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Indefinite Integration

Topic 1 Some Standard Results

Objective Questions (Only one correct option)

1. Let $\alpha \in (0, \pi/2)$ be fixed. If the integral

$\int \frac{\tan x + \tan\alpha}{\tan x - \tan\alpha}\, dx = A(x)\cos 2\alpha + B(x)$

$\sin 2\alpha + C$, where C is a constant of integration, then the functions $A(x)$ and $B(x)$ are respectively

(2019 Main, 12 April II)

(a) $x + \alpha$ and $\log_e|\sin(x+\alpha)|$
(b) $x - \alpha$ and $\log_e|\sin(x-\alpha)|$
(c) $x - \alpha$ and $\log_e|\cos(x-\alpha)|$
(d) $x + \alpha$ and $\log_e|\sin(x-\alpha)|$

2. The integral $\int \frac{2x^3 - 1}{x^4 + x}\, dx$ is equal to

(here C is a constant of integration) **(2019 Main, 12 April I)**

(a) $\frac{1}{2}\log_e \frac{|x^3+1|}{x^2} + C$
(b) $\frac{1}{2}\log_e \frac{(x^3+1)^2}{|x^3|} + C$
(c) $\log_e \left|\frac{x^3+1}{x}\right| + C$
(d) $\log_e \frac{|x^3+1|}{x^2} + C$

3. If $\int \frac{dx}{(x^2 - 2x + 10)^2} = A\left(\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{f(x)}{x^2 - 2x + 10}\right) + C$, where, C is a constant of integration, then

(2019 Main, 10 April I)

(a) $A = \frac{1}{27}$ and $f(x) = 9(x-1)$
(b) $A = \frac{1}{81}$ and $f(x) = 3(x-1)$
(c) $A = \frac{1}{54}$ and $f(x) = 3(x-1)$
(d) $A = \frac{1}{54}$ and $f(x) = 9(x-1)^2$

4. If $\int \frac{dx}{x^3(1+x^6)^{2/3}} = xf(x)(1+x^6)^{\frac{1}{3}} + C$

where, C is a constant of integration, then the function $f(x)$ is equal to **(2019 Main, 8 April II)**

(a) $-\frac{1}{6x^3}$
(b) $-\frac{1}{2x^3}$
(c) $-\frac{1}{2x^2}$
(d) $\frac{3}{x^2}$

5. $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}\, dx$ is equal to

(where, C is a constant of integration) **(2019 Main, 8 April I)**

(a) $2x + \sin x + 2\sin 2x + C$
(b) $x + 2\sin x + 2\sin 2x + C$
(c) $x + 2\sin x + \sin 2x + C$
(d) $2x + \sin x + \sin 2x + C$

6. The integral $\int \frac{3x^{13} + 2x^{11}}{(2x^4 + 3x^2 + 1)^4}\, dx$ is equal to (where C is a constant of integration) **(2019 Main, 12 Jan II)**

(a) $\frac{x^4}{6(2x^4 + 3x^2 + 1)^3} + C$
(b) $\frac{x^{12}}{6(2x^4 + 3x^2 + 1)^3} + C$
(c) $\frac{x^4}{(2x^4 + 3x^2 + 1)^3} + C$
(d) $\frac{x^{12}}{(2x^4 + 3x^2 + 1)^3} + C$

7. If $\int \frac{x+1}{\sqrt{2x-1}}\, dx = f(x)\sqrt{2x-1} + C$, where C is a constant of integration, then $f(x)$ is equal to **(2019 Main, 11 Jan II)**

(a) $\frac{2}{3}(x+2)$
(b) $\frac{1}{3}(x+4)$
(c) $\frac{2}{3}(x-4)$
(d) $\frac{1}{3}(x+1)$

8. If $\int \frac{\sqrt{1-x^2}}{x^4}\, dx = A(x)(\sqrt{1-x^2})^m + C$,

for a suitable chosen integer m and a function $A(x)$, where C is a constant of integration, then $(A(x))^m$ equals **(2019 Main, 11 Jan I)**

(a) $\frac{1}{9x^4}$
(b) $\frac{-1}{3x^3}$
(c) $\frac{-1}{27x^9}$
(d) $\frac{1}{27x^6}$

9. Let $n \geq 2$ be a natural number and $0 < \theta < \frac{\pi}{2}$. Then,

$\int \frac{(\sin^n\theta - \sin\theta)^{\frac{1}{n}}\cos\theta}{\sin^{n+1}\theta}\, d\theta$ is equal to

(where C is a constant of integration) **(2019 Main, 10 Jan I)**

(a) $\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n+1}\theta}\right)^{\frac{n+1}{n}}+C$

(b) $\frac{n}{n^2-1}\left(1+\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

(c) $\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

(d) $\frac{n}{n^2+1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+C$

10. If $f(x)=\int\frac{5x^8+7x^6}{(x^2+1+2x^7)^2}\,dx,\ (x\ge 0)$, and $f(0)=0$, then the value of $f(1)$ is **(2019 Main, 9 Jan I)**

(a) $-\frac{1}{2}$ (b) $-\frac{1}{4}$

(c) $\frac{1}{4}$ (d) $\frac{1}{2}$

11. For $x^2\ne n\pi+1$, $n\in N$ (the set of natural numbers), the integral

$\int x\sqrt{\frac{2\sin(x^2-1)-\sin 2(x^2-1)}{2\sin(x^2-1)+\sin 2(x^2-1)}}\,dx$ is equal to

(where C is a constant of integration) **(2019 Main, 9Jan I)**

(a) $\frac{1}{2}\log_e|\sec(x^2-1)|+C$

(b) $\log_e\left|\sec\left(\frac{x^2-1}{2}\right)\right|+C$

(c) $\log_e\left|\frac{1}{2}\sec^2(x^2-1)\right|+C$

(d) $\frac{1}{2}\log_e\left|\sec^2\left(\frac{x^2-1}{2}\right)\right|+C$

12. The integral

$\int\frac{\sin^2x\cos^2x}{(\sin^5x+\cos^3x\sin^2x+\sin^3x\cos^2x+\cos^5x)^2}\,dx$

is equal to **(2018 Main)**

(a) $\frac{1}{3(1+\tan^3x)}+C$ (b) $\frac{-1}{3(1+\tan^3x)}+C$

(c) $\frac{1}{1+\cot^3x}+C$ (d) $\frac{-1}{1+\cot^3x}+C$

(where C is a constant of integration)

13. The value of $\int\frac{dx}{x^2(x^4+1)^{3/4}}$ is **(2015 Main)**

(a) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$ (b) $(x^4+1)^{\frac{1}{4}}+c$

(c) $-(x^4+1)^{\frac{1}{4}}+c$ (d) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$

14. $\int\frac{\sec^2x}{(\sec x+\tan x)^{9/2}}\,dx$ equals to

(for some arbitrary constant K) **(2012)**

(a) $\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(b) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(c) $\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(d) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

15. If $I=\int\frac{e^x}{e^{4x}+e^{2x}+1}\,dx,\ J=\int\frac{e^{-x}}{e^{-4x}+e^{-2x}+1}\,dx.$

Then, for an arbitrary constant c, the value of $J-I$ equals **(2008, 3M)**

(a) $\frac{1}{2}\log\left|\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right|+c$ (b) $\frac{1}{2}\log\left|\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right|+c$

(c) $\frac{1}{2}\log\left|\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right|+c$ (d) $\frac{1}{2}\log\left|\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right|+c$

16. If $f(x)=\frac{x}{(1+x^n)^{1/n}}$ for $n\ge 2$ and $g(x)=\underbrace{(fofo\ldots of)}_{f\text{ occurs } n \text{ times}}(x)$.

Then, $\int x^{n-2}g(x)\,dx$ equals **(2007, 3M)**

(a) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+c$

(b) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+c$

(c) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+c$

(d) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}}+c$

17. The value of $\int\frac{(x^2-1)\,dx}{x^3\sqrt{2x^4-2x^2+1}}$ is **(2006, 3M)**

(a) $2\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+c$ (b) $2\sqrt{2+\frac{2}{x^2}+\frac{1}{x^4}}+c$

(c) $\frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+c$ (d) None of these

One or More Than One

18. Let $f:R\to R$ and $g:R\to R$ be two non-constant differentiable functions. If $f'(x)=(e^{(f(x)-g(x))})\,g'(x)$ for all $x\in R$ and $f(1)=g(2)=1$, then which of the following statement(s) is (are) TRUE? **(2018 Adv.)**

(a) $f(2)<1-\log_e 2$ (b) $f(2)>1-\log_e 2$

(c) $g(1)>1-\log_e 2$ (d) $g(1)<1-\log_e 2$

Numerical Value

19. Let $f:R\to R$ be a differentiable function with $f(0)=1$ and satisfying the equation $f(x+y)=f(x)f'(y)+f'(x)f(y)$ for all $x,y\in R$.

Then, the value of $\log_e(f(4))$ is .... **(2018 Adv.)**

Fill in the Blank

20. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \log(9e^{2x} - 4) + C$, then $A = \ldots$, $B = \ldots$ and $C = \ldots$. **(1989, 2M)**

Analytical & Descriptive Questions

21. For any natural number m, evaluate $\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx, x > 0$. **(2002, 5M)**

22. Evaluate $\int \left(\frac{1 - \sqrt{x}}{1 + \sqrt{x}}\right)^{1/2} \cdot \frac{dx}{x}$ **(1997C, 3M)**

23. Evaluate $\int \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}} dx$. **(1985, $2\frac{1}{2}$ M)**

24. Evaluate $\int \frac{dx}{x^2(x^4 + 1)^{3/4}}$. **(1984, 2M)**

25. Evaluate the following: **(1980, 4M)**

(i) $\int \sqrt{1 + \sin\left(\frac{1}{2}x\right)} dx$ (ii) $\int \frac{x^2}{\sqrt{1 - x}} dx$

26. Integrate $\frac{x^2}{(a + bx)^2}$. **(1979, 2M)**

27. Integrate $\sin x \cdot \sin 2x \cdot \sin 3x + \sec^2 x \cdot \cos^2 2x + \sin^4 x \cdot \cos^4 x$. **(1979, 1M)**

28. Integrate the curve $\frac{x}{1 + x^4}$. **(1978, 1M)**

29. Integrate $\frac{1}{1 - \cot x}$ or $\frac{\sin x}{\sin x - \cos x}$. **(1978, 2M)**

Topic 2 Some Special Integrals

Objective Question I (Only one correct option)

1. The integral $\int \sec^{2/3} x \operatorname{cosec}^{4/3} x \, dx$ is equal to (here C is a constant of integration) **(2019 Main, 9 April I)**

(a) $3\tan^{-1/3} x + C$ (b) $-3\tan^{-1/3} x + C$

(c) $-3\cot^{-1/3} x + C$ (d) $-\frac{3}{4}\tan^{-4/3} x + C$

2. Let $I_n = \int \tan^n x \, dx \ (n > 1)$. If $I_4 + I_6 = a \tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to **(2017 Main)**

(a) $\left(-\frac{1}{5}, 1\right)$ (b) $\left(\frac{1}{5}, 0\right)$ (c) $\left(\frac{1}{5}, -1\right)$ (d) $\left(-\frac{1}{5}, 0\right)$

Analytical & Descriptive Questions

3. Find the indefinite integral

$$\int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\right) dx.$$ **(1992, 4M)**

4. Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x}) \, dx$. **(1988, 3M)**

5. Evaluate $\int \frac{(\cos 2x)^{1/2}}{\sin x} dx$. **(1987, 6M)**

6. If $f(x)$ is the integral of $\frac{2\sin x - \sin 2x}{x^3}$, where $x \neq 0$, then find $\lim_{x \to 0} f'(x)$. **(1979, 3M)**

Topic 3 Integration by Parts

Objective Questions (Only one correct option)

1. If $\int x^5 e^{-x^2} dx = g(x) e^{-x^2} + C$, where C is a constant of integration, then $g(-1)$ is equal to **(2019 Main, 10 April II)**

(a) -1 (b) 1

(c) $-\frac{1}{2}$ (d) $-\frac{5}{2}$

2. If $\int e^{\sec x}$ $(\sec x \tan x f(x) + (\sec x \tan x + \sec^2 x))$ $dx = e^{\sec x} f(x) + C$, then a possible choice of $f(x)$ is **(2019 Main, 9 April II)**

(a) $x\sec x + \tan x + \frac{1}{2}$ (b) $\sec x + \tan x + \frac{1}{2}$

(c) $\sec x + x\tan x - \frac{1}{2}$ (d) $\sec x - \tan x - \frac{1}{2}$

3. The integral $\int \cos(\log_e x) \, dx$ is equal to (where C is a constant of integration) **(2019 Main, 12 Jan I)**

(a) $\frac{x}{2}[\cos(\log_e x) + \sin(\log_e x)] + C$

(b) $x[\cos(\log_e x) + \sin(\log_e x)] + C$

(c) $x[\cos(\log_e x) - \sin(\log_e x)] + C$

(d) $\frac{x}{2}[\sin(\log_e x) - \cos(\log_e x)] + C$

4. If $\int x^5 e^{-4x^3} dx = \frac{1}{48} e^{-4x^3} f(x) + C$, where C is a constant of integration, then $f(x)$ is equal to **(2019 Main, 10 Jan II)**

(a) $-4x^3 - 1$ (b) $4x^3 + 1$

(c) $-2x^3 - 1$ (d) $-2x^3 + 1$

5. $\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx$ is equal to **(2014 Main)**

(a) $(x - 1) e^{x + \frac{1}{x}} + c$ (b) $x e^{x + \frac{1}{x}} + c$

(c) $(x + 1) e^{x + \frac{1}{x}} + c$ (d) $-x e^{x + \frac{1}{x}} + c$

6. If $\int f(x)\, dx = \psi(x)$, then $\int x^5 f(x^3)\, dx$ is equal to **(2013 Main)**

(a) $\frac{1}{3}[x^3 \psi(x^3) - \int x^2 \psi(x^3) dx] + c$

(b) $\frac{1}{3} x^3 \psi(x^3) - 3 \int x^3 \psi(x^3)\, dx + c$

(c) $\frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3)\, dx + c$

(d) $\frac{1}{3}[x^3 \psi(x^3) - \int x^3 \psi(x^3)\, dx] + c$

Analytical & Descriptive Questions

7. Evaluate $\int \sin^{-1}\left(\frac{2x + 2}{\sqrt{4x^2 + 8x + 13}}\right) dx.$ **(2000, 5M)**

8. Find the indefinite integral
$\int \cos 2\theta \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta.$ **(1994, 5M)**

9. Evaluate $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx.$ **(1986, 2½ M)**

10. Evaluate $\int \frac{(x - 1) e^x}{(x + 1)^3} dx.$ **(1983, 2M)**

11. Evaluate $\int (e^{\log x} + \sin x) \cos x\, dx.$ **(1981, 2M)**

Topic 4 Integration, Irrational Function and Partial Fraction

Objective Questions (Only one correct option)

1. The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$ is equal to **(2016 Main)**

(a) $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$

(b) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$

(c) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$

(d) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + C$

where, C is an arbitrary constant.

2. The value of $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$ is **(1995, 2M)**

(a) $\sin x - 6\tan^{-1}(\sin x) + c$

(b) $\sin x - 2 (\sin x)^{-1} + c$

(c) $\sin x - 2 (\sin x)^{-1} - 6\tan^{-1}(\sin x) + c$

(d) $\sin x - 2 (\sin x)^{-1} + 5\tan^{-1}(\sin x) + c$

Analytical & Descriptive Questions

3. $\int \frac{x^3 + 3x + 2}{(x^2 + 1)^2 (x + 1)} dx.$ **(1999, 5M)**

4. Evaluate $\int \frac{(x + 1)}{x(1 + xe^x)^2} dx.$ **(1996, 2M)**

Answers

Topic 1

1. (b)	2. (c)	3. (c)	4. (b)
5. (c)	6. (b)	7. (b)	8. (c)
9. (c)	10. (c)	11. (b)	12. (b)
13. (d)	14. (c)	15. (c)	16. (a)
17. (c)	18. (b,c)	19. (2)	

20. $A = -\frac{3}{2}$, $B = \frac{35}{36}$ and $C \in R$

21. $\frac{1}{6(m+1)} \cdot (2x^{3m} + 3x^{2m} + 6x^m)^{(m+1)/m} + c$

22. $2[\cos^{-1}\sqrt{x} - \log|1 + \sqrt{1 - x}| - \frac{1}{2}\log|x|] + c$

23. $-2\sqrt{1 - x} + \cos^{-1}\sqrt{x} + \sqrt{x(1 - x)} + c$

24. $-\frac{(x^4 + 1)^{1/4}}{x} + c$

25. (i) $4\sin\frac{x}{4} - 4\cos\frac{x}{4} + c$

(ii) $-2\left\{\sqrt{1 - x} - \frac{2}{3}(1 - x)^{3/2} + \frac{1}{5}(1 - x)^{5/2}\right\} + c$

26. $\frac{1}{b^3}\left(a + bx - 2a\log(a + bx) - \frac{a^2}{a + bx} + c\right)$

27. $-\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24} + \sin 2x + \tan x - 2x + \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$

28. $\frac{1}{2}\tan^{-1}(x^2) + c$ **29.** $\frac{1}{2}\log(\sin x - \cos x) + \frac{x}{2} + c$

Topic 2

1. (b) 2. (b)

3. $\frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}+\frac{4}{3}x^{1/2}-\frac{12}{5}x^{5/12}+\frac{1}{2}x^{1/3}-4x^{1/4}-7x^{1/6}$
$-12x^{1/12}+(2x^{1/2}-3x^{1/3}+6x^{1/6}+11)\ln(1+x^{1/6})$
$+12\ln(1+x^{1/2})-3[\ln(1+x^{1/6})]^2+c$

4. $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+c$

5. $-\log|\cot x+\sqrt{\cot^2 x-1}|+\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right|+c$

6. (1)

Topic 3

1. (d) 2. (b) 3. (a) 4. (a)
5. (b) 6. (c)

7. $(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right)-\frac{3}{4}\log(4x^2+8x+13)+c$

8. $\frac{1}{2}\sin 2\theta\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)+\frac{1}{2}\ln(\cos 2\theta)+c$

9. $\frac{2}{\pi}[\sqrt{x-x^2}-(1-2x)\sin^{-1}\sqrt{x}]-x+c$

10. $\frac{e^x}{(x+1)^2}+c$

11. $x\sin x+\cos x-\frac{\cos 2x}{4}+c$

Topic 4

1. (b) 2. (c)

3. $-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+1|+\frac{3}{2}\tan^{-1}x+\frac{x}{x^2+1}+c$

4. $\log\left|\frac{xe^x}{1+xe^x}\right|+\frac{1}{1+xe^x}+c$

Hints & Solutions

Topic 1 Some Standard Results

1. Let $I=\int\frac{\tan x+\tan\alpha}{\tan x-\tan\alpha}dx, \alpha\in\left(0,\frac{\pi}{2}\right)$

$$=\int\frac{\frac{\sin x}{\cos x}+\frac{\sin\alpha}{\cos\alpha}}{\frac{\sin x}{\cos x}-\frac{\sin\alpha}{\cos\alpha}}dx$$

$$=\int\frac{\sin x\cos\alpha+\sin\alpha\cos x}{\sin x\cos\alpha-\sin\alpha\cos x}dx$$

$$=\int\frac{\sin(x+\alpha)}{\sin(x-\alpha)}dx$$

Now, put $x-\alpha=t\Rightarrow dx=dt$, so

$$I=\int\frac{\sin(t+2\alpha)}{\sin t}dt$$

$$=\int\frac{\sin t\cos 2\alpha+\sin 2\alpha\cos t}{\sin t}dt$$

$$=\int\left(\cos 2\alpha+\sin 2\alpha\frac{\cos t}{\sin t}\right)dt$$

$=t(\cos 2\alpha)+(\sin 2\alpha)\log_e|\sin t|+C$
$=(x-\alpha)\cos 2\alpha+(\sin 2\alpha)\log_e|\sin(x-\alpha)|+C$
$=A(x)\cos 2\alpha+B(x)\sin 2\alpha+C$ (given)

Now on comparing, we get
$A(x)=x-\alpha$ and $B(x)=\log_e|\sin(x-\alpha)|$

2. **Key Idea**
(i) Divide each term of numerator and denominator by x^2.
(ii) Let $x^2+\frac{1}{x}=t$

Let integral is $I=\int\frac{2x^3-1}{x^4+x}dx=\int\frac{2x-1/x^2}{x^2+\frac{1}{x}}dx$

[dividing each term of numerator and denominator by x^2]

Put $x^2+\frac{1}{x}=t\Rightarrow\left(2x+\left(-\frac{1}{x^2}\right)\right)dx=dt$

$\therefore\quad I=\int\frac{dt}{t}=\log_e|(t)|+C$

$$=\log_e\left|\left(x^2+\frac{1}{x}\right)\right|+C$$

$$=\log_e\left|\frac{x^3+1}{x}\right|+C$$

3. Let $I=\int\frac{dx}{(x^2-2x+10)^2}=\int\frac{dx}{((x-1)^2+3^2)^2}$

Now, put $x-1=3\tan\theta\Rightarrow dx=3\sec^2\theta\,d\theta$

So, $I=\int\frac{3\sec^2\theta\,d\theta}{(3^2\tan^2\theta+3^2)^2}=\int\frac{3\sec^2\theta\,d\theta}{3^4\sec^4\theta}$

$$= \frac{1}{27}\int \cos^2\theta \, d\theta = \frac{1}{27}\int \frac{1+\cos 2\theta}{2}\, d\theta \quad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \frac{1}{54}\int (1+\cos 2\theta)\, d\theta = \frac{1}{54}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{54}\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{1}{108}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + C \quad \left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= \frac{1}{54}\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{1}{54}\frac{\left(\frac{x-1}{3}\right)}{1+\left(\frac{x-1}{3}\right)^2} + C$$

$$= \frac{1}{54}\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{1}{18}\left(\frac{x-1}{(x-1)^2+3^2}\right) + C$$

$$= \frac{1}{54}\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{1}{18}\left(\frac{x-1}{x^2-2x+10}\right) + C$$

$$= \frac{1}{54}\left[\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{3(x-1)}{x^2-2x+10}\right] + C$$

It is given, that

$$I = A\left[\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{f(x)}{x^2-2x+10}\right] + C$$

On comparing, we get $A = \frac{1}{54}$ and $f(x) = 3(x-1)$.

4. Let $I = \int \frac{dx}{x^3(1+x^6)^{2/3}}$

$$= \int \frac{dx}{x^3 \cdot x^4\left(\frac{1}{x^6}+1\right)^{2/3}} = \int \frac{dx}{x^7\left(\frac{1}{x^6}+1\right)^{2/3}}$$

Now, put $\frac{1}{x^6} + 1 = t^3$

$\Rightarrow \quad -\frac{6}{x^7}\, dx = 3t^2 dt$

$\Rightarrow \quad \frac{dx}{x^7} = -\frac{t^2}{2}\, dt$

So, $I = \int \frac{-\frac{1}{2}t^2 dt}{t^2} = -\frac{1}{2}\int dt$

$$= -\frac{1}{2}t + C = -\frac{1}{2}\left(\frac{1}{x^6}+1\right)^{1/3} + C \quad \left[\because t^3 = \frac{1}{x^6}+1\right]$$

$$= -\frac{1}{2}\frac{1}{x^2}(1+x^6)^{1/3} + C$$

$$= x\cdot f(x)\cdot(1+x^6)^{1/3} + C \quad \text{[given]}$$

On comparing both sides, we get

$$f(x) = -\frac{1}{2x^3}$$

5. Let $I = \int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}\, dx = \int \frac{2\sin\frac{5x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$

[multiplying by $2\cos\frac{x}{2}$ in numerator and denominator]

$$= \int \frac{\sin 3x + \sin 2x}{\sin x} dx$$

$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)$ and $\sin 2A = 2\sin A\cos A]$

$$= \int \frac{(3\sin x - 4\sin^3 x) + 2\sin x\cos x}{\sin x} dx$$

$[\because \sin 3x = 3\sin x - 4\sin^3 x]$

$$= \int (3 - 4\sin^2 x + 2\cos x) dx$$

$$= \int [3 - 2(1-\cos 2x) + 2\cos x] dx$$

$[\because 2\sin^2 x = 1 - \cos 2x]$

$$= \int [3 - 2 + 2\cos 2x + 2\cos x] dx$$

$$= \int [1 + 2\cos 2x + 2\cos x] dx$$

$$= x + 2\sin x + \sin 2x + C$$

6. Let

$$I = \int \frac{3x^{13} + 2x^{11}}{(2x^4+3x^2+1)^4}\, dx = \int \frac{\frac{3}{x^3} + \frac{2}{x^5}}{\left(2+\frac{3}{x^2}+\frac{1}{x^4}\right)^4}\, dx$$

[on dividing numerator and denominator by x^{16}]

Now, put $2 + \frac{3}{x^2} + \frac{1}{x^4} = t$

$$\Rightarrow \left(\frac{-6}{x^3} - \frac{4}{x^5}\right) dx = dt \Rightarrow \left(\frac{3}{x^3} + \frac{2}{x^5}\right) dx = -\frac{dt}{2}$$

So, $I = \int \frac{-dt}{2t^4} = -\frac{1}{2}\times\frac{t^{-4+1}}{-4+1} + C = \frac{1}{6t^3} + C$

$$= \frac{1}{6\left(2+\frac{3}{x^2}+\frac{1}{x^4}\right)^3} + C \quad \left[\because t = 2 + \frac{3}{x^2} + \frac{1}{x^4}\right]$$

$$= \frac{x^{12}}{6(2x^4+3x^2+1)^3} + C$$

7. We have,

$$\int \frac{x+1}{\sqrt{2x-1}}\, dx = f(x)\sqrt{2x-1} + C \quad \text{...(i)}$$

Let $I = \int \frac{x+1}{\sqrt{2x-1}}\, dx$

Put $2x - 1 = t^2 \Rightarrow 2dx = 2tdt \Rightarrow dx = tdt$

$$I = \int \frac{\frac{t^2+1}{2} + 1}{t}\, tdt = \frac{1}{2}\int (t^2+3)\, dt$$

$$\left[\because 2x - 1 = t^2 \Rightarrow x = \frac{t^2+1}{2}\right]$$

$$= \frac{1}{2}\left(\frac{t^3}{3} + 3t\right) + C = \frac{t}{6}(t^2 + 9) + C$$

$$= \frac{\sqrt{2x-1}}{6}(2x - 1 + 9) + C \qquad [\because t = \sqrt{2x-1}\,]$$

$$= \frac{\sqrt{2x-1}}{6}(2x + 8) + C$$

$$= \frac{x+4}{3}\sqrt{2x-1} + C$$

On comparing it with Eq. (i), we get

$$f(x) = \frac{x+4}{3}$$

8. We have,

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx = A(x)\,(\sqrt{1-x^2})^m + C \qquad \text{... (i)}$$

$$\text{Let } I = \int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int \frac{\sqrt{x^2\left(\frac{1}{x^2} - 1\right)}}{x^4}\,dx$$

$$= \int \frac{x\sqrt{\frac{1}{x^2} - 1}}{x^4}\,dx = \int \frac{1}{x^3}\sqrt{\frac{1}{x^2} - 1}\,dx$$

Put $\frac{1}{x^2} - 1 = t^2 \Rightarrow \frac{-2}{x^3}\,dx = 2t\,dt \Rightarrow \frac{1}{x^3}\,dx = -t\,dt$

$$\therefore \quad I = -\int t^2 dt = -\frac{t^3}{3} + C$$

$$= -\frac{1}{3}\cdot\left(\frac{1-x^2}{x^2}\right)^{3/2} + C\left[\because t = \left(\frac{1}{x^2} - 1\right)^{1/2}\right]$$

$$= -\frac{1}{3}\frac{1}{x^3}(\sqrt{1-x^2})^3 + C \qquad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$A(x) = -\frac{1}{3x^3} \text{ and } m = 3$$

$$\therefore \quad (A(x))^m = (A(x))^3 = -\frac{1}{27x^9}$$

9. Let $I = \int \frac{(\sin^n\theta - \sin\theta)^{1/n}\cos\theta}{\sin^{n+1}\theta}\,d\theta$

Put $\sin\theta = t \Rightarrow \cos\theta\,d\theta = dt$

$$\therefore \quad I = \int \frac{(t^n - t)^{1/n}}{t^{n+1}}\,dt$$

$$= \int \frac{\left[t^n\left(1 - \frac{t}{t^n}\right)\right]^{1/n}}{t^{n+1}}\,dt$$

$$= \int \frac{t(1 - 1/t^{n-1})^{1/n}}{t^{n+1}}\,dt = \int \frac{(1 - 1/t^{n-1})^{1/n}}{t^n}\,dt$$

Put $1 - \frac{1}{t^{n-1}} = u$

or $1 - t^{-(n-1)} = u \Rightarrow \frac{(n-1)}{t^n}\,dt = du$

$$\Rightarrow \quad \frac{dt}{t^n} = \frac{du}{n-1}$$

$$\Rightarrow \quad I = \int \frac{u^{1/n}du}{n-1} = \frac{u^{\frac{1}{n}+1}}{(n-1)\left(\frac{1}{n} + 1\right)} + C$$

$$= \frac{n\left(1 - \frac{1}{t^{n-1}}\right)^{\frac{n+1}{n}}}{(n-1)(n+1)} + C$$

$$= \frac{n\left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}}{n^2 - 1} + C$$

$$\left[\because u = 1 - \frac{1}{t^{n-1}} \text{ and } t = \sin\theta\right]$$

10. We have, $f(x) = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)^2}\,dx$

$$= \int \frac{5\left(\frac{x^8}{x^{14}}\right) + 7\left(\frac{x^6}{x^{14}}\right)}{\left(\frac{x^2}{x^7} + \frac{1}{x^7} + \frac{2x^7}{x^7}\right)^2}\,dx$$

(dividing both numerator and denominator by x^{14})

$$= \int \frac{5x^{-6} + 7x^{-8}}{(x^{-5} + x^{-7} + 2)^2}\,dx$$

Let $x^{-5} + x^{-7} + 2 = t$

$\Rightarrow (-5x^{-6} - 7x^{-8})dx = dt$

$\Rightarrow (5x^{-6} + 7x^{-8})dx = -dt$

$$\therefore \; f(x) = \int -\frac{dt}{t^2} = -\int t^{-2}dt$$

$$= -\frac{t^{-2+1}}{-2+1} + C = -\frac{t^{-1}}{-1} + C = \frac{1}{t} + C$$

$$= \frac{1}{x^{-5} + x^{-7} + 2} + C = \frac{x^7}{2x^7 + x^2 + 1} + C$$

$\because \; f(0) = 0$

$$\therefore \quad 0 = \frac{0}{0 + 0 + 1} + C \Rightarrow C = 0$$

$$\therefore \; f(x) = \frac{x^7}{2x^7 + x^2 + 1}$$

$$\Rightarrow \quad f(1) = \frac{1}{2(1)^7 + 1^2 + 1} = \frac{1}{4}$$

11. Let $I = \int x\sqrt{\frac{2\sin(x^2 - 1) - \sin 2(x^2 - 1)}{2\sin(x^2 - 1) + \sin 2(x^2 - 1)}}\,dx$

Put $\frac{x^2 - 1}{2} = \theta \Rightarrow x^2 - 1 = 2\theta \Rightarrow 2x\,dx = 2\,d\theta$

$$\Rightarrow \quad x\,dx = d\theta$$

Now, $I = \int \sqrt{\frac{2\sin 2\theta - \sin 4\theta}{2\sin 2\theta + \sin 4\theta}}\,d\theta$

$$= \int \sqrt{\frac{2\sin 2\theta - 2\sin 2\theta\cos 2\theta}{2\sin 2\theta + 2\sin 2\theta\cos 2\theta}}\,d\theta$$

$(\because \sin 2A = 2\sin A\cos A)$

$$= \int \sqrt{\frac{2\sin 2\theta(1-\cos 2\theta)}{2\sin 2\theta(1+\cos 2\theta)}}\, d\theta$$

$$= \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\, d\theta = \int \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\, d\theta$$

$[\because 1-\cos 2A = 2\sin^2 A \text{ and } 1+\cos 2A = 2\cos^2 A]$

$$= \int \sqrt{\tan^2\theta}\, d\theta = \int \tan\theta\, d\theta$$

$$= \log_e|\sec\theta| + C = \log_e\left|\sec\left(\frac{x^2-1}{2}\right)\right| + C\left[\because \theta = \frac{x^2-1}{2}\right]$$

12. We have,

$$I = \int \frac{\sin^2 x \cdot \cos^2 x}{(\sin^5 x + \cos^3 x \cdot \sin^2 x + \sin^3 x \cdot \cos^2 x + \cos^5 x)^2}\, dx$$

$$= \int \frac{\sin^2 x \cos^2 x}{\{\sin^3 x(\sin^2 x + \cos^2 x) + \cos^3 x(\sin^2 x + \cos^2 x)\}^2}\, dx$$

$$= \int \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2}\, dx = \int \frac{\sin^2 x \cos^2 x}{\cos^6 x(1+\tan^3 x)^2}\, dx$$

$$= \int \frac{\tan^2 x \sec^2 x}{(1+\tan^3 x)^2}\, dx$$

Put $\tan^3 x = t \quad \Rightarrow \quad 3\tan^2 x \sec^2 x dx = dt$

$\therefore \qquad I = \dfrac{1}{3}\displaystyle\int \frac{dt}{(1+t)^2}$

$\Rightarrow \qquad I = \dfrac{-1}{3(1+t)} + C \;\Rightarrow\; I = \dfrac{-1}{3(1+\tan^3 x)} + C$

13. $\displaystyle\int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$

Put $\quad 1 + \dfrac{1}{x^4} = t^4$

$\Rightarrow \qquad \dfrac{-4}{x^5}\, dx = 4t^3\, dt$

$\Rightarrow \qquad \dfrac{dx}{x^5} = -t^3\, dt$

Hence, the integral becomes

$$\int \frac{-t^3\, dt}{t^3} = -\int dt = -t + c = -\left(1+\frac{1}{x^4}\right)^{1/4} + c$$

14. **PLAN** Integration by Substitution

i.e. $\quad I = \int f\{g(x)\}\cdot g'(x)dx$

Put $\quad g(x) = t \Rightarrow g'(x)dx = dt$

$\therefore \quad I = \int f(t)dt$

Description of Situation Generally, students gets confused after substitution, i.e. $\sec x + \tan x = t$.

Now, for $\sec x$, we should use

$$\sec^2 x - \tan^2 x = 1$$

$\Rightarrow \qquad (\sec x - \tan x)(\sec x + \tan x) = 1$

$\Rightarrow \qquad \sec x - \tan x = \dfrac{1}{t}$

Here, $\qquad I = \displaystyle\int \frac{\sec^2 dx}{(\sec x + \tan x)^{9/2}}$

Put $\sec x + \tan x = t$

$\Rightarrow \quad (\sec x \tan x + \sec^2 x)\, dx = dt$

$\Rightarrow \qquad \sec x \cdot t\, dx = dt \;\Rightarrow\; \sec x\, dx = \dfrac{dt}{t}$

$\therefore \qquad \sec x - \tan x = \dfrac{1}{t} \quad\Rightarrow\quad \sec x = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right)$

$\therefore \qquad I = \displaystyle\int \frac{\sec x \cdot \sec x\, dx}{(\sec x + \tan x)^{9/2}}$

$$\Rightarrow I = \int \frac{\frac{1}{2}\left(t+\frac{1}{t}\right)\cdot\frac{dt}{t}}{t^{9/2}} = \frac{1}{2}\int\left(\frac{1}{t^{9/2}} + \frac{1}{t^{13/2}}\right)dt$$

$$= -\frac{1}{2}\left\{\frac{2}{7\,t^{7/2}} + \frac{2}{11\,t^{11/2}}\right\} + K$$

$$= -\left[\frac{1}{7(\sec x + \tan x)^{7/2}} + \frac{1}{11(\sec x + \tan x)^{11/2}}\right] + K$$

$$= \frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$$

15. Since, $\quad I = \displaystyle\int \frac{e^x}{e^{4x} + e^{2x} + 1}\, dx$ and $J = \displaystyle\int \frac{e^{3x}}{1 + e^{2x} + e^{4x}}\, dx$

$\therefore \quad J - I = \displaystyle\int \frac{(e^{3x} - e^x)}{1 + e^{2x} + e^{4x}}\, dx$

Put $\quad e^x = u \;\Rightarrow\; e^x dx = du$

$$\therefore \quad J - I = \int \frac{(u^2-1)}{1+u^2+u^4}\, du = \int \frac{\left(1-\frac{1}{u^2}\right)}{1+\frac{1}{u^2}+u^2}\, du$$

$$= \int \frac{\left(1-\frac{1}{u^2}\right)}{\left(u+\frac{1}{u}\right)^2 - 1}\, du$$

Put $\qquad u + \dfrac{1}{u} = t$

$\Rightarrow \qquad \left(1 - \dfrac{1}{u^2}\right)du = dt$

$$= \int \frac{dt}{t^2-1} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{u^2-u+1}{u^2+u+1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right| + c$$

16. Given, $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$ for $n \ge 2$

$\therefore \quad ff(x) = \dfrac{f(x)}{[1+f(x)^n]^{1/n}} = \dfrac{x}{(1+2x^n)^{1/n}}$

and $\quad fff(x) = \dfrac{x}{(1+3x^n)^{1/n}}$

$\therefore \quad g(x) = \underbrace{(fofo\ldots of)}_{n\,\text{times}}(x) = \dfrac{x}{(1+nx^n)^{1/n}}$

Let $I = \int x^{n-2} g(x)\, dx = \int \frac{x^{n-1}\, dx}{(1 + nx^n)^{1/n}}$

$$= \frac{1}{n^2} \int \frac{n^2 x^{n-1}\, dx}{(1 + nx^n)^{1/n}} = \frac{1}{n^2} \int \frac{\frac{d}{dx}(1 + nx^n)}{(1 + nx^n)^{1/n}}\, dx$$

$$I = \frac{1}{n(n-1)} (1 + nx^n)^{1 - \frac{1}{n}} + c$$

17. Let $I = \int \frac{(x^2 - 1)\, dx}{x^3 \sqrt{2x^4 - 2x^2 + 1}}$

[dividing numerator and enominator by x^5]

$$= \int \frac{\left(\frac{1}{x^3} - \frac{1}{x^5}\right) dx}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}$$

Put $2 - \frac{2}{x^2} + \frac{1}{x^4} = t$

$\Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5}\right) dx = dt$

$\therefore\ I = \frac{1}{4} \int \frac{dt}{\sqrt{t}} = \frac{1}{4} \cdot \frac{t^{1/2}}{1/2} + c$

$$= \frac{1}{2} \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + c$$

18. We have, $f'(x) = e^{(f(x) - g(x))}\, g'(x)\ \forall\ x \in R$

$\Rightarrow f'(x) = \frac{e^{f(x)}}{e^{g(x)}}\, g'(x)$

$\Rightarrow \frac{f'(x)}{e^{f(x)}} = \frac{g'(x)}{e^{g(x)}}$

$\Rightarrow e^{-f(x)} f'(x) = e^{-g(x)}\, g'(x)$

On integrating both side, we get

$e^{-f(x)} = e^{-g(x)} + C$

At $x = 1$

$e^{-f(1)} = e^{-g(1)} + C$

$e^{-1} = e^{-g(1)} + C$ $\quad [\because f(1) = 1]$ ...(i)

At $x = 2$

$e^{-f(2)} = e^{-g(2)} + C$

$\Rightarrow e^{-f(2)} = e^{-1} + C$ $\quad [\because g(2) = 1]$ ...(ii)

From Eqs. (i) and (ii)

$e^{-f(2)} = 2e^{-1} - e^{-g(1)}$ ...(iii)

$\Rightarrow e^{-f(2)} > 2e^{-1}$

We know that, e^{-x} is decreasing

$\therefore\ -f(2) < \log_e 2 - 1$

$f(2) > 1 - \log_e 2$

$\Rightarrow e^{-g(1)} + e^{-f(2)} = 2e^{-1}$ [from Eq. (iii)]

$\Rightarrow e^{-g(1)} < 2\, e^{-1}$

$- g(1) < \log_e 2 - 1$

$\Rightarrow g(1) > 1 - \log_e 2$

19. Given,

$f(x + y) = f(x)\, f'(y) + f'(x) f(y), \forall x, y \in R$

and $f(0) = 1$

Put $x = y = 0$, we get

$f(0) = f(0)\, f'(0) + f'(0)\, f(0)$

$\Rightarrow 1 = 2f'(0) \Rightarrow f'(0) = \frac{1}{2}$

Put $x = x$ and $y = 0$, we get

$f(x) = f(x)\, f'(0) + f'(x)\, f(0)$

$\Rightarrow f(x) = \frac{1}{2} f(x) + f'(x)$

$\Rightarrow f'(x) = \frac{1}{2} f(x) \Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{2}$

On integrating, we get

$\log f(x) = \frac{1}{2} x + C$

$\Rightarrow f(x) = Ae^{\frac{1}{2}x}$, where $e^C = A$

If $f(0) = 1$, then $A = 1$

Hence, $f(x) = e^{\frac{1}{2}x}$

$\Rightarrow \log_e f(x) = \frac{1}{2} x$

$\Rightarrow \log_e f(4) = \frac{1}{2} \times 4 = 2$

20. Given, $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\, dx = Ax + B \log(9e^{2x} - 4) + c$

LHS $= \int \frac{4e^{2x} + 6}{9e^{2x} - 4}\, dx$

Let $4e^{2x} + 6 = A(9e^{2x} - 4) + B(18\, e^{2x})$

$\Rightarrow 9A + 18B = 4$ and $-4A = 6$

$\Rightarrow A = -\frac{3}{2}$ and $B = \frac{35}{36}$

$\therefore\ \int \frac{A(9e^{2x} - 4) + B(18e^{2x})}{9e^{2x} - 4}\, dx = A\int 1\, dx + B \int \frac{1}{t}\, dt$

where $t = 9e^{2x} - 4$

$= A x + B \log(9e^{2x} - 4) + c$

$= -\frac{3}{2} x + \frac{35}{36} \log(9e^{2x} - 4) + c$

$\therefore\ A = -\frac{3}{2}, B = \frac{35}{36}$

and c = any real number

21. For any natural number m, the given integral can be written as

$$I = \int (x^{3m} + x^{2m} + x^m) \frac{(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}}{x}\, dx$$

$$\Rightarrow I = \int (2\, x^{3m} + 3x^{2m} + 6\, x^m)^{1/m} (x^{3m-1} + x^{2m-1} + x^{m-1})\, dx$$

Put $2\, x^{3m} + 3x^{2m} + 6x^m = t$

$\Rightarrow (6mx^{3m-1} + 6mx^{2m-1} + 6mx^{m-1})\, dx = dt$

$$\therefore\ I = \int t^{1/m} \frac{dt}{6m} = \frac{1}{6m} \cdot \frac{t^{\frac{1}{m} + 1}}{\left(\frac{1}{m} + 1\right)}$$

$$= \frac{1}{6(m+1)} \cdot (2\, x^{3m} + 3x^{2m} + 6\, x^m)^{(m+1)/m} + c$$

22. Let $I=\int\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\cdot\frac{dx}{x}$

Put $x=\cos^2\theta \Rightarrow dx=-2\cos\theta\sin\theta\, d\theta$

$\therefore \quad I=\int\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}\cdot\frac{-2\cos\theta\cdot\sin\theta}{\cos^2\theta}\,d\theta$

$$=\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot\frac{-2\sin\theta}{\cos\theta}\,d\theta$$

$$=-\int\frac{2\sin\frac{\theta}{2}\cdot 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}\cdot\cos\theta}\,d\theta-2\int\frac{2\sin^2\frac{\theta}{2}}{\cos\theta}\,d\theta$$

$$=-2\int\frac{1-\cos\theta}{\cos\theta}\,d\theta$$

$$=2\int(1-\sec\theta)\,d\theta=2\,[\theta-\log|\sec\theta+\tan\theta|]+c$$

$\Rightarrow \quad I=2\left[\cos^{-1}\sqrt{x}-\log\left|\frac{1}{\sqrt{x}}+\sqrt{\frac{1}{x}-1}\right|\right]+c$

$\Rightarrow \quad I=2\left[\cos^{-1}\sqrt{x}-\log|1+\sqrt{1-x}|-\frac{1}{2}\log|x|\right]+c$

23. Let $I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Put $x=\cos^2\theta\Rightarrow dx=-2\sin\theta\cos\theta\,d\theta$

$\therefore \quad I=\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\cdot(-2\sin\theta\cos\theta)\,d\theta$

$$=-\int 2\tan\frac{\theta}{2}\cdot\sin\theta\cos\theta\,d\theta=-2\int 2\sin^2\frac{\theta}{2}\cdot\cos\theta\,d\theta$$

$$=-2\int(1-\cos\theta)\cos\theta\,d\theta=-2\int(\cos\theta-\cos^2\theta)\,d\theta$$

$$=-2\int\cos\theta\,d\theta+\int(1+\cos 2\theta)\,d\theta$$

$$=-2\sin\theta+\theta+\frac{\sin 2\theta}{2}+c$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+c$$

24. Let $I=\int\frac{dx}{x^2(x^4+1)^{3/4}}=\int\frac{dx}{x^2\cdot x^3\left(1+\frac{1}{x^4}\right)^{3/4}}$

Put $1+x^{-4}=t \Rightarrow -\frac{4}{x^5}\,dx=dt$

$\therefore \quad I=-\frac{1}{4}\int\frac{dt}{t^{3/4}}=-\frac{1}{4}\cdot\frac{t^{1/4}}{1/4}+c=-\left(1+\frac{1}{x^4}\right)^{1/4}+c$

$$=-\frac{(x^4+1)^{1/4}}{x}+c$$

25. (i) Let $I=\int\sqrt{1+\sin\frac{x}{2}}\,dx$

$$=\int\sqrt{\cos^2\frac{x}{4}+\sin^2\frac{x}{4}+2\sin\frac{x}{4}\cos\frac{x}{4}}\,dx$$

$$=\int\left(\cos\frac{x}{4}+\sin\frac{x}{4}\right)dx=4\sin\frac{x}{4}-4\cos\frac{x}{4}+c$$

(ii) Let $I=\int\frac{x^2}{\sqrt{1-x}}\,dx$

Put $1-x=t^2 \Rightarrow -dx=2t\,dt$

$\therefore \quad I=\int\frac{(1-t^2)^2\cdot(-2t)}{t}\,dt$

$$=-2\int(1-2t^2+t^4)\,dt$$

$$=-2\left(t-\frac{2t^3}{3}+\frac{t^5}{5}\right)+c$$

$$=-2\left\{\sqrt{1-x}-\frac{2}{3}(1-x)^{3/2}+\frac{1}{5}(1-x)^{5/2}\right\}+c$$

26. Let $I=\frac{x^2}{(a+bx)^2}$

Put $a+bx=t \Rightarrow b\,dx=dt$

$\therefore \quad I=\int\frac{\left(\frac{t-a}{b}\right)^2}{t^2}\cdot\frac{dt}{b}=\frac{1}{b^3}\int\left(\frac{t^2-2at+a^2}{t^2}\right)dt$

$$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt$$

$$=\frac{1}{b^3}\left(t-2a\log t-\frac{a^2}{t}\right)+c$$

$$=\frac{1}{b^3}\left(a+bx-2a\log(a+bx)-\frac{a^2}{a+bx}+c\right)$$

27. Let $I_1=\int\sin x\sin 2x\sin 3x\,dx$

$$=\frac{1}{4}\int(\sin 4x+\sin 2x-\sin 6x)\,dx$$

$$=-\frac{\cos 4x}{16}-\frac{\cos 2x}{8}+\frac{\cos 6x}{24}$$

$I_2=\int\sec^2 x\cdot\cos^2 2x\,dx$

$$=\int\sec^2 x(2\cos^2 x-1)^2dx$$

$$=\int(4\cos^2 x+\sec^2 x-4)\,dx$$

$$=\int(2\cos 2x+\sec^2 x-2)\,dx$$

$$=\sin 2x+\tan x-2x$$

and $I_3=\int\sin^4 x\cos^4 x\,dx$

$$=\frac{1}{128}\int(3-4\cos 4x+\cos 8x)\,dx$$

$$=\frac{3x}{128}-\frac{\sin 4x}{128}+\frac{\sin 8x}{1024}$$

$\therefore \quad I=I_1+I_2+I_3$

$$=-\frac{\cos 4x}{16}-\frac{\cos 2x}{8}+\frac{\cos 6x}{24}+\sin 2x+\tan x-2x+\frac{3x}{128}-\frac{\sin 4x}{128}+\frac{\sin 8x}{1024}$$

28. Let $I=\int \frac{x\,dx}{1+x^4}=\frac{1}{2}\int \frac{2x}{1+(x^2)^2}\,dx$

Put $x^2=u \Rightarrow 2x\,dx=du$

$\therefore \quad I=\frac{1}{2}\int \frac{du}{1+u^2}=\frac{1}{2}\tan^{-1}(u)+c=\frac{1}{2}\tan^{-1}(x^2)+c$

29. Let $I=\int \frac{\sin x}{\sin x-\cos x}\,dx$

Again, let $\sin x=A(\cos x+\sin x)+B(\sin x-\cos x)$, then $A+B=1$ and $A-B=0$

$\Rightarrow \quad A=\frac{1}{2},\ B=\frac{1}{2}$

$$\therefore \quad I=\int \frac{\frac{1}{2}(\cos x+\sin x)+\frac{1}{2}(\sin x-\cos x)}{(\sin x-\cos x)}\,dx$$

$$=\frac{1}{2}\int \frac{\cos x+\sin x}{\sin x-\cos x}\,dx+\frac{1}{2}\int 1\,dx+c$$

$$=\frac{1}{2}\log(\sin x-\cos x)+\frac{1}{2}x+c$$

Topic 2 Some Special Integrals

1. Let $I=\int \sec^{\frac{2}{3}}x\,\mathrm{cosec}^{\frac{4}{3}}x\,dx=\int \frac{dx}{\cos^{\frac{2}{3}}x\sin^{\frac{4}{3}}x}$

$$\int \frac{dx}{\left(\frac{\sin x}{\cos x}\right)^{\frac{4}{3}}\cos^{\frac{4}{3}}x\cos^{\frac{2}{3}}x}$$

[dividing and multiplying by $\cos^{4/3}x$ in denominator]

$$=\int \frac{dx}{\tan^{\frac{4}{3}}x\cos^2 x}=\int \frac{\sec^2 x\,dx}{(\tan x)^{\frac{4}{3}}}$$

Now, put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$$\therefore\ I=\int \frac{dt}{t^{4/3}}=\frac{t^{\frac{-4}{3}+1}}{\frac{-4}{3}+1}+C$$

$$=-3\frac{1}{t^{\frac{1}{3}}}+C=\frac{-3}{(\tan x)^{\frac{1}{3}}}+C=-3\tan^{-\frac{1}{3}}x+C$$

2. We have, $I_n=\int \tan^n x\,dx$

$$\therefore\ I_n+I_{n+2}=\int \tan^n x\,dx+\int \tan^{n+2}x\,dx$$

$$=\int \tan^n x(1+\tan^2 x)\,dx$$

$$=\int \tan^n x\sec^2 x\,dx=\frac{\tan^{n+1}x}{n+1}+C$$

Put $n=4$, we get $I_4+I_6=\frac{\tan^5 x}{5}+C$

$\therefore \quad a=\frac{1}{5}$ and $b=0$

3. Let $I=\int \left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}+\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$

$\therefore \quad I=I_1+I_2$

where, $I_1=\int \left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}\right)dx$,

$$I_2=\int \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\,dx$$

Now, $I_1=\int \left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}\right)dx$

Put $x=t^{12} \Rightarrow dx=12t^{11}dt$

$$\therefore \quad I_1=12\int \frac{t^{11}}{t^4+t^3}\,dt$$

$$=12\int \frac{t^8\,dt}{t+1}$$

$$=12\int (t^7-t^6+t^5-t^4+t^3-t^2+t-1)\,dt+12\int \frac{dt}{t+1}$$

$$=12\left(\frac{t^8}{8}-\frac{t^7}{7}+\frac{t^6}{6}-\frac{t^5}{5}+\frac{t^4}{4}-\frac{t^3}{3}+\frac{t^2}{2}-t\right)+12\ln(t+1)$$

and $I_2=\int \left\{\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right\}dx$

Put $x=u^6 \Rightarrow dx=6u^5\,du$

$$\therefore \quad I_2=\int \frac{\ln(1+u)}{u^2+u^3}6u^5\,du=\int \frac{\ln(1+u)}{u^2(1+u)}.6u^5\,du$$

$$=6\int \frac{u^3}{(u+1)}\ln(1+u)\,du$$

$$=6\int \left(\frac{u^3-1+1}{u+1}\right)\ln(1+u)\,du$$

$$=6\int \left(u^2-u+1-\frac{1}{u+1}\right)\ln(1+u)\,du$$

$$=6\int \underset{\text{II}}{(u^2-u+1)}\underset{\text{I}}{\ln(1+u)}\,du-6\int \frac{\ln(1+u)}{(u+1)}\,du$$

$$=6\left(\frac{u^3}{3}-\frac{u^2}{2}+u\right)\ln(1+u)-\int \frac{2u^3-3u^2+6u}{u+1}\,du-6\,\frac{1}{2}[\ln(1+u)]^2$$

$$=(2u^3-3u^2+6u)\ln(1+u)-\int \left(2u^2-5u+\frac{11u}{u+1}\right)du-3[\ln(1+u)]^2$$

$$=(2u^3-3u^2+6u)\ln(1+u)-\left(\frac{2u^3}{3}-\frac{5}{2}u^2+11u-11\ln(u+1)\right)-3[\ln(1+u)]^2$$

$$\therefore \quad I = \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3} - 4x^{1/4}$$
$$-6x^{1/6} - 12x^{1/12} + 12\ln(x^{1/12}+1)$$
$$+(2x^{1/2} - 3x^{1/3} + 6x^{1/6})\ln(1+x^{1/6})$$
$$-\left[\frac{2}{3}x^{1/2} - \frac{5}{2}x^{1/3}\, 11x^{1/6} - 11\ln(1+x^{1/6})\right]$$
$$-3[\ln(1+x^{1/6})]^2 + c$$
$$= \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + \frac{4}{3}x^{1/2} - \frac{12}{5}x^{5/12}$$
$$+\frac{1}{2}x^{1/3} - 4x^{1/4} - 7x^{1/6} - 12x^{1/12}$$
$$+(2x^{1/2} - 3x^{1/3} + 6x^{1/6} + 11)\ln(1+x^{1/6})$$
$$+12\ln(1+x^{1/12}) - 3[\ln(1+x^{1/6})]^2 + c$$

4. Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \int \frac{\tan x + 1}{\sqrt{\tan x}}\, dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

$\Rightarrow \quad dx = \frac{2t}{1+t^4}\, dt$

$$\therefore \quad I = \int \frac{t^2+1}{\sqrt{t^2}} \cdot \frac{2t}{t^4+1}\, dt = 2\int \frac{t^2+1}{t^4+1}\, dt$$
$$= 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}\, dt = 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2 + (\sqrt{2})^2}\, dt$$

Put $t - \frac{1}{t} = u \Rightarrow 1 + \frac{1}{t_2}\, dt = du$

$$\therefore \quad I = 2\int \frac{du}{u^2 + (\sqrt{2})^2}$$
$$\Rightarrow \quad I = \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$
$$= \sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + c$$

5. Let $I = \int \frac{\sqrt{\cos 2x}}{\sin x}\, dx = \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}\, dx$
$$= \int \sqrt{\cot^2 x - 1}\, dx$$

Put $\cot x = \sec\theta \quad \Rightarrow \quad -\operatorname{cosec}^2 x\, dx = \sec\theta\tan\theta\, d\theta$

$$\therefore \quad I = \int \sqrt{\sec^2\theta - 1} \cdot \frac{\sec\theta \cdot \tan\theta}{-(1+\sec^2\theta)}\, d\theta$$
$$= -\int \frac{\sec\theta \cdot \tan^2\theta}{1+\sec^2\theta}\, d\theta$$
$$= -\int \frac{\sin^2\theta}{\cos\theta + \cos^3\theta}\, d\theta$$
$$= -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\, d\theta$$
$$= -\int \frac{(1+\cos^2\theta) - 2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\, d\theta$$
$$= -\int \sec\theta\, d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta}\, d\theta$$
$$= -\log|\sec\theta + \tan\theta| + 2\int \frac{\cos\theta}{2-\sin^2\theta}\, d\theta$$
$$= -\log|\sec\theta + \tan\theta| + \int \frac{dt}{2-t^2}, \text{ where } \sin\theta = t$$
$$= -\log|\sec\theta + \tan\theta| + 2\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + c$$
$$= -\log|\cot x + \sqrt{\cot^2 x - 1}|$$
$$+ \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + c$$

6. Given, $f(x) = \int \left(\frac{2\sin x - \sin 2x}{x^3}\right) dx$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{2\sin x - \sin 2x}{x^3} = \frac{2\sin x}{x}\left(\frac{1-\cos x}{x^2}\right)$$
$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)\left(\frac{2\sin^2\frac{x}{2}}{x^2}\right)$$
$$= 4\cdot 1\cdot \lim_{x\to 0}\left[\frac{\sin^2\frac{x}{2}}{4\times\left(\frac{x}{2}\right)^2}\right] = 1$$

Topic 3 Integration by Parts

1. Let given integral, $I = \int x^5 e^{-x^2}\, dx$

Put $x^2 = t \Rightarrow 2x dx = dt$

So, $\quad I = \frac{1}{2}\int t^2 e^{-t}\, dt$

$$= \frac{1}{2}[(-t^2 e^{-t}) + \int e^{-t}(2t)\, dt] \quad \text{[Integration by parts]}$$
$$= \frac{1}{2}[-t^2 e^{-t} + 2t(-e^{-t}) + \int 2e^{-t}\, dt]$$
$$= \frac{1}{2}[-t^2 e^{-t} - 2te^{-t} - 2e^{-t}] + C$$
$$= -\frac{e^{-t}}{2}(t^2 + 2t + 2) + C$$
$$= -\frac{e^{-x^2}}{2}(x^4 + 2x^2 + 2) + C \quad [\because t = x^2] \qquad \ldots(\text{i})$$

$\because$ It is given that,
$$I = \int x^5 e^{-x^2}\, dx = g(x)\cdot e^{-x^2} + C$$

By Eq. (i), comparing both sides, we get
$$g(x) = -\frac{1}{2}(x^4 + 2x^2 + 2)$$

So, $\quad g(-1) = -\frac{1}{2}(1+2+2) = -\frac{5}{2}$

2. Given, $\int e^{\sec x}[(\sec x\tan x)f(x)+(\sec x\tan x+\sec^2 x)]dx$

$$= e^{\sec x}\cdot f(x)+C$$

On differentiating both sides w.r.t. x, we get

$e^{\sec x}[(\sec x\tan x)f(x)+(\sec x\tan x+\sec^2 x)]$

$= e^{\sec x}f'(x)+e^{\sec x}(\sec x\tan x)f(x)$

$\Rightarrow \quad e^{\sec x}(\sec x\tan x+\sec^2 x)=e^{\sec x}f'(x)$

$\Rightarrow \quad f'(x)=\sec x\tan x+\sec^2 x$

So, $\quad f(x)=\int f'(x)dx=\int(\sec x\tan x+\sec^2 x)dx$

$$=\sec x+\tan x+C$$

So, possible value of $f(x)$ from options, is

$$f(x)=\sec x+\tan x+\frac{1}{2}.$$

3. Let $I=\int\cos(\log_e x)dx$

$$= x\cos(\log_e x)-\int x(-\sin(\log_e x))\frac{1}{x}\cdot dx$$

[using integration by parts]

$$= x\cos(\log_e x)+\int\sin(\log_e x)\,dx$$

$$= x\cos(\log_e x)+x\sin(\log_e x)-\int x(\cos(\log_e x))\frac{1}{x}\,dx$$

[again, using integration by parts]

$\Rightarrow \quad I = x\cos(\log_e x)+x\sin((\log_e x)-I$

$\Rightarrow \quad I=\frac{x}{2}[\cos(\log_e x)+\sin(\log_e x)]+C.$

4. Given, $\int x^5 e^{-4x^3}\,dx=\frac{1}{48}e^{-4x^3}\,f(x)+C$

In LHS, put $x^3=t$

$\Rightarrow \quad 3x^2dx=dt$

So, $\int x^5e^{-4x^3}dx=\frac{1}{3}\int t\,e^{-4t}dt$

$$=\frac{1}{3}\left[t\,\frac{e^{-4t}}{-4}-\int\frac{e^{-4t}}{-4}\,dt\right]$$

[using integration by parts]

$$=\frac{1}{3}\left[\frac{te^{-4t}}{-4}+\frac{e^{-4t}}{-16}\right]+C$$

$$=-\frac{1}{48}e^{-4t}[4t+1]+C$$

$$=-\frac{e^{-4x^3}}{48}[4x^3+1]+C \qquad [\because t=x^3]$$

$\therefore \quad f(x)=-1-4x^3$ (comparing with given equation)

5. $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

$$=\int e^{x+\frac{1}{x}}dx+\int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx$$

$$=\int e^{x+\frac{1}{x}}dx+xe^{x+\frac{1}{x}}-\int\frac{d}{dx}(x)e^{x+\frac{1}{x}}dx$$

$$=\int e^{x+\frac{1}{x}}dx+xe^{x+\frac{1}{x}}-\int e^{x+\frac{1}{x}}\,dx$$

$$\left[\because\int\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx=e^{x+\frac{1}{x}}\right]$$

$$=\int e^{x+\frac{1}{x}}\,dx+xe^{x+\frac{1}{x}}-\int ex^{x+\frac{1}{x}}\,dx=xe^{x+\frac{1}{x}}+c$$

6. Given, $\int f(x)\,dx=\psi(x)$

Let $\quad I=\int x^5 f(x^3)\,dx$

Put $\quad x^3=t$

$\Rightarrow \quad x^2dx=\frac{dt}{3}$ ...(i)

$\therefore \quad I=\frac{1}{3}\int t\,f(t)\,dt$

$$=\frac{1}{3}\left[t\cdot\int f(t)\,dt-\int\left\{\frac{d}{dt}(t)\int f(t)\,dt\right\}dt\right]$$

[integration by parts]

$$=\frac{1}{3}[t\,\psi(t)-\int\psi(t)\,dt]$$

$$=\frac{1}{3}[x^3\psi(x^3)-3\int x^2\psi(x^3)\,dx]+c \qquad \text{[from Eq. (i)]}$$

$$=\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)\,dx+c$$

7. Let $\quad I=\int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$$=\int\sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+9}}\right)dx$$

Put $2x+2=3\tan\theta\Rightarrow 2\,dx=3\sec^2\theta\,d\theta$

$\therefore \quad I=\int\sin^{-1}\left(\frac{3\tan\theta}{\sqrt{9\tan^2\theta+9}}\right)\cdot\frac{3}{2}\sec^2\theta\,d\theta$

$$=\int\sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$=\int\sin^{-1}\left(\frac{\sin\theta}{\cos\theta\cdot\sec\theta}\right)\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$=\frac{3}{2}\int\sin^{-1}(\sin\theta)\cdot\sec^2\theta\,d\theta$$

$$=\frac{3}{2}\int\theta\cdot\sec^2\theta\,d\theta=\frac{3}{2}[\theta\cdot\tan\theta-\int 1\cdot\tan\theta d\theta]$$

$$=\frac{3}{2}[\theta\tan\theta-\log\sec\theta]+c$$

$$=\frac{3}{2}\left[\tan^{-1}\left(\frac{2x+2}{3}\right)\cdot\left(\frac{2x+2}{3}\right)-\log\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right]+c_1$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log\left[1+\left(\frac{2x+2}{3}\right)^2\right] + c_1$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + c$$

$$\left[\text{let } \frac{3}{2}\log 3 + c_1 = c\right]$$

8. $I = \int \cos 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) d\theta$ [given]

We integrate it by taking parts

$\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$ as first function

$$= \frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

$$-\frac{1}{2}\int \frac{d}{d\theta}\left[\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right]\sin 2\theta d\theta \quad \text{...(i)}$$

But $\frac{d}{d\theta}\left[\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right]$

$$= \frac{d}{d\theta}[\ln(\cos\theta+\sin\theta) - \ln(\cos\theta-\sin\theta)]$$

$$= \frac{1}{(\cos\theta+\sin\theta)}.(-\sin\theta+\cos\theta) - \frac{(-\sin\theta-\cos\theta)}{\cos\theta-\sin\theta}$$

$$= \frac{(\cos\theta-\sin\theta)(\cos\theta-\sin\theta) - (\cos\theta+\sin\theta)(-\sin\theta-\cos\theta)}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}$$

$$= \frac{\cos^2\theta - \cos\theta\sin\theta - \sin\theta\cos\theta + \sin^2\theta + \cos\theta\sin\theta + \cos^2\theta + \sin^2\theta + \cos\theta\cdot\sin\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{2(\cos^2\theta+\sin^2\theta)}{\cos 2\theta} = \frac{2}{\cos 2\theta}$$

Therefore, on putting this value in Eq.(i), we get

$$I = \frac{1}{2}\sin 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \frac{1}{2}\int \sin 2\theta \frac{2}{\cos 2\theta} d\theta$$

$$= \frac{1}{2}\sin 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) + \frac{1}{2}\ln(\cos 2\theta) + c$$

9. Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

$$= \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}} dx$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - x + c \quad \text{...(i)}$$

Now, $\int \sin^{-1}\sqrt{x}\, dx$

Put $x = \sin^2\theta \Rightarrow dx = \sin 2\theta$

$$= \int \theta\cdot\sin 2\theta\, d\theta = -\frac{\theta\cos 2\theta}{2} + \int\frac{1}{2}\cos 2\theta\, d\theta$$

$$= -\frac{\theta}{2}\cos 2\theta + \frac{1}{4}\sin 2\theta$$

$$= -\frac{1}{2}\theta(1-2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}(1-2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x} \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$I = \frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2}\right] - x + c$$

$$= \frac{2}{\pi}[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}] - x + c$$

10. Let $I = \int \frac{(x-1)e^x}{(x+1)^3} dx$

$$I = \int\left\{\frac{x+1-2}{(x+1)^3}\right\}e^x\, dx = \int\left\{\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right\}e^x\, dx$$

$$= \int e^x\cdot\frac{1}{(x+1)^2}dx - 2\int e^x\cdot\frac{1}{(x+1)^3}dx$$

Applying integration by parts,

$$= \left\{\frac{1}{(x+1)^2}\cdot e^x - \int e^x\cdot\frac{-2}{(x+1)^3}dx\right\}$$

$$-2\int e^x\cdot\frac{1}{(x+1)^3}dx = \frac{e^x}{(x+1)^2} + c$$

11. Let $I = \int (e^{\log x} + \sin x)\cos x\, dx$

$$= \int (x + \sin x)\cos x\, dx$$

$$= \int x\cos x\, dx + \frac{1}{2}\int(\sin 2x)\, dx$$

$$= (x\cdot\sin x - \int 1\cdot\sin x\, dx) - \frac{\cos 2x}{4} + c$$

$$= x\sin x + \cos x - \frac{\cos 2x}{4} + c$$

Topic 4 Integration, Irrational Function and Partial Fraction

1. Let $I = \int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx = \int\frac{2x^{12}+5x^9}{x^{15}(1+x^{-2}+x^{-5})^3}dx$

$$= \int\frac{2x^{-3}+5x^{-6}}{(1+x^{-2}+x^{-5})^3}dx$$

Now, put $1 + x^{-2} + x^{-5} = t$

$\Rightarrow (-2x^{-3} - 5x^{-6})\, dx = dt$

$\Rightarrow (2x^{-3} + 5x^{-6})\, dx = -dt$

$\therefore I = -\int\frac{dt}{t^3} = -\int t^{-3}\, dt$

$$= -\frac{t^{-3+1}}{-3+1} + C = \frac{1}{2t^2} + C$$

$$= \frac{x^{10}}{2(x^5+x^3+1)^2} + C$$

2. Let $$I=\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}\,dx$$

$$=\int \frac{(\cos^2 x+\cos^4 x)\cdot \cos x\,dx}{(\sin^2 x+\sin^4 x)}$$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\therefore \quad I=\int \frac{[(1-t^2)+(1-t^2)^2]}{t^2+t^4}\,dt$$

$$\Rightarrow \quad I=\int \frac{1-t^2+1-2t^2+t^4}{t^2+t^4}\,dt$$

$$\Rightarrow \quad I=\int \frac{2-3t^2+t^4}{t^2(t^2+1)}\,dt \qquad \ldots\text{(i)}$$

Using partial fraction for

$$\frac{y^2-3y+2}{y(y+1)}=1+\frac{A}{y}+\frac{B}{y+1} \qquad [\text{where, } y=t^2]$$

$$\Rightarrow \quad A=2, B=-6$$

$$\therefore \quad \frac{y^2-3y+2}{y(y+1)}=1+\frac{2}{y}-\frac{6}{y+1}$$

Now, Eq. (i) reduces to, $I=\int \left(1+\frac{2}{t^2}-\frac{6}{1+t^2}\right)dt$

$$=t-\frac{2}{t}-6\tan^{-1}(t)+c$$

$$=\sin x-\frac{2}{\sin x}-6\tan^{-1}(\sin x)+c$$

3. $$\frac{x^3+3x+2}{(x^2+1)^2(x+1)}=\frac{x^3+2x+x+2}{(x^2+1)^2(x+1)}$$

$$=\frac{x(x^2+1)+2(x+1)}{(x^2+1)^2(x+1)}$$

$$=\frac{x}{(x^2+1)(x+1)}+\frac{2}{(x^2+1)^2}$$

Again, $$\frac{x}{(x^2+1)(x+1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x+1)}$$

$$\Rightarrow \quad x=(Ax+B)(x+1)+C(x^2+1)$$

On putting $x=-1$, we get

$$-1=2C \Rightarrow C=-1/2$$

On equating coefficients of x^2, we get

$$0=A+C$$

$$\Rightarrow \quad A=-C=1/2$$

On putting $x=0$, we get

$$0=B+C$$

$$\Rightarrow \quad B=-C=1/2$$

$$\frac{x^3+3x+2}{(x^2+1)^2(x+1)}=\frac{x+1}{2(x^2+1)}-\frac{1}{2(x+1)}+\frac{2}{(x^2+1)^2}$$

$$\therefore \quad I=\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx$$

$$=-\frac{1}{2}\int \frac{dx}{x+1}+\frac{1}{2}\int \frac{x+1}{x^2+1}\,dx+2\int \frac{dx}{(x^2+1)^2}$$

$$\Rightarrow I=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+1|$$

$$+\frac{1}{2}\tan^{-1}x+2I_1 \qquad \ldots\text{(i)}$$

where, $I_1=\int \frac{dx}{(x^2+1)^2}$

Put $\quad x=\tan\theta$

$$\Rightarrow \quad dx=\sec^2\theta\,d\theta$$

$$\therefore \quad I_1=\int \frac{\sec^2\theta d\theta}{(\tan^2\theta+1)^2}=\int \cos^2\theta\,d\theta=\frac{1}{2}\int (1+\cos 2\theta)d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]$$

$$=\frac{1}{2}\theta+\frac{1}{2}\cdot\frac{\tan\theta}{(1+\tan^2\theta)}$$

$$=\frac{1}{2}\tan^{-1}x+\frac{1}{2}\cdot\frac{x}{(1+x^2)}$$

From Eq. (i),

$$I=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+1|+\frac{3}{2}\tan^{-1}x+\frac{x}{x^2+1}+c$$

4. Let $I=\int \frac{(x+1)}{x(1+xe^x)^2}\,dx=\int \frac{e^x(x+1)}{xe^x(1+xe^x)^2}\,dx$

Put $1+xe^x=t \Rightarrow (e^x+xe^x)\,dx=dt$

$$\therefore \quad I=\int \frac{dt}{(t-1)t^2}=\int \left[\frac{1}{t-1}-\frac{1}{t}-\frac{1}{t^2}\right]dt$$

$$=\log|t-1|-\log|t|+\frac{1}{t}+c$$

$$=\log\left|\frac{t-1}{t}\right|+\frac{1}{t}+c$$

$$=\log\left|\frac{xe^x}{1+xe^x}\right|+\frac{1}{1+xe^x}+c$$