 & x-2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

$$= \begin{vmatrix} x & x & x \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$\Rightarrow \quad \Delta = Ax + B$

where, $A = \begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

and $B = \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

47. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= (a+b+c)\,[-(c-b)^2 - (a-b)(a-c)]$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= -\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca)$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

which is always negative.

48. Given, $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$

$$\Rightarrow \quad x \cdot x^2 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix} = 10$$

Apply $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$x^3 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 0 & 2 & -1+6x^3 \\ 0 & 6 & -2+24x^3 \end{vmatrix} = 10$$

$$\Rightarrow \quad x^3 \cdot \begin{vmatrix} 2 & 6x^3-1 \\ 6 & 24x^3-2 \end{vmatrix} = 10$$

$\Rightarrow \quad x^3(48x^3 - 4 - 36x^3 + 6) = 10$

$\Rightarrow \quad 12x^6 + 2x^3 = 10$

$\Rightarrow \quad 6x^6 + x^3 - 5 = 0$

$\Rightarrow \quad x^3 = \frac{5}{6}, -1$

$$x = \left(\frac{5}{6}\right)^{1/3}, -1$$

Hence, the number of real solutions is 2.

Topic 3 Adjoint and Inverse of a Matrix

1. Given matrix B is the inverse matrix of 3×3 matrix A,

where $B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$

We know that,

$$\det(A) = \frac{1}{\det(B)} \qquad \left[\because \det(A^{-1}) = \frac{1}{\det(A)}\right]$$

Since, $\det(A) + 1 = 0$ (given)

$$\frac{1}{\det(B)} + 1 = 0$$

$\Rightarrow \quad \det(B) = -1$

$\Rightarrow \quad 5(-2-3) - 2\alpha(0-\alpha) + 1(0-2\alpha) = -1$

$\Rightarrow \quad -25 + 2\alpha^2 - 2\alpha = -1$

$\Rightarrow \quad 2\alpha^2 - 2\alpha - 24 = 0$

$\Rightarrow \quad \alpha^2 - \alpha - 12 = 0$

$\Rightarrow \quad (\alpha - 4)(\alpha + 3) = 0$

$\Rightarrow \quad \alpha = -3, 4$

So, required sum of all values of α is $4 - 3 = 1$

2. $|A| = \begin{vmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{vmatrix}$

$$= (e^t)(e^{-t})(e^{-t})\begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\cos t - \sin t & -\sin t + \cos t \\ 1 & 2\sin t & -2\cos t \end{vmatrix}$$

(taking common from each column)

Aplying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$[\because e^{t-t} = e^0 = 1]$

$$= e^{-t}\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -2\cos t - \sin t & -2\sin t + \cos t \\ 0 & 2\sin t - \cos t & -2\cos t - \sin t \end{vmatrix}$$

$= e^{-t}((2\cos t + \sin t)^2 + (2\sin t - \cos t)^2)$

(expanding along column 1)

$= e^{-t}(5\cos^2 t + 5\sin^2 t)$

$= 5e^{-t}$ $\quad (\because \cos^2 t + \sin^2 t = 1)$

$\Rightarrow |A| = 5e^{-t} \neq 0$ for all $t \in R$

$\therefore$ A is invertible for all $t \in R$

[$\because$ If $|A| \neq 0$, then A is invertible]

3. Given, $|ABA^T| = 8$

$\Rightarrow \quad |A||B||A^T| = 8 \quad [\because |XY| = |X||Y|]$

$\therefore \quad |A|^2|B| = 8 \quad \ldots(i) \; [\because |A^T| = |A|]$

Also, we have $|AB^{-1}| = 8 \Rightarrow |A||B^{-1}| = 8$

$\Rightarrow \quad \frac{|A|}{|B|} = 8 \quad \ldots(ii) \left[\because |A^{-1}| = |A|^{-1} = \frac{1}{|A|}\right]$

On multiplying Eqs. (i) and (ii), we get

$|A|^3 = 8 \cdot 8 = 4^3$

$\Rightarrow \quad |A| = 4$

$\Rightarrow \quad |B| = \frac{|A|}{8} = \frac{4}{8} = \frac{1}{2}$

Now, $|BA^{-1}B^T| = |B|\frac{1}{|A|}|B| = \left(\frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{16}$

4. We have, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\therefore \quad |A| = \cos^2\theta + \sin^2\theta = 1$

and $\quad \text{adj } A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\left[\because \text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

$\Rightarrow \quad A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \left(\because A^{-1} = \frac{\text{adj } A}{|A|}\right)$

Note that, $A^{-50} = (A^{-1})^{50}$

Now, $\quad A^{-2} = (A^{-1})(A^{-1})$

$\Rightarrow A^{-2} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ -\cos\theta\sin\theta - \cos\theta\sin\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

Also, $A^{-3} = (A^{-2})(A^{-1})$

$A^{-3} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$

Similarly, $A^{-50} = \begin{bmatrix} \cos 50\theta & \sin 50\theta \\ -\sin 50\theta & \cos 50\theta \end{bmatrix}$

$= \begin{bmatrix} \cos\frac{25}{6}\pi & \sin\frac{25}{6}\pi \\ -\sin\frac{25}{6}\pi & \cos\frac{25}{6}\pi \end{bmatrix} \quad \left(\text{when } \theta = \frac{\pi}{12}\right)$

$= \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}\left[\begin{array}{l} \because \cos\left(\frac{25\pi}{6}\right) = \cos\left(4\pi + \frac{\pi}{6}\right) = \cos\frac{\pi}{6} \\ \text{and } \sin\left(\frac{25\pi}{6}\right) = \sin\left(4\pi + \frac{\pi}{6}\right) = \sin\frac{\pi}{6} \end{array}\right]$

$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

5. We have, $\quad A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4+12 & -6-3 \\ -8-4 & 12+1 \end{bmatrix}$

$= \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$

Now, $\quad 3A^2 + 12A = 3\begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix} + 12\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix} + \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$

$= \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

$\therefore \quad \text{adj }(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

6. Given, $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \text{ adj } A = AA^T$

Clearly, $A(\text{adj } A) = |A| I_2$

[$\because$ if A is square matrix of order n, then $A(\text{adj } A) = (\text{adj } A) \cdot A = |A| I_n$]

$= \begin{vmatrix} 5a & -b \\ 3 & 2 \end{vmatrix} I_2 = (10a + 3b) I_2$

$= (10a + 3b)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix} \quad \ldots(i)$

and $\quad AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$= \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix} \quad \ldots(ii)$

$\because \quad A(\text{adj } A) = AA^T$

$\therefore \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix} = \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix}$

[using Eqs. (i) and (ii)]

$\Rightarrow \quad 15a - 2b = 0$

$\Rightarrow \quad a = \frac{2b}{15} \quad \ldots(iii)$

and $\quad 10a + 3b = 13 \quad \ldots(iv)$

On substituting the value of 'a' from Eq. (iii) in Eq. (iv), we get

$10 \cdot \left(\frac{2b}{15}\right) + 3b = 13$

$\Rightarrow \quad \frac{20b + 45b}{15} = 13$

$\Rightarrow \quad \frac{65b}{15} = 13$

$\Rightarrow \quad b = 3$

Now, substituting the value of b in Eq. (iii), we get

$$5a = 2$$

Hence, $5a + b = 2 + 3 = 5$

7. **PLAN** Use the following properties of transpose $(AB)^T = B^T A^T$, $(A^T)^T = A$ and $A^{-1}A = I$ and simplify. If A is non-singular matrix, then $|A| \neq 0$.

Given, $AA^T = A^T A$ and $B = A^{-1}A^T$

$$\begin{aligned} BB^T &= (A^{-1}A^T)(A^{-1}A^T)^T \\ &= A^{-1}A^T A(A^{-1})^T \quad [\because (AB)^T = B^T A^T] \\ &= A^{-1}AA^T(A^{-1})^T \quad [\because AA^T = A^T A] \\ &= IA^T(A^{-1})^T \quad [\because A^{-1}A = I] \\ &= A^T(A^{-1})^T = (A^{-1}A)^T \quad [\because (AB)^T = B^T A^T] \\ &= I^T = I \end{aligned}$$

8. Given, $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$

$\therefore \quad |P| = 1(12 - 12) - \alpha(4 - 6) + 3(4 - 6) = 2\alpha - 6$

$\because \quad P = \text{adj}(A)$ [given]

$\therefore \quad |P| = |\text{adj } A| = |A|^2 = 16 \quad [\because |\text{adj } A| = |A|^{n-1}]$

$\therefore \quad 2\alpha - 6 = 16$

$\Rightarrow \quad 2\alpha = 22$

$\Rightarrow \quad \alpha = 11$

9. Given, $P^T = 2P + I$ ...(i)

$\therefore \quad (P^T)^T = (2P + I)^T = 2P^T + I$

$\Rightarrow \quad P = 2P^T + I$

$\Rightarrow \quad P = 2(2P + I) + I$

$\Rightarrow \quad P = 4P + 3I$ or $3P = -3I$

$\Rightarrow \quad PX = -IX = -X$

10. $|A| \neq 0$, as non-singular $\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$

$\Rightarrow \quad 1(1 - c\omega) - a(\omega - c\omega^2) + b(\omega^2 - \omega^2) \neq 0$

$\Rightarrow 1 - c\omega - a\omega + ac\omega^2 \neq 0$

$\Rightarrow \quad (1 - c\omega)(1 - a\omega) \neq 0 \Rightarrow a \neq \frac{1}{\omega}, c \neq \frac{1}{\omega}$

$\Rightarrow \quad a = \omega, c = \omega$ and $b \in \{\omega, \omega^2\} \Rightarrow$ 2 solutions

11. Given, $M^T = -M, N^T = -N$ and $MN = NM$ ...(i)

$\therefore \quad M^2N^2(M^TN)^{-1}(MN^{-1})^T$

$\Rightarrow \quad M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^T \cdot M^T$

$\Rightarrow \quad M^2N(NN^{-1})(-M)^{-1}(N^T)^{-1}(-M)$

$\Rightarrow \quad M^2NI(-M^{-1})(-N)^{-1}(-M)$

$\Rightarrow -M^2NM^{-1}N^{-1}M$

$\Rightarrow \quad -M \cdot (MN)M^{-1}N^{-1}M = -M(NM)M^{-1}N^{-1}M$

$\Rightarrow \quad -MN(NM^{-1})N^{-1}M = -M(NN^{-1})M \Rightarrow -M^2$

NOTE Here, non-singular word should not be used, since there is no non-singular 3×3 skew-symmetric matrix.

12. Every square matrix satisfied its characteristic equation,

i.e. $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1 - \lambda)\{(1 - \lambda)(4 - \lambda) + 2\} = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \quad A^3 - 6A^2 + 11A - 6I = O$ ...(i)

Given, $6A^{-1} = A^2 + cA + dI$, multiplying both sides by A, we get

$6I = A^3 + cA^2 + dA \Rightarrow A^3 + cA^2 + dA - 6I = O$ ...(ii)

On comparing Eqs. (i) and (ii), we get

$c = -6$ and $d = 11$

13. Here, $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$

Now, $|P| = 3(5\alpha) + 1(-3\alpha) - 2(-10)$

$= 12\alpha + 20$ ...(i)

$\therefore \quad \text{adj}(P) = \begin{bmatrix} 5\alpha & 2\alpha & -10 \\ -10 & 6 & 12 \\ -\alpha & -(3\alpha + 4) & 2 \end{bmatrix}^T$

$= \begin{bmatrix} 5\alpha & -10 & -\alpha \\ 2\alpha & 6 & -3\alpha - 4 \\ -10 & 12 & 2 \end{bmatrix}$ ...(ii)

As, $PQ = kI$

$\Rightarrow \quad |P||Q| = |kI|$

$\Rightarrow \quad |P||Q| = k^3$

$\Rightarrow \quad |P|\left(\frac{k^2}{2}\right) = k^3 \quad \left[\text{given}, |Q| = \frac{k^2}{2}\right]$

$\Rightarrow \quad |P| = 2k$ ...(iii)

$\because \quad PQ = kI$

$\therefore \quad Q = kp^{-1}I$

$= k \cdot \frac{\text{adj}P}{|P|} = \frac{k(\text{adj } P)}{2k}$ [from Eq. (iii)]

$= \frac{\text{adj } P}{2} = \frac{1}{2}\begin{bmatrix} 5\alpha & -10 & -\alpha \\ 2\alpha & 6 & -3\alpha - 4 \\ -10 & 12 & 2 \end{bmatrix}$

$\therefore \quad q_{23} = \frac{-3\alpha - 4}{2} \quad \left[\text{given}, \ q_{23} = -\frac{k}{8}\right]$

$\Rightarrow \quad -\frac{(3\alpha + 4)}{2} = -\frac{k}{8}$

$\Rightarrow \quad (3\alpha + 4) \times 4 = k$

$\Rightarrow \quad 12\alpha + 16 = k$ ...(iv)

From Eq. (iii), $|P| = 2k$

$\Rightarrow \quad 12\alpha + 20 = 2k$ [from Eq. (i)] ...(v)

On solving Eqs. (iv) and (v), we get

$\alpha = -1$ and $k = 4$ ...(vi)

$\therefore \quad 4\alpha - k + 8 = -4 - 4 + 8 = 0$

$\therefore$ Option (b) is correct.

Now, $|P \text{ adj}(Q)| = |P||\text{adj } Q|$

$$= 2k\left(\frac{k^2}{2}\right)^2 = \frac{k^5}{2} = \frac{2^{10}}{2} = 2^9$$

$\therefore$ Option (c) is correct.

14. **PLAN** A square matrix M is invertible, iff dem (M) or $|M| \neq 0$.

Let $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(a) Given, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \Rightarrow a = b = c = \alpha$ [let]

$\Rightarrow M = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \Rightarrow |M| = 0 \Rightarrow M$ is non-invertible.

(b) Given, $[b \; c] = [a \; b]$

$\Rightarrow a = b = c = \alpha$ [let]

Again, $|M| = 0$

$\Rightarrow M$ is non-invertible.

(c) As given $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M| = ac \neq 0$

[$\because a$ and c are non-zero]

$\Rightarrow M$ is invertible.

(d) $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow |M| = ac - b^2 \neq 0$

$\because$ ac is not equal to square of an integer.

M is invertible.

15. **PLAN** If $|A_{n \times n}| = \Delta$, then $|\text{adj } A| = \Delta^{A-1}$

Here, $\text{adj } P_{3 \times 3} = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$

$\Rightarrow |\text{adj } P| = |P|^2$

$\therefore |\text{adj } P| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = 1(3-7) - 4(6-7) + 4(2-1)$

$= -4 + 4 + 4 = 4 \Rightarrow |P| = \pm 2$

16. $|A| = (2k+1)^3, |B| = 0$

But $\det(\text{adj } A) + \det(\text{adj } B) = 10^6$

$\Rightarrow (2k+1)^6 = 10^6$

$\Rightarrow k = \frac{9}{2} \Rightarrow [k] = 4$

Topic 4 Solving System of Equations

1. Given system of linear equations is

$[\sin\theta]x + [-\cos\theta]y = 0$ ...(i)

and $[\cot\theta]x + y = 0$ ...(ii)

where, $[x]$ denotes the greatest integer $\leq x$.

Here, $\Delta = \begin{vmatrix} [\sin\theta] & [-\cos\theta] \\ [\cot\theta] & 1 \end{vmatrix}$

$\Rightarrow \Delta = [\sin\theta] - [-\cos\theta][\cot\theta]$

When $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

$\sin\theta \in \left(\frac{\sqrt{3}}{2}, 1\right)$

$\Rightarrow [\sin\theta] = 0$ ...(iii)

$-\cos\theta \in \left(0, \frac{1}{2}\right)$

$\Rightarrow [-\cos\theta] = 0$ ...(iv)

and $\cot\theta \in \left(-\frac{1}{\sqrt{3}}, 0\right)$

$\Rightarrow [\cot\theta] = -1$ ...(v)

So, $\Delta = [\sin\theta] - [-\cos\theta][\cot\theta]$

$-(0 \times (-1)) = 0$ [from Eqs. (iii), (iv) and (v)]

Thus, for $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$, the given system have infinitely many solutions.

When $\theta \in \left(\pi, \frac{7\pi}{6}\right), \sin\theta \in \left(-\frac{1}{2}, 0\right)$

$\Rightarrow [\sin\theta] = -1$

$-\cos\theta \in \left(\frac{\sqrt{3}}{2}, 1\right) \Rightarrow [\cos\theta] = 0$

and $\cot\theta \in (\sqrt{3}, \infty) \Rightarrow [\cot\theta] = n, n \in N$.

So, $\Delta = -1 - (0 \times n) = -1$

Thus, for $\theta \in \left(\pi, \frac{7\pi}{6}\right)$, the given system has a unique solution.

2. Given, system of linear equations

$x + y + z = 6$ ... (i)

$4x + \lambda y - \lambda z = \lambda - 2$ ...(ii)

and $3x + 2y - 4z = -5$ ...(iii)

has infinitely many solutions, then $\Delta = 0$

$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 4 & \lambda & -\lambda \\ 3 & 2 & -4 \end{vmatrix} = 0$

$\Rightarrow 1(-4\lambda + 2\lambda) - 1(-16 + 3\lambda) + 1(8 - 3\lambda) = 0$

$\Rightarrow -8\lambda + 24 = 0 \Rightarrow \lambda = 3$

From, the option $\lambda = 3$, satisfy the quadratic equation $\lambda^2 - \lambda - 6 = 0$.

3. Given system of linear equations

$x + y + z = 5$ ...(i)

$x + 2y + 2z = 6$ ...(ii)

$x + 3y + \lambda z = \mu$ ...(iii)

$(\lambda, \mu \in R)$

The above given system has infinitely many solutions, then the plane represented by these equations intersect each other at a line, means $(x + 3y + \lambda z - \mu)$

$= p(x + y + z - 5) + q(x + 2y + 2z - 6)$

$= (p+q)x + (p+2q)y + (p+2q)z - (5p+6q)$

On comparing, we get

$$p+q=1,\ p+2q=3,\ p+2q=\lambda$$

and $5p+6q=\mu$

So, $(p,q)=(-1,2)$

$\Rightarrow \lambda=3$ and $\mu=7$

$\Rightarrow \lambda+\mu=3+7=10$

4. Given system of linear equations

$$2x+3y-z=0,$$
$$x+ky-2z=0$$

and $2x-y+z=0$ has a non-trivial solution (x, y, z).

$$\therefore\ \Delta=0\Rightarrow\begin{vmatrix}2&3&-1\\1&k&-2\\2&-1&1\end{vmatrix}=0$$

$$2(k-2)-3(1+4)-1(-1-2k)=0$$

$\Rightarrow 2k-4-15+1+2k=0$

$\Rightarrow 4k=18\Rightarrow k=\frac{9}{2}$

So, system of linear equations is

$2x+3y-z=0$ ...(i)

$2x+9y-4z=0$ ...(ii)

and $2x-y+z=0$ ...(iii)

From Eqs. (i) and (ii), we get

$$6y-3z=0,\ \frac{y}{z}=\frac{1}{2}$$

From Eqs. (i) and (iii), we get

$$4x+2y=0\Rightarrow\frac{x}{y}=-\frac{1}{2}$$

So, $\frac{x}{z}=\frac{x}{y}\times\frac{y}{z}=-\frac{1}{4}\Rightarrow\frac{z}{x}=-4$ $\left[\because \frac{y}{z}=\frac{1}{2} \text{ and } \frac{x}{y}=-\frac{1}{2}\right]$

$$\therefore\ \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+k=-\frac{1}{2}+\frac{1}{2}-4+\frac{9}{2}=\frac{1}{2}.$$

5. Given system of linear equations

$x-2y+kz=1$ ...(i)

$2x+y+z=2$ ...(ii)

and $3x-y-kz=3$ ...(iii)

has a solution (x, y, z), $z\neq 0$.

On adding Eqs. (i) and (iii), we get

$$x-2y+kz+3x-y-kz=1+3$$
$$4x-3y=4$$

$\Rightarrow 4x-3y-4=0$

This is the required equation of the straight line in which point (x, y) lies.

6. **Key Idea** A homogeneous system of linear equations have non-trivial solutions iff $\Delta=0$

Given system of linear equations is

$$x-cy-cz=0,$$
$$cx-y+cz=0$$

and $cx+cy-z=0$

We know that a homogeneous system of linear equations have non-trivial solutions iff

$$\Delta=0$$

$$\Rightarrow\begin{vmatrix}1&-c&-c\\c&-1&c\\c&c&-1\end{vmatrix}=0$$

$\Rightarrow 1(1-c^2)+c(-c-c^2)-c(c^2+c)=0$

$\Rightarrow 1-c^2-c^2-c^3-c^3-c^2=0$

$\Rightarrow -2c^3-3c^2+1=0$

$\Rightarrow 2c^3+3c^2-1=0$

$\Rightarrow (c+1)[2c^2+c-1]=0$

$\Rightarrow (c+1)[2c^2+2c-c-1]=0$

$\Rightarrow (c+1)(2c-1)(c+1)=0$

$\Rightarrow c=-1$ or $\frac{1}{2}$

Clearly, the greatest value of c is $\frac{1}{2}$.

7. The given system of linear equations is

$$x-2y-2z=\lambda x$$
$$x+2y+z=\lambda y$$
$$-x-y-\lambda z=0,$$

which can be rewritten as

$$(1-\lambda)x-2y-2z=0$$

$\Rightarrow x+(2-\lambda)y+z=0$

$$x+y+\lambda z=0$$

Now, for non-trivial solution, we should have

$$\begin{vmatrix}1-\lambda&-2&-2\\1&2-\lambda&1\\1&1&\lambda\end{vmatrix}=0$$

$\left[\because \text{If } a_1x+b_1y+c_1z=0;\ a_2x+b_2y+c_2z=0\right.$
$a_3x+b_3y+c_3z=0]$
has a non-trivial solution, then $\left.\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}=0\right]$

$\Rightarrow (1-\lambda)[(2-\lambda)\lambda-1]+2[\lambda-1]-2[1-2+\lambda]=0$

$\Rightarrow (\lambda-1)[\lambda^2-2\lambda+1+2-2]=0$

$\Rightarrow (\lambda-1)^3=0$

$\Rightarrow \lambda=1$

8. Given system of linear equations,

$$(1+\alpha)x+\beta y+z=2$$
$$\alpha x+(1+\beta)y+z=3$$
$$\alpha x+\beta y+2z=2$$

has a unique solution, if

$$\begin{vmatrix}1+\alpha&\beta&1\\\alpha&(1+\beta)&1\\\alpha&\beta&2\end{vmatrix}\neq 0$$

Apply $R_1\to R_1-R_3$ and $R_2\to R_2-R_3$

$$\begin{vmatrix}1&0&-1\\0&1&-1\\\alpha&\beta&2\end{vmatrix}\neq 0$$

$\Rightarrow\ 1(2+\beta)-0(0+\alpha)-1(0-\alpha)\neq 0$

$\Rightarrow\quad \alpha+\beta+2\neq 0 \quad\quad \dots\text{(i)}$

Note that, only (2, 4) satisfy the Eq. (i).

9. We know that, if the system of equations

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

has more than one solution, then $D=0$ and $D_1=D_2=D_3=0$. In the given problem,

$$D_1=0\Rightarrow\begin{vmatrix} a & 2 & 3\\ b & -1 & 5\\ c & -3 & 2\end{vmatrix}=0$$

$\Rightarrow\ a(-2+15)-2(2b-5c)+3(-3b+c)=0$

$\Rightarrow\ 13a-4b+10c-9b+3c=0$

$\Rightarrow\ 13a-13b+13c=0$

$\Rightarrow\ a-b+c=0\Rightarrow b-a-c=0$

10. We know that,

the system of linear equations

$$a_1x+b_1y+c_1z=0$$
$$a_2x+b_2y+c_2z=0$$
$$a_3x+b_3y+c_3z=0$$

has a non-trivial solution, if

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0$$

Now, if the given system of linear equations

$$x+3y+7z=0$$
$$-x+4y+7z=0,$$

and $(\sin 3\theta)x+(\cos 2\theta)y+2z=0$

has non-trivial solution, then

$$\begin{vmatrix} 1 & 3 & 7\\ -1 & 4 & 7\\ \sin 3\theta & \cos 2\theta & 2\end{vmatrix}=0$$

$\Rightarrow\ 1(8-7\cos 2\theta)-3(-2-7\sin 3\theta)$
$\quad +7(-\cos 2\theta-4\sin 3\theta)=0$

$\Rightarrow\ 8-7\cos 2\theta+6+21\sin 3\theta$
$\quad -7\cos 2\theta-28\sin 3\theta=0$

$\Rightarrow\ -7\sin 3\theta-14\cos 2\theta+14=0$

$\Rightarrow\ -7(3\sin\theta-4\sin^3\theta)-14(1-2\sin^2\theta)+14=0$

$[\because \sin 3A=3\sin A-4\sin^3 A$ and $\cos 2A=1-2\sin^2 A]$

$\Rightarrow\ 28\sin^3\theta+28\sin^2\theta-21\sin\theta-14+14=0$

$\Rightarrow\ 7\sin\theta\,[4\sin^2\theta+4\sin\theta-3]=0$

$\Rightarrow\ \sin\theta\,[4\sin^2\theta+6\sin\theta-2\sin\theta-3]=0$

$\Rightarrow\ \sin\theta\,[2\sin\theta\,(2\sin\theta+3)-1\,(2\sin\theta+3)]=0$

$\Rightarrow\ (\sin\theta)(2\sin\theta-1)(2\sin\theta+3)=0$

Now, either $\sin\theta=0$ or $\frac{1}{2}$

$$\left[\because \sin\theta\neq-\frac{3}{2}\text{ as }-1\le\sin\theta\le 1\right]$$

In given interval $(0,\pi)$,

$$\sin\theta=\frac{1}{2}$$

$\Rightarrow\quad \theta=\frac{\pi}{6},\frac{5\pi}{6}\quad [\because \sin\theta\neq 0,\theta\in(0,\pi)]$

Hence, 2 solutions in $(0,\pi)$

11. Since, the system of equations has infinitely many solution, therefore $D=D_1=D_2=D_3=0$

Here,

$$D=\begin{vmatrix} 1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & \alpha\end{vmatrix}=1(2\alpha-9)-1(\alpha-3)+1(3-2)$$

$=\alpha-5$

and $$D_3=\begin{vmatrix} 1 & 1 & 5\\ 1 & 2 & 9\\ 1 & 3 & \beta\end{vmatrix}=1(2\beta-27)-1(\beta-9)+5(3-2)$$

$=\beta-13$

Now, $\quad D=0$

$\Rightarrow\quad \alpha-5=0\ \Rightarrow\ \alpha=5$

and $\quad D_3=0\ \Rightarrow\ \beta-13=0$

$\Rightarrow\quad \beta=13$

$\therefore\quad \beta-\alpha=13-5=8$

12. *(a)* Here, $$D=\begin{vmatrix} 1 & -4 & 7\\ 0 & 3 & -5\\ -2 & 5 & -9\end{vmatrix}$$

$=1(-27+25)+4(0-10)+7(0+6)$

[expanding along R_1]

$=-2-40+42=0$

$\therefore$ The system of linear equations have infinite many solutions.

[$\because$ system is consistent and does not have unique solution as $D=0$]

$\Rightarrow\quad D_1=D_2=D_3=0$

Now, $$D_1=0\Rightarrow\begin{vmatrix} g & -4 & 7\\ h & 3 & -5\\ k & 5 & -9\end{vmatrix}=0$$

$\Rightarrow g(-27+25)+4(-9h+5k)+7(5h-3k)=0$

$\Rightarrow -2g-36h+20k+35h-21k=0$

$\Rightarrow -2g-h-k=0\Rightarrow 2g+h+k=0$

13 According to Cramer's rule, here

$$D=\begin{vmatrix} 1 & 1 & 1\\ 2 & 3 & 2\\ 2 & 3 & a^2-1\end{vmatrix}=\begin{vmatrix} 1 & 0 & 0\\ 2 & 1 & 0\\ 2 & 1 & a^2-3\end{vmatrix}$$

(Applying $C_2\to C_2-C_1$ and $C_3\to C_3-C_1$)

$=a^2-3$ (Expanding along R_1)

and $$D_1=\begin{vmatrix} 2 & 1 & 1\\ 5 & 3 & 2\\ a+1 & 3 & a^2-1\end{vmatrix}=\begin{vmatrix} 2 & 1 & 0\\ 5 & 3 & -1\\ a+1 & 3 & a^2-1-3\end{vmatrix}$$

(Applying $C_3\to C_3-C_2$)

$$= \begin{vmatrix} 2 & 0 & 0 \\ 5 & 3-\frac{5}{2} & -1 \\ a+1 & 3-\frac{(a+1)}{2} & a^2-1-3 \end{vmatrix}$$

(Applying $C_2 \to C_2 - \frac{1}{2}C_1$)

$$= \begin{vmatrix} 2 & 0 & 0 \\ 5 & \frac{1}{2} & -1 \\ a+1 & \frac{5}{2}-\frac{a}{2} & a^2-4 \end{vmatrix}$$

$$= 2\left[\frac{1}{2}(a^2-4) + \left(\frac{5}{2}-\frac{a}{2}\right)\right] \quad \text{[Expanding along } R_1\text{]}$$

$$= 2\left[\frac{a^2}{2} - 2 + \frac{5}{2} - \frac{a}{2}\right] = a^2 - 4 + 5 - a = a^2 - a + 1$$

Clearly, when $a = 4$, then $D = 13 \neq 0 \Rightarrow$ unique solution and

when $|a| = \sqrt{3}$, then $D = 0$ and $D_1 \neq 0$.

$\therefore$ When $|a| = \sqrt{3}$, then the system has no solution i.e. system is inconsistent.

14. We have,

$x + ky + 3z = 0;\ 3x + ky - 2z = 0;\ 2x + 4y - 3z = 0$

System of equation has non-zero solution, if

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$$

$\Rightarrow (-3k + 8) - k(-9 + 4) + 3(12 - 2k) = 0$

$\Rightarrow \quad -3k + 8 + 9k - 4k + 36 - 6k = 0$

$\Rightarrow \quad -4k + 44 = 0 \Rightarrow k = 11$

Let $z = \lambda$, then we get

$x + 11y + 3\lambda = 0$ ...(i)

$3x + 11y - 2\lambda = 0$ ...(ii)

and $2x + 4y - 3\lambda = 0$ ...(iii)

Solving Eqs. (i) and (ii), we get

$$x = \frac{5\lambda}{2},\ y = \frac{-\lambda}{2},\ z = \lambda \Rightarrow \frac{xz}{y^2} = \frac{5\lambda^2}{2 \times \left(-\frac{\lambda}{2}\right)^2} = 10$$

15. Given, system of linear equation is

$x + \lambda y - z = 0;\ \lambda x - y - z = 0;\ x + y - \lambda z = 0$

Note that, given system will have a non-trivial solution only if determinant of coefficient matrix is zero,

i.e.
$$\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 1(\lambda + 1) - \lambda(-\lambda^2 + 1) - 1(\lambda + 1) = 0$

$\Rightarrow \quad \lambda + 1 + \lambda^3 - \lambda - \lambda - 1 = 0$

$\Rightarrow \quad \lambda^3 - \lambda = 0 \Rightarrow \lambda(\lambda^2 - 1) = 0$

$\Rightarrow \quad \lambda = 0$ or $\lambda = \pm 1$

Hence, given system of linear equation has a non-trivial solution for exactly three values of λ.

16. Given system of linear equations

$2x_1 - 2x_2 + x_3 = \lambda x_1$ ...(i)

$\Rightarrow \quad (2 - \lambda)x_1 - 2x_2 + x_3 = 0$ ...(ii)

$2x_1 - 3x_2 + 2x_3 = \lambda x_2$ ...(iii)

$\Rightarrow \quad 2x_1 - (3 + \lambda)x_2 + 2x_3 = 0$

$-x_1 + 2x_2 = \lambda x_3$

$\Rightarrow \quad -x_1 + 2x_2 - \lambda x_3 = 0$

Since, the system has non-trivial solution.

$$\therefore \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & \lambda \end{bmatrix} = 0$$

$\Rightarrow (2 - \lambda)(3\lambda + \lambda^2 - 4) + 2(-2\lambda + 2) + 1(4 - 3) - \lambda) = 0$

$\Rightarrow \quad (2 - \lambda)(\lambda^2 + 3\lambda - 4) + 4(1 - \lambda) + (1 - \lambda) = 0$

$\Rightarrow \quad (2 - \lambda)(\lambda + 4)(\lambda - 1) + 5(1 - \lambda) = 0$

$\Rightarrow \quad (\lambda - 1)[(2 - \lambda)(\lambda + 4) - 5] = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda^2 + 2\lambda - 3) = 0$

$\Rightarrow \quad (\lambda - 1)[(\lambda - 1)(\lambda + 8)] = 0$

$\Rightarrow \quad (\lambda - 1)^2(\lambda + 3) = 0$

$\Rightarrow \quad \lambda = 1, 1, -3$

Hence, λ contains two elements.

17. Given equations can be written in matrix form

$AX = B$

where, $A = \begin{bmatrix} k+1 & 8 \\ k & k+3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \dfrac{4k}{3k-1}$

For no solution, $|A| = 0$ and (adj A) $B \neq 0$

Now, $|A| = \begin{bmatrix} k+1 & 8 \\ k & k+3 \end{bmatrix} = 0$

$\Rightarrow \quad (k^2 + 1)(k + 3) - 8k = 0$

$k^2 + 4k + 3 - 8k = 0$

$\Rightarrow \quad k^2 - 4k \times 3 = 0$

$\Rightarrow \quad (k - 1)(k - 3) = 0$

$\Rightarrow \quad k = 1, k = 3,$

Now $\quad \text{adj } A = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix}$

Now, $\quad (\text{adj } A)B = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix}\begin{bmatrix} 4k \\ 3k+1 \end{bmatrix}$

$$= \begin{bmatrix} (k+3)(4k) - 8(3k-1) \\ -4k^2 + (k+1)(3k-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4k^2 - 12k + 8 \\ -k^2 + 2k - 1 \end{bmatrix}$$

Put $k = 1$

$$(\text{adj } A)\ B = \begin{bmatrix} 4 - 12 + 8 \\ -1 + 2 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ not true}$$

Put $k=3$

$$(\text{adj } A)\, B=\begin{bmatrix}36 & -36+8\\ -9+ & 6-1\end{bmatrix}=\begin{bmatrix}8\\ -4\end{bmatrix}\neq 0 \text{ true}$$

Hence, required value of k is 3.

Alternate Solution

Condition for the system of equations has no solution is

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$$

$$\therefore \qquad \frac{k+1}{k}=\frac{8}{k+3}\neq\frac{4k}{3k-1}$$

Take $\dfrac{k+1}{k}=\dfrac{8}{k+3}$

$\Rightarrow \quad k^2+4k+3=8k$

$\Rightarrow \quad k^2-4k+3$

$\Rightarrow \quad (k-1)(k-3)=0$

$k=1, 3$

If $k-1$, then $\dfrac{8}{1+3}\neq\dfrac{4.1}{2}$, false

And, if $k=3$, then $\dfrac{8}{6}\neq\dfrac{4.3}{9-1}$, true

Therefore, $k=3$

Hence, only one value of k exist.

18. Since, $A\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$ is linear equation in three variables and that could have only unique, no solution or infinitely many solution.

$\therefore$ It is not possible to have two solutions.

Hence, number of matrices A is zero.

19. Since, given system has no solution.

$\therefore\ \Delta=0$ and any one amongst $\Delta_x, \Delta_y, \Delta_z$ is non-zero.

Let $\begin{vmatrix}2 & -1 & 2\\ 1 & -2 & 1\\ 1 & 1 & \lambda\end{vmatrix}=0$ and $\Delta_z=\begin{vmatrix}2 & -1 & 2\\ 1 & -2 & -4\\ 1 & 1 & 4\end{vmatrix}=6\neq 0$

$\Rightarrow \qquad \lambda=1$

20. For infinitely many solutions, we must have

$$\frac{k+1}{k}=\frac{8}{k+3}=\frac{4k}{3k-1} \Rightarrow k=1$$

21. Given equations $x+ay=0, az+y=0, ax+z=0$ has infinite solutions.

$$\therefore \qquad \begin{vmatrix}1 & a & 0\\ 0 & 1 & a\\ a & 0 & 1\end{vmatrix}=0$$

$\Rightarrow \quad 1+a^3=0$ or $a=-1$

22. Since, the given system has non-zero solution.

$$\therefore \qquad \begin{vmatrix}1 & -k & -1\\ k & -1 & -1\\ 1 & 1 & -1\end{vmatrix}=0$$

Applying $C_1\to C_1-C_2, C_2\to C_2+C_3$

$$\Rightarrow \begin{vmatrix}1+k & -k-1 & -1\\ 1+k & -2 & -1\\ 0 & 0 & -1\end{vmatrix}=0$$

$\Rightarrow \quad 2(k+1)-(k+1)^2=0$

$\Rightarrow \quad (k+1)(2-k-1)=0 \quad\Rightarrow\quad k=\pm 1$

NOTE There is a golden rule in determinant that n one's $\Rightarrow$ $(n-1)$ zero's or n (constant) $\Rightarrow (n-1)$ zero's for all constant should be in a single row or a single column.

23. The given system of equations can be expressed as

$$\begin{bmatrix}1 & -2 & 3\\ 1 & -3 & 4\\ -1 & 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ 1\\ k\end{bmatrix}$$

Applying $R_2\to R_2-R_1, R_3\to R_3+R_1$

$$\sim\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ k-1\end{bmatrix}$$

Applying $R_3\to R_3-R_2$

$$\sim\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ k-3\end{bmatrix}$$

When $k\neq 3$, the given system of equations has no solution.

$\Rightarrow$ Statement I is true. Clearly, Statement II is also true as it is rearrangement of rows and columns of

$$\begin{bmatrix}1 & -2 & 3\\ 1 & -3 & 4\\ -1 & 1 & -2\end{bmatrix}.$$

24. We have,

$$-x+2y+5z=b_1$$
$$2x-4y+3z=b_2$$
$$x-2y+2z=b_3$$

has at least one solution.

$$\therefore \qquad D=\begin{vmatrix}-1 & 2 & 5\\ 2 & -4 & 3\\ 1 & -2 & 2\end{vmatrix}$$

and $\qquad D_1=D_2=D_3=0$

$$\Rightarrow \qquad D_1=\begin{vmatrix}b_1 & 2 & 5\\ b_2 & -4 & 3\\ b_3 & -2 & 2\end{vmatrix}$$

$$=-2b_1-14b_2+26b_3=0$$

$\Rightarrow \quad b_1+7b_2=13b_3 \qquad$ ...(i)

(a) $D=\begin{vmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 1 & 2 & 6\end{vmatrix}=1(24-10)+1(10-12)$

$=14-2=12\neq 0$

Here, $D\neq 0\Rightarrow$ unique solution for any b_1, b_2, b_3.

(b) $D=\begin{vmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{vmatrix}$

$=1(-6+6)-1(-15+12)+3(-5+4)=0$

For atleast one solution

$D_1 = D_2 = D_3 = 0$

Now, $D_1 = \begin{vmatrix} b_1 & 1 & 3 \\ b_2 & 2 & 6 \\ b_3 & -1 & -3 \end{vmatrix}$

$= b_1(-6+6) - b_2(-3+3) + b_3(6-6)$
$=0$

$D_2 = \begin{vmatrix} 1 & b_1 & 3 \\ 5 & b_2 & 6 \\ -2 & b_3 & -3 \end{vmatrix}$

$= -b_1(-15+12) + b_2(-3+6) - b_3(6-15)$
$= 3b_1 + 3b_2 + 9b_3 = 0 \Rightarrow b_1 + b_2 + 3b_3 = 0$

not satisfies the Eq. (i)

It has no solution.

(c) $D=\begin{vmatrix} -1 & 2 & -5 \\ 2 & -4 & 10 \\ 1 & -2 & 5 \end{vmatrix}$

$= -1(-20+20) - 2(10-10) - 5(-4+4)$
$=0$

Here, $b_2 = -2b_1$ and $b_3 = -b_1$ satisfies the Eq. (i)

Planes are parallel.

(d) $D=\begin{vmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & -5 \end{vmatrix} = 1(0-12) - 2(-10-3) + 5(8-0)$

$=54$

$D \neq 0$

It has unique solution for any b_1, b_2, b_3.

25. Given system $\lambda x + y + z = 0, -x + \lambda y + z = 0$

and $-x - y + \lambda z = 0$

will have non-zero solution, if

$$\begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(\lambda^2+1) - 1(-\lambda+1) + 1(1+\lambda) = 0$

$\Rightarrow \lambda^3 + \lambda + \lambda - 1 + 1 + \lambda = 0$

$\Rightarrow \lambda^3 + 3\lambda = 0$

$\Rightarrow \lambda(\lambda^2+3) = 0 \Rightarrow \lambda = 0$

26. Since, $AX = U$ has infinitely many solutions.

$\Rightarrow |A| = 0 \Rightarrow \begin{vmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{vmatrix} = 0$

$\Rightarrow a(bc - bd) + 1(d-c) = 0 \Rightarrow (d-c)(ab-1) = 0$

$\therefore ab = 1$ or $d = c$

Again, $|A_3| = \begin{vmatrix} a & 0 & f \\ 1 & c & g \\ 1 & d & h \end{vmatrix} = 0 \Rightarrow g = h$

$\Rightarrow |A_2| = \begin{vmatrix} a & f & 1 \\ 1 & g & b \\ 1 & h & b \end{vmatrix} = 0 \Rightarrow g = h$

and $|A_1| = \begin{vmatrix} f & 0 & 1 \\ g & c & b \\ h & d & b \end{vmatrix} = 0 \Rightarrow g = h$

$\therefore g = h, c = d$ and $ab = 1$ …(i)

Now, $BX = V$

$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$ [from Eq. (i)]

[since, C_2 and C_3 are equal]

$\therefore BX = V$ has no solution.

$|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0$ [from Eq. (i)]

[since, $c = d$ and $g = h$]

$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df$ [$\because c = d$]

Since, $adf \neq 0 \Rightarrow |B_2| \neq 0$

$|B| = 0$ and $|B_2| \neq 0$

$\therefore BX = V$ has no solution.

27. Given, $\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$

$x + (\cos\alpha)y + (\sin\alpha)z = 0$

and $-x + (\sin\alpha)y - (\cos\alpha)z = 0$ has non-trivial solution.

$\therefore \Delta = 0$

$\Rightarrow \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$

$\Rightarrow \lambda(-\cos^2\alpha - \sin^2\alpha) - \sin\alpha(-\cos\alpha + \sin\alpha)$
$+ \cos\alpha(\sin\alpha + \cos\alpha) = 0$

$\Rightarrow -\lambda + \sin\alpha\cos\alpha + \sin\alpha\cos\alpha - \sin^2\alpha + \cos^2\alpha = 0$

$\Rightarrow \lambda = \cos 2\alpha + \sin 2\alpha$

$\left[\because -\sqrt{a^2+b^2} \leq a\sin\theta + b\cos\theta \leq \sqrt{a^2+b^2}\right]$

$\therefore -\sqrt{2} \leq \lambda \leq \sqrt{2}$ …(i)

Again, when $\lambda = 1$, $\cos 2\alpha + \sin 2\alpha = 1$

$\Rightarrow \frac{1}{\sqrt{2}}\cos 2\alpha + \frac{1}{\sqrt{2}}\sin 2\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos(2\alpha - \pi/4) = \cos \pi/4$

$\therefore 2\alpha - \pi/4 = 2n\pi \pm \pi/4$

$\Rightarrow 2\alpha = 2n\pi - \pi/4 + \pi/4$ or $2\alpha = 2n\pi + \pi/4 + \pi/4$

$\therefore \alpha = n\pi$ or $n\pi + \pi/4$

28. Since, α_1, α_2 are the roots of $ax^2 + bx + c = 0$.

$$\Rightarrow \quad \alpha_1 + \alpha_2 = -\frac{b}{a} \quad \text{and} \quad \alpha_1\alpha_2 = \frac{c}{a} \quad \text{...(i)}$$

Also, β_1, β_2 are the roots of $px^2 + qx + r = 0$.

$$\Rightarrow \quad \beta_1 + \beta_2 = -\frac{q}{p} \quad \text{and} \quad \beta_1\beta_2 = \frac{r}{p} \quad \text{...(ii)}$$

Given system of equations

$$\alpha_1 y + \alpha_2 z = 0$$

and $\beta_1 \ y + \beta_2 \ z = 0$, has non-trivial solution.

$$\therefore \quad \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0 \Rightarrow \frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$$

Applying componendo-dividendo, $\dfrac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} = \dfrac{\beta_1 + \beta_2}{\beta_1 - \beta_2}$

$$\Rightarrow (\alpha_1 + \alpha_2)(\beta_1 - \beta_2) = (\alpha_1 - \alpha_2)(\beta_1 + \beta_2)$$

$$\Rightarrow (\alpha_1 + \alpha_2)^2 \{(\beta_1 + \beta_2)^2 - 4\beta_2\beta_2\} = (\beta_1 + \beta_2)^2\{(\alpha_1 + \alpha_2)^2 - 4\alpha_1\alpha_2\}$$

From Eqs. (i) and (ii), we get

$$\frac{b^2}{a^2}\left(\frac{q^2}{p^2} - \frac{4r}{p}\right) = \frac{q^2}{p^2}\left(\frac{b^2}{a^2} - \frac{4c}{a}\right)$$

$$\Rightarrow \frac{b^2q^2}{a^2p^2} - \frac{4b^2r}{a^2p} = \frac{b^2q^2}{a^2p^2} - \frac{4q^2c}{ap^2}$$

$$\Rightarrow \frac{b^2r}{a} = \frac{q^2c}{p} \Rightarrow \frac{b^2}{q^2} = \frac{ac}{pr}$$

29. The system of equations has non-trivial solution, if $\Delta = 0$.

$$\Rightarrow \quad \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Expanding along C_1, we get

$$\sin 3\theta \cdot (28 - 21) - \cos 2\theta(-7 - 7) + 2(-3 - 4) = 0$$

$$\Rightarrow \quad 7\sin 3\theta + 14\cos 2\theta - 14 = 0$$

$$\Rightarrow \quad \sin 3\theta + 2\cos 2\theta - 2 = 0$$

$$\Rightarrow 3\sin\theta - 4\sin^3\theta + 2(1 - 2\sin^2\theta) - 2 = 0$$

$$\Rightarrow \quad \sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$$

$$\Rightarrow \quad \sin\theta(2\sin\theta - 1)(2\sin\theta + 3) = 0$$

$$\Rightarrow \quad \sin\theta = 0, \sin\theta = \frac{1}{2}$$

[neglecting $\sin\theta = -3/2$]

$$\Rightarrow \quad \theta = n\pi, n\pi + (-1)^n \frac{\pi}{6}, n \in Z$$

30. The given system of equations

$$3x - y + 4z = 3$$
$$x + 2y - 3z = -2$$
$$6x + 5y + \lambda z = -3$$

has atleast one solution, if $\Delta \neq 0$.

$$\therefore \quad \Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} \neq 0$$

$$\Rightarrow \quad 3(2\lambda + 15) + 1(\lambda + 18) + 4(5 - 12) \neq 0$$

$$\Rightarrow \quad 7(\lambda + 5) \neq 0$$

$$\Rightarrow \quad \lambda \neq -5$$

Let $z = -k$, then equations become

$$3x - y = 3 - 4k$$

and $\quad x + 2y = 3k - 2$

On solving, we get

$$x = \frac{4 - 5k}{7}, \ y = \frac{13k - 9}{7}, \ z = k$$

31. Given system of equations are

$$3x + my = m \quad \text{and} \quad 2x - 5y = 20$$

Here, $\quad \Delta = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = -15 - 2m$

and $\quad \Delta_x = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix} = -25m$

$$\Delta_y = \begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix} = 60 - 2m$$

If $\Delta = 0$, then system is inconsistent, i.e. it has no solution.

If $\Delta \neq 0$, i.e. $m \neq \dfrac{15}{2}$, the system has a unique solution for any fixed value of m.

We have, $\quad x = \dfrac{\Delta_x}{\Delta} = \dfrac{-25m}{-15 - 2m} = \dfrac{25m}{15 + 2m}$

and $\quad y = \dfrac{\Delta_y}{\Delta} = \dfrac{60 - 2m}{-15 - 2m} = \dfrac{2m - 60}{15 + 2m}$

For $x > 0$, $\dfrac{25m}{15 + 2m} > 0$

$\Rightarrow \ m > 0$

or $\quad m < -\dfrac{15}{2}$ ...(i)

and $y > 0$, $\dfrac{2m - 60}{2m + 15} > 0 \Rightarrow m > 30$ or $m < -\dfrac{15}{2}$ ...(ii)

From Eqs. (i) and (ii), we get $m < -\dfrac{15}{2}$ or $m > 30$

32. Since, the given system of equations posses non-trivial solution, if $\begin{vmatrix} 0 & 1 & -2 \\ 0 & -3 & 1 \\ k & -5 & 4 \end{vmatrix} = 0 \Rightarrow k = 0$

On solving the equations $x = y = z = \lambda$ [say]

$\therefore$ For $k = 0$, the system has infinite solutions of $\lambda \in R$.

33. Given systems of equations can be rewritten as

$-x + cy + by = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$

Above system of equations are homogeneous equation. Since, x, y and z are not all zero, so it has non-trivial solution.

Therefore, the coefficient of determinant must be zero.

$$\therefore \quad \begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow -1(1-a^2) - c(-c-ab) + b(ca+b) = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc - 1 = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$

34. $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{vmatrix} = 0$

$\Rightarrow \alpha^4 - 2\alpha^2 + 1 = 0$

$\Rightarrow \quad \alpha^2 = 1$

$\Rightarrow \quad \alpha = \pm 1$

But $\alpha = 1$ not possible [Not satisfying equation]

$\therefore \quad \alpha = -1$

Hence, $1 + \alpha + \alpha^2 = 1$

35. Let $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$$\therefore \quad M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$M \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} a_1 - a_2 \\ b_1 - b_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

$\Rightarrow \quad a_2 = -1, b_2 = 2, c_2 = 3, a_1 - a_2 = 1,$
$b_1 - b_2 = 1, c_1 - c_2 = -1$

$\Rightarrow \quad a_1 + a_2 + a_3 = 0, b_1 + b_2 + b_3 = 0$
$c_1 + c_2 + c_3 = 12$

$\therefore \quad a_1 = 0, b_2 = 2, c_3 = 7$

$\Rightarrow$ Sum of diagonal elements $= 0 + 2 + 7 = 9$