9

Limit, Continuity and Differentiability

Topic 1 $\frac{0}{0}$ and $\frac{\infty}{\infty}$ Form

Objective Questions I (Only one correct option)

1. $\lim_{x \to 0} \frac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}}$ is **(2019 Main, 12 April II)**

(a) 6 (b) 2 (c) 3 (d) 1

2. If $\lim_{x \to 1} \frac{x^2 - ax + b}{x - 1} = 5$, then $a + b$ is equal to **(2019 Main, 10 April II)**

(a) -4 (b) 1 (c) -7 (d) 5

3. If $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$, then k is **(2019 Main, 10 April I)**

(a) $\frac{4}{3}$ (b) $\frac{3}{8}$ (c) $\frac{3}{2}$ (d) $\frac{8}{3}$

4. $\lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$ equals **(2019 Main, 8 April I)**

(a) $4\sqrt{2}$ (b) $\sqrt{2}$
(c) $2\sqrt{2}$ (d) 4

5. $\lim_{x \to \frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ is **(2019 Main, 12 Jan I)**

(a) $4\sqrt{2}$ (b) 4 (c) 8 (d) $8\sqrt{2}$

6. $\lim_{x \to 0} \frac{x \cot(4x)}{\sin^2 x \cot^2(2x)}$ is equal to **(2019 Main, 11 Jan II)**

(a) 0 (b) 1 (c) 4 (d) 2

7. $\lim_{y \to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$ **(2019 Main, 9 Jan I)**

(a) exists and equals $\frac{1}{4\sqrt{2}}$
(b) does not exist
(c) exists and equals $\frac{1}{2\sqrt{2}}$
(d) exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2} + 1)}$

8. $\lim_{x \to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals **(2017 Main)**

(a) $\frac{1}{24}$ (b) $\frac{1}{16}$
(c) $\frac{1}{8}$ (d) $\frac{1}{4}$

9. $\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to **(2014 Main)**

(a) $\frac{\pi}{2}$ (b) 1 (c) $-\pi$ (d) π

10. $\lim_{x \to 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to **(2013 Main)**

(a) 4 (b) 3 (c) 2 (d) $\frac{1}{2}$

11. If $\lim_{x \to \infty} \left(\frac{x^2 + x + 1}{x + 1} - ax - b\right) = 4$, then **(2012)**

(a) $a = 1, b = 4$ (b) $a = 1, b = -4$
(c) $a = 2, b = -3$ (d) $a = 2, b = 3$

12. $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$, given that $f'(2) = 6$ and $f'(1) = 4$, **(2003, 2M)**

(a) does not exist (b) is equal to $-3/2$
(c) is equal to $3/2$ (d) is equal to 3

13. If $\lim_{x \to 0} \frac{\{(a - n)nx - \tan x\} \sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to **(2003, 2M)**

(a) 0 (b) $\frac{n + 1}{n}$ (c) n (d) $n + \frac{1}{n}$

14. The integer n for which $\lim_{x \to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number, is **(2002, 2M)**

(a) 1 (b) 2
(c) 3 (d) 4

15. $\lim_{x \to 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ is **(1999, 2M)**

(a) 2 (b) -2
(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

16. $\lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}$ **(1998, 2M)**

(a) exists and it equals $\sqrt{2}$

(b) exists and it equals $-\sqrt{2}$

(c) does not exist because $x-1\to 0$

(d) does not exist because left hand limit is not equal to right hand limit

17. The value of $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos^2 x)}}{x}$ is **(1991, 2M)**

(a) 1 (b) –1

(c) 0 (d) None of these

18. If $f(x)=\begin{cases}\frac{\sin[x]}{[x]}, & [x]\neq 0\\ 0, & [x]\neq 0\end{cases}$

where, $[x]$ denotes the greatest integer less than or equal to x, then $\lim_{x\to 0} f(x)$ equals **(1985, 2M)**

(a) 1 (b) 0

(c) –1 (d) None of these

19. $\lim_{n\to\infty}\left(\frac{1}{1-n^2}+\frac{2}{1-n^2}+\ldots+\frac{n}{1-n^2}\right)$ is equal to **(1984, 2M)**

(a) 0 (b) $-\frac{1}{2}$

(c) $\frac{1}{2}$ (d) None of these

20. If $f(a)=2,\ f'(a)=1,\ g(a)=-1,\ g'(a)=2$,

then the value of $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$ is **(1983, 1M)**

(a) – 5 (b) $\frac{1}{5}$

(c) 5 (d) None of these

21. If $G(x)=-\sqrt{25-x^2}$, then $\lim_{x\to 1}\frac{G(x)-G(1)}{x-1}$ has the value **(1983, 1M)**

(a) $\frac{1}{\sqrt{24}}$ (b) $\frac{1}{5}$

(c) $-\sqrt{24}$ (d) None of these

Objective Question II

(One or more than one correct option)

22. For any positive integer n, define $f_n:(0,\infty)\to R$ as $f_n(x)=\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$ for all $x\in(0,\infty)$. (Here, the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$). Then, which of the following statement(s) is (are) TRUE? **(2018 Adv.)**

(a) $\sum_{j=1}^{5}\tan^2(f_j(0))=55$

(b) $\sum_{j=1}^{10}(1+f'_j(0))\sec^2(f_j(0))=10$

(c) For any fixed positive integer n, $\lim_{x\to\infty}\tan(f_n(x))=\frac{1}{n}$

(d) For any fixed positive integer n, $\lim_{x\to\infty}\sec^2(f_n(x))=1$

23. Let $L=\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}$, $a>0$. If L is finite, then **(2009)**

(a) $a=2$

(b) $a=1$

(c) $L=\frac{1}{64}$

(d) $L=\frac{1}{32}$

Fill in the Blanks

24. $\lim_{h\to 0}\frac{\log(1+2h)-2\log(1+h)}{h^2}=\ldots$. **(1997C, 2M)**

25. If $f(x)=\begin{cases}\sin x, & x\neq n\pi, n=0,\pm 1,\pm 2,\ldots\\ 2, & \text{otherwise}\end{cases}$

and $g(x)=\begin{cases}x^2+1, & x\neq 0, 2\\ 4, & x=0\\ 5, & x=2\end{cases}$, then $\lim_{x\to 0} g[f(x)]$ is ……… **(1996, 2M)**

26. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB=AC$ and h is the altitude from A to BC, then the ΔABC has perimeter $P=2(\sqrt{2hr-h^2}+\sqrt{2hr})$ and area $A=\ldots$. Also, $\lim_{h\to 0}\frac{A}{P^3}=\ldots$ **(1989, 2M)**

27. $\lim_{x\to-\infty}\left[\frac{\left(x^4\sin\left(\frac{1}{x}\right)+x^2\right)}{(1+|x|^3)}\right]=\ldots$ **(1987, 2M)**

28. Let $f(x)=\begin{cases}(x^3+x^2-16x+20)/(x-2)^2, & \text{if } x\neq 2\\ k, & \text{if } x=2\end{cases}$.

If $f(x)$ is continuous for all x, then $k=\ldots$. **(1981, 2M)**

29. $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}=\ldots$. **(1978, 2M)**

True/False

30. If $\lim_{x\to a}[f(x)g(x)]$ exists, then both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. **(1981, 2M)**

Analytical & Descriptive Questions

31. Use the formula $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$, to find $\lim_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}$. **(1983, 3M)** **(1982, 2M)**

32. Evaluate $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$. **(1980, 3M)**

33. Evaluate $\lim_{x\to 0}\sqrt{\frac{x-\sin x}{x+\cos^2 x}}$. **(1979, 3M)**

34. Evaluate $\lim_{x\to 1}\left(\frac{x-1}{2x^2-7x+5}\right)$. **(1978, 3M)**

Integer Type Questions

35. Let $\alpha, \beta \in R$ be such that $\lim_{x \to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. Then, $6(\alpha + \beta)$ equals **(2016 Adv)**

36. Let m and n be two positive integers greater than 1. If $\lim_{\alpha \to 0} \left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m} \right) = -\left(\frac{e}{2}\right)$, then the value of $\frac{m}{n}$ is **(2015 Adv.)**

Topic 2 1^∞ Form, RHL and LHL

Objective Questions I (Only one correct option)

1. Let $f: R \to R$ be a differentiable function satisfying $f'(3) + f'(2) = 0$. Then $\lim_{x \to 0} \left(\frac{1 + f(3+x) - f(3)}{1 + f(2-x) - f(2)} \right)^{\frac{1}{x}}$ is equal to **(2019 Main, 8 April II)**

(a) e (b) e^{-1} (c) e^2 (d) 1

2. $\lim_{x \to 1-} \frac{\sqrt{\pi} - \sqrt{2 \sin^{-1} x}}{\sqrt{1-x}}$ is equal to **(2019 Main, 12 Jan II)**

(a) $\sqrt{\frac{\pi}{2}}$ (b) $\sqrt{\frac{2}{\pi}}$ (c) $\sqrt{\pi}$ (d) $\frac{1}{\sqrt{2\pi}}$

3. Let $[x]$ denote the greatest integer less than or equal to x. Then,

$\lim_{x \to 0} \frac{\tan(\pi \sin^2 x) + (|x| - \sin(x[x]))^2}{x^2}$ **(2019 Main, 11 Jan I)**

(a) equals π (b) equals $\pi + 1$
(c) equals 0 (d) does not exist

4. For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then,

$\lim_{x \to 1+} \frac{(1 - |x| + \sin|1-x|) \sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|\,[1-x]}$ **(2019 Main, 10 Jan I)**

(a) equals 0 (b) does not exist
(c) equals -1 (d) equals 1

5. For each $x \in R$, let $[x]$ be the greatest integer less than or equal to x. Then,

$\lim_{x \to 0^-} \frac{x([x] + |x|) \sin[x]}{|x|}$ is equal to **(2019 Main, 9 Jan II)**

(a) 0 (b) sin 1
(c) – sin 1 (d) 1

6. For each $t \in \mathbf{R}$, let $[t]$ be the greatest integer less than or equal to t. Then,

$$\lim_{x \to 0^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \dots + \left[\frac{15}{x}\right]\right)$$ **(2018 Main)**

(a) is equal to 0 (b) is equal to 15
(c) is equal to 120 (d) does not exist (in $\mathbf{R}$)

7. Let $f(x) = \frac{1 - x(1 + |1-x|)}{|1-x|} \cos\left(\frac{1}{1-x}\right)$

for $x \neq 1$. Then

(a) $\lim_{x \to 1^+} f(x) = 0$
(b) $\lim_{x \to 1^-} f(x)$ does not exist
(c) $\lim_{x \to 1^-} f(x) = 0$
(d) $\lim_{x \to 1^+} f(x)$ does not exist

8. Let $p = \lim_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{1/2x}$, then $\log p$ is equal to **(2016 Main)**

(a) 2 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

9. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a} - 1)x^2 - (\sqrt{1+a} - 1)x + (\sqrt[6]{1+a} - 1) = 0$, where $a > -1$. Then, $\lim_{a \to 0^+} \alpha(a)$ and $\lim_{a \to 0^+} \beta(a)$ are **(2012)**

(a) $-\frac{5}{2}$ and 1 (b) $-\frac{1}{2}$ and -1
(c) $-\frac{7}{2}$ and 2 (d) $-\frac{9}{2}$ and 3

10. If $\lim_{x \to 0} [1 + x \log(1 + b^2)]^{\frac{1}{x}} = 2b \sin^2 \theta$, $b > 0$

and $\theta \in (-\pi, \pi]$, then the value of θ is **(2011)**

(a) $\pm \frac{\pi}{4}$ (b) $\pm \frac{\pi}{3}$ (c) $\pm \frac{\pi}{6}$ (d) $\pm \frac{\pi}{2}$

11. For $x > 0$, $\lim_{x \to 0} \left[(\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x} \right]$ is **(2006, 3M)**

(a) 0 (b) – 1 (c) 1 (d) 2

12. Let $f: R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then, $\lim_{x \to 0} \left[\frac{f(1+x)}{f(1)} \right]^{1/x}$ equals **(2002, 2M)**

(a) 1 (b) $e^{\frac{1}{2}}$ (c) e^2 (d) e^3

13. For $x \in R$, $\lim_{x \to \infty} \left(\frac{x-3}{x+2} \right)^x$ is equal to **(2000, 2M)**

(a) e (b) e^{-1} (c) e^{-5} (d) e^5

Fill in the Blanks

14. $\lim_{x \to 0} \left(\frac{1 + 5x^2}{1 + 3x^2} \right)^{1/x^2} = \dots$. **(1996, 1M)**

15. $\lim_{x \to \infty} \left(\frac{x+6}{x+1} \right)^{x+4} = \dots\dots$. **(1991, 2M)**

Analytical & Descriptive Question

16. Find $\lim_{x \to 0} \left\{ \tan\left(\frac{\pi}{4} + x\right) \right\}^{1/x}$. **(1993, 2M)**

Integer Answer Type Question

17. The largest value of the non-negative integer a for which $\lim_{x \to 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$ is **(2014 Adv)**

Topic 3 Squeeze, Newton-Leibnitz's Theorem and Limit Based on Converting infinite Series into Definite Integrals

Objective Questions I (Only one correct option)

1. If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then $\lim_{n\to\infty} \sum_{r=1}^{n} \alpha^r + \lim_{n\to\infty} \sum_{r=1}^{n} \beta^r$ is equal to **(2019 Main, 12 April I)**

(a) $\frac{21}{346}$ (b) $\frac{29}{358}$

(c) $\frac{1}{12}$ (d) $\frac{7}{116}$

2. $\lim_{x\to\frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\, dt}{x^2 - \frac{\pi^2}{16}}$ equals **(2007, 3M)**

(a) $\frac{8}{\pi} f(2)$

(b) $\frac{2}{\pi} f(2)$

(c) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$

(d) $4f(2)$

3. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$ equals **(1999, 2M)**

(a) $1 + \sqrt{5}$ (b) $\sqrt{5} - 1$

(c) $-1 + \sqrt{2}$ (d) $1 + \sqrt{2}$

Objective Questions II

(One more than one correct option)

4. Let $f(x) = \lim_{n\to\infty} \left[\frac{n^n (x+n)\left(x+\frac{n}{2}\right)\ldots\left(x+\frac{n}{n}\right)}{n!\,(x^2+n^2)\left(x^2+\frac{n^2}{4}\right)\ldots\left(x^2+\frac{n^2}{n^2}\right)}\right]^{\frac{x}{n}}$,

for all $x = 0$. Then **(2016 Adv.)**

(a) $f\left(\frac{1}{2}\right) \geq f(1)$ (b) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$

(c) $f'(2) \leq 0$ (d) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

Numerical Value

5. For each positive integer n, let

$$y_n = \frac{1}{n}\left((n+1)(n+2)\ldots(n+n)\right)^{\frac{1}{n}}.$$

For $x \in R$, let $[x]$ be the greatest integer less than or equal to x. If $\lim_{n\to\infty} y_n = L$, then the value of $[L]$ is .... **(2018 Adv.)**

Fill in the Blank

6. $\lim_{x\to 0} \frac{\int_0^{x^2} \cos^2 t\, dt}{x \sin x} = \ldots$ **(1997C, 2M)**

Topic 4 Continuity at a Point

Objective Questions I (Only one correct option)

1. If the function f defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by

$$f(x) = \begin{cases} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \neq \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$$ is continuous,

then k is equal to **(2019 Main, 9 April I)**

(a) $\frac{1}{2}$ (b) 2

(c) 1 (d) $\frac{1}{\sqrt{2}}$

2. The function $f(x) = [x]^2 - [x^2]$ (where, $[x]$ is the greatest integer less than or equal to x), is discontinuous at

(a) all integers **(1999, 2M)**

(b) all integers except 0 and 1

(c) all integers except 0

(d) all integers except 1

3. Let [.] denotes the greatest integer function and $f(x) = [\tan^2 x]$, then **(1993, 1M)**

(a) $\lim_{x\to 0} f(x)$ does not exist

(b) $f(x)$ is continuous at $x = 0$

(c) $f(x)$ is not differentiable at $x = 0$

(d) $f'(0) = 1$

4. The function $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, [·] denotes the greatest integer function, is discontinuous at

(a) all x **(1993, 1M)**

(b) all integer points

(c) no x

(d) x which is not an integer

5. If $f(x) = x(\sqrt{x} + \sqrt{(x+1)})$, then **(1985, 2M)**

(a) $f(x)$ is continuous but not differentiable at $x = 0$

(b) $f(x)$ is differentiable at $x = 0$

(c) $f(x)$ is not differentiable at $x = 0$

(d) None of the above

6. The function $f(x) = \frac{\log(1 + ax) - \log(1 - bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$, so that it is continuous at $x = 0$, is **(1983, 1M)**

(a) $a - b$ (b) $a + b$
(c) $\log a + \log b$ (d) None of these

Objective Questions II

(One or more than one correct option)

7. Let $[x]$ be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous ? **(2017 Adv.)**

(a) $x = -1$ (b) $x = 1$
(c) $x = 0$ (d) $x = 2$

8. For every pair of continuous function $f, g : [0,1] \to R$ such that $\max\{f(x) : x \in [0,1]\} = \max\{g(x) : x \in [0,1]\}$.

The correct statement(s) is (are) **(2014 Adv.)**

(a) $[f(c)]^2 + 3f(c) = [g(c)]^2 + 3g(c)$ for some $c \in [0,1]$
(b) $[f(c)]^2 + f(c) = [g(c)]^2 + 3g(c)$ for some $c \in [0,1]$
(c) $[f(c)]^2 + 3f(c) = [g(c)]^2 + g(c)$ for some $c \in [0,1]$
(d) $[f(c)]^2 = [g(c)]^2$ for some $c \in [0,1]$

9. For every integer n, let a_n and b_n be real numbers. Let function $f : R \to R$ be given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases},$$

for all integers n.

If f is continuous, then which of the following hold(s) for all n ? **(2012)**

(a) $a_{n-1} - b_{n-1} = 0$ (b) $a_n - b_n = 1$
(c) $a_n - b_{n+1} = 1$ (d) $a_{n-1} - b_n = -1$

Fill in the Blank

10. A discontinuous function $y = f(x)$ satisfying $x^2 + y^2 = 4$ is given by $f(x) = ...$. **(1982, 2M)**

Analytical & Descriptive Questions

11. Let $f(x) = \begin{cases} \{1 + |\sin x|\}^{a/|\sin x|}, & -\frac{\pi}{6} < x < 0 \\ b, & x = 0 \\ e^{\tan 2x/\tan 3x}, & 0 < x < \frac{\pi}{6} \end{cases}$

Determine a and b such that $f(x)$ is continuous at $x = 0$. **(1994, 4M)**

12. Let $f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & x > 0 \end{cases}$

Determine the value of a if possible, so that the function is continuous at $x = 0$. **(1990, 4M)**

13. Find the values of a and b so that the function **(1989)**

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x \le \pi/4 \\ 2x\cot x + b, & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x, & \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$.

14. Let $g(x)$ be a polynomial of degree one and $f(x)$ be defined by $f(x) = \begin{cases} g(x), & x \le 0 \\ \left[\frac{(1+x)}{(2+x)}\right]^{1/x}, & x > 0 \end{cases}$

Find the continuous function $f(x)$ satisfying $f'(1) = f(-1)$. **(1987, 6M)**

15. Determine the values a, b, c, for which the function

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x}, & \text{for } x < 0 \\ c, & \text{for } x = 0 \\ \frac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}, & \text{for } x > 0 \end{cases}$$

is continuous at $x = 0$. **(1982, 3M)**

Match the Columns

16. Let $f_1 : R \to R$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$, $f_3 : (-1, e^{\pi/2} - 2) \to R$ and $f_4 : R \to R$ be functions defined by

(i) $f_1(x) = \sin(\sqrt{1 - e^{-x^2}})$,

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where for $t \in R$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

	List-I		List-II
P.	The function f_1 is	1.	NOT continuous at $x = 0$
Q.	The function f_2 is	2.	continuous at $x = 0$ and NOT differentiable at $x = 0$
R.	The function f_3 is	3.	differentiable at $x = 0$ and its derivative is NOT continuous at $x = 0$
S.	The function f_4 is	4.	differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is
(a) P → 2; Q → 3; R → 1; S → 4
(b) P → 4; Q → 1; R → 2; S → 3
(c) P → 4; Q → 2; R → 1; S → 3
(d) P → 2; Q → 1; R → 4; S → 3

Topic 5 Continuity in a Domain

Objective Question I (Only one correct option)

1. Let $f : R \to R$ be a continuously differentiable function such that $f(2) = 6$ and $f'(2) = \frac{1}{48}$. If $\int_6^{f(x)} 4t^3 \, dt = (x-2) g(x)$, then $\lim_{x \to 2} g(x)$ is equal to **(2019 Main, 12 April I)**

(a) 18 (b) 24
(c) 12 (d) 36

2. If $f(x) = \begin{cases} \dfrac{\sin (p+1) x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$

is continuous at $x = 0$, then the ordered pair (p, q) is equal to **(2019 Main, 10 April I)**

(a) $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ (b) $\left(-\frac{1}{2}, \frac{3}{2}\right)$
(c) $\left(\frac{5}{2}, \frac{1}{2}\right)$ (d) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

3. If the function $f(x) = \begin{cases} a|\pi - x| + 1, & x \le 5 \\ b|x - \pi| + 3, & x > 5 \end{cases}$ is continuous at $x = 5$, then the value of $a - b$ is **(2019 Main, 9 April II)**

(a) $\frac{-2}{\pi + 5}$ (b) $\frac{2}{\pi + 5}$
(c) $\frac{2}{\pi - 5}$ (d) $\frac{2}{5 - \pi}$

4. If $f(x) = [x] - \left[\frac{x}{4}\right], x \in R$ where $[x]$ denotes the greatest integer function, then **(2019 Main, 9 April II)**

(a) $\lim_{x \to 4+} f(x)$ exists but $\lim_{x \to 4-} f(x)$ does not exist
(b) f is continuous at $x = 4$
(c) Both $\lim_{x \to 4-} f(x)$ and $\lim_{x \to 4+} f(x)$ exist but are not equal
(d) $\lim_{x \to 4-} f(x)$ exists but $\lim_{x \to 4+} f(x)$ does not exist

5. Let $f : [-1, 3] \to R$ be defined as

$f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3, \end{cases}$ **(2019 Main, 8 April II)**

where, $[t]$ denotes the greatest integer less than or equal to t. Then, f is discontinuous at

(a) four or more points (b) only two points
(c) only three points (d) only one point

6. Let $f : R \to R$ be a function defined as

$f(x) = \begin{cases} 5, & \text{if } x \le 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \le x < 5 \\ 30, & \text{if } x \ge 5 \end{cases}$

Then, f is **(2019 Main, 9 Jan I)**

(a) continuous if $a = -5$ and $b = 10$
(b) continuous if $a = 5$ and $b = 5$
(c) continuous if $a = 0$ and $b = 5$
(d) not continuous for any values of a and b

7. If $f(x) = \frac{1}{2} x - 1$, then on the interval $[0, \pi]$ **(1989, 2M)**

(a) $\tan [f(x)]$ and $1/f(x)$ are both continuous
(b) $\tan [f(x)]$ and $1/f(x)$ are both discontinuous
(c) $\tan [f(x)]$ and $f^{-1}(x)$ are both continuous
(d) $\tan [f(x)]$ is continuous but $1/f(x)$ is not continuous

Objective Questions II

(One or more than one correct option)

8. The following functions are continuous on $(0, \pi)$

(a) $\tan x$ (b) $\int_0^x t \sin \frac{1}{t} \, dt$ **(1991, 2M)**

(c) $\begin{cases} 1, & 0 \le x \le 3\pi/4 \\ 2 \sin \frac{2}{9} x, & \frac{3\pi}{4} < x < \pi \end{cases}$ (d) $\begin{cases} x \sin x, & 0 < x \le \pi/2 \\ \frac{\pi}{2} \sin (\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$

9. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is **(1986, 2M)**

(a) continuous at $x = 0$ (b) continuous in $(-1, 0)$
(c) differentiable at $x = 1$ (d) differentiable in $(-1, 1)$

Fill in the Blank

10. Let $f(x) = [x] \sin \left(\frac{\pi}{[x+1]}\right)$, where $[\cdot]$ denotes the greatest integer function. The domain of f is...... and the points of discontinuity of f in the domain are...... . **(1996, 2M)**

Analytical & Descriptive Question

11. Let $f(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \le x < 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & 1 \le x \le 2 \end{cases}$

Discuss the continuity of f, f' and f'' on $[0, 2]$. **(1983, 2M)**

Topic 6 Continuity for Composition and Function

Objective Question II

(One or more than one correct option)

1. For the function $f(x) = x\cos\frac{1}{x}$, $x \ge 1$, **(2009)**

(a) for atleast one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$

(b) $\lim_{x\to\infty} f'(x) = 1$

(c) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$

(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

Analytical & Descriptive Questions

2. Let $f(x) = \begin{cases} x + a, & \text{if } x < 0 \\ |x - 1|, & \text{if } x \ge 0 \end{cases}$ and

$g(x) = \begin{cases} x + 1, & \text{if } x < 0 \\ (x-1)^2 + b, & \text{if } x \ge 0 \end{cases}$

where, a and b are non-negative real numbers. Determine the compositie function gof. If $(gof)(x)$ is continuous for all real x determine the values of a and b. Further, for these values of a and b, is gof differentiable at $x = 0$? Justify your answer. **(2002, 5M)**

3. Let $f(x)$ be a continuous and $g(x)$ be a discontinuous function. Prove that $f(x) + g(x)$ is a discontinuous function. **(1987, 2M)**

4. Let $f(x) = \begin{cases} 1 + x, & 0 \le x \le 2 \\ 3 - x, & 2 < x \le 3 \end{cases}$

Determine the form of $g(x) = f[f(x)]$ and hence find the points of discontinuity of g, if any **(1983, 2M)**

5. Let $f(x + y) = f(x) + f(y)$ for all x and y. If the function $f(x)$ is continuous at $x = 0$, then show that $f(x)$ is continuous at all x. **(1981, 2M)**

Topic 7 Differentiability at a Point

Objective Questions I (Only one correct option)

1. Let $f : R \to R$ be differentiable at $c \in R$ and $f(c) = 0$. If $g(x) = |f(x)|$, then at $x = c$, g is **(2019 Main, 10 April I)**

(a) not differentiable

(b) differentiable if $f'(c) \ne 0$

(c) not differentiable if $f'(c) = 0$

(d) differentiable if $f'(c) = 0$

2. If $f : R \to R$ is a differentiable function and $f(2) = 6$, then $\lim_{x \to 2} \int_6^{f(x)} \frac{2t\,dt}{(x-2)}$ is **(2019 Main, 9 April II)**

(a) $12f'(2)$ (b) 0

(c) $24f'(2)$ (d) $2f'(2)$

3. Let $f(x) = 15 - |x - 10|$; $x \in \mathbf{R}$. Then, the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable, is **(2019 Main, 9 April I)**

(a) $\{5, 10, 15, 20\}$ (b) $\{5, 10, 15\}$

(c) $\{10\}$ (d) $\{10, 15\}$

4. Let S be the set of all points in $(-\pi, \pi)$ at which the function, $f(x) = \min\{\sin x, \cos x\}$ is not differentiable. Then, S is a subset of which of the following? **(2019 Main, 12 Jan I)**

(a) $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$ (b) $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$

(c) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$ (d) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$

5. Let K be the set of all real values of x, where the function $f(x) = \sin|x| - |x| + 2(x - \pi)\cos|x|$ is not differentiable. Then, the set K is equal to **(2019 Main, 11 Jan II)**

(a) $\{0\}$ (b) ϕ (an empty set)

(c) $\{\pi\}$ (d) $\{0, \pi\}$

6. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$ and

$g(x) = |f(x)| + f(|x|)$. Then, in the interval $(-2, 2)$, g is **(2019 Main, 11 Jan I)**

(a) not differentiable at one point

(b) not differentiable at two points

(c) differentiable at all points

(d) not continuous

7. Let $f : (-1, 1) \longrightarrow R$ be a function defined by $f(x) = \max\{-|x|, -\sqrt{1 - x^2}\}$. If K be the set of all points at which f is not differentiable, then K has exactly **(2019 Main, 10 Jan II)**

(a) three elements (b) five elements

(c) two elements (d) one element

8. Let $f(x) = \begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$

Let S be the set of points in the interval $(-4, 4)$ at which f is not differentiable. Then, S **(2019 Main, 10 Jan I)**

(a) equals $\{-2, -1, 0, 1, 2\}$ (b) equals $\{-2, 2\}$

(c) is an empty set (d) equals $\{-2, -1, 1, 2\}$

9. Let f be a differentiable function from R to R such that $|f(x) - f(y)| \le 2|x - y|^{\frac{3}{2}}$, for all $x, y \in R$. If $f(0) = 1$, then $\int_0^1 f^2(x)\,dx$ is equal to **(2019 Main, 9 Jan II)**

(a) 2 (b) $\frac{1}{2}$ (c) 1 (d) 0

10. Let $S = (t \in \mathbf{R} : f(x) = |x - \pi| \cdot (e^{|x|} - 1)\sin|x|$ is not differentiable at $t\}$. Then, the set S is equal to **(2018 Main)**

(a) ϕ (an empty set) (b) $\{0\}$

(c) $\{\pi\}$ (d) $\{0, \pi\}$

11. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then **(2016 Main)**

(a) g is not differentiable at $x = 0$

(b) $g'(0) = \cos(\log 2)$

(c) $g'(0) = -\cos(\log 2)$

(d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

12. If f and g are differentiable functions in $(0, 1)$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$ **(2014 Main)**

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$

(c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$

13. Let $f(x) = \begin{cases} x^2 \left|\cos \dfrac{\pi}{x}\right|, & x \neq 0, x \in R, \\ 0, & x = 0 \end{cases}$ then f is **(2012)**

(a) differentiable both at $x = 0$ and at $x = 2$

(b) differentiable at $x = 0$ but not differentiable at $x = 2$

(c) not differentiable at $x = 0$ but differentiable at $x = 2$

(d) differentiable neither at $x = 0$ nor at $x = 2$

14. Let $g(x) = \dfrac{(x-1)^n}{\log \cos^m (x-1)}$; $0 < x < 2$, m and n are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim_{x \to 1^+} g(x) = p$, then

(a) $n = 1, m = 1$ (b) $n = 1, m = -1$ **(2008, 3M)**

(c) $n = 2, m = 2$ (d) $n > 2, m = n$

15. If f is a differentiable function satisfying $f\left(\dfrac{1}{n}\right) = 0, \forall\ n \geq 1, n \in I$, then **(2005, 2M)**

(a) $f(x) = 0, x \in (0, 1]$

(b) $f'(0) = 0 = f(0)$

(c) $f(0) = 0$ but $f'(0)$ not necessarily zero

(d) $|f(x)| \leq 1, x \in (0, 1]$

16. Let $f(x) = ||x| - 1|$, then points where, $f(x)$ is not differentiable is/are **(2005, 2M)**

(a) $0, \pm 1$ (b) ± 1

(c) 0 (d) 1

17. The domain of the derivative of the functions $f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \leq 1 \\ \dfrac{1}{2}(|x| - 1), & \text{if } |x| > 1 \end{cases}$ is **(2002, 2M)**

(a) $R - \{0\}$ (b) $R - \{1\}$

(c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

18. Which of the following functions is differentiable at $x = 0$? **(2001, 2M)**

(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$

(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

19. The left hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, k is an integer, is **(2001, 2M)**

(a) $(-1)^k (k - 1)\pi$ (b) $(-1)^{k-1}(k - 1)\pi$

(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

20. Let $f : R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points, where $f(x)$ is not differentiable, is

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$ **(2001, 2M)**

(c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

21. Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|, \forall\ x$. Then, g is **(2000, 2M)**

(a) onto if f is onto

(b) one-one if f is one-one

(c) continuous if f is continuous

(d) differentiable if f is differentiable

22. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is not differentiable at **(1999, 2M)**

(a) -1 (b) 0 (c) 1 (d) 2

23. The set of all points, where the function $f(x) = \dfrac{x}{1 + |x|}$ is differentiable, is **(1987, 2M)**

(a) $(-\infty, \infty)$ (b) $[0, \infty)$

(c) $(-\infty, 0) \cup (0, \infty)$ (d) $(0, \infty)$

24. There exists a function $f(x)$ satisfying $f(0) = 1$, $f'(0) = -1$, $f(x) > 0, \forall\ x$ and **(1982, 2M)**

(a) $f''(x) < 0, \forall\ x$ (b) $-1 < f''(x) < 0, \forall\ x$

(c) $-2 \leq f''(x) \leq -1, \forall\ x$ (d) $f''(x) < -2, \forall\ x$

25. For a real number y, let $[y]$ denotes the greatest integer less than or equal to y. Then, the function $f(x) = \dfrac{\tan \pi [(x - \pi)]}{1 + [x]^2}$ is **(1981, 2M)**

(a) discontinuous at some x

(b) continuous at all x, but the derivative $f'(x)$ does not exist for some x

(c) $f'(x)$ exists for all x, but the derivative $f''(x)$ does not exist for some x

(d) $f'(x)$ exists for all x

Objective Questions II

(One or more than one correct option)

26. For every twice differentiable function $f : R \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ? **(2018 Adv.)**

(a) There exist $r, s \in R$, where $r < s$, such that f is one-one on the open interval (r, s)

(b) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \leq 1$

(c) $\lim_{x \to \infty} f(x) = 1$

(d) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

27. Let $f : (0, \pi) \to R$ be a twice differentiable function such that $\lim_{t \to x} \dfrac{f(x) \sin t - f(t) \sin x}{t - x} = \sin^2 x$ for all $x \in (0, \pi)$.

If $f\left(\dfrac{\pi}{6}\right) = -\dfrac{\pi}{12}$, then which of the following statement(s) is (are) TRUE? **(2018 Adv.)**

(a) $f\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{4\sqrt{2}}$

(b) $f(x) < \dfrac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(c) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(d) $f''\left(\dfrac{\pi}{2}\right) + f\left(\dfrac{\pi}{2}\right) = 0$

28. Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then, **(2016 Adv.)**

(a) $g'(2) = \frac{1}{15}$
(b) $h'(1) = 666$
(c) $h(0) = 16$
(d) $h(g(3)) = 36$

29. Let $a, b \in R$ and $f : R \to R$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then, f is **(2016 Adv.)**

(a) differentiable at $x = 0$, if $a = 0$ and $b = 1$
(b) differentiable at $x = 1$, if $a = 1$ and $b = 0$
(c) not differentiable at $x = 0$, if $a = 1$ and $b = 0$
(d) not differentiable at $x = 1$, if $a = 1$ and $b = 1$

30. Let $f : \left[-\frac{1}{2}, 2\right] \to R$ and $g : \left[-\frac{1}{2}, 2\right] \to R$ be functions defined by $f(x) = [x^2 - 3]$ and $g(x) = |x| f(x) + |4x - 7| f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in R$. Then, **(2016 Adv.)**

(a) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$
(b) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$
(c) g is not differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$
(d) g is not differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$

31. Let $g : R \to R$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$. **(2015 Adv.)**

Let $f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

and $h(x) = e^{|x|}$ for all $x \in R$. Let $(foh)(x)$ denotes $f\{h(x)\}$ and $(hof)(x)$ denotes $h\{f(x)\}$. Then, which of the following is/are true?

(a) f is differentiable at $x = 0$
(b) h is differentiable at $x = 0$
(c) foh is differentiable at $x = 0$
(d) hof is differentiable at $x = 0$

32. Let $f, g : [-1, 2] \to R$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points $-1, 0$ and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$, the function $(f - 3g)''$ never vanishes. Then, the correct statement(s) is/are **(2015 Adv.)**

(a) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$
(b) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$
(c) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$
(d) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

33. Let $f : [a, b] \to [1, \infty)$ be a continuous function and $g : R \to R$ be defined as $g(x) = \begin{cases} 0, & \text{if } x < a \\ \int_a^x f(t)\,dt, & \text{if } a \le x \le b \\ \int_a^b f(t)\,dt, & \text{if } x > b \end{cases}$.

Then, **(2013)**

(a) $g(x)$ is continuous but not differentiable at a
(b) $g(x)$ is differentiable on R
(c) $g(x)$ is continuous but not differentiable at b
(d) $g(x)$ is continuous and differentiable at either a or b but not both

34. If $f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \le -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \le 0 \\ x - 1, & 0 < x \le 1 \\ \ln x, & x > 1 \end{cases}$, then **(2011)**

(a) $f(x)$ is continuous at $x = -\frac{\pi}{2}$
(b) $f(x)$ is not differentiable at $x = 0$
(c) $f(x)$ is differentiable at $x = 1$
(d) $f(x)$ is differentiable at $x = -\frac{3}{2}$

35. Let $f : R \to R$ be a function such that $f(x + y) = f(x) + f(y)$, $\forall x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then **(2011)**

(a) $f(x)$ is differentiable only in a finite interval containing zero
(b) $f(x)$ is continuous for all $x \in R$
(c) $f'(x)$ is constant for all $x \in R$
(d) $f(x)$ is differentiable except at finitely many points

36. If $f(x) = \min\{1, x^2, x^3\}$, then **(2006, 3M)**

(a) $f(x)$ is continuous everywhere
(b) $f(x)$ is continuous and differentiable everywhere
(c) $f(x)$ is not differentiable at two points
(d) $f(x)$ is not differentiable at one point

37. Let $h(x) = \min\{x, x^2\}$ for every real number of x, then **(1998, 2M)**

(a) h is continuous for all x
(b) h is differentiable for all x
(c) $h'(x) = 1, \forall x > 1$
(d) h is not differentiable at two values of x

38. The function $f(x) = \begin{cases} |x - 3|, & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$ is **(1988, 2M)**

(a) continuous at $x = 1$
(b) differentiable at $x = 1$
(c) discontinuous at $x = 1$
(d) differentiable at $x = 3$

39. The function $f(x) = 1 + |\sin x|$ is **(1986, 2M)**

(a) continuous no where
(b) continuous everywhere
(c) differentiable at $x = 0$
(d) not differentiable at infinite number of points

40. If $x + |y| = 2y$, then y as a function of x is **(1984, 2M)**

(a) defined for all real x (b) continuous at $x = 0$
(c) differentiable for all x (d) such that $\frac{dy}{dx} = \frac{1}{3}$ for $x < 0$

Assertion and Reason

For the following questions, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows.

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

41. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x) \sin x$.

Statement I $\lim_{x \to 0} [g(x) \cos x - g(0) \operatorname{cosec} x] = f''(0)$. and

Statement II $f'(0) = g(0)$. **(2008, 3M)**

Match the Columns

42. In the following, $[x]$ denotes the greatest integer less than or equal to x.

	Column I		Column II
A.	$x\|x\|$	p.	continuous in $(-1, 1)$
B.	$\sqrt{\|x\|}$	q.	differentiable in $(-1, 1)$
C.	$x + [x]$	r.	strictly increasing $(-1, 1)$
D.	$\|x-1\| + \|x+1\|$, in $(-1, 1)$	s.	not differentiable atleast at one point in $(-1, 1)$

(2007, 6M)

43. Match the conditions/expressions in Column I with statement in Column II **(1992, 2M)**

	Column I		Column II
A.	$\sin(\pi [x])$	p.	differentiable everywhere
B.	$\sin\{\pi(x - [x])\}$	q.	no where differentiable
		r.	not differentiable at 1 and -1

Fill in the Blanks

44. Let $F(x) = f(x)\, g(x)\, h(x)$ for all real x, where $f(x)$, $g(x)$ and $h(x)$ are differentiable functions. At same point x_0, $F'(x_0) = 21 F(x_0)$, $f'(x_0) = 4 f(x_0)$, $g'(x_0) = -7 g(x_0)$ and $h'(x_0) = k h(x_0)$, then $k = \ldots$. **(1997C, 2M)**

45. For the function $f(x) = \begin{cases} \dfrac{x}{1 + e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$;

the derivative from the right, $f'(0^+) = \ldots$ and the derivative from the left, $f'(0^-) = \ldots$. **(1983, 2M)**

46. Let $f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{(x-1)} - |x|, & \text{if } x \neq 1 \\ -1, & \text{if } x = 1 \end{cases}$ be a real valued function. Then, the set of points, where $f(x)$ is not differentiable, is .... . **(1981, 2M)**

True/False

47. The derivative of an even function is always an odd function. **(1983, 1M)**

Analytical & Descriptive Questions

48. $f(x) = \begin{cases} b \sin^{-1}\left(\dfrac{x + c}{2}\right), & -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2}, & x = 0 \\ \dfrac{e^{ax/2} - 1}{x}, & 0 < x < \dfrac{1}{2} \end{cases}$

If $f(x)$ is differentiable at $x = 0$ and $|c| < \frac{1}{2}$, then find the value of a and prove that $64b^2 = (4 - c^2)$. **(2004, 4M)**

49. If $f : [-1, 1] \to R$ and $f'(0) = \lim_{n \to \infty} nf\left(\frac{1}{n}\right)$ and $f(0) = 0$.

Find the value of $\lim_{n \to \infty} \frac{2}{\pi}(n+1) \cos^{-1}\left(\frac{1}{n}\right) - n$, given that

$$0 < \left| \lim_{n \to \infty} \cos^{-1}\left(\frac{1}{n}\right) \right| < \frac{\pi}{2}.$$ **(2004, 2M)**

50. Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α if and only if there is a function $g : R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$, $\forall\, x \in R$. **(2001, 5M)**

51. Determine the values of x for which the following function fails to be continuous or differentiable

$f(x) = \begin{cases} 1 - x, & x < 1 \\ (1 - x)(2 - x), & 1 \le x \le 2 \\ 3 - x, & x > 2 \end{cases}$. Justify your answer. **(1997, 5M)**

52. Let $f(x) = \begin{cases} x\, e^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Test whether

(i) $f(x)$ is continuous at $x = 0$.
(ii) $f(x)$ is differentiable at $x = 0$. **(1997C, 5M)**

53. Let $f[(x + y)/2] = \{f(x) + f(y)\}/2$ for all real x and y, if $f'(0)$ exists and equals -1 and $f(0) = 1$, find $f(2)$. **(1995, 5M)**

54. A function $f: R \to R$ satisfies the equation $f(x+y) = f(x) f(y), \forall x, y$ in R and $f(x) \neq 0$ for any x in R. Let the function be differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$, $\forall x$ in R. Hence, determine $f(x)$. **(1990, 4M)**

55. Draw a graph of the function

$$y = [x] + |1 - x|, -1 \le x \le 3.$$

Determine the points if any, where this function is not differentiable. **(1989, 4M)**

56. Let R be the set of real numbers and $f: R \to R$ be such that for all x and y in R, $f(x) - f(y)|^2 \le (x-y)^3$. Prove that $f(x)$ is a constant. **(1988, 2M)**

57. Let $f(x)$ be a function satisfying the condition $f(-x) = f(x), \forall x$. If $f'(0)$ exists, find its value. **(1987, 2M)**

58. Let $f(x)$ be defined in the interval $[-2, 2]$ such that

$$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x-1, & 0 < x \le 2 \end{cases}$$

and $g(x) = f(|x|) + |f(x)|$.

Test the differentiability of $g(x)$ in $(-2, 2)$. **(1986, 5M)**

59. Let $f(x) = x^3 - x^2 - x + 1$

and $g(x) \begin{cases} = \max\{f(t); 0 \le t \le x\}, & 0 \le x \le 1 \\ = 3 - x, & 1 < x \le 2 \end{cases}$

Discuss the continuity and differentiability of the function $g(x)$ in the interval $(0, 2)$. **(1985, 5M)**

60. Find $f'(1)$, if $f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5}, & \text{when } x \neq 1 \\ -\dfrac{1}{3}, & \text{when } x = 1 \end{cases}$. **(1979, 3M)**

61. If $f(x) = x \tan^{-1} x$, find $f'(1)$ from first principle. **(1978, 3M)**

Integer Answer Type Questions

62. Let $f: R \to R$ be a differentiable function such that $f(0) = 0, f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$.

If $g(x) = \int_x^{\frac{\pi}{2}} [f'(t) \operatorname{cosec} t - \cot t \operatorname{cosec} t \, f(t)] \, dt$

for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \to 0} g(x) =$ **(2017 Adv.)**

63. Let $f: R \to R$ and $g: R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h: R \to R$ by

$$h(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } x \le 0. \\ \min\{f(x), g(x)\}, & \text{if } x > 0. \end{cases}$$

The number of points at which $h(x)$ is not differentiable is **(2014 Adv.)**

64. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1$, 2 and $\lim_{x \to 0} \left[1 + \frac{p(x)}{x^2}\right] = 2$. Then, the value of $p(2)$ is .... **(2010)**

Topic 8 Differentiation

Objective Questions I (Only one correct option)

1. If ${}^{20}C_1 + (2^2)\, {}^{20}C_2 + (3^2)\, {}^{20}C_3 + \ldots + (20^2)\, {}^{20}C_{20} = A(2^\beta)$, then the ordered pair (A, β) is equal to **(2019 Main, 12 April II)**

(a) (420, 19) (b) (420, 18) (c) (380, 18) (d) (380, 19)

2. The derivative of $\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x \in \left(0, \frac{\pi}{2}\right)\right)$ is **(2019 Main, 12 April II)**

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) 2

3. If $e^y + xy = e$, the ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ at $x = 0$ is equal to **(2019 Main, 12 April I)**

(a) $\left(\frac{1}{e}, -\frac{1}{e^2}\right)$ (b) $\left(-\frac{1}{e}, \frac{1}{e^2}\right)$ (c) $\left(\frac{1}{e}, \frac{1}{e^2}\right)$ (d) $\left(-\frac{1}{e}, -\frac{1}{e^2}\right)$

4. If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is **(2019 Main, 8 April II)**

(a) 12 (b) 9 (c) 15 (d) 33

5. If $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$, $x \in \left(0, \frac{\pi}{2}\right)$ then $\frac{dy}{dx}$ is equal to **(2019 Main, 8 April I)**

(a) $\frac{\pi}{6} - x$ (b) $x - \frac{\pi}{6}$ (c) $\frac{\pi}{3} - x$ (d) $2x - \frac{\pi}{3}$

6. For $x > 1$, if $(2x)^{2y} = 4e^{2x - 2y}$, then $(1 + \log_e 2x)^2 \frac{dy}{dx}$ is equal to **(2019 Main, 12 Jan I)**

(a) $\frac{x \log_e 2x + \log_e 2}{x}$ (b) $\frac{x \log_e 2x - \log_e 2}{x}$

(c) $x \log_e 2x$ (d) $\log_e 2x$

7. If $x \log_e(\log_e x) - x^2 + y^2 = 4 (y > 0)$, then $\frac{dy}{dx}$ at $x = e$ is equal to **(2019 Main, 11 Jan I)**

(a) $\frac{e}{\sqrt{4 + e^2}}$ (b) $\frac{(2e - 1)}{2\sqrt{4 + e^2}}$ (c) $\frac{(1 + 2e)}{\sqrt{4 + e^2}}$ (d) $\frac{(1 + 2e)}{2\sqrt{4 + e^2}}$

8. Let $f: R \to R$ be a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3), x \in R$. Then, $f(2)$ equals **(2019 Main, 10 Jan I)**

(a) 30 (b) -4 (c) -2 (d) 8

9. If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$, is **(2019 Main, 9 Jan II)**

(a) $\frac{1}{6}$ (b) $\frac{1}{6\sqrt{2}}$

(c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{3}{2\sqrt{2}}$

10. For $x \in \left(0, \frac{1}{4}\right)$, if the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then $g(x)$ equals **(2017 Main)**

(a) $\frac{9}{1+9x^3}$ (b) $\frac{3x\sqrt{x}}{1-9x^3}$ (c) $\frac{3x}{1-9x^3}$ (d) $\frac{3}{1+9x^3}$

11. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to **(2015)**

(a) $1+x^5$ (b) $5x^4$

(c) $\frac{1}{1+\{g(x)\}^5}$ (d) $1+\{g(x)\}^5$

12. If $y = \sec(\tan^{-1} x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to **(2013)**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt{2}$

13. Let $g(x) = \log f(x)$, where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = x f(x)$. Then, for $N = 1, 2, 3, \ldots, g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right)$ is equal to **(2008, 3M)**

(a) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N+1)^2}\right\}$

14. $\frac{d^2x}{dy^2}$ equals **(2007, 3M)**

(a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

15. If $f''(x) = -f(x)$, where $f(x)$ is a continuous double differentiable function and $g(x) = f'(x)$.

If $F(x) = \left\{f\left(\frac{x}{2}\right)\right\}^2 + \left\{g\left(\frac{x}{2}\right)\right\}^2$ and $F(5) = 5$, then $F(10)$ is **(2006, 3M)**

(a) 0 (b) 5 (c) 10 (d) 25

16. Let f be twice differentiable function satisfying $f(1) = 1, f(2) = 4, f(3) = 9$, then **(2005, 2M)**

(a) $f''(x) = 2, \forall x \in (R)$

(b) $f'(x) = 5 = f''(x)$, for some $x \in (1, 3)$

(c) there exists atleast one $x \in (1, 3)$ such that $f''(x) = 2$

(d) None of the above

17. If y is a function of x and $\log(x+y) = 2xy$, then the value of $y'(0)$ is **(2004, 1M)**

(a) 1 (b) -1 (c) 2 (d) 0

18. If $x^2 + y^2 = 1$, then **(2000, 1M)**

(a) $yy'' - 2(y')^2 + 1 = 0$ (b) $yy'' + (y')^2 + 1 = 0$

(c) $yy'' + (y')^2 - 1 = 0$ (d) $yy'' + 2(y')^2 + 1 = 0$

19. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is constant.

Then, $\frac{d^3}{dx^3} f(x)$ at $x = 0$ is **(1997, 2M)**

(a) p (b) $p + p^2$

(c) $p + p^3$ (d) independent of p

20. If $y^2 = P(x)$ is a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$ equals **(1988, 2M)**

(a) $P'''(x) + P'(x)$ (b) $P''(x) \cdot P'''(x)$

(c) $P(x)P'''(x)$ (d) a constant

Fill in the Blanks

21. If $x e^{xy} = y + \sin^2 x$, then at $x = 0$, $\frac{dy}{dx} = \ldots\ldots$. **(1996, 2M)**

22. Let $f(x) = x|x|$. The set of points, where $f(x)$ is twice differentiable, is … . **(1992, 2M)**

23. If $f(x) = |x-2|$ and $g(x) = f[f(x)]$, then $g'(x) = \ldots\ldots$ for $x > 2$. **(1990, 2M)**

24. The derivative of $\sec^{-1}\left(-\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x = \frac{1}{2}$ is …… . **(1986, 2M)**

25. If $f(x) = \log_x(\log x)$, then $f'(x)$ at $x = e$ is …… . **(1985, 2M)**

26. If $f_r(x), g_r(x), h_r(x), r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a), r = 1, 2, 3$

and $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$,

then $F'(x)$ at $x = a$ is …… . **(1985, 2M)**

27. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin^2 x$, then $\frac{dy}{dx} = \ldots\ldots$. **(1982, 2M)**

Analytical & Descriptive Questions

28. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{(x-c)} + 1$,

Prove that $\frac{y'}{y} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$. **(1998, 8M)**

29. Find $\frac{dy}{dx}$ at $x = -1$, when

$(\sin y)^{\sin \frac{\pi}{2} x} + \frac{\sqrt{3}}{2} \sec^{-1}(2x) + 2^x \tan \ln(x+2) = 0$. **(1991, 4M)**

30. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then show that

$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$. **(1989, 2M)**

31. If α be a repeated roots of a quadratic equation $f(x)=0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3, 4 and 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where prime denotes the derivatives. **(1984, 4M)**

32. Find the derivative with respect to x of the function

$$y=\left\{(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1}+\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\}$$

at $x=\frac{\pi}{4}$. **(1984, 4M)**

33. If $(a+bx)e^{y/x}=x$, then prove that

$$x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2.$$

(1983, 3M)

34. Let f be a twice differentiable function such that **(1983, 3M)**

$$f''(x)=-f(x),\ f'(x)=g(x)\ \text{and}$$
$$h(x)=[f(x)]^2+[g(x)]^2$$

Find $h(10)$, if $h(5)=11$.

35. Let $y=e^{x\sin x^3}+(\tan x)^x$, find $\frac{dy}{dx}$. **(1981, 2M)**

36. Given, $y=\frac{5x}{3\sqrt{(1-x)^2}}+\cos^2(2x+1)$, find $\frac{dy}{dx}$. **(1980)**

Integer Type Questions

37. Let $f:R\to R$ be a continuous odd function, which vanishes exactly at one point and $f(1)=\frac{1}{2}$.

Suppose that $F(x)=\int_{-1}^{x}f(t)\,dt$ for all $x\in[-1,2]$ and $G(x)=\int_{-1}^{x}t\,|f\{f(t)\}|\,dt$ for all $x\in[-1,2]$. If $\lim_{x\to1}\frac{F(x)}{G(x)}=\frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **(2015 Adv.)**

Answers

Topic 1

1. (b)	2. (c)	3. (d)	4. (a)
5. (c)	6. (b)	7. (a)	8. (b)
9. (d)	10. (c)	11. (b)	12. (d)
13. (d)	14. (c)	15. (c)	16. (c)
17. (d)	18. (d)	19. (b)	20. (c)
21. (a)	22. (d)	23. (a, c)	24. -1
25. 1	26. $h\sqrt{2hr-h^2},\frac{1}{128r}$		27. -1
28. 7	29. $\frac{2}{\pi}$	30. False	31. $\log_e 4$
32. $a^2\cos\alpha+2a\sin\alpha$		33. 0	
34. $\frac{-1}{3}$	35. (7)	36. (2)	

Topic 2

1. (d)	2. (b)	3. (d)	4. (a)
5. (c)	6. (c)	7. (d)	8. (c)
9. (b)	10. (d)	11. (c)	12. (c)
13. (c)	14. e^2	15. e^5	16. e^2
17. $a=2$			

Topic 3

1. (c)	2. (a)	3. (b)	4. (b,c)
5. (1)	6. 1		

Topic 4

1. (a)	2. (b)	3. (b)	4. (c)
5. (c)	6. (b)	7. (a,b,d)	8. (a, d)
9. (b,d)	10. $f(x)=\sqrt{4-x^2}$		

11. $a=\frac{2}{3},b=e^{2/3}$ **12.** $a=8$ **13.** $a=\frac{\pi}{6},b=\frac{-\pi}{12}$

14. $f(x)=\begin{cases}\frac{2}{3}\left(\log\left(\frac{2}{3}\right)-\frac{1}{6}\right)x, & x\le 0\\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x>0\end{cases}$

15. $a=\frac{-3}{2},c=\frac{1}{2}$ and $b\in R$ **16.** (d)

Topic 5

1. (a)	2. (d)	3. (d)	4. (b)
5. (c)	6. (d)	7. (b)	8. (b, c)
9. (a,b, d)	10. $x\in(-\infty,-1)\cup[0,\infty),[-1,0)$		

11. f and f'' are continuous and f' is discontinuous at $x=\{1,2\}$.

Topic 6

1. (b, c, d)

2. $g\{f(x)\}=\begin{cases}x+a+1, & \text{if } x<-a\\ (x+a-1)^2, & \text{if } a\le x<c\\ x^2+b, & \text{if } 0\le x\le 1\\ (x-2)^2+b, & \text{if } x>1\end{cases}$

$a=1$, $b=0$

gof is differentiable at $x=0$

4. $g(x)=\begin{cases}4-x, & 2<x\le 3\\ 2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\end{cases}$, discontinuous at $x=\{1,2\}$

Discontinuity of g at $x=\{1,2\}$

Topic 7

1. (b) **2.** (a) **3.** (b) **4.** (c)
5. (b) **6.** (a) **7.** (a) **8.** (a)
9. (c) **10.** (a) **11.** (b) **12.** (d)
13. (b) **14.** (c) **15.** (b) **16.** (a)
17. (d) **18.** (d) **19.** (a) **20.** (d)
21. (c) **22.** (d) **23.** (a) **24.** (a)
25. (d) **26.** (a,b,d) **27.** (b,c,d) **28.** (b,c)
29. (a,b) **30.** (b,c) **31.** (a,d) **32.** (b,c)
33. (b, c) **34.** (a, b, c, d) **35.** (b, c) **36.** (a, d)
37. (a, c, d) **38.** (a, b) **39.** (b, d) **40.** (a, b, d)
41. (b)
42. (A) → p, q, r, s; (B) → p, s; (C) → r, s; (D) → p, s
43. (A) → p; (B) → r **44.** (24)
45. $f'(0^+) = 0,\ f'(0^-) = 1$
46. $x = 0$ **47.** True **48.** $(a = 1)$ **49.** $\left(1 - \frac{2}{\pi}\right)$
51. (1, 2) **52.** (i) Yes (ii) No
53. (-1) **54.** e^{2x} **55.** {0, 1, 2} **57.** $f'(0) = 0$
58. $g(x)$ is differentiable for all $x \in (-2, 2) - \{0, 1\}$
59. $g(x)$ is continuous for all $x \in (0, 2) - \{1\}$ and $g(x)$ is differentiable for all $x \in (0, 2) - \{1\}$
60. $\left(-\frac{2}{9}\right)$ **61.** $\left(\frac{1}{2} + \frac{\pi}{4}\right)$ **62.** (2) **63.** (3)
64. $p(2) = 0$

Topic 8

1. (b) **2.** (d) **3.** (b) **4.** (d)
5. (b) **6.** (b) **7.** (b) **8.** (c)
9. (b) **10.** (a) **11.** (d) **12.** (a)
13. (a) **14.** (d) **15.** (b) **16.** (c)
17. (a) **18.** (b) **19.** (d) **20.** (c)
21. 1 **22.** $x \in R - \{0\}$ **23.** 1 **24.** -4
25. $\frac{1}{e}$ **26.** 0 **27.** $\frac{-2(x^2 - x - 1)}{(x^2+1)^2} \cdot \sin^2\left(\frac{2x-1}{x^2+1}\right)$
29. $\frac{3}{\pi\sqrt{\pi^2 - 3}}$ **32.** $\frac{-8}{\log e^2} + \frac{32}{16 + \pi^2}$ **33.** 11
35. $e^{x \sin x^3}(3x^3 \cos x^3 + \sin x^3) + (\tan x)^x[2x \operatorname{cosec} 2x + \log(\tan x)]$
36. $\begin{cases} \dfrac{5}{3(1-x)^2} - 2\sin(4x+2), & x < 1 \\ \dfrac{-5}{3(x-1)^2} - 2\sin(4x+2), & x > 1 \end{cases}$ **37.** (7)

Hints & Solutions

Topic 1 $\frac{0}{0}$ and $\frac{\infty}{\infty}$ Form

1. Let

$$P = \lim_{x \to 0} \frac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}} \qquad \left[\frac{0}{0} \text{ form}\right]$$

On rationalization, we get

$$P = \lim_{x \to 0} \frac{(x + 2\sin x)}{x^2 + 2\sin x + 1 - \sin^2 x + x - 1} \times \left(\sqrt{x^2 + 2\sin x + 1} + \sqrt{\sin^2 x - x + 1}\right)$$

$$= \lim_{x \to 0} \left(\sqrt{x^2 + 2\sin x + 1} + \sqrt{\sin^2 x - x + 1}\right) \times \lim_{x \to 0} \frac{x + 2\sin x}{x^2 - \sin^2 x + 2\sin x + x}$$

$$= 2 \times \lim_{x \to 0} \frac{x + 2\sin x}{x^2 - \sin^2 x + 2\sin x + x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Now applying the L' Hopital's rule, we get

$$P = 2 \times \lim_{x \to 0} \frac{1 + 2\cos x}{2x - \sin 2x + 2\cos x + 1}$$

$$= 2\,\frac{(1+2)}{0 - 0 + 2 + 1} \qquad \text{[on applying limit]}$$

$$= 2 \times \frac{3}{3} = 2$$

$$\Rightarrow \lim_{x \to 0} \frac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}} = 2$$

2. It is given that $\lim_{x \to 1} \frac{x^2 - ax + b}{x - 1} = 5$ ...(i)

Since, limit exist and equal to 5 and denominator is zero at $x = 1$, so numerator $x^2 - ax + b$ should be zero at $x = 1$,

So $1 - a + b = 0 \Rightarrow a = 1 + b$ ...(ii)

On putting the value of 'a' from Eq. (ii) in Eq. (i), we get

$$\lim_{x \to 1} \frac{x^2 - (1+b)x + b}{x - 1} = 5 \Rightarrow \lim_{x \to 1} \frac{(x^2 - x) - b(x - 1)}{x - 1} = 5$$

$$\Rightarrow \lim_{x \to 1} \frac{(x-1)(x-b)}{x-1} = 5 \Rightarrow \lim_{x \to 1}(x - b) = 5$$

$$\Rightarrow 1 - b = 5$$

$$\Rightarrow b = -4 \qquad \text{...(iii)}$$

On putting value of 'b' from Eq. (iii) to Eq. (ii), we get

$$a = -3$$

So, $a + b = -7$

3. Given,

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$$

$$\Rightarrow \lim_{x \to 1} \frac{(x-1)(x+1)(x^2+1)}{x-1}$$

$$= \lim_{x \to k} \frac{(x-k)(x^2 + k^2 + xk)}{(x-k)(x+k)}$$

$$\Rightarrow \quad 2 \times 2 = \frac{3k^2}{2k}$$

$$\Rightarrow \quad k = \frac{8}{3}$$

4. Given limit is $\lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{2}\cos\frac{x}{2}} \quad \left[\because 1 + \cos x = 2\cos^2\frac{x}{2}\right]$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2}\left(1 - \cos\frac{x}{2}\right)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2} \times 2\sin^2\left(\frac{x}{4}\right)} \quad \left[\because 1 - \cos\frac{x}{2} = 2\sin^2\frac{x}{4}\right]$$

$$= \lim_{x\to 0} \frac{x^2}{2\sqrt{2}\left(\frac{x}{4}\right)^2} = \frac{16}{2\sqrt{2}} = 4\sqrt{2} \quad [\lim_{x\to 0} \sin x = \lim_{x\to 0} x]$$

5. Given, limit $= \lim_{x\to\pi/4} \frac{\cot^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$

$$= \lim_{x\to\pi/4} \frac{1 - \tan^4 x}{\frac{1}{\sqrt{2}}(\cos x - \sin x)} \times \frac{1}{\tan^3 x} \quad \left[\because \cot x = \frac{1}{\tan x}\right]$$

$$= \lim_{x\to\pi/4} \frac{(1 - \tan^2 x)}{\cos x - \sin x} \times \frac{\sqrt{2}(1 + \tan^2 x)}{\tan^3 x}$$

$$= \lim_{x\to\pi/4} \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} \times \frac{\sqrt{2}(\sec^2 x)}{\cos^2 x \tan^3 x}$$

$[\because 1 + \tan^2 x = \sec^2 x]$

$$= \lim_{x\to\pi/4} \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x - \sin x)} \times \frac{\sqrt{2}\sec^4 x}{\tan^3 x}$$

$[\because (a^2 - b^2) = (a - b)(a + b)]$

$$= \lim_{x\to\pi/4} \frac{\sqrt{2}\sec^4 x}{\tan^3 x}(\cos x + \sin x)$$

$$= \frac{\sqrt{2}\,(\sqrt{2})^4}{(1)^3}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \quad \text{[on applying limit]}$$

$$= 4\sqrt{2}\left(\frac{2}{\sqrt{2}}\right) = 8.$$

6. $\lim_{x\to 0} \frac{x\cot 4x}{\sin^2 x.\cot^2 2x} = \lim_{x\to 0} \frac{x}{\tan 4x} \cdot \frac{1}{\sin^2 x} \frac{\tan^2 2x}{1}$

$$= \lim_{x\to 0} \frac{1}{4} \frac{4x}{(\tan 4x)} \frac{x^2}{\sin^2 x} \cdot \frac{\tan^2 2x}{x^2}$$

$$= \lim_{x\to 0} \frac{1}{4} \frac{4x}{(\tan 4x)} \left(\frac{x}{\sin x}\right)^2 \cdot \left(\frac{\tan 2x}{2x}\right)^2 \cdot \frac{4}{1}$$

$$= \frac{1}{4} \cdot 1 \cdot 1 \cdot 1 \cdot \frac{4}{1} = 1 \quad \left[\because \lim_{x\to 0} \frac{x}{\sin x} = 1 = \lim_{x\to 0} \frac{\tan x}{x}\right]$$

7. Clearly,

$$\lim_{y\to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$$

$$= \lim_{y\to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4} \times \frac{\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}}{\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}}$$

[rationalising the numerator]

$$= \lim_{y\to 0} \frac{(1 + \sqrt{1 + y^4}) - 2}{y^4(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2})} \quad [\because (a + b)(a - b) = a^2 - b^2]$$

$$= \lim_{y\to 0} \frac{\sqrt{1 + y^4} - 1}{y^4(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2})} \times \frac{\sqrt{1 + y^4} + 1}{\sqrt{1 + y^4} + 1}$$

[again, rationalising the numerator]

$$= \lim_{y\to 0} \frac{y^4}{y^4(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2})(\sqrt{1 + y^4} + 1)}$$

$$= \frac{1}{2\sqrt{2} \times 2}$$

(by cancelling y^4 and then by direct substitution).

$$= \frac{1}{4\sqrt{2}}.$$

8. $\lim_{x\to\pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3} = \lim_{x\to\pi/2} \frac{1}{8} \cdot \frac{\cos x(1 - \sin x)}{\sin x\left(\frac{\pi}{2} - x\right)^3}$

$$= \lim_{h\to 0} \frac{1}{8} \cdot \frac{\cos\left(\frac{\pi}{2} - h\right)\left[1 - \sin\left(\frac{\pi}{2} - h\right)\right]}{\sin\left(\frac{\pi}{2} - h\right)\left(\frac{\pi}{2} - \frac{\pi}{2} + h\right)^3}$$

$$= \frac{1}{8} \lim_{h\to 0} \frac{\sin h\,(1 - \cos h)}{\cos h \cdot h^3}$$

$$= \frac{1}{8} \lim_{h\to 0} \frac{\sin h\left(2\sin^2\frac{h}{2}\right)}{\cos h \cdot h^3}$$

$$= \frac{1}{4} \lim_{h\to 0} \frac{\sin h \cdot \sin^2\left(\frac{h}{2}\right)}{h^3 \cos h}$$

$$= \frac{1}{4} \lim_{h\to 0} \left(\frac{\sin h}{h}\right)\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 \cdot \frac{1}{\cos h} \cdot \frac{1}{4} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

9. $\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0} \frac{\sin\pi(1 - \sin^2 x)}{x^2}$

$$= \lim_{x\to 0} \frac{\sin(\pi - \pi\sin^2 x)}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{x^2} \quad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= \lim_{x\to 0} \frac{\sin\pi\sin^2 x}{\pi\sin^2 x} \times (\pi)\left(\frac{\sin^2 x}{x^2}\right) = \pi \quad \left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

10. We have, $\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}=\lim_{x\to 0}\frac{2\sin^2 x(3+\cos x)}{x\times\frac{\tan 4x}{4x}\times 4x}$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}\times\lim_{x\to 0}\frac{(3+\cos x)}{4}\times\frac{1}{\lim_{x\to 0}\frac{\tan 4x}{4x}}$$

$$=2\times\frac{4}{4}\times 1 \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \text{ and } \lim_{\theta\to 0}\frac{\tan}{\theta}=1\right]$$

$$=2$$

11. **PLAN** $\left(\frac{\infty}{\infty}\right)$ form

$$\lim_{x\to\infty}\frac{a_0x^n+a_1x^{n-1}+\ldots+a_n}{b_0x^m+b_1x^{m-1}+\ldots+b_m}=\begin{cases}0, & \text{if } n<m\\ \frac{a_0}{b_0}, & \text{if } n=m\\ +\infty, & \text{if } n>m \text{ and } a_0b_0>0\\ -\infty, & \text{if } n>m \text{ and } a_0b_0<0\end{cases}$$

Description of Situation As to make degree of numerator equal to degree of denominator.

$$\therefore \quad \lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{x^2+x+1-ax^2-ax-bx-b}{x+1}=4$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{x^2(1-a)+x(1-a-b)+(1-b)}{x+1}=4$$

Here, we make degree of numerator = degree of denominator

$$\therefore \quad 1-a=0 \quad\Rightarrow\quad a=1$$

and $\lim_{x\to\infty}\frac{x(1-a-b)+(1-b)}{x+1}=4$

$$\Rightarrow \quad 1-a-b=4$$

$$\Rightarrow \quad b=-4 \quad [\because (1-a)=0]$$

12. Here, $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

$[\because f'(2)=6 \text{ and } f'(1)=4, \text{ given}]$

Applying L'Hospital's rule,

$$=\lim_{h\to 0}\frac{\{f'(2h+2+h^2)\}\cdot(2+2h)-0}{\{f'(h-h^2+1)\}\cdot(1-2h)-0}=\frac{f'(2)\cdot 2}{f'(1)\cdot 1}$$

$$=\frac{6\cdot 2}{4\cdot 1}=3 \qquad [\text{using } f'(2)=6 \text{ and } f'(1)=4]$$

13. Given, $\lim_{x\to 0}\frac{\{(a-n)nx-\tan x\}\sin nx}{x^2}=0$

$$\Rightarrow \quad \lim_{x\to 0}\left\{(a-n)n-\frac{\tan x}{x}\right\}\frac{\sin nx}{nx}\times n=0$$

$$\Rightarrow \quad \{(a-n)n-1\}n=0$$

$$\Rightarrow \quad (a-n)n=1$$

$$\Rightarrow \quad a=n+\frac{1}{n}$$

14. $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$

$$=\lim_{x\to 0}\frac{\left(-2\sin^2\frac{x}{2}\right)\left\{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\right\}}{x^n}$$

$$=\lim_{x\to 0}\frac{\left(-2\sin^2\frac{x}{2}\right)\left(-x-\frac{2x^2}{2!}-\frac{x^3}{3!}-\ldots\right)}{4\left(\frac{x}{2}\right)^2 x^{n-2}}$$

$$=\lim_{x\to 0}\frac{\sin^2\frac{x}{2}\left(1+x+\frac{x^2}{3!}+\ldots\right)}{2\left(\frac{x}{2}\right)^2 x^{n-3}}$$

Above limit is finite, if $n-3=0$, i.e. $n=3$.

15. $\lim_{x\to 0}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$

NOTE In trigonometry try to make all trigonometric functions in same angle. It is called 3rd Golden rule of trigonometry.

$$=\lim_{x\to 0}\frac{x\frac{2\tan x}{1-\tan^2 x}-2x\tan x}{(2\sin^2 x)^2}$$

$$=\lim_{x\to 0}\frac{2x\tan x\left[\frac{1}{1-\tan^2 x}-1\right]}{4\sin^4 x}$$

$$=\lim_{x\to 0}\frac{2x\tan x\left[\frac{1-1+\tan^2 x}{1-\tan^2 x}\right]}{4\sin^4 x}$$

$$=\lim_{x\to 0}\frac{2x\tan^3 x}{2\sin^4 x(1-\tan^2 x)}=\lim_{x\to 0}\frac{1}{2}\frac{x\left(\frac{\tan x}{x}\right)^3\cdot x^3}{\sin^4 x(1-\tan^2 x)}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{\left(\frac{\tan x}{x}\right)^3}{\left(\frac{\sin x}{x}\right)^4(1-\tan^2 x)}=\frac{1\cdot(1)^3}{2(1)^4(1-0)}=\frac{1}{2}$$

16. LHL $=\lim_{x\to 1^-}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}$

$$=\lim_{x\to 1^-}\frac{\sqrt{2\sin^2(x-1)}}{x-1}=\sqrt{2}\lim_{x\to 1^-}\frac{|\sin(x-1)|}{x-1}$$

Put $x=1-h, h>0$, for $x\to 1^-, h\to 0$

$$=\sqrt{2}\lim_{h\to 0}\frac{|\sin(-h)|}{-h}$$

$$=\sqrt{2}\lim_{h\to 0}\frac{\sin h}{-h}=-\sqrt{2}$$

Again, RHL $=\lim_{x\to 1^+}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}$

$$=\lim_{x\to 1^+}\sqrt{2}\frac{|\sin(x-1)|}{x-1}$$

Put $x = 1 + h, h > 0$

For $x \to 1^+$, $h \to 0$

$$= \lim_{h \to 0} \sqrt{2}\,\frac{|\sin h|}{h} = \lim_{h \to 0} \sqrt{2}\,\frac{\sin h}{h} = \sqrt{2}$$

$\therefore$ LHL $\neq$ RHL.

Hence, $\lim_{x \to 1} f(x)$ does not exist.

17. $$\lim_{x \to 0} \frac{\sqrt{\frac{1}{2}(1 - \cos^2 x)}}{x} = \lim_{x \to 0} \frac{1}{\sqrt{2}} \cdot \frac{|\sin x|}{x}$$

At $x = 0$

$$\text{RHL} = \lim_{h \to 0} \frac{1}{\sqrt{2}} \cdot \frac{\sin h}{h} = \frac{1}{\sqrt{2}}$$

and $$\text{LHL} = \lim_{h \to 0} \frac{1}{\sqrt{2}} \cdot \frac{\sin h}{-h} = -\frac{1}{\sqrt{2}}$$

Here, RHL $\neq$ LHL

$\therefore$ Limit does not exist.

18. Since, $f(x) = \begin{cases} \dfrac{\sin [x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{\sin [x]}{[x]}, & x \in R - [0, 1) \\ 0, & 0 \leq x < 1 \end{cases}$$

At $x = 0$,

$$\text{RHL} = \lim_{x \to 0^+} 0 = 0$$

and $$\text{LHL} = \lim_{x \to 0^-} \frac{\sin [x]}{[x]} = \lim_{h \to 0} \frac{\sin [0 - h]}{[0 - h]}$$

$$= \lim_{h \to 0} \frac{\sin (-1)}{-1} = \sin 1$$

Since, RHL $\neq$ LHL

$\therefore$ Limit does not exist.

19. $$\lim_{n \to \infty} \left(\frac{1}{1 - n^2} + \frac{2}{1 - n^2} + \ldots + \frac{n}{1 - n^2} \right)$$

$$= \lim_{n \to \infty} \frac{1 + 2 + 3 + \ldots + n}{(1 - n^2)} = \lim_{n \to \infty} \frac{n(n + 1)}{2(1 - n)(1 + n)}$$

$$\Rightarrow \lim_{n \to \infty} \frac{n}{2(1 - n)} = -\frac{1}{2}$$

20. Given, $f(a) = 2, f'(a) = 1, g(a) = -1, g'(a) = 2$

$$\therefore \quad \lim_{x \to a} \frac{g(x) f(a) - g(a) f(x)}{x - a}$$

$$= \lim_{x \to a} \frac{g'(x) f(a) - g(a) f'(x)}{1 - 0},$$

[using L' Hospital's rule]

$$= g'(a) f(a) - g(a) f'(a)$$

$$= 2(2) - (-1)(1) = 5$$

21. Given, $G(x) = -\sqrt{25 - x^2}$

$$\therefore \quad \lim_{x \to 1} \frac{G(x) - G(1)}{x - 1} = \lim_{x \to 1} \frac{G'(x) - 0}{1 - 0}$$

[using L' Hospital's rule]

$$= G'(1) = \frac{1}{\sqrt{24}}$$

$$\left[\because \quad G(x) = -\sqrt{25 - x^2} \Rightarrow G'(x) = \frac{2x}{2\sqrt{25 - x^2}}\right]$$

22. We have,

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1 + (x + j)(x + j - 1)}\right) \text{for all } x \in (0, \infty)$$

$$\Rightarrow \quad f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{(x + j) - (x + j - 1)}{1 + (x + j)(x + j - 1)}\right)$$

$$\Rightarrow \quad f_n(x) = \sum_{j=1}^{n} [\tan^{-1}(x + j) - \tan^{-1}(x + j - 1)]$$

$$\Rightarrow \quad f_n(x) = (\tan^{-1}(x + 1) - \tan^{-1} x) + (\tan^{-1}(x + 2) - \tan^{-1}(x + 1)) + (\tan^{-1}(x + 3) - \tan^{-1}(x + 2)) + \ldots + (\tan^{-1}(x + n) - \tan^{-1}(x + n - 1))$$

$$\Rightarrow \quad f_n(x) = \tan^{-1}(x + n) - \tan^{-1} x$$

This statement is false as $x \neq 0$. i.e., $x \in (0, \infty)$.

(b) This statement is also false as $0 \notin (0, \infty)$

(c) $f_n(x) = \tan^{-1}(x + n) - \tan^{-1} x$

$$\lim_{x \to \infty} \tan(f_n(x)) = \lim_{x \to \infty} \tan(\tan^{-1}(x + n) - \tan^{-1} x)$$

$$\Rightarrow \quad \lim_{x \to \infty} \tan(f_n(x)) = \lim_{x \to \infty} \tan\left(\tan^{-1} \frac{n}{1 + nx + x^2}\right)$$

$$= \lim_{x \to \infty} \frac{n}{1 + nx + x^2} = 0$$

$\therefore$ (c) statement is false.

(d) $$\lim_{x \to \infty} \sec^2(f_n(x)) = \lim_{x \to \infty} (1 + \tan^2 f_n(x))$$

$$= 1 + \lim_{x \to \infty} \tan^2(f_n(x)) = 1 + 0 = 1$$

$\therefore$ (d) statement is true.

23. $$L = \lim_{x \to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}, a > 0$$

$$= \lim_{x \to 0} \frac{a - a \cdot \left[1 - \frac{1}{2} \cdot \frac{x^2}{a^2} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2} \cdot \frac{x^4}{a^4} - \ldots\right] - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2a} + \frac{1}{8} \cdot \frac{x^4}{a^3} + \ldots - \frac{x^2}{4}}{x^4}$$

Since, L is finite

$$\Rightarrow \quad 2a = 4 \Rightarrow a = 2$$

$$\therefore \quad L = \lim_{x \to 0} \frac{1}{8 \cdot a^3} = \frac{1}{64}$$

24. $$\lim_{h \to 0} \frac{\log(1 + 2h) - 2\log(1 + h)}{h^2} \quad \left[\frac{0}{0} \text{ form}\right]$$

Applying L'Hospital's rule, we get

$$= \lim_{h \to 0} \frac{\frac{2}{1 + 2h} - \frac{2}{1 + h}}{2h}$$

$$= \lim_{h \to 0} \frac{2 + 2h - 2 - 4h}{2h(1+2h)(1+h)}$$

$$= \lim_{h \to 0} \frac{-1}{(1+2h)(1+h)} = -1$$

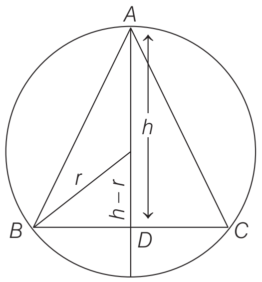
25. Given, $f(x) = \begin{cases} \sin x, & x \neq n\pi, \quad n = 0, \pm 1, \pm 2, \dots \\ 2, & \text{otherwise} \end{cases}$

$$g[f(x)] = \begin{cases} \{f(x)\}^2 + 1 & , f(x) \neq 0, 2 \\ 4 & , f(x) = 0 \\ 5 & , f(x) = 2 \end{cases}$$

$$\therefore \quad g[f(x)] = \begin{cases} (\sin^2 x) + 1, \ x \neq, n\pi = 0, \pm 1, \dots \\ 5, \ x = n\pi \end{cases}$$

Now, $\lim_{x \to 0} g[f(x)] = \lim_{x \to 0} (\sin^2 x) + 1 = 1$

26. Given, $P = 2(\sqrt{2hr - h^2} + \sqrt{2hr})$



Here, $BD = \sqrt{r^2 - (h-r)^2} = \sqrt{2hr - h^2}$

$$\therefore \quad A = \frac{1}{2} \cdot 2BD \cdot h = (\sqrt{2hr - h^2})\, h$$

$$\therefore \lim_{h \to 0} \frac{A}{P^3} = \lim_{h \to 0} \frac{h\sqrt{2hr - h^2}}{8(\sqrt{2hr - h^2} + \sqrt{2hr})^3}$$

$$= \lim_{h \to 0} \frac{h^{3/2}(\sqrt{2r - h})}{8\, h^{3/2}(\sqrt{2r - h} + \sqrt{2r})^3}$$

$$= \frac{1}{8} \cdot \frac{\sqrt{2r}}{(\sqrt{2r} + \sqrt{2r})^3} = \frac{1}{128\, r}$$

27. $\lim_{x \to -\infty} \left(\frac{x^4 \sin \frac{1}{x} + x^2}{1 + |x|^3} \right) = \lim_{x \to -\infty} \frac{x^4 \sin \frac{1}{x} + x^2}{1 - x^3}$

On dividing by x^3, we get

$$\lim_{x \to -\infty} \frac{\frac{\sin(1/x)}{\frac{1}{x}} + \frac{1}{x}}{\frac{1}{x^3} - 1} = \frac{1+0}{0-1} = -1$$

28. $f(x) = \begin{cases} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2} & , \text{if } x \neq 2 \\ k & , \text{if } x = 2 \end{cases}$

Since, continuous at $x = 2$.

$\Rightarrow f(2) = \lim_{x \to 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$, [using L'Hospital's rule]

$$= \lim_{x \to 2} \frac{3x^2 + 2x - 16}{2(x-2)} = \lim_{x \to 2} \frac{6x + 2}{2} = 7$$

$\therefore \quad k = 7$

29. $\lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2}$

Put $\quad x - 1 = y$

$$\therefore \; -\lim_{y \to 0} y \tan \frac{\pi}{2}(y+1) = -\lim_{y \to 0} y\left[-\cot\left(\frac{\pi}{2} y\right)\right]$$

$$= \lim_{y \to 0} \left(\frac{y \frac{\pi}{2}}{\tan \frac{\pi}{2} y} \right) \cdot \frac{2}{\pi} = \frac{2}{\pi}$$

30. If $\lim_{x \to a} [f(x) g(x)]$ exists, then both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ may or may not exist. Hence, it is a false statement.

31. $\lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} = \lim_{x \to 0} \frac{(2^x - 1)(\sqrt{1+x} + 1)}{x}$

$$= \log_e(2) \cdot (2)$$

$$= 2 \log_e 2 = \log_e 4$$

32. Here, $\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

$$= \lim_{h \to 0} \frac{a^2[\sin(a+h) - \sin a]}{h} + \frac{h[2a \sin(a+h) + h \sin(a+h)]}{h}$$

$$= \lim_{h \to 0} \frac{a^2 \cdot 2 \cos\left(a + \frac{h}{2}\right) \cdot \sin \frac{h}{2}}{2 \cdot \frac{h}{2}} + (2a + h) \sin(a + h)$$

$$= a^2 \cos a + 2a \sin a$$

33. $\lim_{x \to 0} \sqrt{\frac{x - \sin x}{x + \cos^2 x}} = \frac{\lim_{x \to 0}(x - \sin x)^{1/2}}{\lim_{x \to 0}(x + \cos^2 x)^{1/2}}$

$$= \frac{\lim_{x \to 0}\left[x\left(1 - \frac{\sin x}{x}\right)\right]^{1/2}}{\lim_{x \to 0}(0+1)^{1/2}}$$

$$= \frac{0 \cdot 0}{1} = 0$$

34. $\lim_{x \to 1} \left\{ \frac{x - 1}{(x-1)(2x-5)} \right\} = \lim_{x \to 1} \frac{1}{(2x-5)}$

$$= -\frac{1}{3}$$

35. Here, $\lim_{x \to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$

$$\Rightarrow \quad \lim_{x \to 0} \frac{x^2\left(\beta x - \frac{(\beta x)^3}{3!} + \frac{(\beta x)^5}{5!} - \dots\right)}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{x^3\left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \dots\right)}{(\alpha-1)x + \frac{x^3}{3!} + \frac{x^5}{5!} - \dots} = 1$$

Limit exists only, when $\alpha - 1 = 0$

$$\Rightarrow \alpha = 1 \quad \dots(i)$$

$$\therefore \lim_{x\to 0} \frac{x^3\left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \dots\right)}{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} - \dots\right)} = 1$$

$$\Rightarrow 6\beta = 1 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$6(\alpha+\beta) = 6\alpha + 6\beta = 6 + 1 = 7$$

36. Given, $\lim_{\alpha\to 0}\left[\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right] = -\frac{e}{2}$

$$\Rightarrow \lim_{\alpha\to 0} \frac{e\{e^{\cos(\alpha^n)-1} - 1\}}{\cos(\alpha^n) - 1} \cdot \frac{\cos(\alpha^n) - 1}{\alpha^m} = \frac{-e}{2}$$

$$\Rightarrow \lim_{\alpha\to 0} e\left\{\frac{e^{\cos(\alpha^n)-1} - 1}{\cos(\alpha^n) - 1}\right\} \cdot \lim_{\alpha\to 0} \frac{-2\sin^2\frac{\alpha^n}{2}}{\alpha^m} = -e/2$$

$$\Rightarrow e \times 1 \times (-2) \lim_{\alpha\to 0} \frac{\sin^2\left(\frac{\alpha^n}{2}\right)}{\frac{\alpha^{2n}}{4}} \cdot \frac{\alpha^{2n}}{4\alpha^m} = \frac{-e}{2}$$

$$\Rightarrow e \times 1 \times -2 \times 1 \times \lim_{\alpha\to 0} \frac{\alpha^{2n-m}}{4} = \frac{-e}{2}$$

For this to be exists, $2n - m = 0$

$$\Rightarrow \frac{m}{n} = 2$$

Topic 2 1^∞ Form, RHL and LHL

1. Let $l = \lim_{x\to 0}\left(\frac{1 + f(3+x) - f(3)}{1 + f(2-x) - f(2)}\right)^{\frac{1}{x}}$ [1^∞ form]

$$\Rightarrow l = e^{\lim_{x\to 0}\frac{1}{x}\left(1 - \frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)}$$

$$= e^{\lim_{x\to 0}\left[\frac{1+f(2-x)-f(2)-1-f(3+x)+f(3)}{x(1+f(2-x)-f(2))}\right]}$$

$$= e^{\lim_{x\to 0}\left[\frac{f(2-x)-f(3+x)+f(3)-f(2)}{x(1+f(2-x)-f(2))}\right]}$$

On applying L'Hopital rule, we get

$$l = e^{\lim_{x\to 0}\left[\frac{-f'(2-x)-f'(3+x)}{1-xf'(2-x)+f(2-x)-f(2)}\right]}$$

On applying limit, we get

$$l = e^{\left(\frac{-f'(2)-f'(3)}{1-0+f(2)-f(2)}\right)} = e^0 = 1$$

So, $\lim_{x\to 0}\left(\frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)^{\frac{1}{x}} = 1$

2. Let $L = \lim_{x\to 1^-} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1}x}}{\sqrt{1-x}}$, then

$$L = \lim_{x\to 1^-} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1}x}}{\sqrt{1-x}} \times \frac{\sqrt{\pi} + \sqrt{2\sin^{-1}x}}{\sqrt{\pi} + \sqrt{2\sin^{-1}x}}$$

[on rationalization]

$$= \lim_{x\to 1^-} \frac{\pi - 2\sin^{-1}x}{\sqrt{1-x}} \times \frac{1}{\sqrt{\pi} + \sqrt{2\sin^{-1}x}}$$

$$= \lim_{x\to 1^-} \frac{\pi - 2\left(\frac{\pi}{2} - \cos^{-1}x\right)}{\sqrt{1-x}} \times \frac{1}{\sqrt{\pi} + \sqrt{2\sin^{-1}x}}$$

$$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$= \lim_{x\to 1^-} \frac{2\cos^{-1}x}{\sqrt{1-x}} \times \lim_{x\to 1^-} \frac{1}{\sqrt{\pi} + \sqrt{2\sin^{-1}x}}$$

$$= \frac{1}{2\sqrt{\pi}} \lim_{x\to 1^-} \frac{2\cos^{-1}x}{\sqrt{1-x}} \quad \left[\because \lim_{x\to 1^-} \sin^{-1}x = \frac{\pi}{2}\right]$$

Put $x = \cos\theta$, then as $x \to 1^-$, therefore $\theta \to 0^+$

Now, $L = \frac{1}{2\sqrt{\pi}} \lim_{\theta\to 0^+} \frac{2\theta}{\sqrt{1-\cos\theta}}$

$$= \frac{1}{2\sqrt{\pi}} \lim_{\theta\to 0^+} \frac{2\theta}{\sqrt{2}\sin\left(\frac{\theta}{2}\right)} \quad \left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{2\sqrt{\pi}} \cdot \sqrt{2} \lim_{\theta\to 0^+} \frac{2\cdot\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{1}{2\sqrt{\pi}} 2\sqrt{2} = \sqrt{\frac{2}{\pi}} \quad \left[\because \lim_{x\to 0^+} \frac{\theta}{\sin\theta} = 1\right]$$

3.

> **Key Idea** $\lim_{x\to a} f(x)$ exist iff
> $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

At $x = 0$,

$$\text{RHL} = \lim_{x\to 0^+} \frac{\tan(\pi\sin^2 x) + (|x| - \sin(x[x]))^2}{x^2}$$

$$= \lim_{x\to 0^+} \frac{\tan(\pi\sin^2 x) + (x - \sin(x\cdot 0))^2}{x^2}$$

$$\left[\begin{array}{c}\because |x| = x \text{ for } x > 0 \\ \text{and } [x] = 0 \text{ for } 0 < x < 1\end{array}\right]$$

$$= \lim_{x\to 0^+} \frac{\tan(\pi\sin^2 x) + x^2}{x^2}$$

$$= \lim_{x\to 0^+} \left(\frac{\tan(\pi\sin^2 x)}{\pi\sin^2 x} \cdot \frac{\pi\sin^2 x}{x^2} + 1\right)$$

$$= \pi \lim_{x\to 0^+} \frac{\tan(\pi \sin^2 x)}{\pi \sin^2 x} \cdot \lim_{x\to 0^+} \frac{\sin^2 x}{x^2} + 1$$

$$= \pi + 1 \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1 \text{ and } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

and LHL

$$= \lim_{x\to 0^-} \frac{\tan(\pi \sin^2 x) + (|x| - \sin(x[x]))^2}{x^2}$$

$$= \lim_{x\to 0^-} \frac{\tan(\pi \sin^2 x) + (-x - \sin(x(-1)))^2}{x^2}$$

$$\left[\because |x| = -x \text{ for } x<0 \text{ and } [x] = -1 \text{ for } -1<x<0\right]$$

$$= \lim_{x\to 0^-} \frac{\tan(\pi \sin^2 x) + (x + \sin(-x))^2}{x^2}$$

$$= \lim_{x\to 0^-} \frac{\tan(\pi \sin^2 x) + (x - \sin x)^2}{x^2} \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= \lim_{x\to 0^-} \left(\frac{\tan(\pi \sin^2 x) + x^2 + \sin^2 x - 2x\sin x}{x^2}\right)$$

$$= \lim_{x\to 0^-} \left(\frac{\tan(\pi \sin^2 x)}{x^2} + 1 + \frac{\sin^2 x}{x^2} - \frac{2x\sin x}{x^2}\right)$$

$$= \lim_{x\to 0^-} \left(\frac{\tan(\pi \sin^2 x)}{\pi \sin^2 x} \cdot \frac{\pi \sin^2 x}{x^2} + 1 + \frac{\sin^2 x}{x^2} - 2\frac{\sin x}{x}\right)$$

$$= \lim_{x\to 0^-} \frac{\tan(\pi \sin^2 x)}{\pi \sin^2 x} \cdot \lim_{x\to 0^-} \frac{\pi \sin^2 x}{x^2} + 1 + \lim_{x\to 0^-}\frac{\sin^2 x}{x^2} - 2\lim_{x\to 0^-}\frac{\sin x}{x}$$

$$= \pi + 1 + 1 - 2 = \pi$$

$\because$ RHL $\neq$ LHL

$\therefore$ Limit does not exist

4. Given,

$$\lim_{x\to 1^+} \frac{(1 - |x| + \sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|\,[1-x]}$$

Put $x = 1 + h$, then

$x \to 1^+ \Rightarrow h \to 0^+$

$$\therefore \lim_{x\to 1^+} \frac{(1 - |x| + \sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|\,[1-x]}$$

$$= \lim_{h\to 0^+} \frac{(1 - |h+1| + \sin|-h|)\sin\left(\frac{\pi}{2}[-h]\right)}{|-h|\,[-h]}$$

$$= \lim_{h\to 0^+} \frac{(1 - (h+1) + \sin h)\sin\left(\frac{\pi}{2}[-h]\right)}{h\,[-h]}$$

$(\because |-h| = h \text{ and } |h+1| = h+1 \text{ as } h>0)$

$$= \lim_{h\to 0^+} \frac{(-h + \sin h)\sin\left(\frac{\pi}{2}(-1)\right)}{h(-1)}$$

$(\because [x] = -1 \text{ for } -1 < x < 0 \text{ and } h \to 0^+ \Rightarrow -h \to 0^-)$

$$= \lim_{h\to 0^+} \frac{(-h + \sin h)}{-h}\sin\left(\frac{-\pi}{2}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{\sin h}{h}\right) - \lim_{h\to 0^+}\left(\frac{h}{h}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{\sin h}{h}\right) - \lim_{h\to 0^+}\left(\frac{h}{h}\right) = 1 - 1 = 0 \left[\because \lim_{h\to 0^+}\frac{\sin h}{h} = 1\right]$$

5. $$\lim_{x\to 0^-} \frac{x([x] + |x|)\sin[x]}{|x|} = \lim_{x\to 0^-} \frac{x([x] - x)\sin[x]}{-x}$$

$(\because |x| = -x, \text{ if } x < 0)$

$$= \lim_{x\to 0^-} \frac{x(-1-x)\sin(-1)}{-x} \qquad (\because \lim_{x\to 0^-}[x] = -1)$$

$$= \lim_{x\to 0^-} \frac{-x(x+1)\sin(-1)}{-x} = \lim_{x\to 0^-}(x+1)\sin(-1)$$

$= (0+1)\sin(-1)$ (by direct substitution)

$= -\sin 1 \qquad (\because \sin(-\theta) = -\sin\theta)$

6. **Key Idea** Use property of greatest integer function $[x] = x - \{x\}$.

$$\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \ldots + \left[\frac{15}{x}\right]\right)$$

We know, $[x] = x - \{x\}$

$$\therefore \qquad \left[\frac{1}{x}\right] = \frac{1}{x} - \left\{\frac{1}{x}\right\}$$

Similarly, $$\left[\frac{n}{x}\right] = \frac{n}{x} - \left\{\frac{n}{x}\right\}$$

$$\therefore \text{Given limit} = \lim_{x\to 0^+} x\left(\frac{1}{x} - \left\{\frac{1}{x}\right\} + \frac{2}{x} - \left\{\frac{2}{x}\right\} + \ldots \frac{15}{x} - \left\{\frac{15}{x}\right\}\right)$$

$$= \lim_{x\to 0^+} (1 + 2 + 3 + \ldots + 15) - x\left(\left\{\frac{1}{x}\right\} + \left\{\frac{2}{x}\right\} + \ldots + \left\{\frac{15}{x}\right\}\right)$$

$$= 120 - 0 = 120 \qquad \left[\because 0 \le \left\{\frac{n}{x}\right\} < 1, \text{ therefore } 0 \le x\left\{\frac{n}{x}\right\} < x \Rightarrow \lim_{x\to 0^+} x\left\{\frac{n}{x}\right\} = 0\right]$$

7. $$f(x) = \frac{1 - x(1 + |1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$$

Now, $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1 - x(1 + 1 - x)}{1-x}\cos\left(\frac{1}{1-x}\right)$$

$$= \lim_{x\to 1^-} (1-x)\cos\left(\frac{1}{1-x}\right) = 0$$

and $$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1 - x(1 - 1 + x)}{x-1}\cos\left(\frac{1}{1-x}\right)$$

$$= \lim_{x\to 1^+} -(x+1)\cdot\cos\left(\frac{1}{x+1}\right), \text{ which does not exist.}$$

8. Given, $p = \lim_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{\frac{1}{2x}}$ (1^∞ form)

$$= e^{\lim_{x \to 0^+} \frac{\tan^2 \sqrt{x}}{2x}} = e^{\frac{1}{2} \lim_{x \to 0^+} \left(\frac{\tan \sqrt{x}}{\sqrt{x}}\right)^2} = e^{\frac{1}{2}}$$

$\therefore \quad \log p = \log e^{\frac{1}{2}} = \frac{1}{2}$

9. **PLAN** To make the quadratic into simple form we should eliminate radical sign.

Description of Situation As for given equation, when $a \to 0$ the equation reduces to identity in x.

i.e. $ax^2 + bx + c = 0, \forall x \in R$ or $a = b = c \to 0$

Thus, first we should make above equation independent from coefficients as 0.

Let $a + 1 = t^6$. Thus, when $a \to 0, t \to 1$.

$\therefore \quad (t^2 - 1) x^2 + (t^3 - 1)x + (t - 1) = 0$

$\Rightarrow \quad (t - 1) \{(t + 1) x^2 + (t^2 + t + 1) x + 1\} = 0$, as $t \to 1$

$$2x^2 + 3x + 1 = 0$$

$\Rightarrow \quad 2x^2 + 2x + x + 1 = 0$

$\Rightarrow \quad (2x + 1)(x + 1) = 0$

Thus, $\quad x = -1, -1/2$

or $\quad \lim_{a \to 0^+} \alpha(a) = -1/2$

and $\quad \lim_{a \to 0^+} \beta(a) = -1$

10. Here, $\lim_{x \to 0} \{1 + x \log (1 + b^2)\}^{1/x}$ $\quad [1^\infty$ form]

$$= e^{\lim_{x \to 0} \{x \log (1 + b^2)\} \cdot \frac{1}{x}}$$

$$= e^{\log (1 + b^2)} = (1 + b^2) \quad \ldots \text{(i)}$$

Given, $\lim_{x \to 0} \{1 + x \log (1 + b^2)\}^{1/x} = 2b \sin^2 \theta$

$\Rightarrow \quad (1 + b^2) = 2b \sin^2 \theta$

$\therefore \quad \sin^2 \theta = \frac{1 + b^2}{2b} \quad \ldots$(ii)

By AM $\ge$ GM, $\frac{b + \frac{1}{b}}{2} \ge \left(b \cdot \frac{1}{b}\right)^{1/2}$

$\Rightarrow \quad \frac{b^2 + 1}{2b} \ge 1 \quad \ldots$(iii)

From Eqs. (ii) and (iii),

$$\sin^2 \theta = 1$$

$\Rightarrow \quad \theta = \pm \frac{\pi}{2}$, as $\theta \in (-\pi, \pi]$

11. Here, $\lim_{x \to 0} (\sin x)^{1/x} + \lim_{x \to 0} \left(\frac{1}{x}\right)^{\sin x}$

$$= 0 + \lim_{x \to 0} e^{\log\left(\frac{1}{x}\right)^{\sin x}} = e^{\lim_{x \to 0} \frac{\log (1/x)}{\operatorname{cosec} x}} \quad \left[\begin{array}{l}\lim_{x \to 0} (\sin x)^{1/x} \to 0 \\ \text{as, (decimal)}^\infty \to 0\end{array}\right]$$

Applying L'Hospital's rule, we get

$$e^{\lim_{x \to 0} \frac{x\left(-\frac{1}{x^2}\right)}{-\operatorname{cosec} x \cot x}} = e^{\lim_{x \to 0} \frac{\sin x}{x} \tan x} = e^0 = 1$$

12. Let $y = \left[\frac{f(1 + x)}{f(1)}\right]^{1/x} \Rightarrow \log y = \frac{1}{x} [\log f(1 + x) - \log f(1)]$

$\Rightarrow \quad \lim_{x \to 0} \log y = \lim_{x \to 0} \left[\frac{1}{f(1 + x)} \cdot f'(1 + x)\right]$

[using L' Hospital's rule]

$$= \frac{f(1)}{f(1)} = \frac{6}{3}$$

$\Rightarrow \log\left(\lim_{x \to 0} y\right) = 2 \quad \Rightarrow \quad \lim_{x \to 0} y = e^2$

13. For $x \in R$, $\lim_{x \to \infty} \left(\frac{x - 3}{x + 2}\right)^x = \lim_{x \to \infty} \frac{(1 - 3/x)^x}{(1 + 2/x)^x} = \frac{e^{-3}}{e^2} = e^{-5}$

14. $\lim_{x \to 0} \left(\frac{1 + 5x^2}{1 + 3x^2}\right)^{1/x^2} = \frac{\lim_{x \to 0} [(1 + 5x^2)^{1/5x^2}]^5}{\lim_{x \to 0} [(1 + 3x^2)^{1/3x^2}]^3} = \frac{e^5}{e^3} = e^2$

15. $\lim_{x \to \infty} \left(\frac{x + 6}{x + 1}\right)^{x + 4} = \lim_{x \to \infty} \left(1 + \frac{5}{x + 1}\right)^{x + 4}$ $\quad [1^\infty$ form]

$$= e^{\lim_{x \to \infty} \frac{5(x + 4)}{x + 1}} = e^5$$

16. $\lim_{x \to 0} \left\{\tan\left(\frac{\pi}{4} + x\right)\right\}^{1/x}$

$$= \lim_{x \to 0} \left\{\frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan \frac{\pi}{4} \tan x}\right\}^{1/x} = \lim_{x \to 0} \left\{\frac{1 + \tan x}{1 - \tan x}\right\}^{1/x}$$

$$= \lim_{x \to 0} \frac{[(1 + \tan x)^{1/\tan x}]^{\tan x/x}}{[(1 - \tan x)^{-1/\tan x}]^{-\tan x/x}} = \frac{e^1}{e^{-1}} = e^2$$

17. **PLAN** $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Given, $\lim_{x \to 1} \left\{\frac{\sin (x - 1) + a(1 - x)}{(x - 1) + \sin (x - 1)}\right\}^{\frac{(1 + \sqrt{x})(1 - \sqrt{x})}{1 - \sqrt{x}}} = \frac{1}{4}$

$$\lim_{x \to 1} \left\{\frac{\frac{\sin (x - 1)}{(x - 1)} - a}{1 + \frac{\sin(x - 1)}{(x - 1)}}\right\}^{1 + \sqrt{x}} = \frac{1}{4}$$

$\Rightarrow \quad \left(\frac{1 - a}{2}\right)^2 = \frac{1}{4} \quad \Rightarrow \quad (a - 1)^2 = 1$

$\Rightarrow \quad a = 2$ or 0

Hence, the maximum value of a is 2.

Topic 3 Squeeze, Newton-Leibnitz's Theorem and Limit Based on Coverting infinite Serie s into Definite Integrals

1. Given α and β are roots of quadratic equation $375x^2 - 25x - 2 = 0$

$$\therefore \quad \alpha + \beta = \frac{25}{375} = \frac{1}{15} \quad \dots \text{(i)}$$

$$\text{and} \quad \alpha\beta = -\frac{2}{375} \quad \dots \text{(ii)}$$

Now, $\lim_{n \to \infty} \sum_{r=1}^{n} \alpha^r + \lim_{n \to \infty} \sum_{r=1}^{n} \beta^r$

$= (\alpha + \alpha^2 + \alpha^3 + \dots + \text{upto infinite terms}) + (\beta + \beta^2 + \beta^3 + \dots + \text{upto infinite terms})$

$$= \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} \quad \left[\because S_\infty = \frac{a}{1-r} \text{ for GP}\right]$$

$$= \frac{\alpha(1-\beta) + \beta(1-\alpha)}{(1-\alpha)(1-\beta)} = \frac{\alpha - \alpha\beta + \beta - \alpha\beta}{1 - \alpha - \beta + \alpha\beta}$$

$$= \frac{(\alpha+\beta) - 2\alpha\beta}{1 - (\alpha+\beta) + \alpha\beta}$$

On substituting the value $\alpha + \beta = \frac{1}{15}$ and $\alpha\beta = \frac{-2}{375}$ from Eqs. (i) and (ii) respectively,
we get

$$= \frac{\frac{1}{15} + \frac{4}{375}}{1 - \frac{1}{15} - \frac{2}{375}} = \frac{29}{375 - 25 - 2} = \frac{29}{348} = \frac{1}{12}$$

2. $$\lim_{x \to \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \pi/4} \frac{f(\sec^2 x)\, 2\sec x \,\sec x \tan x}{2x}$$

[using L' Hospital's rule]

$$= \frac{2f(2)}{\pi/4} = \frac{8}{\pi} f(2)$$

3. Let $I = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}} = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{n\sqrt{1 + (r/n)^2}}$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r/n}{\sqrt{1 + (r/n)^2}}$$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx = [\sqrt{1+x^2}]_0^2 = \sqrt{5} - 1$$

4. Here,

$$f(x) = \lim_{n \to \infty} \left[\frac{n^n (x+n)\left(x + \frac{n}{2}\right)\dots\left(x + \frac{n}{n}\right)}{n!\,(x^2 + n^2)\left(x^2 + \frac{n^2}{4}\right)\dots\left(x^2 + \frac{n^2}{n^2}\right)}\right]^{\frac{x}{n}}, x > 0$$

Taking log on both sides, we get

$$\log_e \{f(x)\} = \lim_{n \to \infty} \log\left[\frac{n^n (x+n)\left(x + \frac{n}{2}\right)\dots\left(x + \frac{n}{n}\right)}{n!(x^2 + n^2)\left(x^2 + \frac{n^2}{4}\right)\dots\left(x^2 + \frac{n^2}{n^2}\right)}\right]^{\frac{x}{n}}$$

$$= \lim_{n \to \infty} \frac{x}{n} \cdot \log\left[\frac{\prod_{r=1}^{n}\left(x + \frac{1}{r/n}\right)}{\prod_{r=1}^{n}\left(x^2 + \frac{1}{(r/n)^2}\right)\prod_{r=1}^{n}(r/n)}\right]$$

$$= x \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \log\left[\frac{x + \frac{n}{r}}{\left(x^2 + \frac{n^2}{r^2}\right)\frac{r}{n}}\right]$$

$$= x \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(\frac{\frac{r}{n} \cdot x + 1}{\frac{r^2}{n^2} \cdot x^2 + 1}\right)$$

Converting summation into definite integration, we get

$$\log_e\{f(x)\} = x\int_0^1 \log\left(\frac{xt+1}{x^2t^2+1}\right)dt$$

Put, $\quad t\,x = z$

$\Rightarrow \quad x\,dt = dz$

$$\therefore \quad \log_e\{f(x)\} = x\int_0^x \log\left(\frac{1+z}{1+z^2}\right)\frac{dz}{x}$$

$$\Rightarrow \quad \log_e\{f(x)\} = \int_0^x \log\left(\frac{1+z}{1+z^2}\right)dz$$

Using Newton-Leibnitz formula, we get

$$\frac{1}{f(x)} \cdot f'(x) = \log\left(\frac{1+x}{1+x^2}\right) \quad \dots \text{(i)}$$

Here, at $x = 1$,

$$\frac{f'(1)}{f(1)} = \log (1) = 0$$

$$\therefore \quad f'(1) = 0$$

Now, sign scheme of $f'(x)$ is shown below

+   −
x=1

∴ At $x = 1$, function attains maximum.
Since, $f(x)$ increases on (0, 1).

$$\therefore \quad f(1) > f(1/2)$$

∴ Option (a) is incorrect.

$$f(1/3) < f(2/3)$$

∴ Option (b) is correct.

Also, $f'(x) < 0$, when $x > 1$

$\Rightarrow f'(2) < 0$

$\therefore$ Option (c) is correct.

Also, $\dfrac{f'(x)}{f(x)} = \log\left(\dfrac{1+x}{1+x^2}\right)$

$$\therefore \quad \frac{f'(3)}{f(3)} - \frac{f'(2)}{f(2)} = \log\left(\frac{4}{10}\right) - \log\left(\frac{3}{5}\right)$$

$$= \log(2/3) < 0$$

$$\Rightarrow \quad \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$$

$\therefore$ Option (d) is incorrect.

5. We have,

$$y_n = \frac{1}{n}[(n+1)(n+2)\ldots(n+n)]^{1/n}$$

and $\lim_{n\to\infty} y_n = L$

$$\Rightarrow L = \lim_{n\to\infty} \frac{1}{n}[(n+1)(n+2)(n+3)\ldots(n+n)]^{1/n}$$

$$\Rightarrow L = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$$

$$\Rightarrow \log L = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)\ldots\log\left(1+\frac{n}{n}\right)\right]$$

$$\Rightarrow \log L = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \underset{\text{II}}{1} \times \underset{\text{I}}{\log(1+x)}\,dx$$

$$\Rightarrow \log L = (x\cdot\log(1+x))_0^1 - \int_0^1\left[\frac{d}{dx}(\log(1+x)\int dx\right]dx$$

[by using integration by parts]

$$\Rightarrow \log L = [x\log(1+x)]_0^1 - \int_0^1\frac{x}{1+x}\,dx$$

$$\Rightarrow \log L = \log 2 - \int_0^1\left(\frac{x+1}{x+1}-\frac{1}{x+1}\right)dx$$

$$\Rightarrow \log L = \log 2 - [x]_0^1 + [\log(x+1)]_0^1$$

$$\Rightarrow \log L = \log 2 - 1 + \log 2 - 0$$

$$\Rightarrow \log L = \log 4 - \log e = \log\frac{4}{e} \Rightarrow L = \frac{4}{e} \Rightarrow$$

$$[L] = \left[\frac{4}{e}\right] = 1$$

6. $\lim_{x\to 0}\dfrac{\int_0^{x^2}\cos^2 t\,dt}{x\sin x}$ $\left[\dfrac{0}{0}\text{ form}\right]$

Applying L'Hospital's rule, we get

$$= \lim_{x\to 0}\frac{\cos^2(x^2)\cdot 2x - 0}{x\cos x+\sin x} = \lim_{x\to 0}\frac{2\cdot\cos^2(x^2)}{\cos x+\dfrac{\sin x}{x}} = \frac{2}{1+1} = 1$$

Topic 4 Continuity at a Point

1. Given function is

$$f(x) = \begin{cases}\dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \neq \dfrac{\pi}{4}\\ k, & x = \dfrac{\pi}{4}\end{cases}$$

$\because$ Function $f(x)$ is continuous, so it is continuous at $x = \dfrac{\pi}{4}$.

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} f(x)$$

$$\Rightarrow \quad k = \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

Put $x = \dfrac{\pi}{4}+h$, when $x\to\dfrac{\pi}{4}$, then $h\to 0$

$$k = \lim_{h\to 0}\frac{\sqrt{2}\cos\left(\frac{\pi}{4}+h\right)-1}{\cot\left(\frac{\pi}{4}+h\right)-1}$$

$$= \lim_{h\to 0}\frac{\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos h - \frac{1}{\sqrt{2}}\sin h\right]-1}{\dfrac{\cot h - 1}{\cot h + 1} - 1}$$

$\left[\because \cos(x+y) = \cos x\cos y - \sin x\sin y \text{ and } \cot(x+y) = \dfrac{\cot x\cot y - 1}{\cot y + \cot x}\right]$

$$= \lim_{h\to 0}\frac{\cos h - \sin h - 1}{\dfrac{-2}{1+\cot h}}$$

$$= \lim_{h\to 0}\left[\frac{(1-\cos h)+\sin h}{2\sin h}(\sin h+\cos h)\right]$$

$$= \lim_{h\to 0}\left[\frac{2\sin^2\frac{h}{2} + 2\sin\frac{h}{2}\cos\frac{h}{2}}{4\sin\frac{h}{2}\cos\frac{h}{2}}(\sin h+\cos h)\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin\frac{h}{2}+\cos\frac{h}{2}}{2\cos\frac{h}{2}}\times(\sin h+\cos h)\right] \Rightarrow k = \frac{1}{2}$$

2. **NOTE** All integers are critical point for greatest integer function.

***Case* I** When $x \in I$

$$f(x) = [x]^2 - [x^2] = x^2 - x^2 = 0$$

***Case* II** When $x \notin I$

If $0 < x < 1$, then $[x] = 0$

and $\quad 0 < x^2 < 1$, then $[x^2] = 0$

Next, if $\quad 1 \le x^2 < 2 \Rightarrow 1 \le x < \sqrt{2}$

$\Rightarrow \quad [x] = 1$ and $[x^2] = 1$

Therefore, $f(x) = [x]^2 - [x^2] = 0$, if $1 \le x < \sqrt{2}$

Therefore, $f(x) = 0$, if $0 \le x < \sqrt{2}$

This shows that $f(x)$ is continuous at $x = 1$.

Therefore, $f(x)$ is discontinuous in $(-\infty, 0) \cup [\sqrt{2}, \infty)$ on many other points. Therefore, (b) is the answer.

3. Given, $f(x) = [\tan^2 x]$

Now, $-45° < x < 45°$

$\Rightarrow \quad \tan(-45°) < \tan x < \tan(45°)$

$\Rightarrow \quad -\tan 45° < \tan x < \tan(45°)$

$\Rightarrow \quad -1 < \tan x < 1$

$\Rightarrow \quad 0 < \tan^2 x < 1$

$\Rightarrow \quad [\tan^2 x] = 0$

i.e. $f(x)$ is zero for all values of x from $x = -45°$ to $45°$. Thus, $f(x)$ exists when $x \to 0$ and also it is continuous at $x = 0$. Also, $f(x)$ is differentiable at $x = 0$ and has a value of zero.

Therefore, (b) is the answer.

4. Here, $f(x) = [x] \cos\left(\dfrac{2x-1}{2}\right)\pi$

$$\therefore \quad f(x) = \begin{cases} -\cos\left(\dfrac{2x-1}{2}\right)\pi & , -1 \le x < 0 \\ 0 & , 0 \le x < 1 \\ \cos\left(\dfrac{2x-1}{2}\right)\pi & , 1 \le x < 2 \\ 2\cos\left(\dfrac{2x-1}{2}\right)\pi & , 2 \le x < 3 \end{cases}$$

which shows RHL = LHL at $x = n \in$ Integer as if $x = 1$

$$\Rightarrow \quad \lim_{x \to 1^+} \cos\left(\frac{2x-1}{2}\right)\pi = 0 \quad \text{and} \quad \lim_{x \to 1^-} 0 = 0$$

Also, $\quad f(1) = 0$

$\therefore$ Continuous at $x = 1$.

Similarly, when $x = 2$,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 0$$

Thus, function is discontinuous at no x.

Hence, option (c) is the correct answer.

5. Given, $f(x) = x(\sqrt{x} + \sqrt{x+1})$

$\Rightarrow f(x)$ would exists when $x \ge 0$ and $x + 1 \ge 0$.

$\Rightarrow f(x)$ would exists when $x \ge 0$.

$\therefore \quad f(x)$ is not continuous at $x = 0$,

because LHL does not exist.

Hence, option (c) is correct.

6. For $f(x)$ to be continuous, we must have

$$f(0) = \lim_{x\to 0} f(x)$$
$$= \lim_{x\to 0} \frac{\log(1+ax) - \log(1-bx)}{x}$$
$$= \lim_{x\to 0} \frac{a\log(1+ax)}{ax} + \frac{b\log(1-bx)}{-bx}$$
$$= a \cdot 1 + b \cdot 1 \qquad \left[\text{using } \lim_{x\to 0} \frac{\log(1+x)}{x} = 1\right]$$
$$= a + b$$

$\therefore \quad f(0) = (a + b)$

7. $f(x) = x\cos(\pi(x + [x]))$

At $x = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\cos(\pi(x + [x]) = 0$$

and $\quad f(x) = 0$

$\therefore$ It is continuous at $x = 0$ and clearly discontinuous at other integer points.

8. **PLAN** If a continuous function has values of opposite sign inside an interval , then it has a root in that interval.

$f, g : [0, 1] \to R$

We take two cases.

***Case* I** Let f and g attain their common maximum value at p.

$\Rightarrow \quad f(p) = g(p)$,

where $p \in [0, 1]$

***Case* II** Let f and g attain their common maximum value at different points.

$\Rightarrow \quad f(a) = M$ and $g(b) = M$

$\Rightarrow \quad f(a) - g(a) > 0$ and $f(b) - g(b) < 0$

$\Rightarrow f(c) - g(c) = 0$ for some $c \in [0, 1]$ as f and g are continuous functions.

$\Rightarrow f(c) - g(c) = 0$ for some $c \in [0, 1]$ for all cases. ...(i)

Option (a) $\Rightarrow f^2(c) - g^2(c) + 3[f(c) - g(c)] = 0$

which is true from Eq. (i).

Option (d) $\Rightarrow f^2(c) - g^2(c) = 0$ which is true from Eq. (i)

Now, if we take $f(x) = 1$ and $g(x) = 1, \ \forall x \in [0, 1]$

Options (b) and (c) does not hold. Hence, options (a) and (d) are correct.

9. $\quad f(2n) = a_n, \ f(2n^+) = a_n$

$f(2n^-) = b_n + 1$

$\Rightarrow \quad a_n - b_n = 1$

$f(2n + 1) = a_n$

$f\{(2n + 1)^-\} = a_n$

$f\{(2n + 1)^+\} = b_{n+1} - 1$

$\Rightarrow \quad a_n = b_{n+1} - 1$ or $a_n - b_{n+1} = -1$

or $\quad a_{n-1} - b_n = -1$

10. Given, $\quad x^2 + y^2 = 4 \Rightarrow y = \sqrt{4 - x^2}$

or $\quad f(x) = \sqrt{4 - x^2}$

11. $$f(x) = \begin{cases} \{1 + |\sin x|\}^{a/|\sin x|} & , \ \pi/6 < x < 0 \\ b & , \ x = 0 \\ e^{\tan 2x/\tan 3x} & , \ 0 < x < \pi/6 \end{cases}$$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ RHL (at $x=0$) = LHL (at $x=0$) = $f(0)$

$\Rightarrow \lim_{h\to 0} e^{\tan 2h/\tan 3h} = \lim_{h\to 0} \{1+|\sin h|\}^{a/|\sin h|} = b$

$\Rightarrow e^{2/3} = e^a = b$

$\therefore a = 2/3$

and $b = e^{2/3}$

12. Since, $f(x)$ is continuous at $x=0$.

$\therefore f(0) = \text{LHL}$

$\Rightarrow a = \lim_{h\to 0} \frac{1-\cos 4h}{h^2}$

$\Rightarrow a = \lim_{h\to 0} \frac{2\sin^2 2h}{h^2} \times \frac{4}{4}$

$\Rightarrow a = 8$

13. Since, $f(x)$ is continuous for $0 \le x \le \pi$

$\therefore \text{RHL}\left(\text{at } x=\frac{\pi}{4}\right) = \text{LHL}\left(\text{at } x=\frac{\pi}{4}\right)$

$\Rightarrow \left(2\cdot\frac{\pi}{4}\cot\frac{\pi}{4}+b\right) = \left(\frac{\pi}{4}+a\sqrt{2}\cdot\sin\frac{\pi}{4}\right)$

$\Rightarrow \frac{\pi}{2}+b = \frac{\pi}{4}+a \Rightarrow a-b=\frac{\pi}{4}$ ...(i)

Also, $\text{RHL}\left(\text{at } x=\frac{\pi}{2}\right) = \text{LHL}\left(\text{at } x=\frac{\pi}{2}\right)$

$\Rightarrow \left(a\cos\frac{2\pi}{2} - b\sin\frac{\pi}{2}\right) = \left(2\cdot\frac{\pi}{2}\cdot\cot\frac{\pi}{2}+b\right)$

$\Rightarrow -a-b=b$

$\Rightarrow a+2b=0$ ...(ii)

On solving Eqs. (i) and (ii), we get

$a=\frac{\pi}{6}$ and $b=-\frac{\pi}{12}$

14. Let $g(x)=ax+b$ be a polynomial of degree one.

$$\Rightarrow f(x) = \begin{cases} ax+b, & x\le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x>0 \end{cases}$$

Since, $f(x)$ is continuous and $f'(1)=f(-1)$

$\therefore$ (LHL at $x=0$) = (RHL at $x=0$)

$\Rightarrow \lim_{x\to 0}(ax+b) = \lim_{x\to 0}\left(\frac{x+1}{x+2}\right)^{1/x}$

$\Rightarrow b=0$ ...(i)

Also, $f'(1)=f(-1)$

$\Rightarrow f(x) = \left(\frac{1+x}{2+x}\right)^{1/x}, x>0$

$\Rightarrow \log f(x) = \frac{1}{x}[\log(1+x)-\log(2+x)]$

On differentiating both sides, we get

$$\frac{f'(x)}{f(x)} = \frac{x\left[\frac{1}{1+x}-\frac{1}{2+x}\right]-1\left[\log\left(\frac{1+x}{2+x}\right)\right]}{x^2}$$

$$\therefore f'(x) = \left(\frac{1+x}{2+x}\right)^{1/x}\left[\frac{\frac{x}{(1+x)(2+x)}-\log\left(\frac{1+x}{2+x}\right)}{x^2}\right]$$

$\Rightarrow f'(1) = \frac{2}{3}\left\{\frac{1}{6}-\log\left(\frac{2}{3}\right)\right\}$

and $f(-1)=-a+b=-a$ [from Eq. (i)]

$\therefore -a = \frac{2}{3}\left(\frac{1}{6}-\log\left(\frac{2}{3}\right)\right)$

$$\text{Thus, } f(x) = \begin{cases} \frac{2}{3}\left(\log\left(\frac{2}{3}\right)-\frac{1}{6}\right)x, & x\le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x>0 \end{cases}$$

Now, to check continuity of $f(x)$ (at $x=0$).

$\text{RHL} = \lim_{x\to 0}\left(\frac{1+x}{2+x}\right)^{1/x} = 0$

$\therefore \text{LHL} = \lim_{x\to 0}\frac{2}{3}\left[\log\left(\frac{2}{3}\right)-\frac{1}{6}\right]x = 0$

Hence, $f(x)$ is continuous for all x.

15. Given, $f(x) = \begin{cases} \frac{\sin(a+1)x+\sin x}{x}, & x<0 \\ c, & x=0 \\ \frac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}}, & x>0 \end{cases}$

is continuous at $x=0$.

$\Rightarrow$ (LHL at $x=0$) = (RHL at $x=0$) = $f(0)$

$\Rightarrow \lim_{x\to 0}\left[\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right]$

$= \lim_{x\to 0}\frac{(1+bx)^{1/2}-1}{bx} = c$

$\Rightarrow (a+1)+1 = \lim_{x\to 0}\frac{bx}{bx}\cdot\frac{1}{\sqrt{1+bx}+1} = c$

$\Rightarrow a+2 = \frac{1}{2} = c$

$\therefore a = -\frac{3}{2}, c=\frac{1}{2}$

and $b\in R$

16. (i) Given, $f_1: R\to R$ and $f_1(x) = \sin(\sqrt{1-e^{-x^2}})$

$\therefore f_1(x)$ is continuous at $x=0$

Now, $f_1'(x) = \cos\sqrt{1-e^{-x^2}}\cdot\frac{1}{2\sqrt{1-e^{-x^2}}}(2xe^{-x^2})$

At $x=0$

$f_1'(x)$ does not exists.

$\therefore f_1(x)$ is not differential at $x=0$

Hence, option (2) for P.

(ii) Given, $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1}x}, & \text{if } x\ne 0 \\ 1, & \text{if } x=0 \end{cases}$

$$\Rightarrow \quad f_2(x) = \begin{cases} \dfrac{-\sin x}{\tan^{-1} x} & x < 0 \\ \dfrac{\sin x}{\tan^{-1} x} & x > 0 \\ 1 & x = 0 \end{cases}$$

Clearly, $f_2(x)$ is not continuous at $x = 0$.

$\therefore$ Option (1)for Q.

(iii) Given, $f_3(x) = [\sin(\log_e(x+2))]$, where [ ] is G.I.F.

and $f_3 : (-1, e^{\pi/2} - 2) \to R$

It is given $\quad -1 < x < e^{\pi/2} - 2$

$\Rightarrow \quad -1 + 2 < x + 2 < e^{\pi/2} - 2 + 2$

$\Rightarrow \quad 1 < x + 2 < e^{\pi/2}$

$\Rightarrow \quad \log_e 1 < \log_e (x+2) < \log_e e^{\pi/2}$

$\Rightarrow \quad 0 < \log_e (x+2) < \dfrac{\pi}{2}$

$\Rightarrow \quad \sin 0 < \sin \log_e(x+2) < \sin \dfrac{\pi}{2}$

$\Rightarrow \quad 0 < \sin \log_e (x+2) < 1$

$\therefore \quad [\sin \log_e (x+2)] = 0$

$\therefore \quad f_3(x) = 0, f'_3(x) = f_3''(x) = 0$

It is differentiable and continuous at $x = 0$.

$\therefore$ Option (4) for R

(iv) Given, $f_4(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Now, $\lim\limits_{x \to 0} f_4(x) = \lim\limits_{x \to 0} x^2 \sin\left(\dfrac{1}{x}\right) = 0$

$$f_4'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

For $x = 0$, $f_4'(x) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$$\Rightarrow \quad f_4'(x) = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$\Rightarrow \quad f_4'(x) = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

Thus, $\quad f_4'(x) = \begin{cases} 2x \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Again, $\lim\limits_{x \to 0} f'(x) = \lim\limits_{x \to 0} \left(2x \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right)\right)$

does not exists.

Since, $\lim\limits_{x \to 0} \cos\left(\dfrac{1}{x}\right)$ does not exists.

Hence, $f'(x)$ is not continuous at $x = 0$.

$\therefore$ Option (3) for S.

Topic 5 Continuity in a Domain

1. Given $\int_6^{f(x)} 4t^3\, dt = (x-2)\, g(x)$

$$\Rightarrow \quad g(x) = \frac{\int_6^{f(x)} 4t^3\, dt}{(x-2)} \quad \text{[provided } x \neq 2]$$

So, $\lim\limits_{x \to 2} g(x) = \lim\limits_{x \to 2} \dfrac{\int_6^{f(x)} 4t^3\, dt}{x-2}$

$$\left[\because \frac{0}{0} \text{ form as } x \to 2 \Rightarrow f(2) = 6\right]$$

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{4(f(x))^3 f'(x)}{1}$$

$$\left[\because \frac{d}{dx} \int_{\phi_1(x)}^{\phi_2(x)} f(t)\, dt = f(\phi_2(x)), \phi_2'(x) - f(\phi_1(x)) \cdot \phi_1'(x)\right]$$

On applying limit, we get

$$\lim_{x \to 2} g(x) = 4(f(2))^3 f'(2) = 4 \times (6)^3 \frac{1}{48},$$

$$\left[\because f(2) = 6 \text{ and } f'(2) = \frac{1}{48}\right]$$

$$= \frac{4 \times 216}{48} = 18$$

2. Given function

$$f(x) = \begin{cases} \dfrac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$$

is continuous at $x =$`0, then

$$f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \quad \ldots \text{(i)}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin(p+1)x + \sin x}{x}$$

$$= p + 1 + 1 = p + 2 \quad \left[\because \lim_{x \to 0} \frac{\sin(ax)}{x} = a\right]$$

and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}$

$$= \lim_{x \to 0^+} \frac{\sqrt{x}[(1+x)^{1/2} - 1]}{x\sqrt{x}}$$

$$= \lim_{x \to 0^+} \frac{\left(1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} x^2 + \ldots - 1\right)}{x}$$

$$\left[\because (1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1(n-2))}{1 \cdot 2 \cdot 3} x^3 + \ldots, |x| < 1\right]$$

$$= \lim_{x\to 0^+}\left(\frac{1}{2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x+\dots\right)=\frac{1}{2}$$

From Eq. (i), we get

$$f(0)=q=\frac{1}{2} \text{ and } \lim_{x\to 0^-} f(x)=p+2=\frac{1}{2}$$

$$\Rightarrow \quad p=-\frac{3}{2}$$

So, $\quad (p, q)=\left(-\frac{3}{2}, \frac{1}{2}\right)$

3. Given function

$$f(x)=\begin{cases} a|\pi - x|+1, & x\le 5\\ b|x-\pi|+3, & x>5\end{cases}$$

and it is also given that $f(x)$ is continuous at $x=5$.

Clearly, $\quad f(5)=a(5-\pi)+1 \quad \dots(i)$

$$\lim_{x\to 5^-} f(x)=\lim_{h\to 0}[a|\pi-(5-h)|+1]$$
$$=a(5-\pi)+1 \quad \dots(ii)$$

and $\lim_{x\to 5^+} f(x)=\lim_{h\to 0}[b|(5+h)-\pi|+3]$
$$=b(5-\pi)+3 \quad \dots(iii)$$

$\because$ Function $f(x)$ is continuous at $x=5$.

$\therefore f(5)=\lim_{x\to 5^+} f(x)=\lim_{x\to 5^-} f(x)$

$\Rightarrow a(5-\pi)+1=b(5-\pi)+3$

$\Rightarrow \quad (a-b)(5-\pi)=2$

$$\Rightarrow \quad a-b=\frac{2}{5-\pi}$$

4. Given function $f(x)=[x]-\left[\frac{x}{4}\right], x\in R$

Now, $\lim_{x\to 4^+} f(x)=\lim_{h\to 0}\left([4+h]-\left[\frac{4+h}{4}\right]\right)$

[$\because$ put $x=4+h$, when $x\to 4^+$, then $h\to 0$]

$$=\lim_{h\to 0}(4-1)=3$$

and $\lim_{x\to 4^-} f(x)=\lim_{h\to 0}\left([4-h]-\left[\frac{4-h}{4}\right]\right)$

[$\because$ put $x=4-h$, when $x\to 4^-$ then $h\to 0$]

$$=\lim_{h\to 0}(3-0)=3$$

and $\quad f(4)=[4]-\left[\frac{4}{4}\right]=4-1=3$

$\because \lim_{x\to 4^-} f(x)=f(4)=\lim_{x\to 4^+} f(x)=3$

So, function $f(x)$ is continuous at $x=4$.

5. Given function $f:[-1,3]\to R$ is defined as

$$f(x)=\begin{cases} |x|+[x], & -1\le x<1\\ x+|x|, & 1\le x<2\\ x+[x], & 2\le x\le 3\end{cases}$$

$$=\begin{cases} -x-1, & -1\le x<0\\ x, & 0\le x<1\\ 2x, & 1\le x<2\\ x+2, & 2\le x<3\\ 6, & x=3\end{cases}$$

[$\because$ if $n\le x<n+1, \forall\, n\in$ Integer, $[x]=n$]

$\because \quad \lim_{x\to 0^-} f(x)=-1\ne f(0) \quad [\because f(0)=0]$

$\because \quad \lim_{x\to 1^-} f(x)=1\ne f(1) \quad [\because f(1)=2]$

$\because \quad \lim_{x\to 2^-} f(x)=4=f(2)=\lim_{x\to 2^+} f(x)=4 \quad [\because f(2)=4]$

and $\quad \lim_{x\to 3^-} f(x)=5\ne f(3) \quad [\because f(3)=6]$

$\therefore$ Function $f(x)$ is discontinuous at points 0, 1 and 3.

6. **Key Idea** A function is said to be continuous if it is continuous at each point of the domain.

We have,

$$f(x)=\begin{cases} 5 & \text{if } x\le 1\\ a+bx & \text{if } 1<x<3\\ b+5x & \text{if } 3\le x<5\\ 30 & \text{if } x\ge 5\end{cases}$$

Clearly, for $f(x)$ to be continuous, it has to be continuous at $x=1$, $x=3$ and $x=5$

[$\because$ In rest portion it is continuous everywhere]

$\therefore \quad \lim_{x\to 1^+}(a+bx)=a+b=5 \quad \dots(i)$

$[\because \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)]$

$\lim_{x\to 5^-}(b+5x)=b+25=30 \quad \dots(ii)$

$[\because \lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)]$

On solving Eqs. (i) and (ii), we get $b=5$ and $a=0$

Now, let us check the continuity of $f(x)$ at $x=3$.

Here, $\quad \lim_{x\to 3^-}(a+bx)=a+3b=15$

and $\quad \lim_{x\to 3^+}(b+5x)=b+15=20$

Hence, for $a=0$ and $b=5$, $f(x)$ is not continuous at $x=3$

$\therefore f(x)$ cannot be continuous for any values of a and b.

7. Given, $f(x)=\frac{1}{2}x-1$ for $0\le x\le\pi$

$$\therefore \quad [f(x)]=\begin{cases} -1, & 0\le x<2\\ 0, & 2\le x\le\pi\end{cases}$$

$$\Rightarrow \quad \tan[f(x)]=\begin{cases} \tan(-1), & 0\le x<2\\ \tan 0, & 2\le x\le\pi\end{cases}$$

$$\therefore \quad \lim_{x\to 2^-}\tan[f(x)]=-\tan 1$$

and $\lim_{x \to 2^+} \tan [f(x)] = 0$

So, $\tan f(x)$ is not continuous at $x = 2$.

Now, $f(x) = \frac{1}{2}x - 1 \Rightarrow f(x) = \frac{x-2}{2} \Rightarrow \frac{1}{f(x)} = \frac{2}{x-2}$

Clearly, $1/f(x)$ is not continuous at $x = 2$.

So, $\tan [f(x)]$ and $\left[\frac{1}{f(x)}\right]$ are both discontinuous at $x = 2$.

8. The function $f(x) = \tan x$ is not defined at $x = \frac{\pi}{2}$, so $f(x)$ is not continuous on $(0, \pi)$.

(b) Since, $g(x) = x \sin \frac{1}{x}$ is continuous on $(0, \pi)$ and the integral function of a continuous function is continuous,

$\therefore \quad f(x) = \int_0^x t\left(\sin \frac{1}{t}\right) dt$ is continuous on $(0, \pi)$.

(c) Also, $f(x) = \begin{cases} 1, & 0 < x \le \frac{3\pi}{4} \\ 2\sin\left(\frac{2x}{9}\right), & \frac{3\pi}{4} < x < \pi \end{cases}$

We have, $\lim_{x \to \frac{3\pi}{4}^-} f(x) = 1$

$$\lim_{x \to \frac{3\pi}{4}^+} f(x) = \lim_{x \to \frac{3\pi}{4}} 2\sin\left(\frac{2x}{9}\right) = 1$$

So, $f(x)$ is continuous at $x = 3\pi/4$.

$\Rightarrow \quad f(x)$ is continuous at all other points.

(d) Finally, $f(x) = \frac{\pi}{2}\sin(x + \pi) \Rightarrow f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$

$$\lim_{x \to \left(\frac{\pi}{2}\right)^-} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right) = \lim_{h \to 0} \frac{\pi}{2}\sin\left(\frac{3\pi}{2} - h\right) = \frac{\pi}{2}$$

and $\lim_{x \to (\pi/2)^+} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right)$

$$= \lim_{h \to 0} \frac{\pi}{2}\sin\left(\frac{3\pi}{2} + h\right) = \frac{\pi}{2}$$

So, $f(x)$ is not continuous at $x = \pi/2$.

9. We have, for $-1 < x < 1$

$\Rightarrow \quad 0 \le x \sin \pi x \le 1/2$

$\therefore \quad [x \sin \pi x] = 0$

Also, $x \sin \pi x$ becomes negative and numerically less than 1 when x is slightly greater than 1 and so by definition of $[x]$.

$$f(x) = [x \sin \pi x] = -1, \text{ when } 1 < x < 1 + h$$

Thus, $f(x)$ is constant and equal to 0 in the closed interval $[-1, 1]$ and so $f(x)$ is continuous and differentiable in the open interval $(-1, 1)$.

At $x = 1$, $f(x)$ is discontinuous, since $\lim_{h \to 0} (1 - h) = 0$

and $\lim_{h \to 0} (1 + h) = -1$

$\therefore f(x)$ is not differentiable at $x = 1$.

Hence, (a), (b) and (d) are correct answers.

10. $f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$

We know that, $[x]$ is continuous on $R \sim I$, where I denotes the set of integers and $\sin\left(\frac{\pi}{[x+1]}\right)$ is discontinuous for $[x + 1] = 0$.

$\Rightarrow \quad 0 \le x + 1 < 1 \Rightarrow -1 \le x < 0$

Thus, the function is defined in the interval.

11. Given, $f(x) = \begin{cases} \frac{x^2}{2}, & 0 \le x < 1 \\ 2x^2 - 3x + \frac{3}{2}, & 1 \le x \le 2 \end{cases}$ ...(i)

Clearly, RHL (at $x = 1$) $= 1/2$ and LHL (at $x = 1$) $= 1/2$

Also, $f(x) = 1/2$

$\therefore f(x)$ is continuous for all $x \in [0, 2]$.

On differentiating Eq. (i), we get

$$f'(x) = \begin{cases} x, & 0 \le x < 1 \\ 4x - 3, & 1 \le x \le 2 \end{cases} \quad \text{...(ii)}$$

Clearly, RHL (at $x = 1$) for $f'(x) = 1$
and LHL (at $x = 1$) for $f'(x) = 1$
Also, $f(1) = 1$
Thus, $f'(x)$ is continuous for all $x \in [0, 2]$.
Again, differentiating Eq. (ii), we get

$$f''(x) = \begin{cases} 1, & 0 \le x < 1 \\ 4, & 1 \le x \le 2 \end{cases}$$

Clearly, RHL (at $x = 1$) $\ne$ LHL (at $x = 1$)
Thus, $f''(x)$ is not continuous at $x = 1$.
or $f''(x)$ is continuous for all $x \in [0, 2] - \{1\}$.

Topic 6 Continuity for Composition and Function

1. Given, $f(x) = x \cos \frac{1}{x}$, $x \ge 1 \Rightarrow f'(x) = \frac{1}{x}\sin\frac{1}{x} + \cos\frac{1}{x}$

$\Rightarrow \quad f''(x) = -\frac{1}{x^3}\cos\left(\frac{1}{x}\right)$

Now, $\lim_{x \to \infty} f'(x) = 0 + 1 = 1 \Rightarrow$ Option (b) is correct.

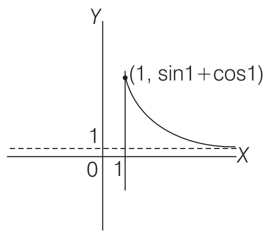
Now, $x \in [1, \infty) \Rightarrow \frac{1}{x} \in (0, 1] \Rightarrow f''(x) < 0$

Option (d) is correct.

As $f'(1) = \sin 1 + \cos 1 > 1$

$f'(x)$ is strictly decreasing and $\lim_{x \to \infty} f'(x) = 1$

So, graph of $f'(x)$ is shown as below.



Now, in $[x, x+2]$, $x \in [1, \infty)$, $f(x)$ is continuous and differentiable so by LMVT,

$$f'(x) = \frac{f(x+2) - f(x)}{2}$$

As, $f'(x) > 1$

For all $x \in [1, \infty)$

$$\Rightarrow \frac{f(x+2) - f(x)}{2} > 1 \Rightarrow f(x+2) - f(x) > 2$$

For all $x \in [1, \infty)$

2. $$gof(x) = \begin{cases} f(x) + 1, & \text{if } f(x) < 0 \\ \{f(x) - 1\}^2 + b, & \text{if } f(x) \ge 0 \end{cases}$$

$$= \begin{cases} x + a + 1, & \text{if } x < -a \\ (x + a - 1)^2 + b, & \text{if } -a \le x < 0 \\ (|x - 1| - 1)^2 + b, & \text{if } x \ge 0 \end{cases}$$

As $gof(x)$ is continuous at $x = -a$

$$gof(-a) = gof(-a^+) = gof(-a^-)$$

$$\Rightarrow 1 + b = 1 + b = 1 \Rightarrow b = 0$$

Also, $gof(x)$ is continuous at $x = 0$

$$\Rightarrow gof(0) = gof(0^+) = gof(0^-)$$

$$\Rightarrow b = b = (a-1)^2 + b \Rightarrow a = 1$$

Hence, $$gof(x) = \begin{cases} x + 2, & \text{if } x < -1 \\ x^2, & \text{if } -1 \le x < 0 \\ (|x - 1| - 1)^2, & \text{if } x \ge 0 \end{cases}$$

In the neighbourhood of $x = 0$, $gof(x) = x^2$, which is differentiable at $x = 0$.

3. As, $f(x)$ is continuous and $g(x)$ is discontinuous.

***Case* I** $g(x)$ is discontinuous as limit does not exist at $x = k$.

$\therefore \quad \phi(x) = f(x) + g(x)$

$\Rightarrow \lim_{x \to k} \phi(x) = \lim_{x \to k} \{f(x) + g(x)\}$ = does not exist.

$\therefore$ $\phi(x)$ is discontinuous.

***Case* II** $g(x)$ is discontinuous as, $\lim_{x \to k} g(x) \ne g(k)$.

$\therefore \quad \phi(x) = f(x) + g(x)$.

$\Rightarrow \lim_{x \to k} \phi(x) = \lim_{x \to k} \{f(x) + g(x)\}$ = exists and is a finite quantity

but $\quad \phi(k) = f(k) + g(k) \ne \lim_{x \to k} \{f(x) + g(x)\}$

$\therefore \quad \phi(x) = f(x) + g(x)$ is discontinuous, whenever $g(x)$ is discontinuous.

4. Given, $$f(x) = \begin{cases} 1 + x, & 0 \le x \le 2 \\ 3 - x, & 2 < x \le 3 \end{cases}$$

$$\therefore \quad fof(x) = f[f(x)] = \begin{cases} 1 + f(x), & 0 \le f(x) \le 2 \\ 3 - f(x), & 2 < f(x) \le 3 \end{cases}$$

$$\Rightarrow fof = \begin{cases} 1 + f(x), & 0 \le f(x) \le 1 \\ 1 + f(x), & 1 < f(x) \le 2 \\ 3 - f(x), & 2 < f(x) \le 3 \end{cases} = \begin{cases} 1 + (3 - x), & 2 < x \le 3 \\ 1 + (1 + x), & 0 \le x \le 1 \\ 3 - (1 + x), & 1 < x \le 2 \end{cases}$$

$$\Rightarrow (fof)(x) = \begin{cases} 4 - x, & 2 < x \le 3 \\ 2 + x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$$

Now, RHL (at $x = 2$) $= 2$ and LHL (at $x = 2$) $= 0$

Also, RHL (at $x = 1$) $= 1$ and LHL (at $x = 1$) $= 3$

Therefore, $f(x)$ is discontinuous at $x = 1, 2$

$\therefore$ $f[f(x)]$ is discontinuous at $x = \{1, 2\}$.

5. Since, $f(x)$ is continuous at $x = 0$.

$$\Rightarrow \lim_{x \to 0} f(x) = f(0)$$

$$\Rightarrow f(0^+) = f(0^-) = f(0) = 0 \qquad \text{...(i)}$$

To show, continuous at $x = k$

$$\text{RHL} = \lim_{h \to 0} f(k + h) = \lim_{h \to 0} [f(k) + f(h)] = f(k) + f(0^+)$$

$$= f(k) + f(0)$$

$$\text{LHL} = \lim_{h \to 0} f(k - h) = \lim_{h \to 0} [f(k) + f(-h)]$$

$$= f(k) + f(0^-) = f(k) + f(0)$$

$$\therefore \quad \lim_{x \to k} f(x) = f(k)$$

$\Rightarrow$ $f(x)$ is continuous for all $x \in R$.

Topic 7 Differentiability at a Point

1. Given function, $g(x) = |f(x)|$

where $f: R \to R$ be differentiable at $c \in R$ and $f(c) = 0$, then for function 'g' at $x = c$

$$g'(c) = \lim_{h \to 0} \frac{g(c + h) - g(c)}{h} \quad [\text{where } h > 0]$$

$$= \lim_{h \to 0} \frac{|f(c + h)| - |f(c)|}{h} = \lim_{h \to 0} \frac{|f(c + h)|}{h}$$

[as $f(c) = 0$ (given)]

$$= \lim_{h \to 0} \left| \frac{f(c + h) - f(c)}{h} \right| \quad [\because h > 0]$$

$$= \left| \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \right|$$

$$= |f'(c)| \quad [\because f \text{ is differentiable at } x = c]$$

Now, if $f'(c) = 0$, then $g(x)$ is differentiable at $x = c$, otherwise LHD (at $x = c$) and RHD (at $x = c$) is different.

2.

> **Key Idea** (i) First use L' Hopital rule
>
> (ii) Now, use formula
>
> $$\frac{d}{dx} \int_{\phi_1(x)}^{\phi_2(x)} f(t)\,dt = f[\phi_2(x)] \cdot \phi'_2(x) - f[\phi_1(x)] \cdot \phi'_1(x)$$

Let $$l = \lim_{x \to 2} \int_6^{f(x)} \frac{2t\,dt}{(x - 2)} = \lim_{x \to 2} \frac{\int_6^{f(x)} 2t\,dt}{(x - 2)} \quad \left[\frac{0}{0} \text{ form, as } f(2) = 6\right]$$

On applying the L' Hopital rule, we get

$$l = \lim_{x \to 2} \frac{2f(x)f'(x)}{1} \quad \left[\because \frac{d}{dx} \int_{\phi_1(x)}^{\phi_2(x)} f(t)\,dt = f(\phi_2(x)) \cdot \phi_2'(x) - f(\phi_1(x)) \cdot \phi_1'(x)\right]$$

So, $l = 2f(2) \cdot f'(2) = 12f'(2)$ $\quad [\because f(2) = 6]$

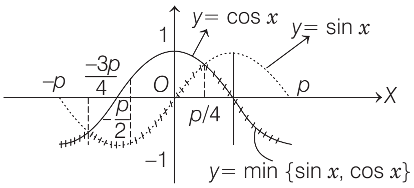
$$\therefore \lim_{x \to 2} \int_6^{f(x)} \frac{2t\,dt}{x - 2} = 12f'(2)$$

3. Given function is $f(x)=15-|x-10|, x\in\mathbf{R}$ and

$$g(x)=f(f(x))$$
$$=f(15-|x-10|)$$
$$=15-|15-|x-10|-10|$$
$$=15-|5-|x-10||$$
$$=\begin{cases}15-|5-(x-10)| & ,x\ge 10\\ 15-|5+(x-10)| & ,x<10\end{cases}$$
$$=\begin{cases}15-|15-x| & ,x\ge 10\\ 15-|x-5| & ,x<10\end{cases}$$
$$=\begin{cases}15+(x-5)=10+x, & x<5\\ 15-(x-5)=20-x, & 5\le x<10\\ 15+(x-15)=x, & 10\le x<15\\ 15-(x-15)=30-x, & x\ge 15\end{cases}$$

From the above definition it is clear that $g(x)$ is not differentiable at $x=5, 10, 15$.

4. Let us draw the graph of $y=f(x)$, as shown below



Clearly, the function $f(x)=\min\{\sin x,\cos x\}$ is not differentiable at $x=\frac{-3\pi}{4}$ and $\frac{\pi}{4}$ [these are point of intersection of graphs of sin x and cos x in $(-\pi,\pi)$, on which function has sharp edges]. So, $S=\left\{\frac{-3\pi}{4},\frac{\pi}{4}\right\}$, which is a subset of $\left\{\frac{-3\pi}{4},\frac{-\pi}{4},\frac{3\pi}{4},\frac{\pi}{4}\right\}$

5. We have,

$$f(x)=\sin|x|-|x|+2(x-\pi)\cos|x|$$
$$f(x)=\begin{cases}-\sin x+x+2(x-\pi)\cos x, & \text{if } x<0\\ \sin x-x+2(x-\pi)\cos x, & \text{if } x\ge 0\end{cases}$$

[$\because \sin(-\theta)=-\sin\theta$ and $\cos(-\theta)=\cos\theta$]

$$\therefore\ f'(x)=\begin{cases}-\cos x+1+2\cos x-2(x-\pi)\sin x; & \text{if } x<0\\ \cos x-1+2\cos x-2(x-\pi)\sin x, & \text{if } x>0\end{cases}$$

Clearly, $f(x)$ is differentiable everywhere except possibly at $x=0$

[$\because f'(x)$ exist for $x<0$ and $x>0$]

Here, $Rf'(0)=\lim\limits_{x\to 0^+}(3\cos x-1-2(x-\pi)\sin x)$
$=3-1-0=2$

and $Lf'(0)=\lim\limits_{x\to 0^-}(\cos x+1-2(x-\pi)\sin x)$
$=1+1-0=2$

$\because\ Rf'(0)=Lf'(0)$

So, $f(x)$ is differentiable at all values of x.

$\Rightarrow\ K=\phi$

6. **Key Idea** This type of problem can be solved graphically.

We have, $f(x)=\begin{cases}-1, & -2\le x<0\\ x^2-1, & 0\le x\le 2\end{cases}$

and $g(x)=|f(x)|+f(|x|)$

Clearly, $|f(x)|=\begin{cases}1, & -2\le x<0\\ |x^2-1|, & 0\le x\le 2\end{cases}$

$$=\begin{cases}1, & -2\le x<0\\ -(x^2-1), & 0\le x<1\\ x^2-1, & 1\le x\le 2\end{cases}$$

and $f(|x|)=|x|^2-1,\ 0\le|x|\le 2$

[$\because f(|x|)=-1$ is not possible as $|x|\nless 0$]

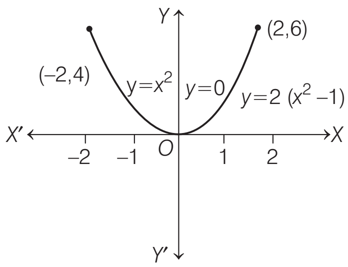
$=x^2-1,\ |x|\le 2$ [$\because |x|^2=x^2$]

$=x^2-1,\ -2\le x\le 2$

$\therefore\ g(x)=|f(x)|+f(|x|)$

$$=\begin{cases}1+x^2-1, & -2\le x<0\\ -(x^2-1)+x^2-1, & 0\le x<1\\ x^2-1+x^2-1, & 1\le x\le 2\end{cases}$$
$$=\begin{cases}x^2, & -2\le x<0\\ 0, & 0\le x<1\\ 2(x^2-1), & 1\le x\le 2\end{cases}$$

Now, let us draw the graph of $y=g(x)$, as shown in the figure.



[Here, $y=2(x^2-1)$ or $x^2=\frac{1}{2}(y+2)$ represent a parabola with vertex $(0,-2)$ and it open upward]

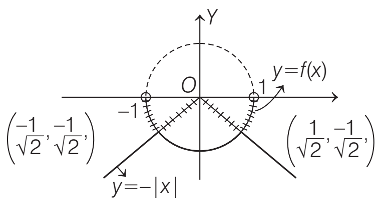
Note that there is a sharp edge at $x=1$ only, so $g(x)$ is not differentiable at $x=1$ only.

7. **Key Idea** This type of questions can be solved graphically.

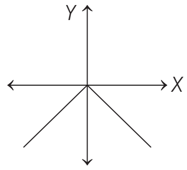
Given, $f:(-1,1)\longrightarrow R$, such that

$$f(x)=\max\left\{-|x|,-\sqrt{1-x^2}\right\}$$

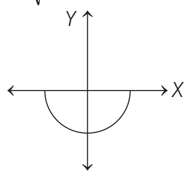
On drawing the graph, we get the follwong figure.



[$\because$ graph of $y = -|x|$ is



and graph of $y = -\sqrt{1-x^2}$



\[$\because$ $x^2 + y^2 = 1$ represent a complete circle]

$$\Rightarrow f(x) = \begin{cases} -\sqrt{1-x^2}, & -1 < x \le -\frac{1}{\sqrt{2}} \\ -|x|, & -\frac{1}{\sqrt{2}} < x \le \frac{1}{\sqrt{2}} \\ -\sqrt{1-x^2}, & \frac{1}{\sqrt{2}} < x < 1 \end{cases}$$

From the figure, it is clear that function have sharp edges, at $x = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

$\therefore$ Function is not differentiable at 3 points.

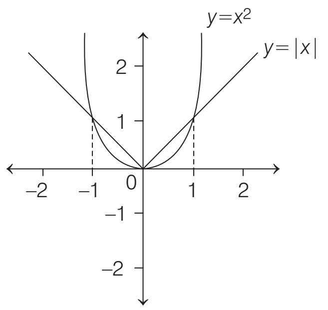
8. **Key Idea** This type of problem can be solved graphically

We have, $f(x) = \begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$

Let us draw the graph of $y = f(x)$

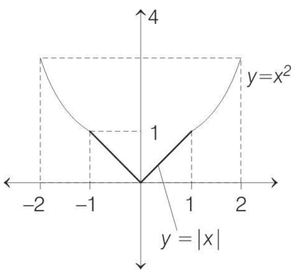
For $|x| \le 2$ $f(x) = \max\{|x|, x^2\}$

Let us first draw the graph of $y = |x|$ and $y = x^2$ as shown in the following figure.



Clearly, $y = |x|$ and $y = x^2$ intersect at $x = -1, 0, 1$

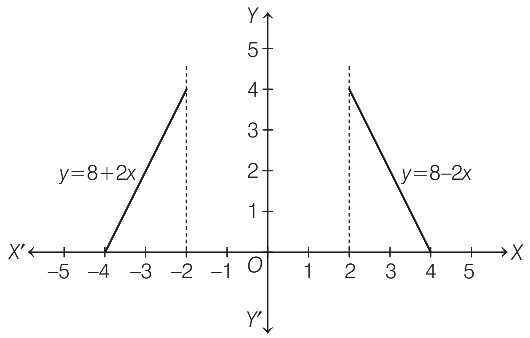
Now, the graph of $y = \max\{|x|, x^2\}$ for $|x| \le 2$ is



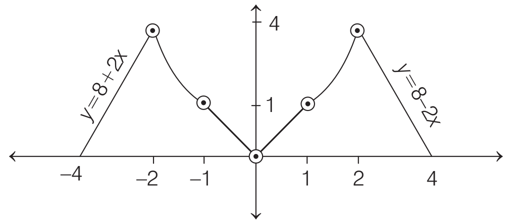
For $|x| \in (2, 4]$

$$f(x) = 8 - 2|x| = \begin{cases} 8 - 2x, & x \in (2, 4] \\ 8 + 2x, & x \in [-4, -2) \end{cases}$$

$$\left[\begin{array}{c} \because 2 < |x| \le 4 \\ \Rightarrow |x| > 2 \text{ and } |x| \le 4 \end{array}\right]$$



Hence, the graph of $y = f(x)$ is



From the graph it is clear that at $x = -2, -1, 0, 1, 2$ the curve has sharp edges and hence at these points f is not differentiable.

9. Given, $|f(x) - f(y)| \le 2|x - y|^{\frac{3}{2}}, \forall x, y \in R$

$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \le 2|x - y|^{\frac{1}{2}}$$

(dividing both sides by $|x - y|$)

Put $x = x + h$ and $y = x$, where h is very close to zero.

$$\Rightarrow \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{(x+h) - x} \right| \le \lim_{h \to 0} 2|(x+h) - x|^{\frac{1}{2}}$$

$$\Rightarrow \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \le \lim_{h \to 0} 2|h|^{\frac{1}{2}}$$

$$\Rightarrow \left| \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right| \le 0$$

[substituting limit directly on right hand side and using $\lim_{x \to a} |f(x)| = \left| \lim_{x \to a} f(x) \right|$]

$\Rightarrow |f'(x)| \le 0$ $\left(\because \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)\right)$

$\Rightarrow |f'(x)| = 0$ ($\because |f'(x)|$ can not be less than zero)

$\Rightarrow f'(x) = 0$ [$\because |x| = 0 \Leftrightarrow x = 0$]

$\Rightarrow f(x)$ is a constant function.

Since, $f(0) = 1$, therefore $f(x)$ is always equal to 1.

Now, $\int_0^1 (f(x))^2 dx = \int_0^1 dx = [x]_0^1 = (1 - 0) = 1$

10. We have,

$$f(x) = |x-\pi| \cdot (e^{|x|}-1)\sin|x|$$

$$f(x) = \begin{cases} (x-\pi)(e^{-x}-1)\sin x, & x<0 \\ -(x-\pi)(e^{x}-1)\sin x, & 0 \le x < \pi \\ (x-\pi)(e^{x}-1)\sin x, & x \ge \pi \end{cases}$$

We check the differentiability at $x=0$ and π.

We have,

$$f'(x) = \begin{cases} (x-\pi)(e^{-x}-1)\cos x + (e^{-x}-1)\sin x \\ \quad + (x-\pi)\sin x e^{-x}(-1), x<0 \\ -[(x-\pi)(e^{x}-1)\cos x + (e^{x}-1)\sin x \\ \quad + (x-\pi)\sin x e^{x}], 0<x<\pi \\ (x-\pi)(e^{x}-1)\cos x + (e^{x}-1)\sin x \\ \quad + (x-\pi)\sin x e^{x}, x>\pi \end{cases}$$

Clearly,

$$\lim_{x\to 0^-} f'(x) = 0 = \lim_{x\to 0^+} f'(x)$$

and $$\lim_{x\to \pi^-} f'(x) = 0 = \lim_{x\to \pi^+} f'(x)$$

$\therefore$ f is differentiable at $x=0$ and $x=\pi$

Hence, f is differentiable for all x.

11. We have, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x)), x \in R$

Note that, for $x \to 0$, $\log 2 > \sin x$

$\therefore$ $f(x) = \log 2 - \sin x$

$\Rightarrow$ $g(x) = \log 2 - \sin(f(x))$

$= \log 2 - \sin(\log 2 - \sin x)$

Clearly, $g(x)$ is differentiable at $x=0$ as $\sin x$ is differentiable.

Now, $g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$

$= \cos x \cdot \cos(\log 2 - \sin x)$

$\Rightarrow$ $g'(0) = 1 \cdot \cos(\log 2)$

12. Given, $f(0) = 2 = g(1), g(0) = 0$ and $f(1) = 6$

f and g are differentiable in (0, 1).

Let $h(x) = f(x) - 2g(x)$ ...(i)

$h(0) = f(0) - 2g(0) = 2 - 0 = 2$

and $h(1) = f(1) - 2g(1) = 6 - 2(2) = 2$

$h(0) = h(1) = 2$

Hence, using Rolle's theorem,

$h'(c) = 0$, such that $c \in (0,1)$

Differentiating Eq. (i) at c, we get

$\Rightarrow$ $f'(c) - 2g'(c) = 0$

$\Rightarrow$ $f'(c) = 2g'(c)$

13. **PLAN** To check differentiability at a point we use RHD and LHD at a point and if RHD = LHD, then $f(x)$ is differentiable at the point.

Description of Situation

As, $$R\{f'(x)\} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

and $$L\{f'(x)\} = \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h}$$

Here, students generally gets confused in defining modulus. To check differentiable at $x=0$,

$$R\{f'(0)\} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{h} = \lim_{h\to 0} h \cdot \left|\cos\frac{\pi}{h}\right| = 0$$

$$L\{f'(0)\} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{h^2\left|\cos\left(-\frac{\pi}{h}\right)\right| - 0}{-h} = 0$$

So, $f(x)$ is differentiable at $x=0$.

To check differentiability at $x=2$,

$$R\{f'(2)\} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2\left|\cos\left(\frac{\pi}{2+h}\right)\right| - 0}{h} = \lim_{h\to 0} \frac{(2+h)^2 \cdot \cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2+h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2 \cdot \sin\left(\frac{\pi h}{2(2+h)}\right)}{h \cdot \frac{\pi}{2(2+h)}} \cdot \frac{\pi}{2(2+h)} = \pi$$

and $$L\{f'(2)\} = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 \cdot \left|\cos\frac{\pi}{2-h}\right| - 2^2 \cdot \left|\cos\frac{\pi}{2}\right|}{-h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 - \left(-\cos\frac{\pi}{2-h}\right) - 0}{-h}$$

$$= \lim_{h\to 0} \frac{-(2-h)^2 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2-h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 \cdot \sin\left(-\frac{\pi h}{2(2-h)}\right)}{h \times \frac{-\pi}{2(2-h)}} \times \frac{-\pi}{2(2-h)} = -\pi$$

Thus, $f(x)$ is differentiable at $x=0$ but not at $x=2$.

14. Given, $g(x) = \dfrac{(x-1)^n}{\log \cos^m(x-1)}$; $0 < x < 2, m \ne 0$, n are integers and $|x-1| = \begin{cases} x-1, & x \ge 1 \\ 1-x, & x < 1 \end{cases}$

The left hand derivative of $|x-1|$ at $x=1$ is $p=-1$.

Also, $\lim_{x\to 1^+} g(x) = p = -1$

$$\Rightarrow \lim_{h\to 0} \frac{(1+h-1)^n}{\log \cos^m(1+h-1)} = -1$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{h^n}{\log \cos^m h} = -1$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{h^n}{m \log \cos h} = -1$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{n \cdot h^{n-1}}{m \dfrac{1}{\cos h}(-\sin h)} = -1$$

[using L'Hospital's rule]

$$\Rightarrow \lim_{h\to 0}\left(-\frac{n}{m}\right)\cdot \frac{h^{n-2}}{\left(\dfrac{\tan h}{h}\right)} = -1 \Rightarrow \left(\frac{n}{m}\right)\lim_{h\to 0}\frac{h^{n-2}}{\left(\dfrac{\tan h}{h}\right)} = 1$$

$$\Rightarrow \quad n = 2 \quad \text{and} \quad \frac{n}{m} = 1 \quad \Rightarrow \quad m = n = 2$$

15. Given, $f(1) = f\left(\frac{1}{2}\right) = f\left(\frac{1}{3}\right) = \ldots = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0$

as $\quad f\left(\frac{1}{n}\right) = 0$; $n \in$ integers and $n \geq 1$.

$$\Rightarrow \quad \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0 \quad \Rightarrow \quad f(0) = 0$$

Since, there are infinitely many points in neighbourhood of $x = 0$.

$\therefore \quad f(x) = 0$

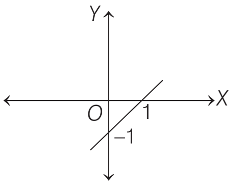
$\Rightarrow \quad f'(x) = 0$

$\Rightarrow \quad f'(0) = 0$

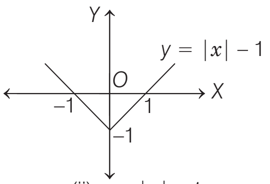
Hence, $\quad f(0) = f'(0) = 0$

16. Using graphical transformation.

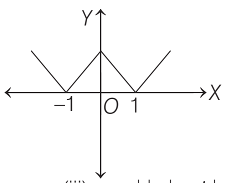
As, we know that, the function is not differentiable at sharp edges.



(i) $y = x - 1$



(ii) $y = |x| - 1$



(iii) $y = ||x| - 1|$

In function,

$y = || x| - 1|$ we have 3 sharp edges at $x = -1, 0, 1$.

Hence, $f(x)$ is not differentiable at $\{0, \pm 1\}$.

17. Given, $f(x) = \begin{cases} \dfrac{1}{2}(-x-1), & \text{if } x < -1 \\ \tan^{-1} x, & \text{if } -1 \leq x \leq 1 \\ \dfrac{1}{2}(x-1), & \text{if } x > 1 \end{cases}$

$f(x)$ is discontinuous at $x = -1$ and $x = 1$.

$\therefore$ Domain of $f'(x) \in R - \{-1, 1\}$

18. RHD of $\sin(|x|) - |x| = \lim_{h\to 0} \dfrac{\sin h - h}{h} = 1 - 1 = 0$

[$\because f(0) = 0$]

LHD of $\sin(|x|) - |x|$

$$= \lim_{h\to 0} \frac{\sin|-h| - |-h|}{-h} = \frac{\sin h - h}{-h} = 0$$

Therefore, (d) is the answer.

19. Given, $f(x) = [x] \sin \pi x$

If x is just less than k, $[x] = k - 1$

$\therefore \quad f(x) = (k-1) \sin \pi x.$

LHD of $f(x) = \lim_{x\to k} \dfrac{(k-1)\sin \pi x - k \sin \pi k}{x - k}$

$$= \lim_{x\to k} \frac{(k-1)\sin \pi x}{x - k},$$

$$= \lim_{h\to 0} \frac{(k-1)\sin \pi (k-h)}{-h}$$

[where $x = k - h$]

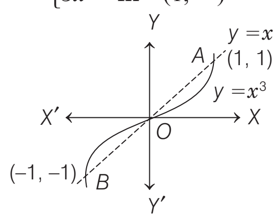
$$= \lim_{h\to 0} \frac{(k-1)(-1)^{k-1}\cdot \sin h\pi}{-h} = (-1)^k (k-1)\pi$$

20. Given, $f(x) = \max\{x, x^3\}$ considering the graph separately, $y = x^3$ and $y = x$

NOTE $y = x^3$ is odd order parabola and $y = x$ is always intersect at (1, 1) and (–1, – 1).

Now, $\quad f(x) = \begin{cases} x & \text{in } (-\infty, -1] \\ x^3 & \text{in } (-1, 0] \\ x & \text{in } (0, 1] \\ x^3 & \text{in } (1, \infty) \end{cases}$

$\Rightarrow \quad f'(x) = \begin{cases} 1 & \text{in } (-\infty, -1] \\ 3x^2 & \text{in } (-1, 0] \\ 1 & \text{in } (0, 1] \\ 3x^2 & \text{in } (1, \infty) \end{cases}$



The point of consideration are

$f'(-1^-) = 1 \quad$ and $\quad f'(-1^+) = 3$

$f'(-0^-) = 0 \quad$ and $\quad f'(0^+) = 1$

$f'(1^-) = 1 \quad$ and $\quad f'(1^+) = 3$

Hence, f is not differentiable at $-1, 0, 1$.

21. Let $h(x) = |x|$, then $\quad g(x) = |f(x)| = h\{f(x)\}$

Since, composition of two continuous functions is continuous, g is continuous if f is continuous. So, answer is (c).

(a) Let $f(x) = x \Rightarrow g(x) = |x|$

Now, $f(x)$ is an onto function. Since, co-domain of x is R and range of x is R. But g (x) is into function. Since, range of $g(x)$ is $[0, \infty)$ but co-domain is given R. Hence, (a) is wrong.

(b) Let $f(x) = x \Rightarrow g(x) = |x|$. Now, $f(x)$ is one-one function but $g(x)$ is many-one function. Hence, (b) is wrong.

(d) Let $f(x) = x \Rightarrow g(x) = |x|$. Now, $f(x)$ is differentiable for all $x \in R$ but $g(x) = |x|$ is not differentiable at $x = 0$ Hence, (d) is wrong.

22. Function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ ...(i)

NOTE In differentiable of $|f(x)|$ we have to consider critical points for which $f(x) = 0$.

$|x|$ is not differentiable at $x = 0$

but $$\cos|x| = \begin{cases} \cos(-x), & \text{if } x < 0 \\ \cos x, & \text{if } x \ge 0 \end{cases}$$

$$\Rightarrow \quad \cos|x| = \begin{cases} \cos x, & \text{if } x < 0 \\ \cos x, & \text{if } x \ge 0 \end{cases}$$

Therefore, it is differentiable at $x = 0$.

Now, $|x^2 - 3x + 2| = |(x-1)(x-2)|$

$$= \begin{cases} (x-1)(x-2), & \text{if } x < 1 \\ (x-1)(2-x), & \text{if } 1 \le x < 2 \\ (x-1)(x-2), & \text{if } 2 \le x \end{cases}$$

Therefore,

$$f(x) = \begin{cases} (x^2-1)(x-1)(x-2) + \cos x, & \text{if } -\infty < x < 1 \\ -(x^2-1)(x-1)(x-2) + \cos x, & \text{if } 1 \le x < 2 \\ (x^2-1)(x-1)(x-2) + \cos x, & \text{if } 2 \le x < \infty \end{cases}$$

Now, $x = 1, 2$ are critical point for differentiability. Because $f(x)$ is differentiable on other points in its domain.

Differentiability at $x = 1$

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \left[(x^2-1)(x-2) + \frac{\cos x - \cos 1}{x - 1}\right]$$

$$= 0 - \sin 1 = -\sin 1$$

$\left[\because \lim_{x \to 1^-} \frac{\cos x - \cos 1}{x - 1} = \frac{d}{dx}(\cos x) \text{ at } x = 1 - 0\right.$

$\left. = -\sin x \text{ at } x = 1 - 0 = -\sin x \text{ at } x = 1 = -\sin 1\right]$

and $$Rf'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^+} \left[-(x^2-1)(x-2) + \frac{\cos x - \cos 1}{x - 1}\right]$$

$= 0 - \sin 1 = -\sin 1$ [same approach]

$\because Lf'(1) = Rf'(1)$. Therefore, function is differentiable at $x = 1$.

Again, $$Lf'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \left[-(x^2-1)(x-1) + \frac{\cos x - \cos 2}{x - 2}\right]$$

$$= -(4-1)(2-1) - \sin 2 = -3 - \sin 2$$

and $$Rf'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^+} \left[(x^2-1)(x-1) + \frac{\cos x - \cos 2}{x - 2}\right]$$

$$= (2^2 - 1)(2-1) - \sin 2 = 3 - \sin 2$$

So, $Lf'(2) \ne Rf'(2)$, f is not differentiable at $x = 2$

Therefore, (d) is the answer.

23. Given, $$f(x) = \frac{x}{1 + |x|} = \begin{cases} \dfrac{x}{1+x}, & x \ge 0 \\ \dfrac{x}{1-x}, & x < 0 \end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} \dfrac{(1+x)\cdot 1 - x \cdot 1}{(1+x)^2}, & x \ge 0 \\ \dfrac{(1-x)\cdot 1 - x(-1)}{(1-x)^2}, & x < 0 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} \dfrac{1}{(1+x)^2}, & x \ge 0 \\ \dfrac{1}{(1-x)^2}, & x < 0 \end{cases}$$

$\therefore$ RHD at $x = 0 \Rightarrow \lim_{x \to 0} \frac{1}{(1+x)^2} = 1$

and LHD at $x = 0 \Rightarrow \lim_{x \to 0} \frac{1}{(1-x)^2} = 1$

Hence, $f(x)$ is differentiable for all x.

24. Since, $f(x)$ is continuous and differentiable where $f(0) = 1$ and $f'(0) = -1$, $f(x) > 0$, $\forall x$.

Thus, $f(x)$ is decreasing for $x > 0$ and concave down.

$\Rightarrow \quad f''(x) < 0$

Therefore, (a) is answer.

25. Here, $$f(x) = \frac{\tan \pi [(x - \pi)]}{1 + [x]^2}$$

Since, we know $\pi[(x - \pi)] = n\pi$ and $\tan n\pi = 0$

$\because \quad 1 + [x]^2 \ne 0$

$\therefore \quad f(x) = 0, \forall x$

Thus, $f(x)$ is a constant function.

$\therefore f'(x), f''(x), \ldots$ all exist for every x, their value being 0.

$\Rightarrow f'(x)$ exists for all x.

26. We have,

$$(f(0))^2 + (f'(0))^2 = 85$$

and $$f : R \to [-2, 2]$$

(a) Since, f is twice differentiable function, so f is continuous function.

$\therefore$ This is true for every continuous function.

Hence, we can always find $x \in (r, s)$, where $f(x)$ is one-one.

$\therefore$ This statement is true.

(b) By L.M.V.T

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow |f'(c)| = \left|\frac{f(b)-f(a)}{b-a}\right|$$

$$\Rightarrow \quad |f'(x_0)| = \left|\frac{f(0)-f(-4)}{0+4}\right| = \left|\frac{f(0)-f(-4)}{4}\right|$$

Range of f is $[-2, 2]$

$$\therefore -4 \le f(0) - f(-4) \le 4 \Rightarrow 0 \le \left|\frac{f(0)-f(-4)}{4}\right| \le 1$$

Hence, $|f'(x_0)| \le 1$.

Hence, statement is true.

(c) As no function is given, then we assume

$$f(x) = 2\sin\left(\frac{\sqrt{85}\,x}{2}\right)$$

$$\therefore \quad f'(x) = \sqrt{85}\cos\left(\frac{\sqrt{85}\,x}{2}\right)$$

Now, $(f(0))^2 + (f'(0))^2 = (2\sin 0)^2 + (\sqrt{85}\cos 0)^2$

$(f(0))^2 + (f'(0))^2 = 85$

and $\lim\limits_{x\to\infty} f(x)$ does not exists.

Hence, statement is false.

(d) From option b, $|f'(x_0)| \le 1$ and $x_0 \in (-4, 0)$

$\therefore \quad (f'(x_0))^2 \le 1$

Hence, $g(x_0) = (f(x_0))^2 + (f'(x_0))^2 \le 4 + 1$ $\quad [\because f(x_0) \in [-2, 2]]$

$\Rightarrow \quad g(x_0) \le 5$

Now, let $p \in (-4, 0)$ for which $g(p) = 5$

Similarly, let q be smallest positive number $q \in (0, 4)$ such that $g(q) = 5$

Hence, by Rolle's theorem is (p, q)

$g'(c) = 0$ for $\alpha \in (-4, 4)$ and since $g(x)$ is greater than 5 as we move form $x = p$ to $x = q$

and $f(x))^2 \le 4 \Rightarrow (f'(x))^2 \ge 1$ in (p, q)

Thus, $\quad g'(c) = 0$

$\Rightarrow \quad f'f + f'f'' = 0$

So, $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \ne 0$

Hence, statement is true.

27. Given, $\lim\limits_{t\to x} \frac{f(x)\sin t - f(t)\sin x}{t-x} = \sin^2 x$

Using L' Hospital rules

$$\lim_{t\to x} \frac{f(x)\cos t - f'(t)\sin x}{1} = \sin^2 x$$

$$\Rightarrow \quad f(x)\cos x - f'(x)\sin x = \sin^2 x$$

$$\Rightarrow \quad f'(x)\sin x - f(x)\cos x = -\sin^2 x$$

$$\Rightarrow \quad \frac{f'(x)\sin x - f(x)\cos x}{\sin^2 x} = -1$$

$$\Rightarrow \quad d\left(\frac{f(x)}{\sin x}\right) = -1$$

On integrating, we get

$$\frac{f(x)}{\sin x} = -x + C \Rightarrow f(x) = -x\sin x + C\sin x$$

It is given that $x = \frac{\pi}{6}$, $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$

$$\therefore \quad f\left(\frac{\pi}{6}\right) = -\frac{\pi}{6}\sin\frac{\pi}{6} + C\sin\frac{\pi}{6}$$

$$= -\frac{\pi}{12} = -\frac{\pi}{12} + \frac{1}{2}C$$

$\Rightarrow C = 0$

$\therefore \quad f(x) = -x\sin x$

(a) $f(x) = -x\sin x$

$$f\left(\frac{\pi}{4}\right) = -\frac{\pi}{4}\sin\frac{\pi}{4} = -\frac{\pi}{4\sqrt{2}} \quad \text{false}$$

(b) $f(x) = -x\sin x$

$$\sin x > x - \frac{x^3}{6}, \forall x \in (0, \pi)$$

$$\Rightarrow \quad -x\sin x < -x^2 + \frac{x^4}{6}$$

$$\Rightarrow \quad f(x) < \frac{x^4}{6} - x^2, \forall x \in (0, \pi)$$

It is true

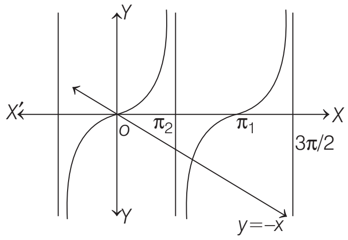
(c) $f(x) = -x\sin x$

$f'(x) = -\sin x - x\cos x$

$f'(x) = 0$

$\Rightarrow -\sin x - x\cos x = 0$

$\tan x = -x$



$\Rightarrow$ Their exists $\alpha \in (0, \pi)$ for which $f'(\alpha) = 0$

It is true

(d) $f(x) = -x\sin x$

$f'(x) = -\sin x - x\cos x$

$f''(x) = -2\cos x + x\sin x$

$$f''\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, \quad f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

$$\therefore \quad f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$$

It is true.

28. As, $g(f(x)) = x$

Thus, $g(x)$ is inverse of $f(x)$.

$\Rightarrow \quad g(f(x)) = x$

$\Rightarrow \quad g'(f(x)) \cdot f'(x) = 1$

$$\therefore \quad g'(f(x)) = \frac{1}{f'(x)} \quad \ldots(i)$$

[where, $f'(x) = 3x^2 + 3$]

When $\quad f(x) = 2$, then

$$x^3 + 3x + 2 = 2$$

$\Rightarrow \quad x = 0$

i.e. when $x = 0$, then $f(x) = 2$

$\therefore \quad g'(f(x)) = \dfrac{1}{3x^2 + 3}$ at (0, 2)

$\Rightarrow \quad g'(2) = \dfrac{1}{3}$

$\therefore$ Option (a) is incorrect.

Now, $\quad h(g(g(x))) = x$

$\Rightarrow \quad h(g(g(f(x))) = f(x)$

$\Rightarrow \quad h(g(x)) = f(x) \quad \ldots$(ii)

As $\quad g(f(x)) = x$

$\therefore \quad h(g(3)) = f(3) = 3^3 + 3(3) + 2 = 38$

$\therefore$ Option (d) is incorrect.

From Eq. (ii), $\quad h(g(x)) = f(x)$

$\Rightarrow \quad h(g(f(x))) = f(f(x))$

$\Rightarrow \quad h(x) = f(f(x)) \quad \ldots$(iii)

[using $g(f(x)) = x$]

$\Rightarrow \quad h'(x) = f'(f(x)) \cdot f'(x) \quad \ldots$(iv)

Putting $x = 1$, we get

$$h'(1) = f'(f(1)) \cdot f'(1) = (3 \times 36 + 3) \times (6)$$
$$= 111 \times 6 = 666$$

$\therefore$ Option (b) is correct.

Putting $x = 0$ in Eq. (iii), we get

$$h(0) = f(f(0)) = f(2) = 8 + 6 + 2 = 16$$

$\therefore$ Option (c) is correct.

29. Here, $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$

If $x^3 - x \geq 0$

$\Rightarrow \quad \cos|x^3 - x| = \cos(x^3 - x)$

$x^3 - x \leq 0$

$\Rightarrow \quad \cos|x^3 - x| = \cos(x^3 - x)$

$\therefore \quad \cos(|x^3 - x|) = \cos(x^3 - x), \forall x \in R \quad \ldots$(i)

Again, if $x^3 + x \geq 0$

$\Rightarrow \quad |x|\sin(|x^3 + x|) = x\sin(x^3 + x)$

$x^3 + x \leq 0$

$\Rightarrow \quad |x|\sin(|x^3 + x|) = -x\sin\{-(x^3 + x)\}$

$\therefore \quad |x|\sin(|x^3 + x|) = x\sin(x^3 + x), \forall x \in R \quad \ldots$(ii)

$\Rightarrow \quad f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$

$\therefore \quad f(x) = a\cos(x^3 - x) + bx\sin(x^3 + x) \quad \ldots$(iii)

which is clearly sum and composition of differential functions.

Hence, $f(x)$ is always continuous and differentiable.

30. Here,

$$f(x) = [x^2 - 3] = [x^2] - 3 = \begin{cases} -3, & -1/2 \leq x < 1 \\ -2, & 1 \leq x < \sqrt{2} \\ -1, & \sqrt{2} \leq x < \sqrt{3} \\ 0, & \sqrt{3} \leq x < 2 \\ 1, & x = 2 \end{cases}$$

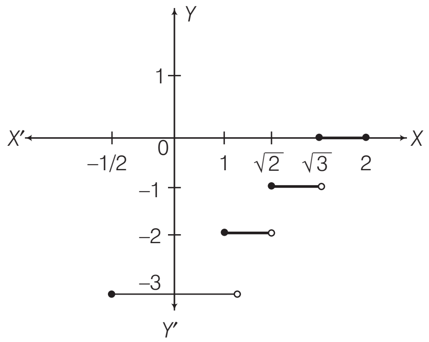
and $\quad g(x) = |x|f(x) + |4x - 7|f(x)$

$$= (|x| + |4x - 7|)f(x)$$
$$= (|x| + |4x - 7|)[x^2 - 3]$$
$$= \begin{cases} (-x - 4x - 7)(-3), & -1/2 \leq x < 0 \\ (x - 4x + 7)(-3), & 0 \leq x < 1 \\ (x - 4x + 7)(-2), & 1 \leq x < \sqrt{2} \\ (x - 4x + 7)(-1), & \sqrt{2} \leq x < \sqrt{3} \\ (x - 4x + 7)(0), & \sqrt{3} \leq x < 7/4 \\ (x + 4x - 7)(0), & 7/4 \leq x < 2 \\ (x + 4x - 7)(1), & x = 2 \end{cases}$$

$$\therefore \quad g(x) = \begin{cases} 15x + 21, & -1/2 \leq x < 0 \\ 9x - 21, & 0 \leq x < 1 \\ 6x - 14, & 1 \leq x < \sqrt{2} \\ 3x - 7, & \sqrt{2} \leq x < \sqrt{3} \\ 0, & \sqrt{3} \leq x < 2 \\ 5x - 7, & x = 2 \end{cases}$$

Now, the graphs of $f(x)$ and $g(x)$ are shown below.

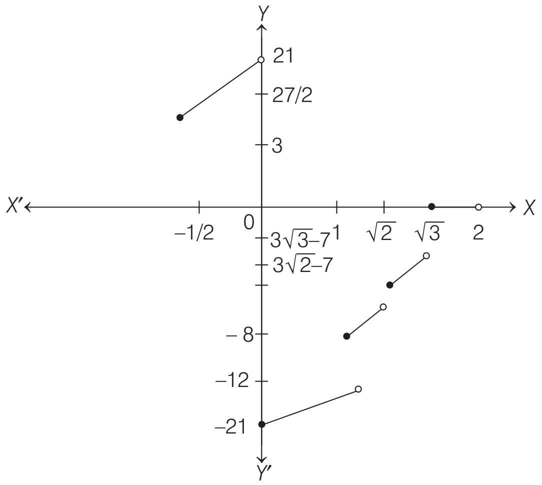
Graph for $f(x)$



Clearly, $f(x)$ is discontinuous at 4 points.

$\therefore$ Option (b) is correct.

Graph for $g(x)$



Clearly, $g(x)$ is not differentiable at 4 points, when $x \in (-1/2, 2)$.

$\therefore$ Option (c) is correct.

31. Here, $f(x)=\begin{cases} g(x), & x>0 \\ 0, & x=0 \\ -g(x), & x<0 \end{cases}$

$$f'(x)=\begin{cases} g'(x), & x\ge 0 \\ -g'(x), & x<0 \end{cases}$$

$\therefore$ Option (a) is correct.

(b) $h(x)=e^{|x|}=\begin{cases} e^{x}, & x\ge 0 \\ e^{-x}, & x<0 \end{cases}$

$\Rightarrow$ $h'(x)=\begin{cases} e^{x}, & x\ge 0 \\ -e^{-x}, & x<0 \end{cases}$

$\Rightarrow$ $h'(0^+)=1$ and $h'(0^-)=-1$

So, $h(x)$ is not differentiable at $x=0$.

$\therefore$ Option (b) is not correct.

(c) $(foh)(x)=f\{h(x)\}$, as $h(x)>0$

$$=\begin{cases} g(e^{x}), & x\ge 0 \\ g(e^{-x}), & x<0 \end{cases}$$

$\Rightarrow$ $(foh)'(x)=\begin{cases} e^{x}g'(e^{x}), & x\ge 0 \\ -e^{-x}g'(e^{-x}), & x<0 \end{cases}$

$\Rightarrow$ $(foh)'(0^+)=g'(1),\ (foh)'(0^-)=-g'(1)$

So, $(foh)(x)$ is not differentiable at $x=0$.

$\therefore$ Option (c) is not correct.

(d) $(hof)(x)=e^{|f(x)|}=\begin{cases} e^{|g(x)|}, & x\ne 0 \\ e^{0}=1, & x=0 \end{cases}$

Now, $(hof)'(0)=\lim_{h\to 0}\dfrac{e^{|g(x)|}-1}{x}$

$$=\lim_{h\to 0}\frac{e^{|g(x)|}-1}{|g(x)|}\cdot\frac{|g(x)|}{x}$$

$$=\lim_{h\to 0}\frac{e^{|g(x)|}-1}{|g(x)|}\cdot\lim_{h\to 0}\frac{|g(x)-0|}{|x|}\cdot\lim_{h\to 0}\frac{|x|}{x}$$

$$=1\cdot g'(0)\cdot\lim_{h\to 0}\frac{|x|}{x}=0,\ \text{as } g'(0)=0$$

$\therefore$ Option (d) is correct.

32. Let $F(x)=f(x)-3g(x)$

$\therefore$ $F(-1)=3,\ F(0)=3$ and $F(2)=3$

So, $F'(x)$ will vanish atleast twice in $(-1,0)\cup(0,2)$.

$\because F''(x)>0$ or <0, $\forall\ x\in(-1,0)\cup(0,2)$

Hence, $f'(x)-3g'(x)=0$ has exactly one solution in $(-1,0)$ and one solution in $(0,2)$.

33. A function $f(x)$ is continuous at $x=a$,

if $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=f(a)$.

Also, a function $f(x)$ is differentiable at $x=a$, if

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

i.e. $f'(a^-)=f'(a^+)$

Given that, $f:[a,b]\to[1,\infty)$

an $g(x)=\begin{cases} 0, & x<a \\ \int_a^x f(t)dt, & a\le x\le b \\ \int_a^b f(t)dt, & x>b \end{cases}$

Now, $g(a^-)=0=g(a^+)=g(a)$

[as $g(a^+)=\lim_{x\to a^+}\int_a^x f(t)dt=0$

and $g(a)=\int_a^a f(t)dt=0$]

$$g(b^-)=g(b^+)=g(b)=\int_a^b f(t)dt$$

$\Rightarrow g$ is continuous for all $x\in R$.

Now, $g'(x)=\begin{cases} 0, & x<a \\ f(x), & a<x<b \\ 0, & x>b \end{cases}$

$g'(a^-)=0$

but $g'(a^+)=f(a)\ge 1$

[$\because$ range of $f(x)$ is $[1,\infty)$, $\forall\ x\in[a,b]$]

$\Rightarrow g$ is non-differentiable at $x=a$

and $g'(b^+)=0$

but $g'(b^-)=f(b)\ge 1$

$\Rightarrow$ g is not differentiable at $x=b$.

34. $f(x)=\begin{cases} -x-\frac{\pi}{2}, & x\le -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2}<x\le 0 \\ x-1, & 0<x\le 1 \\ \log x, & x>1 \end{cases}$

Continuity at $x=-\frac{\pi}{2}$,

$$f\left(-\frac{\pi}{2}\right)=-\left(-\frac{\pi}{2}\right)-\frac{\pi}{2}=0$$

$$\text{RHL}=\lim_{h\to 0}-\cos\left(-\frac{\pi}{2}+h\right)=0$$

$\therefore$ Continuous at $x=\left(-\frac{\pi}{2}\right)$.

Continuity at $x=0$

$$f(0)=-1$$

$$\text{RHL}=\lim_{h\to 0}(0+h)-1=-1$$

$\therefore$ Continuous at $x=0$.

Continuity at $x=1$,

$$f(1)=0$$

$$\text{RHL}=\lim_{h\to 0}\log(1+h)=0$$

$\therefore$ Continuous at $x=1$

$$f'(x)=\begin{cases}-1, & x\le -\frac{\pi}{2}\\ \sin x, & -\frac{\pi}{2}<x\le 0\\ 1, & 0<x\le 1\\ \frac{1}{x}, & x>1\end{cases}$$

Differentiable at $x=0$, LHD $=0$, RHD $=1$

$\therefore$ Not differentiable at $x=0$

Differentiable at $x=1$, LHD $=1$, RHD $=1$

$\therefore$ Differentiable at $x=1$.

Also, for $x=-\frac{3}{2}$

$\Rightarrow \quad f(x)=-x-\frac{\pi}{2}$

$\therefore$ Differentiable at $x=-\frac{3}{2}$

35. $f(x+y)=f(x)+f(y)$, as $f(x)$ is differentiable at $x=0$.

$\Rightarrow \quad f'(0)=k \quad$ ...(i)

Now, $\quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f(h)}{h} \quad \left[\frac{0}{0}\text{ form}\right]$

Given, $f(x+y)=f(x)+f(y),\ \forall\ x, y$

$\therefore \quad f(0)=f(0)+f(0)$,

when $\quad x=y=0 \quad \Rightarrow f(0)=0)$

Using L'Hospital's rule,

$=\lim_{h\to 0}\frac{f'(h)}{1}=f'(0)=k \quad$ ...(ii)

$\Rightarrow f'(x)=k$, integrating both sides,

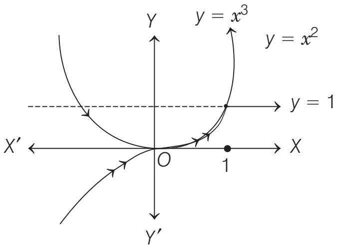
$f(x)=kx+C$, as $f(0)=0$

$\Rightarrow \quad C=0 \quad \therefore \quad f(x)=kx$

$\therefore f(x)$ is continuous for all $x\in R$ and $f'(x)=k$, i.e. constant for all $x\in R$.

Hence, (b) and (c) are correct.

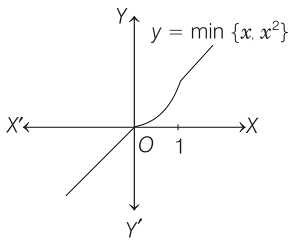
36. Here, $f(x)=\min\{1, x^2, x^3\}$ which could be graphically shown as



$\Rightarrow f(x)$ is continuous for $x\in R$ and not differentiable at $x=1$ due to sharp edge.

Hence, (a) and (d) are correct answers.

37. From the figure,



$h(x)$ is continuous all x, but $h(x)$ is not differentiable at two points $x=0$ and $x=1$. (due to sharp edges). Also $h'(x)=1, \forall\ x>1$.

Hence, (a), (c) and (d) is correct answers.

38. Here, $f(x)=\begin{cases}|x-3|, & x\ge 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$

$\therefore$ RHL at $x=1$, $\lim_{h\to 0}|1+h-3|=2$

LHL at $x=1$,

$\lim_{h\to 0}\frac{(1-h)^2}{4}-\frac{3(1-h)}{2}+\frac{13}{4}=\frac{1}{4}-\frac{3}{2}+\frac{13}{4}=\frac{14}{4}-\frac{3}{2}=2$

$\therefore \quad f(x)$ is continuous at $x=1$

Again, $\quad f(x)=\begin{cases}-(x-3), & 1\le x<3\\ (x-3), & x\ge 3\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$

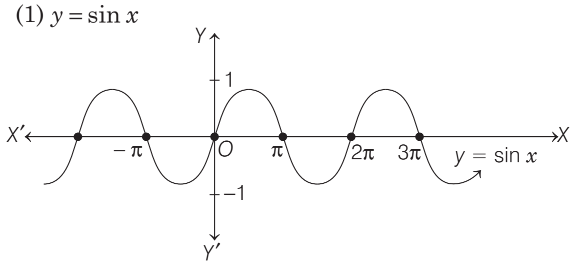
$\therefore \quad f'(x)=\begin{cases}-1, & 1\le x<3\\ 1, & x\ge 3\\ \frac{x}{2}-\frac{3}{2}, & x<1\end{cases}$

$\therefore$ $\left.\begin{array}{l}\text{RHD at } x=1 \Rightarrow -1\\ \text{LHD at } x=1 \Rightarrow \frac{1}{2}-\frac{3}{2}=-1\end{array}\right]$ differentiable at $x=1$.

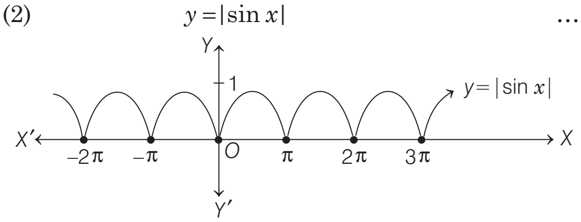
Again, $\left.\begin{array}{l}\text{RHD at } x=3 \Rightarrow 1\\ \text{LHD at } x=3 \Rightarrow -1\end{array}\right]$ not differentiable at $x=3$.

39. We know that, $f(x)=1+|\sin x|$ could be plotted as,

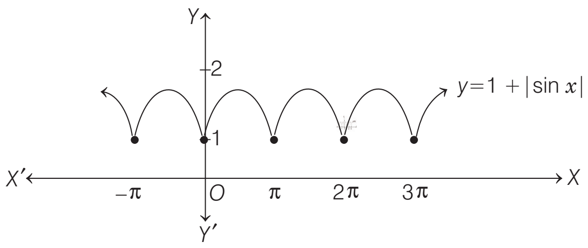
(1) $y=\sin x$ ...(i)



(2) $\quad y=|\sin x|$ ...(ii)



(3) $y = 1 + |\sin x|$ ...(iii)

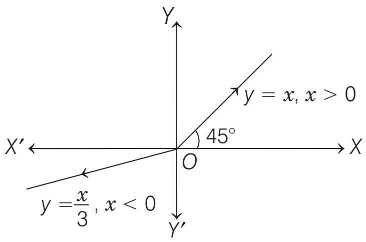


Clearly, $y = 1 + |\sin x|$ is continuous for all x, but not differentiable at infinite number of points..

40. Since, $x + |y| = 2y \Rightarrow \begin{cases} x + y = 2y, & \text{when } y > 0 \\ x - y = 2y, & \text{when } y < 0 \end{cases}$

$\Rightarrow \begin{cases} y = x, & \text{when } y > 0 \Rightarrow x > 0 \\ y = x/3, & \text{when } y < 0 \Rightarrow x < 0 \end{cases}$

which could be plotted as,



Clearly, y is continuous for all x but not differentiable at $x = 0$.

Also, $\dfrac{dy}{dx} = \begin{cases} 1, & x > 0 \\ 1/3, & x < 0 \end{cases}$

Thus, $f(x)$ is defined for all x, continuous at $x = 0$, differentiable for all $x \in R - \{0\}$, $\dfrac{dy}{dx} = \dfrac{1}{3}$ for $x < 0$.

41. We have, $\lim_{x \to 0} \dfrac{g(x)\cos x - g(0)}{\sin x}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \dfrac{g'(x)\cos x - g(x)\sin x}{\cos x} = 0$

Since, $f(x) = g(x)\sin x$

$f'(x) = g'(x)\sin x + g(x)\cos x$

and $f''(x) = g''(x)\sin x + 2g'(x)\cos x - g(x)\sin x$

$\Rightarrow f''(0) = 0$

Thus, $\lim_{x \to 0} [g(x)\cos x - g(0)\operatorname{cosec} x] = 0 = f''(0)$

$\Rightarrow$ Statement I is true.

Statement II $f'(x) = g'(x)\sin x + g(x)\cos x$

$\Rightarrow f'(0) = g(0)$

Statement II is not a correct explanation of Statement I.

42. A. $x|x|$ is continuous, differentiable and strictly increasing in $(-1, 1)$.

B. $\sqrt{|x|}$ is continuous in $(-1, 1)$ and not differentiable at $x = 0$.

C. $x + [x]$ is strictly increasing in $(-1, 1)$ and discontinuous at $x = 0$

$\Rightarrow$ not differentiable at $x = 0$.

D. $|x - 1| + |x + 1| = 2$ in $(-1, 1)$

$\Rightarrow$ The function is continuous and differentiable in $(-1, 1)$.

43. We know, $[x] \in I, \forall x \in R$.

Therefore, $\sin(\pi[x]) = 0, \forall x \in R$. By theory, we know that $\sin(\pi[x])$ is differentiable everywhere, therefore (A) $\leftrightarrow$ (p).

Again, $f(x) = \sin\{\pi(x - [x])\}$

Now, $x - [x] = \{x\}$

then $\pi(x - [x]) = \pi\{x\}$

which is not differentiable at $x \in I$.

Therefore, (B) $\leftrightarrow$ (r) is the answer.

44. Given, $F(x) = f(x) \cdot g(x) \cdot h(x)$

On differentiating at $x = x_0$, we get

$F'(x_0) = f'(x_0) \cdot g(x_0)\, h(x_0) + f(x_0) \cdot g'(x_0)\, h(x_0) + f(x_0)\, g(x_0) h'(x_0)$ ...(i)

where, $F'(x_0) = 21\, F(x_0), f'(x_0) = 4 f(x_0)$

$g'(x_0) = -7\, g(x_0)$ and $h'(x_0) = k\, h(x_0)$

On substituting in Eq. (i), we get

$21\, F(x_0) = 4\, f(x_0)\, g(x_0) h(x_0) - 7\, f(x_0)\, g(x_0)\, h(x_0) + k\, f(x) g(x_0) h(x_0)$

$\Rightarrow 21 = 4 - 7 + k$, [using $F(x_0) = f(x_0)\, g(x_0)\, h(x_0)$]

$\therefore k = 24$

45. Given, $f(x) = \begin{cases} \dfrac{x}{1 + e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\therefore Rf'(0) = f'(0^+) = \lim_{h \to 0} \dfrac{\dfrac{h}{1 + e^{1/h}} - 0}{h} = \lim_{h \to 0} \dfrac{1}{1 + e^{1/h}} = 0$

and $Lf'(0) = f'(0^-) = \lim_{h \to 0} \dfrac{\dfrac{-h}{1 + e^{-1/h}} - 0}{-h}$

$= \lim_{h \to 0} \dfrac{1}{1 + \dfrac{1}{e^{1/h}}} = \dfrac{1}{1 + 0} = 1$

$\therefore f'(0^+) = 0$ and $f'(0^-) = 1$

46. Given, $f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{(x-1)} - |x|, & \text{if } x \neq 1 \\ -1, & \text{if } x = 1 \end{cases}$

As, $f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{(x-1)} - x, & 0 \le x - \{1\} \\ (x-1)^2 \sin \dfrac{1}{(x-1)} + x, & x < 0 \\ -1, & x = 1 \end{cases}$

Here, $f(x)$ is not differentiable at $x = 0$ due to $|x|$.

Thus, $f(x)$ is not differentiable at $x = 0$.

47. It is always true that differential of even function is and odd function.

48. Since, $f(x)$ is differentiable at $x = 0$.

$\Rightarrow$ It is continuous at $x = 0$.

i.e. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

Here, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \frac{e^{ah/2}-1}{h} = \lim_{h\to 0} \frac{e^{ah/2}-1}{a\frac{h}{2}} \cdot \frac{a}{2} = \frac{a}{2}$

Also, $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} b\sin^{-1}\left(\frac{c-h}{2}\right) = b\sin^{-1}\frac{c}{2}$

$\therefore \quad b\sin^{-1}\frac{c}{2} = \frac{a}{2} = \frac{1}{2}$

$\Rightarrow \quad a = 1$

Also, it is differentiable at $x = 0$

$Rf'(0^+) = Lf'(0^-)$

$$Rf'(0^+) = \lim_{h\to 0} \frac{\frac{e^{h/2}-1}{h} - \frac{1}{2}}{h} \quad [\because a = 1]$$

$$= \lim_{h\to 0} \frac{2e^{h/2}-2-h}{2h^2} = \frac{1}{8}$$

and $$Lf'(0^-) = \lim_{h\to 0} \frac{b\sin^{-1}\left(\frac{c-h}{2}\right) - \frac{1}{2}}{-h} = \frac{b/2}{\sqrt{1-\frac{c^2}{4}}}$$

$\therefore \quad \frac{b}{\sqrt{4-c^2}} = \frac{1}{8}$

$\Rightarrow \quad 64b^2 = (4-c^2)$

$\Rightarrow \quad a = 1 \quad \text{and} \quad 64b^2 = (4-c^2)$

49. Here, $\lim_{n\to\infty} \frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n$

$$= \lim_{n\to\infty} n\left\{\frac{2}{\pi}\left(1+\frac{1}{n}\right)\cos^{-1}\left(\frac{1}{n}\right) - 1\right\} = \lim_{n\to\infty} nf\left(\frac{1}{n}\right)$$

where, $f\left(\frac{1}{n}\right) = \frac{2}{\pi}\left(1+\frac{1}{n}\right)\cos^{-1}\left(\frac{1}{n}\right) - 1 = f'(0)$

$\left[\text{given, } f'(0) = \lim_{n\to\infty} nf\left(\frac{1}{n}\right)\right]$

$$\therefore \quad \lim_{n\to\infty} \frac{2}{\pi}(n+1)\cos^{-1}\frac{1}{n} - n = f'(0) \quad \text{...(i)}$$

where, $f(x) = \frac{2}{\pi}(1+x)\cos^{-1}x - 1,\ f(0) = 0$

$$\Rightarrow \quad f'(x) = \frac{2}{\pi}\left\{(1+x)\frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x\right\}$$

$$\Rightarrow \quad f'(0) = \frac{2}{\pi}\left\{-1 + \frac{\pi}{2}\right\} = 1 - \frac{2}{\pi} \quad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii), we get

$$\lim_{n\to\infty} \frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n = 1 - \frac{2}{\pi}$$

50. Since, $g(x)$ is continuous at $x = \alpha \Rightarrow \lim_{x\to\alpha} g(x) = g(\alpha)$

and $f(x) - f(\alpha) = g(x)(x-\alpha),\ \forall x \in R$ [given]

$$\Rightarrow \quad \frac{f(x)-f(\alpha)}{(x-\alpha)} = g(x)$$

$$\Rightarrow \quad \lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha} = \lim_{x\to\alpha} g(x)$$

$$\Rightarrow \quad f'(\alpha) = \lim_{x\to\alpha} g(x) \Rightarrow f'(\alpha) = g(\alpha)$$

$\Rightarrow$ $f(x)$ is differentiable at $x = \alpha$.

Conversely, suppose f is differentiable at α, then

$\lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha}$ exists finitely.

Let $$g(x) = \begin{cases} \frac{f(x)-f(\alpha)}{x-\alpha}, & x \neq \alpha \\ f'(\alpha), & x = \alpha \end{cases}$$

Clearly, $\lim_{x\to\alpha} g(x) = f'(\alpha)$

$\Rightarrow$ $g(x)$ is continuous at $x = \alpha$.

Hence, $f(x)$ is differentiable at $x = \alpha$, iff $g(x)$ is continuous at $x = \alpha$.

51. It is clear that the given function

$$f(x) = \begin{cases} (1-x), & x < 1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ (3-x), & x > 2 \end{cases}$$

continuous and differentiable at all points except possibly at $x = 1$ and 2.

Continuity at $x = 1$,

$$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x)$$

$$= \lim_{h\to 0} [1-(1-h)] = \lim_{h\to 0} h = 0$$

and $$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1-x)(2-x)$$

$$= \lim_{h\to 0} [1-(1+h)][2-(1+h)]$$

$$= \lim_{h\to 0} -h\cdot(1-h) = 0$$

$\therefore \quad \text{LHL} = \text{RHL} = f(1) = 0$

Therefore, f is continuous at $x = 1$

Differentiability at $x = 1$,

$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{1-(1-h)-0}{-h} = \lim_{h\to 0}\left(\frac{h}{-h}\right) = -1$$

and $$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{[1-(1+h)][(2-(1+h)]-0}{h}$$

$$= \lim_{h\to 0} \frac{-h(1-h)}{h} = \lim_{h\to 0}(h-1) = -1$$

Since, $L[f'(1)] = Rf'(1)$, therefore f is differentiable at $x = 1$.

Continuity at $x = 2$,

$$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (1-x)(2-x)$$

$$= \lim_{h\to 0} [1-(2-h)][(2-(2-h)]$$

$= \lim_{h \to 0} (-1+h)\, h = 0$

and RHL $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3-x)$

$= \lim_{h \to 0} [3-(2+h)] = \lim_{h \to 0} (1-h) = 1$

Since, LHL $\neq$ RHL, therefore f is not continuous at $x=2$ as such f cannot be differentiable at $x=2$.
Hence, f is continuous and differentiable at all points except at $x=2$.

52. Given, $f(x) = \begin{cases} xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)} &, x>0 \\ xe^{-\left(-\frac{1}{x}+\frac{1}{x}\right)} &, x<0 \\ 0 &, x=0 \end{cases}$

$= \begin{cases} xe^{-\frac{2}{x}} &, x>0 \\ x &, x<0 \\ 0 &, x=0 \end{cases}$

(i) To check continuity at $x=0$,

LHL (at $x=0$) $= \lim_{h \to 0} -h = 0$

RHL $= \lim_{h \to 0} \frac{h}{e^{2/h}} = 0$

Also, $f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x=0$

(ii) To check differentiability at $x=0$,

$$L f'(0) = \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(0-h)-0}{-h} = 1$$

$$R f'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{he^{-2/h}-0}{h} = 0$$

$\therefore$ $f(x)$ is not differentiable at $x=0$.

53. Given, $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \forall x, y \in R$

On putting $y=0$, we get

$$f\left(\frac{x}{2}\right) = \frac{f(x)+f(0)}{2} = \frac{1}{2}[1+f(x)] \quad [\because f(0)=1]$$

$$\Rightarrow \quad 2f\left(\frac{x}{2}\right) = f(x)+1$$

$$\Rightarrow \quad f(x) = 2f\left(\frac{x}{2}\right) - 1, \forall x, y \in R \quad \ldots\text{(i)}$$

Since, $f'(0) = -1$, we get

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = -1$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{f(h)-1}{h} = -1 \quad \ldots\text{(ii)}$$

Again, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left(\frac{2x+2h}{2}\right)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(2x)+f(2h)}{2}-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}\left[2f\left(\frac{2x}{2}\right)-1+2f\left(\frac{2h}{2}\right)-1\right]-f(x)}{h}$$

[from Eq. (i)]

$$= \lim_{h\to 0} \frac{\frac{1}{2}[2f(x)-1+2f(h)-1]-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)+f(h)-1-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-1}{h} = -1 \quad \text{[from Eq. (ii)]}$$

$\therefore$ $f'(x) = -1, \forall x \in R$

$\Rightarrow$ $\int f'(x)dx = \int -1\, dx$

$\Rightarrow$ $f(x) = -x+k$, where, k is a constant.

But $f(0) = 1$,

therefore $f(0) = -0+k$

$\Rightarrow$ $1 = k$

$\Rightarrow$ $f(x) = 1-x, \forall x \in R \Rightarrow f(2) = -1$

54. We have, $f(x+y) = f(x)\cdot f(y), \forall x, y \in R$.

$\therefore$ $f(0) = f(0)\cdot f(0) \Rightarrow f(0)\{f(0)-1\} = 0$

$\Rightarrow$ $f(0) = 1$ $[\because f(0) \neq 0]$

Since, $f'(0) = 2 \Rightarrow \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 2$

$$\Rightarrow \quad \lim_{h\to 0} \frac{f(h)-1}{h} = 2 \quad [\because f(0)=1] \quad \ldots\text{(i)}$$

Also, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x)\cdot f(h)-f(x)}{h},$$

[using, $f(x+y) = f(x)\cdot f(y)$]

$$= f(x)\left\{\lim_{h\to 0} \frac{f(h)-1}{h}\right\}$$

$\therefore$ $f'(x) = 2f(x)$ [from Eq. (i)]

$\Rightarrow$ $\frac{f'(x)}{f(x)} = 2$

On integrating both sides between 0 to x, we get

$$\int_0^x \frac{f'(x)}{f(x)}\, dx = 2x$$

$\Rightarrow$ $\log_e |f(x)| - \log_e |f(0)| = 2x$

$\Rightarrow$ $\log_e |f(x)| = 2x$ $[\because f(0)=1]$

$\Rightarrow$ $\log_e |f(0)| = 0$

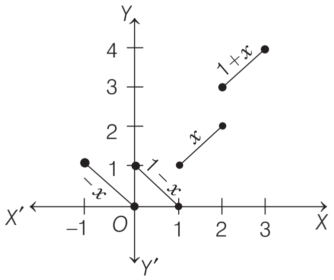
$\Rightarrow$ $f(x) = e^{2x}$

55. $y = [x] + |1 - x|, -1 \le x \le 3$

$$\Rightarrow \qquad y = \begin{cases} -1 + 1 - x, & -1 \le x < 0 \\ 0 + 1 - x, & 0 \le x < 1 \\ 1 + x - 1, & 1 \le x < 2 \\ 2 + x - 1, & 2 \le x \le 3 \end{cases}$$

$$\Rightarrow \qquad y = \begin{cases} -x, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ +x, & 1 \le x < 2 \\ x + 1, & 2 \le x < 3 \end{cases}$$

which could be shown as,



Clearly, from above figure, y is not continuous and not differentiable at $x = \{0, 1, 2\}$.

56. Since, $\quad |f(y) - f(x)|^2 \le (x - y)^3$

$$\Rightarrow \qquad \frac{|f(y) - f(x)|^2}{(y - x)^2} \le (x - y)$$

$$\Rightarrow \qquad \left|\frac{f(y) - f(x)}{y - x}\right|^2 \le x - y \qquad \ldots\text{(i)}$$

$$\Rightarrow \qquad \lim_{y \to x} \left|\frac{f(y) - f(x)}{y - x}\right|^2 \le \lim_{y \to x} (x - y)$$

$$\Rightarrow \qquad |f'(x)|^2 \le 0$$

which is only possible, if $|f'(x)| = 0$

$\therefore \quad f'(x) = 0$

or $\quad f'(x) = \text{Constant}$

57. Since, $f(-x) = f(x)$

$\therefore f(x)$ is an even function.

$$\therefore \quad f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} \qquad [\because f(-h) = f(h)]$$

Since, $f'(0)$ exists.

$$\therefore \qquad Rf'(0) = Lf'(0)$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{-h}$$

$$\Rightarrow \quad 2 \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$$

$$\therefore \qquad f'(0) = 0$$

58. Given that, $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ (x - 1), & 0 < x \le 2 \end{cases}$

Since, $x \in [-2, 2]$. Therefore, $|x| \in [0, 2]$

$$\Rightarrow \qquad f(|x|) = |x| - 1, \forall x \in [-2, 2]$$

$$\Rightarrow \qquad f(|x|) = \begin{cases} x - 1, & 0 \le x \le 2 \\ -x - 1, & -2 \le x \le 0 \end{cases}$$

Also, $\quad |f(x)| = \begin{cases} 1, & -2 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2 \end{cases}$

Also, $\quad g(x) = f(|x|) + |f(x)|$

$$= \begin{cases} -x - 1 + 1, & -2 \le x \le 0 \\ x - 1 + 1 - x, & 0 \le x < 1 \\ x - 1 + x - 1, & 1 \le x \le 2 \end{cases}$$

$$g(x) = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 \le x < 1 \\ 2(x - 1), & 1 \le x \le 2 \end{cases}$$

$$\therefore \qquad g'(x) = \begin{cases} -1, & -2 \le x \le 0 \\ 0, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$$

$\therefore$ RHD (at $x = 1$) $= 2$, LHD (at $x = 1$) $= 0$

$\Rightarrow g(x)$ is not differentiable at $x = 1$.

Also, RHD (at $x = 0$) $= 0$, LHD at ($x = 0$) $= -1$

$\Rightarrow g(x)$ is not differentiable at $x = 0$.

Hence, $g(x)$ is differentiable for all $x \in (-2, 2) - \{0, 1\}$

59. Given, $\quad f(x) = x^3 - x^2 - x + 1$

$$\Rightarrow \qquad f'(x) = 3x^2 - 2x - 1 = (3x + 1)(x - 1)$$

$\therefore \quad f(x)$ is increasing for $x \in (-\infty, -1/3) \cup (1, \infty)$

and decreasing for $x \in (-1/3, 1)$

Also, given $g(x) = \begin{cases} \max\{f(t); 0 \le t \le x\}, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$

$$\Rightarrow \quad g(x) = \begin{cases} f(x), & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} x^3 - x^2 - x + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

At $x = 1$,

$$\text{RHL} = \lim_{x \to 1} (3 - x) = 2$$

and $\quad \text{LHL} = \lim_{x \to 1} (x^3 - x^2 - x + 1) = 0$

$\therefore$ It is discontinuous at $x = 1$.

Also, $\quad g'(x) = \begin{cases} 3x^2 - 2x - 1, & 0 \le x \le 1 \\ -1, & 1 < x \le 2 \end{cases}$

$$\Rightarrow \qquad g'(1^+) = -1$$

and $\quad g'(1^-) = 3 - 2 - 1 = 0$

$\therefore g(x)$ is continuous for all $x \in (0, 2) - \{1\}$ and $g(x)$ is differentiable for all $x \in (0, 2) - \{1\}$.

60. Given that, $f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5}, & \text{when } x\neq 1\\ -\dfrac{1}{3}, & \text{when } x=1\end{cases}$

$$\text{RHD}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{\left[\dfrac{1+h-1}{2(1+h)^2-7(1+h)+5}-\left(-\dfrac{1}{3}\right)\right]}{h}$$
$$=\lim_{h\to 0}\left[\frac{3h+2(1+h)^2-7(1+h)+5}{3h\{2(1+h)^2-7(1+h)+5\}}\right]$$
$$=\lim_{h\to 0}\left(\frac{2h^2}{3h(-3h+2h^2)}\right)=-\frac{2}{9}$$
$$\text{LHD}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$
$$=\lim_{h\to 0}\frac{\left[\dfrac{1-h-1}{2(1-h)^2-7(1-h)+5}-\left(-\dfrac{1}{3}\right)\right]}{-h}$$
$$=\lim_{h\to 0}\frac{-3h+2(1+h^2-2h)-7(1-h)+5}{-3h[2(1-h)^2-7(1-h)+5]}$$
$$=\lim_{h\to 0}\frac{2h^2}{-3h(2h^2+3h)}=-\frac{2}{9}\quad\therefore\ \text{LHD}=\text{RHD}$$

Hence, required value of $f'(1)=-\dfrac{2}{9}$.

61. Given, $f(x)=x\tan^{-1}x$

Using first principle,
$$f'(1)=\lim_{h\to 0}\left[\frac{f(1+h)-f(1)}{h}\right]$$
$$=\lim_{h\to 0}\left[\frac{(1+h)\tan^{-1}(1+h)-\tan^{-1}(1)}{h}\right]$$
$$=\lim_{h\to 0}\left[\frac{\tan^{-1}(1+h)-\tan^{-1}(1)}{h}+\frac{h\tan^{-1}(1+h)}{h}\right]$$
$$=\lim_{h\to 0}\left[\frac{1}{h}\tan^{-1}\left(\frac{h}{2+h}\right)+\tan^{-1}(1+h)\right]$$
$$=\lim_{h\to 0}\left[\frac{\tan^{-1}\left(\dfrac{h}{2+h}\right)}{(2+h)\cdot\dfrac{h}{2+h}}\right]+\frac{\pi}{4}$$
$$=\lim_{h\to 0}\frac{1}{2+h}\left(\frac{\tan^{-1}\left(\dfrac{h}{2+h}\right)}{\dfrac{h}{(2+h)}}\right)+\frac{\pi}{4}=\frac{1}{2}+\frac{\pi}{4}$$

62. $g(x)=\int_x^{\frac{\pi}{2}}(f'(t)\operatorname{cosec}t-\cot t\operatorname{cosec}t\,f(t))dt$

$\therefore\quad g(x)=f\left(\dfrac{\pi}{2}\right)\operatorname{cosec}\dfrac{\pi}{2}-f(x)\operatorname{cosec}x$

$\Rightarrow\quad g(x)=3-\dfrac{f(x)}{\sin x}$

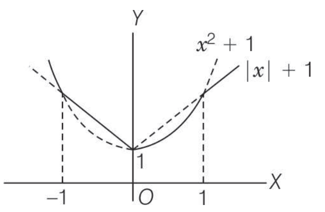
$$\lim_{x\to 0}g(x)=\lim_{x\to 0}\left(\frac{3\sin x-f(x)}{\sin x}\right)$$
$$=\lim_{x\to 0}\frac{3\cos x-f'(x)}{\cos x}$$
$$=\frac{3-1}{1}=2$$

63. PLAN

(i) In these type of questions, we draw the graph of the function.

(ii) The points at which the curve taken a sharp turn, are the points of non-differentiability.

Curve of $f(x)$ and $g(x)$ are



$h(x)$ is not differentiable at $x=\pm 1$ and 0.
As, $h(x)$ take sharp turns at $x=\pm 1$ and 0.

Hence, number of points of non-differentiability of $h(x)$ is 3.

64. Let $p(x)=ax^4+bx^3+cx^2+dx+e$

$\Rightarrow\quad p'(x)=4ax^3+3bx^2+2cx+d$

$\therefore\quad p'(1)=4a+3b+2c+d=0$ ... (i)

and $p'(2)=32a+12b+4c+d=0$ ... (ii)

Since, $\lim_{x\to 0}\left(1+\dfrac{p(x)}{x^2}\right)=2$ [given]

$\therefore\quad \lim_{x\to 0}\dfrac{ax^4+bx^3+(c+1)x^2+dx+e}{x^2}=2$

$\Rightarrow\quad c+1=2,\ d=0,\ e=0$

$\Rightarrow\quad c=1$

From Eqs. (i) and (ii), we get

$4a+3b=-2$

and $32a+12b=-4$

$\Rightarrow\quad a=\dfrac{1}{4}$ and $b=-1$.

$\therefore\quad p(x)=\dfrac{x^4}{4}-x^3+x^2$

$\Rightarrow\quad p(2)=\dfrac{16}{4}-8+4$

$\Rightarrow\quad p(2)=0$

Topic 8 Differentiation

1. We know,

$(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n$

On differentiating both sides w.r.t. x, we get

$n(1+x)^{n-1}={}^nC_1+2\,{}^nC_2x+\ldots+n\,{}^nC_nx^{n-1}$

On multiplying both sides by x, we get

$$n\,x(1+x)^{n-1} = {}^nC_1x + 2\,{}^nC_2x^2 + \ldots + n\,{}^nC_nx^n$$

Again on differentiating both sides w.r.t. x, we get

$$n\,[(1+x)^{n-1} + (n-1)\,x\,(1+x)^{n-2}]$$
$$= {}^nC_1 + 2^2\,{}^nC_2x + \ldots + n^2\,{}^nC_nx^{n-1}$$

Now putting $x = 1$ in both sides, we get

$${}^nC_1 + (2^2)\,{}^nC_2 + (3^2)\,{}^nC_3 + \ldots + (n^2)\,{}^nC_n$$
$$= n(2^{n-1} + (n-1)\,2^{n-2})$$

For $n = 20$, we get

$${}^{20}C_1 + (2^2)\,{}^{20}C_2 + (3^2)\,{}^{20}C_3 + \ldots + (20)^2\,{}^{20}C_{20}$$
$$= 20(2^{19} + (19)\,2^{18})$$
$$= 20\,(2+19)\,2^{18} = 420\,(2^{18})$$
$$= A(2^B) \text{ (given)}$$

On comparing, we get

$$(A, B) = (420, 18)$$

2. Let $f(x) = \tan^{-1}\left(\dfrac{\sin x - \cos x}{\sin x + \cos x}\right) = \tan^{-1}\left(\dfrac{\tan x - 1}{\tan x + 1}\right)$

[dividing numerator and denominator by $\cos x > 0$, $x \in \left(0, \dfrac{\pi}{2}\right)$]

$$= \tan^{-1}\left(\frac{\tan x - \tan\dfrac{\pi}{4}}{1 + \left(\tan\dfrac{\pi}{4}\right)(\tan x)}\right)$$

$$= \tan^{-1}\left[\tan\left(x - \frac{\pi}{4}\right)\right]$$

$$\left[\because \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B)\right]$$

Since, it is given that $x \in \left(0, \dfrac{\pi}{2}\right)$, so

$$x - \frac{\pi}{4} \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Also, for $\left(x - \dfrac{\pi}{4}\right) \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$,

Then,

$$f(x) = \tan^{-1}\left(\tan\left(x - \frac{\pi}{4}\right)\right) = x - \frac{\pi}{4}$$

$$\left[\because \tan^{-1}\tan\theta = \theta, \text{ for } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

Now, derivative of $f(x)$ w.r.t. $\dfrac{x}{2}$ is

$$\frac{d(f(x))}{d(x/2)} = 2\,\frac{df(x)}{d(x)}$$
$$= 2 \times \frac{d}{dx}\left(x - \frac{\pi}{4}\right) = 2$$

3. **Key Idea** Differentiating the given equation twice w.r.t. 'x'.

Given equation is

$$e^y + xy = e \qquad \ldots\text{(i)}$$

On differentiating both sides w.r.t. x, we get

$$e^y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0 \qquad \ldots\text{(ii)}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{y}{e^y + x}\right) \qquad \ldots\text{(iii)}$$

Again differentiating Eq. (ii) w.r.t. 'x', we get

$$e^y\frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0 \qquad \ldots\text{(iv)}$$

Now, on putting $x = 0$ in Eq. (i), we get

$$e^y = e^1$$
$$\Rightarrow \quad y = 1$$

On putting $x = 0$, $y = 1$ in Eq. (iii), we get

$$\frac{dy}{dx} = -\frac{1}{e+0} = -\frac{1}{e}$$

Now, on putting $x = 0$, $y = 1$ and $\dfrac{dy}{dx} = -\dfrac{1}{e}$ in Eq. (iv), we get

$$e^1\frac{d^2y}{dx^2} + e^1\left(-\frac{1}{e}\right)^2 + 0\left(\frac{d^2y}{dx^2}\right) + \left(-\frac{1}{e}\right) + \left(-\frac{1}{e}\right) = 0$$

$$\Rightarrow \quad \left.\frac{d^2y}{dx^2}\right|_{(0,\,1)} = \frac{1}{e^2}$$

So, $\left(\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}\right)$ at $(0, 1)$ is $\left(-\dfrac{1}{e}, \dfrac{1}{e^2}\right)$

4. Let $y = f(f(f(x))) + (f(x))^2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = f'(f(f(x)))\cdot f'(f(x))\cdot f'(x) + 2f(x)f'(x)$$

[by chain rule]

So, $\left.\dfrac{dy}{dx}\right|_{\text{at } x=1} = f'(f(f(1)))\cdot f'(f(1))\cdot f'(1) + 2f(1)f'(1)$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=1} = f'(f(1))\cdot f'(1)\cdot(3) + 2(1)(3)$$

$[\because f(1) = 1 \text{ and } f'(1) = 3]$

$$= f'(1)\cdot(3)\cdot(3) + 6$$
$$= (3 \times 9) + 6 = 27 + 6 = 33$$

5. Given expression is

$$2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2 = \left(\cot^{-1}\left(\frac{\sqrt{3}\cot x + 1}{\cot x - \sqrt{3}}\right)\right)^2$$

[dividing each term of numerator and denominator by $\sin x$]

$$= \left(\cot^{-1}\left(\frac{\cot\dfrac{\pi}{6}\cot x + 1}{\cot x - \cot\dfrac{\pi}{6}}\right)\right)^2 \qquad \left[\because \cot\frac{\pi}{6} = \sqrt{3}\right]$$

$$= \left(\cot^{-1}\left(\cot\left(\frac{\pi}{6} - x\right)\right)\right)^2 \qquad \left[\because \cot(A - B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}\right]$$

$$= \begin{cases} \left(\frac{\pi}{6} - x\right)^2, & 0 < x < \frac{\pi}{6} \\ \left(\pi + \left(\frac{\pi}{6} - x\right)\right)^2, & \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$$

$$\left[\because \cot^{-1}(\cot\theta) = \begin{cases} \pi + \theta, & -\pi < \theta < 0 \\ \theta, & 0 < \theta < \pi \\ \theta - \pi, & \pi < \theta < 2\pi \end{cases}\right]$$

$$\Rightarrow 2y = \begin{cases} \left(\frac{\pi}{6} - x\right)^2, & 0 < x < \frac{\pi}{6} \\ \left(\frac{7\pi}{6} - x\right)^2, & \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$$

$$\Rightarrow 2\frac{dy}{dx} = \begin{cases} 2\left(\frac{\pi}{6} - x\right)(-1), & 0 < x < \frac{\pi}{6} \\ 2\left(\frac{7\pi}{6} - x\right)(-1), & \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} x - \frac{\pi}{6}, & 0 < x < \frac{\pi}{6} \\ x - \frac{7\pi}{6}, & \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$$

6. Given equation is

$$(2x)^{2y} = 4 \cdot e^{2x - 2y} \quad \dots \text{(i)}$$

On applying '$\log_e$' both sides, we get

$$\log_e (2x)^{2y} = \log_e 4 + \log_e e^{2x - 2y}$$

$$2y \log_e (2x) = \log_e (2)^2 + (2x - 2y)$$

$[\because \log_e n^m = m \log_e n$ and $\log_e e^{f(x)} = f(x)]$

$$\Rightarrow \quad (2\log_e(2x) + 2)y = 2x + 2\log_e(2)$$

$$\Rightarrow \quad y = \frac{x + \log_e 2}{1 + \log_e(2x)}$$

On differentiating 'y' w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{(1 + \log_e(2x))1 - (x + \log_e 2)\frac{2}{2x}}{(1 + \log_e(2x))^2}$$

$$= \frac{1 + \log_e(2x) - 1 - \frac{1}{x}\log_e 2}{(1 + \log_e(2x))^2}$$

So, $(1 + \log_e(2x))^2 \frac{dy}{dx} = \left(\frac{x\log_e(2x) - \log_e 2}{x}\right)$

7. We have, $x\log_e(\log_e x) - x^2 + y^2 = 4$, which can be written as

$$y^2 = 4 + x^2 - x\log_e(\log_e x) \quad \dots \text{(i)}$$

Now, differentiating Eq. (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 2x - x\frac{1}{\log_e x} \cdot \frac{1}{x} - 1 \cdot \log_e(\log_e x)$$

[by using product rule of derivative]

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \frac{2x - \frac{1}{\log_e x} - \log_e(\log_e x)}{2y} \quad \dots \text{(ii)}$$

Now, at $x = e$, $y^2 = 4 + e^2 - e\log_e(\log_e e)$ [using Eq. (i)]

$$= 4 + e^2 - e\log_e(1) = 4 + e^2 - 0$$

$$= e^2 + 4$$

$$\Rightarrow \quad y = \sqrt{e^2 + 4} \quad [\because y > 0]$$

$\therefore$ At $x = e$ and $y = \sqrt{e^2 + 4}$,

$$\frac{dy}{dx} = \frac{2e - 1 - 0}{2\sqrt{e^2 + 4}} = \frac{2e - 1}{2\sqrt{e^2 + 4}} \quad \text{[using Eq. (ii)]}$$

8. We have,

$$f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$$

$$\Rightarrow f'(x) = 3x^2 + 2xf'(1) + f''(2) \quad \dots \text{(i)}$$

$$\Rightarrow f''(x) = 6x + 2f'(1) \quad \dots \text{(ii)}$$

$$\Rightarrow f'''(x) = 6 \quad \dots \text{(iii)}$$

$$\Rightarrow f'''(3) = 6$$

Putting $x = 1$ in Eq. (i), we get

$$f'(1) = 3 + 2f'(1) + f''(2) \quad \dots \text{(iv)}$$

and putting $x = 2$ in Eq. (ii), we get

$$f''(2) = 12 + 2f'(1) \quad \dots \text{(v)}$$

From Eqs. (iv) and (v), we get

$$f'(1) = 3 + 2f'(1) + (12 + 2f'(1))$$

$$\Rightarrow \quad 3f'(1) = -15$$

$$\Rightarrow \quad f'(1) = -5$$

$$\Rightarrow \quad f''(2) = 12 + 2(-5) = 2 \quad \text{[using Eq. (v)]}$$

$$\therefore \quad f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$$

$$\Rightarrow \quad f(x) = x^3 - 5x^2 + 2x + 6$$

$$\Rightarrow \quad f(2) = 2^3 - 5(2)^2 + 2(2) + 6 = 8 - 20 + 4 + 6 = -2$$

9. We have, $x = 3\tan t$ and $y = 3\sec t$

Clearly, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(3\sec t)}{\frac{d}{dt}(3\tan t)}$$

$$= \frac{3\sec t \tan t}{3\sec^2 t} = \frac{\tan t}{\sec t} = \sin t$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(\sin t)}{\frac{d}{dt}(3\tan t)} = \frac{\cos t}{3\sec^2 t} = \frac{\cos^3 t}{3}$$

Now, $$\frac{d^2y}{dx^2}\left(\text{at } t = \frac{\pi}{4}\right) = \frac{\cos^3 \frac{\pi}{4}}{3} = \frac{1}{3(2\sqrt{2})} = \frac{1}{6\sqrt{2}}$$

10. Let $$y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1 - 9x^3}\right) = \tan^{-1}\left[\frac{2 \cdot (3x^{3/2})}{1 - (3x^{3/2})^2}\right]$$

$$= 2\tan^{-1}(3x^{3/2}) \left[\because 2\tan^{-1} x = \tan^{-1}\frac{2x}{1 - x^2}\right]$$

$$\therefore \quad \frac{dy}{dx} = 2 \cdot \frac{1}{1 + (3x^{3/2})^2} \cdot 3 \times \frac{3}{2}(x)^{1/2} = \frac{9}{1 + 9x^3} \cdot \sqrt{x}$$

$$\therefore \quad g(x) = \frac{9}{1 + 9x^3}$$

11. Here, g is the inverse of $f(x)$.

$\Rightarrow \quad fog(x) = x$

On differentiating w.r.t. x, we get

$$f'\{g(x)\} \times g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

$$= \frac{1}{\dfrac{1}{1+\{g(x)\}^5}} \qquad \left[\because f'(x) = \frac{1}{1+x^5}\right]$$

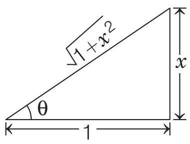
$\Rightarrow \quad g'(x) = 1 + \{g(x)\}^5$

12. Given, $y = \sec(\tan^{-1} x)$

Let $\quad \tan^{-1} x = \theta$

$\Rightarrow \quad x = \tan\theta$

$\therefore \quad y = \sec\theta = \sqrt{1+x^2}$



On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

At $x = 1$, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{2}}$

13. Since, $f(x) = e^{g(x)} \Rightarrow \; e^{g(x+1)} = f(x+1) = xf(x) = xe^{g(x)}$

and $\quad g(x+1) = \log x + g(x)$

i.e. $\quad g(x+1) - g(x) = \log x \qquad \text{... (i)}$

Replacing x by $x - \dfrac{1}{2}$, we get

$$g\left(x+\frac{1}{2}\right) - g\left(x-\frac{1}{2}\right) = \log\left(x-\frac{1}{2}\right) = \log(2x-1) - \log 2$$

$$\therefore \quad g''\left(x+\frac{1}{2}\right) - g''\left(x-\frac{1}{2}\right) = \frac{-4}{(2x-1)^2} \qquad \text{... (ii)}$$

On substituting, $x = 1, 2, 3, \ldots, N$ in Eq. (ii) and adding, we get

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}.$$

14. Since, $\quad \dfrac{dx}{dy} = \dfrac{1}{dy/dx} = \left(\dfrac{dy}{dx}\right)^{-1}$

$$\Rightarrow \quad \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1}\frac{dx}{dy}$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}\left(\frac{dx}{dy}\right) = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

15. Since, $\quad f''(x) = -f(x)$

$$\Rightarrow \quad \frac{d}{dx}\{f'(x)\} = -f(x)$$

$\Rightarrow \quad g'(x) = -f(x) \qquad [\because g(x) = f'(x), \text{ given}] \ldots\text{(i)}$

Also, $\quad F(x) = \left\{f\left(\dfrac{x}{2}\right)\right\}^2 + \left\{g\left(\dfrac{x}{2}\right)\right\}^2$

$$\Rightarrow \quad F'(x) = 2\left(f\left(\frac{x}{2}\right)\right) \cdot f'\left(\frac{x}{2}\right) \cdot \frac{1}{2}$$

$$+ 2\left(g\left(\frac{x}{2}\right)\right) \cdot g'\left(\frac{x}{2}\right) \cdot \frac{1}{2} = 0 \text{ [from Eq.(i)]}$$

$\therefore \quad F(x)$ is constant $\Rightarrow F(10) = F(5) = 5$

16. Let, $g(x) = f(x) - x^2$

$\Rightarrow g(x)$ has atleast 3 real roots which are $x = 1, 2, 3$ [by mean value theorem]

$\Rightarrow \; g'(x)$ has atleast 2 real roots in $x \in (1, 3)$

$\Rightarrow \; g''(x)$ has atleast 1 real roots in $x \in (1, 3)$

$\Rightarrow \; f''(x) - 2 \cdot 1 = 0$. for atleast 1 real root in $x \in (1, 3)$

$\Rightarrow \; f''(x) = 2$, for atleast one root in $x \in (1, 3)$

17. Given that, $\log(x+y) = 2xy \qquad \ldots\text{(i)}$

$\therefore$ At $x = 0, \Rightarrow \; \log(y) = 0 \; \Rightarrow \; y = 1$

$\therefore$ To find $\dfrac{dy}{dx}$ at $(0, 1)$

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{1}{x+y}\left(1 + \frac{dy}{dy}\right) = 2x\frac{dy}{dx} + 2y \cdot 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y(x+y) - 1}{1 - 2(x+y)x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,1)} = 1$$

18. Given, $x^2 + y^2 = 1$

On differentiating w.r.t. x, we get

$$2x + 2yy' = 0$$

$\Rightarrow \quad x + yy' = 0.$

Again, differentiating w.r.t. x, we get

$$1 + y'y' + yy'' = 0$$

$\Rightarrow \quad 1 + (y')^2 + yy'' = 0$

19. Given, $\quad f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

On differentiating w.r.t. x, we get

$$f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} x^3 & \sin x & \cos x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix} + \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow \quad f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\Rightarrow \quad f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + 0 + 0$$

and $f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} + 0 + 0$

$\therefore \quad f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0 =$ independent of p

20. Since, $y^2 = P(x)$

On differentiating both sides, we get

$$2yy_1 = P'(x),$$

Again, differentiating, we get

$$2yy_2 + 2y_1^2 = P''(x)$$

$\Rightarrow \quad 2y^3 y_2 + 2y^2 y_1^2 = y^2 P''(x)$

$\Rightarrow \quad 2y^3 y_2 = y^2 P''(x) - 2(yy_1)^2$

$\Rightarrow \quad 2y^3 y_2 = P(x) \cdot P''(x) - \dfrac{\{P'(x)\}^2}{2}$

Again, differentiating, we get

$$2\frac{d}{dx}(y^3 y_2) = P'(x) \cdot P''(x) + P(x) \cdot P'''(x) - \frac{2P'(x) \cdot P''(x)}{2}$$

$\Rightarrow \quad 2\dfrac{d}{dx}(y^3 y_2) = P(x) \cdot P'''(x)$

$\Rightarrow \quad 2\dfrac{d}{dx}\left(y^3 \cdot \dfrac{d^2y}{dx^2}\right) = P(x) \cdot P'''(x)$

21. Given, $xe^{xy} = y + \sin^2 x$ ...(i)

On putting $x = 0$, we get

$$0 \cdot e^0 = y + 0$$

$\Rightarrow \quad y = 0$

On differentiating Eq. (i) both sides w.r.t. x, we get

$$1 \cdot e^{xy} + x \cdot e^{xy}\left(x \cdot \frac{dy}{dx} + y\right) = \frac{dy}{dx} + 2\sin x \cos x$$

On putting $x = 0$, $y = 0$, we get

$$e^0 + 0(0 + 0) = \left[\frac{dy}{dx}\right]_{(0,\,0)} + 2\sin 0 \cos 0$$

$\Rightarrow \quad \left[\dfrac{dy}{dx}\right]_{0,\,0} = 1$

22. Given, $f(x) = x|x|$

$\Rightarrow \quad f(x) = \begin{cases} x^2, & \text{if } x \ge 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$f(x)$ is not differentiable at $x = 0$ but all $R - \{0\}$.

Therefore, $\quad f'(x) = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases}$

$\Rightarrow \quad f''(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases}$

Therefore, $f(x)$ is twice differentiable for all $x \in R - \{0\}$.

23. Given, $\quad f(x) = |x - 2|$

$\therefore \quad g(x) = f[f(x)] = ||x - 2| - 2|$

When, $\quad x > 2$

$$g(x) = |(x - 2) - 2| = |x - 4| = x - 4$$

$\therefore \quad g'(x) = 1$ when $x > 2$

24. Let $u = \sec^{-1}\left(-\dfrac{1}{2x^2 - 1}\right)$ and $v = \sqrt{1 - x^2}$

Put $\quad x = \cos\theta$

$\therefore \quad u = \sec^{-1}(-\sec 2\theta)$ and $v = \sin\theta$

$\Rightarrow \quad u = \pi - 2\theta \quad [\because \sec^{-1}(-x) = \pi - \sec^{-1} x]$

and $\quad v = \sin\theta$

$\Rightarrow \quad \dfrac{du}{d\theta} = -2$

an d $\quad \dfrac{dv}{d\theta} = \cos\theta$

$\Rightarrow \quad \dfrac{du}{dv} = -\dfrac{2}{\cos\theta}, \left(\dfrac{du}{dv}\right)_{\theta = \pi/3} = -4$

25. Given, $\quad f(x) = \log_x(\log x)$

$\therefore \quad f(x) = \dfrac{\log(\log x)}{\log x}$

On differentiating both sides, we get

$$f'(x) = \frac{(\log x)\left(\dfrac{1}{\log x} \cdot \dfrac{1}{x}\right) - \log(\log x) \cdot \dfrac{1}{x}}{(\log x)^2}$$

$\therefore \quad f'(e) = \dfrac{1 \cdot \left(\dfrac{1}{1} \cdot \dfrac{1}{e}\right) - \log(1) \cdot \dfrac{1}{e}}{(1)^2}$

$\Rightarrow \quad f'(e) = \dfrac{1}{e}$

26. Given, $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

$\therefore \quad F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

$$+ \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

$\Rightarrow \quad F'(a) = 0 + 0 + 0 = 0$

$[\because f_r(a) = g_r(a) = h_r(a);\ r = 1, 2, 3]$

27. Given, $\quad y = f\left(\dfrac{2x - 1}{x^2 + 1}\right)$

and $\quad f'(x) = \sin^2 x$

$\therefore \quad \dfrac{dy}{dx} = f'\left(\dfrac{2x - 1}{x^2 + 1}\right) \cdot \dfrac{d}{dx}\left(\dfrac{2x - 1}{x^2 + 1}\right)$

$$= \sin^2\left(\frac{2x - 1}{x^2 + 1}\right) \cdot \left\{\frac{(x^2 + 1) \cdot 2 - (2x - 1)(2x)}{(x^2 + 1)^2}\right\}$$

$$= \sin^2\left(\frac{2x-1}{x^2+1}\right)\cdot\frac{-2x^2+2x+2}{(x^2+1)^2}$$
$$= \frac{-2\,(x^2-x-1)}{(x^2+1)^2}\sin^2\left(\frac{2x-1}{x^2+1}\right)$$

28. $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{(x-c)} + 1$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{x}{(x-c)}$$
$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x}{(x-c)}\left(\frac{b}{x-b}+1\right)$$
$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x}{(x-c)}\cdot\frac{x}{(x-b)}$$
$$= \frac{x^2}{(x-c)(x-b)}\left(\frac{a}{x-1}+1\right) \Rightarrow y = \frac{x^3}{(x-a)(x-b)(x-c)}$$

$\Rightarrow \log y = \log x^3 - \log (x-a)(x-b)(x-c)$

$\Rightarrow \log y = 3\log x - \log(x-a) - \log(x-b) - \log(x-c)$

On differentiating, we get

$$\frac{y'}{y} = \frac{3}{x} - \frac{1}{x-a} - \frac{1}{x-b} - \frac{1}{x-c}$$
$$\Rightarrow \frac{y'}{y} = \left(\frac{1}{x}-\frac{1}{x-a}\right) + \left(\frac{1}{x}-\frac{1}{x-b}\right) + \left(\frac{1}{x}-\frac{1}{x-c}\right)$$
$$\Rightarrow \frac{y'}{y} = \frac{-a}{x(x-a)} - \frac{b}{x(x-b)} - \frac{c}{x(x-c)}$$
$$\Rightarrow \frac{y'}{y} = \frac{a}{x(a-x)} + \frac{b}{x(b-x)} + \frac{c}{x(c-x)}$$
$$\Rightarrow \frac{y'}{y} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$$

29. Here, $(\sin y)^{\sin\frac{\pi}{2}x} + \dfrac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x \tan\{\log(x+2)\} = 0$

On differentiating both sides, we get

$$(\sin y)^{\sin\frac{\pi}{2}x}\cdot\log(\sin y)\cdot\cos\frac{\pi}{2}x\cdot\frac{\pi}{2}$$
$$+\left(\sin\frac{\pi}{2}x\right)(\sin y)^{\left(\sin\frac{\pi}{2}x\right)-1}\cdot\cos y\cdot\frac{dy}{dx}$$
$$+\frac{\sqrt{3}}{2}\cdot\frac{2}{(2|x|)\sqrt{4x^2-1}} + \frac{2^x\cdot\sec^2\{\log(x+2)\}}{(x+2)}$$
$$+2^x\log 2\cdot\tan\{\log(x+2)\} = 0$$

Putting $\left(x=-1,\ y=-\dfrac{\sqrt{3}}{\pi}\right)$, we get

$$\frac{dy}{dx} = \frac{\left(-\frac{\sqrt{3}}{\pi}\right)^2}{\sqrt{1-\left(\frac{\sqrt{3}}{\pi}\right)^2}} = \frac{3}{\pi\sqrt{\pi^2-3}}$$

30. Given, $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$

On differentiating w.r.t. θ respectively, we get

$$\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$$

and $\dfrac{dy}{d\theta} = n\sec^{n-1}\theta\cdot\sec\theta\tan\theta - n\cos^{n-1}\theta\cdot(-\sin\theta)$

$\Rightarrow \dfrac{dx}{d\theta} = \tan\theta(\sec\theta + \cos\theta)$

and $\dfrac{dy}{d\theta} = n\tan\theta(\sec^n\theta + \cos^n\theta)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{n(\sec^n\theta + \cos^n\theta)}{\sec\theta + \cos\theta}$

$\therefore \left(\dfrac{dy}{dx}\right)^2 = \dfrac{n^2(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$

$$= \frac{n^2\{(\sec^n\theta - \cos^n\theta)^2 + 4\}}{\{(\sec\theta - \cos\theta)^2 + 4\}} = \frac{n^2(y^2+4)}{(x^2+4)}$$

$\Rightarrow (x^2+4)\left(\dfrac{dy}{dx}\right)^2 = n^2(y^2+4)$

31. Let $\phi(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ ...(i)

Given that, α is repeated root of quadratic equation $f(x) = 0$.

$\therefore$ We must have $f(x) = (x-\alpha)^2\cdot g(x)$

$$\therefore \quad \phi'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$
$$\Rightarrow \quad \phi'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$

$\Rightarrow$ $x = \alpha$ is root of $\phi'(x)$.

$\Rightarrow$ $(x-\alpha)$ is a factor of $\phi'(x)$ also.

or we can say $(x-\alpha)^2$ is a factor of $f(x)$.

$\Rightarrow$ $\phi(x)$ is divisible by $f(x)$.

32. Given, $y = \left\{(\log_{\cos x}\sin x)\cdot(\log_{\sin x}\cos x)^{-1} + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)\right\}$

$$\therefore \quad y = \left(\frac{\log_e(\sin x)}{\log_e(\cos x)}\right)^2 + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \frac{dy}{dx} = 2\left\{\frac{\log_e(\sin x)}{\log_e(\cos x)}\cdot\frac{(\log_e(\cos x)\cdot\cot x + \log_e(\text{ in } x)\cdot\tan x)}{\{\log_e(\cos x)\}^2}\right\} + \frac{2}{1+x^2}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(x=\frac{\pi}{4}\right)} = 2\left\{1\cdot\frac{2\cdot\log\left(\frac{1}{\sqrt{2}}\right)}{\left(\log\frac{1}{\sqrt{2}}\right)^2}\right\} + \frac{2}{1+\frac{\pi^2}{16}}$$
$$= -\frac{8}{\log_e 2} + \frac{32}{16+\pi^2}$$

33. Given, $(a+bx)\,e^{y/x} = x \Rightarrow y = x\log\left(\dfrac{x}{a+bx}\right)$

$\Rightarrow \quad y = x[\log(x) - \log(a+bx)]$ ...(i)

On differentiating both sides, we get

$$\frac{dy}{dx} = x\left(\frac{1}{x} - \frac{b}{a+bx}\right) + 1\,[\log(x) - \log(a+bx)]$$

$$\Rightarrow x\frac{dy}{dx} = x^2\left(\frac{a}{x(a+bx)}\right) + y$$

$$\Rightarrow x\,y_1 = \frac{a\,x}{a+bx} + y \quad \text{...(ii)}$$

Again, differentiating both sides, we get

$$x\,y_2 + y_1 = a\left\{\frac{(a+bx)\cdot 1 - x\cdot b}{(a+bx)^2}\right\} + y_1$$

$$\Rightarrow x^3 y_2 = \frac{a^2x^2}{(a+bx)^2}$$

$$\Rightarrow x^3 y_2 = \left(\frac{ax}{(a+bx)}\right)^2 \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow x^3 y_2 = (xy_1 - y)^2$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

34. Given, $h(x) = [f(x)]^2 + [g(x)]^2$

$$\Rightarrow h'\,x = 2f(x)\cdot f'(x) + 2g(x)\cdot g'(x)$$

$$= 2\,[f(x)\cdot g(x) - g(x)\cdot f(x)]$$

$$= 0 \quad [\because f'(x) = g(x) \text{ and } g'(x) = -f(x)]$$

$\therefore$ $h(x)$ is constant.

$\Rightarrow h(10) = h(5) = 11$

35. Since, $y = e^{x\sin x^3} + (\tan x)^x$, then

$y = u + v$, where $u = e^{x\sin x^3}$ and $v = (\tan x)^x$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{du}{dx} + \frac{dv}{dx}\right) \quad \text{...(i)}$$

Here, $u = e^{x\sin x^3}$ and $\log v = x\log(\tan x)$

On differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} = e^{x\sin x^3}\cdot(3x^3\cos x^3 + \sin x^3) \quad \text{...(ii)}$$

and $$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{x\cdot\sec^2 x}{\tan x} + \log(\tan x)$$

$$\frac{dv}{dx} = (\tan x)^x\,[2x\cdot\operatorname{cosec}(2x) + \log(\tan x)] \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), wet get

$$\frac{dy}{dx} = e^{x\sin x^3}(3x^3\cdot\cos x^3 + \sin x^3) + (\tan x)^x\,[2x\operatorname{cosec}2x + \log(\tan x)]$$

36. Given, $y = \dfrac{5x}{3|1-x|} + \cos^2(2x+1)$

$$\Rightarrow y = \begin{cases}\dfrac{5x}{3(1-x)} + \cos^2(2x+1), & x<1\\[2mm] \dfrac{5x}{3(x-1)} + \cos^2(2x+1), & x>1\end{cases}$$

The function is not defined at $x = 1$.

$$\Rightarrow \frac{dy}{dx} = \begin{cases}\dfrac{5}{3}\left\{\dfrac{(1-x)-x(-1)}{(1-x)^2}\right\} - 2\sin(4x+2), & x<1\\[2mm] \dfrac{5}{3}\left\{\dfrac{(x-1)-x(1)}{(x-1)^2}\right\} - 2\sin(4x+2), & x>1\end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases}\dfrac{5}{3(1-x)^2} - 2\sin(4x+2), & x<1\\[2mm] -\dfrac{5}{3(x-1)^2} - 2\sin(4x+2), & x>1\end{cases}$$

37. Here, $\lim\limits_{x\to 1}\dfrac{F(x)}{G(x)} = \dfrac{1}{14}$

$$\Rightarrow \lim_{x\to 1}\frac{F'(x)}{G'(x)} = \frac{1}{14} \quad \text{[using L'Hospital's rule]...(i)}$$

As $$F(x) = \int_{-1}^{x} f(t)\,dt \Rightarrow F'(x) = f(x) \quad \text{...(ii)}$$

and $$G(x) = \int_{-1}^{x} t\,|f\{f(t)\}|\,dt$$

$$\Rightarrow G'(x) = x\,|f\{f(x)\}| \quad \text{...(iii)}$$

$$\therefore \lim_{x\to 1}\frac{F(x)}{G(x)} = \lim_{x\to 1}\frac{F'(x)}{G'(x)} = \lim_{x\to 1}\frac{f(x)}{x\,|f\{f(x)\}|}$$

$$= \frac{f(1)}{1\,|f\{f(1)\}|} = \frac{1/2}{|f(1/2)|} \quad \text{...(iv)}$$

Given, $\lim\limits_{x\to 1}\dfrac{F(x)}{G(x)} = \dfrac{1}{14}$

$$\therefore \frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14} \Rightarrow \left|f\left(\frac{1}{2}\right)\right| = 7$$