7

Gravitation

Topic 1 Gravitational Force and Acceleration due to Gravity

Objective Questions I (Only one correct option)

1 The ratio of the weights of a body on the earth's surface, so that on the surface of a planet is 9 : 4. The mass of the planet is $\frac{1}{9}$th of that of the earth. If R is the radius of the earth, what is the radius of the planet? (Take, the planets to have the same mass density) **(2019 Main, 12 April II)**

(a) $\frac{R}{3}$ (b) $\frac{R}{4}$

(c) $\frac{R}{9}$ (d) $\frac{R}{2}$

2 The time dependence of the position of a particle of mass $m = 2$ is given by $\mathbf{r}(t) = 2t\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}}$. Its angular momentum, with respect to the origin, at time $t = 2$ is **(2019 Main, 10 April II)**

(a) $36\,\hat{\mathbf{k}}$ (b) $-34\,(\hat{\mathbf{k}} - \hat{\mathbf{i}})$

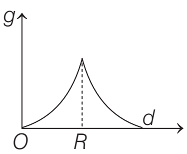
(c) $-48\,\hat{\mathbf{k}}$ (d) $48\,(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

3 The value of acceleration due to gravity at earth's surface is 9.8 ms^{-2}. The altitude above its surface at which the acceleration due to gravity decreases to 4.9 ms^{-2}, is close to (Take, radius of earth $= 6.4 \times 10^6$ m) **(2019 Main, 10 April I)**

(a) 9.0×10^6 m (b) 2.6×10^6 m

(c) 6.4×10^6 m (d) 1.6×10^6 m

4. The variation of acceleration due to gravity g with distance d from centre of the Earth is best represented by (R = Earth's radius) **(2017 Main)**

(a)


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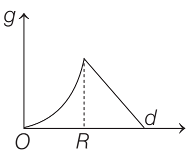

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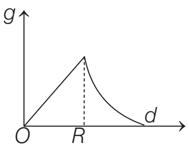

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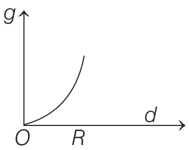


5. A planet of radius $R = \frac{1}{10} \times$ (radius of earth) has the same mass density as earth. Scientists dig a well of depth $\frac{R}{5}$ on it and lower a wire of the same length and of linear mass density 10^{-3} kg m^{-1} into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is (take the radius of earth $= 6 \times 10^6$ m and the acceleration due to gravity of earth is $10\,\text{ms}^{-2}$) **(2014 Adv.)**

(a) 96 N (b) 108 N (c) 120 N (d) 150 N

6. If the radius of the earth were to shrink by one per cent, its mass remaining the same, the acceleration due to gravity on the earth's surface would **(1981, 2M)**

(a) decrease (b) remain unchanged

(c) increase (d) be zero

Objective Question II (One or more correct option)

7. The magnitudes of the gravitational field at distance r_1 and r_2 from the centre of a uniform sphere of radius R and mass M are F_1 and F_2, respectively. Then **(1994, 2M)**

(a) $\frac{F_1}{F_2} = \frac{r_1}{r_2}$ if $r_1 < R$ and $r_2 < R$

(b) $\frac{F_1}{F_2} = \frac{r_2^2}{r_1^2}$ if $r_1 > R$ and $r_2 > R$

(c) $\frac{F_1}{F_2} = \frac{r_1^3}{r_2^3}$ if $r_1 < R$ and $r_2 < R$

(d) $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2}$ if $r_1 < R$ and $r_2 < R$

Fill in the Blank

8. The numerical value of the angular velocity of rotation of the earth should be ……… rad/s in order to make the effective acceleration due to gravity at equator equal to zero. **(1984, 2M)**

Topic 2 Field Strength, Potential, Potential Energy and Escape Velocity

Objective Questions I (Only one correct option)

1 A spaceship orbits around a planet at a height of 20 km from its surface. Assuming that only gravitational field of the planet acts on the spaceship, what will be the number of complete revolutions made by the spaceship in 24 hours around the planet? **(2019 Main, 10 April II)**

[Take, mass of planet $= 8 \times 10^{22}$ kg,

radius of planet $= 2 \times 10^6$ m,

gravitational constant $G = 6.67 \times 10^{-11}$ N-m^2/kg^2]

(a) 11 (b) 17
(c) 13 (d) 9

2 A solid sphere of mass M and radius a is surrounded by a uniform concentric spherical shell of thickness $2a$ and $2M$. The gravitational field at distance $3a$ from the centre will be **(2019 Main, 9 April I)**

(a) $\frac{GM}{9a^2}$ (b) $\frac{2GM}{9a^2}$
(c) $\frac{GM}{3a^2}$ (d) $\frac{2GM}{3a^2}$

3 A rocket has to be launched from earth in such a way that it never returns. If E is the minimum energy delivered by the rocket launcher, what should be the minimum energy that the launcher should have, if the same rocket is to be launched from the surface of the moon? Assume that the density of the earth and the moon are equal and that the earth's volume is 64 times the volume of the moon. **(Main 2019, 8 April II)**

(a) $\frac{E}{64}$ (b) $\frac{E}{16}$ (c) $\frac{E}{32}$ (d) $\frac{E}{4}$

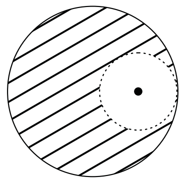
4. A rocket is launched normal to the surface of the Earth, away from the Sun, along the line joining the Sun and the Earth. The Sun is 3×10^5 times heavier than the Earth and is at a distance 2.5×10^4 times larger than the radius of Earth. The escape velocity from Earth's gravitational field is $v_e = 11.2 \text{ km s}^{-1}$. The minimum initial velocity (v_s) required for the rocket to be able to leave the Sun-Earth system is closest to (Ignore the rotation and revolution of the Earth and the presence of any other planet) **(2017 Adv.)**

(a) $v_s = 72 \text{ km s}^{-1}$ (b) $v_s = 22 \text{ km s}^{-1}$
(c) $v_s = 42 \text{ km s}^{-1}$ (d) $v_s = 62 \text{ km s}^{-1}$

5. A satellite is revolving in a circular orbit at a height h from the Earth's surface (radius of earth $R, h << R$). The minimum increase in its orbital velocity required, so that the satellite could escape from the Earth's gravitational field, is close to (Neglect the effect of atmosphere) **(2016 Main)**

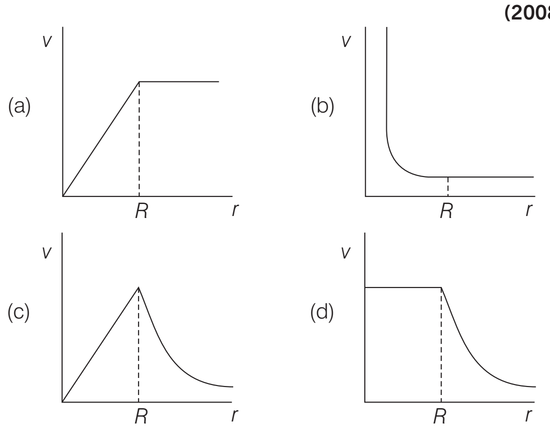
(a) $\sqrt{2gR}$ (b) $\sqrt{gR}$
(c) $\sqrt{gR/2}$ (d) $\sqrt{gR}\,(\sqrt{2}-1)$

6. From a solid sphere of mass M and radius R, a spherical portion of radius $\left(\frac{R}{2}\right)$ is removed as shown in the figure. Taking gravitational potential $V = 0$ at $r = \infty$, the potential at the centre of the cavity thus formed is (G = gravitational constant) **(2015 Main)**



(a) $\frac{-GM}{R}$ (b) $\frac{-GM}{2R}$
(c) $\frac{-2GM}{3R}$ (d) $\frac{-2GM}{R}$

7. A spherically symmetric gravitational system of particles has a mass density $\rho = \begin{cases} \rho_0 \text{ for } r \le R \\ 0 \text{ for } r > R \end{cases}$, where ρ_0 is a constant. A test mass can undergo circular motion under the influence of the gravitational field of particles. Its speed v as a function of distance r from the centre of the system is represented by **(2008, 3M)**



8. If g is the acceleration due to gravity on the earth's surface, the gain in the potential energy of an object of mass m raised from the surface of the earth to a height equal to the radius R of the Earth, is **(1983, 1M)**

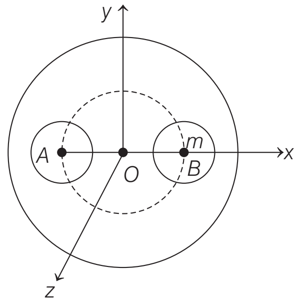
(a) $\frac{1}{2}mgR$ (b) $2\,mgR$
(c) mgR (d) $\frac{1}{4}mgR$

Objective Questions II (One or more correct option)

9. Two spherical planets P and Q have the same uniform density ρ, masses M_P and M_Q, and surface areas A and $4A$, respectively. A spherical planet R also has uniform density ρ and its mass is $(M_P + M_Q)$. The escape velocities from the planets P, Q and R, are v_P, v_Q and v_R, respectively. Then **(2012)**

(a) $v_Q > v_R > v_P$ (b) $v_R > v_Q > v_P$
(c) $v_R/v_P = 3$ (d) $v_P/v_Q = 1/2$

10. A solid sphere of uniform density and radius 4 units is located with its centre at the origin O of coordinates. Two spheres of equal radii 1 unit, with their centres at A $(-2, 0, 0)$ and B $(2, 0, 0)$ respectively, are taken out of the solid leaving behind spherical cavities as shown in figure. **(1993, 2M)**



Then,

(a) the gravitational field due to this object at the origin is zero

(b) the gravitational field at the point B (2, 0, 0) is zero

(c) the gravitational potential is the same at all points of circle $y^2 + z^2 = 36$

(d) the gravitational potential is the same at all points on the circle $y^2 + z^2 = 4$

Fill in the Blanks

11. A particle is projected vertically upwards from the surface of Earth (radius R) with a kinetic energy equal to half of the minimum value needed for it to escape. The height to which it rises above the surface of Earth is .... **(1997, 2M)**

12. The masses and radii of the Earth and the Moon are M_1, R_1 and M_2, R_2 respectively. Their centres are a distance d apart. The minimum speed with which a particle of mass m should be projected from a point midway between the two centres so as to escape to infinity is .... **(1988, 2M)**

Integer Answer Type Questions

13. A bullet is fired vertically upwards with velocity v from the surface of a spherical planet. When it reaches its maximum height, its acceleration due to the planet's gravity is $\frac{1}{4}$ th of its value at the surface of the planet. If the escape velocity from the planet is $v_{\text{sec}} = v\sqrt{N}$, then the value of N is (ignore energy loss due to atmosphere) **(2015 Adv.)**

14. Gravitational acceleration on the surface of a planet is $\frac{\sqrt{6}}{11} g$, where g is the gravitational acceleration on the surface of the earth. The average mass density of the planet is 1/3 times that of the earth. If the escape speed on the surface of the earth is taken to be 11 kms^{-1}, the escape speed on the surface of the planet in kms^{-1} will be **(2010)**

Analytical & Descriptive Questions

15. There is a crater of depth $\frac{R}{100}$ on the surface of the moon (radius R). A projectile is fired vertically upward from the crater with velocity, which is equal to the escape velocity v from the surface of the moon. Find the maximum height attained by the projectile. **(2003, 4M)**

Topic 3 Motion of Satellites

Objective Questions I (Only one correct option)

1. A test particle is moving in a circular orbit in the gravitational field produced by mass density $\rho(r) = \frac{K}{r^2}$. Identify the correct relation between the radius R of the particle's orbit and its period T **(Main 2019, 9 April II)**

(a) $\frac{T^2}{R^3}$ is a constant (b) $\frac{T}{R^2}$ is a constant

(c) TR is a constant (d) $\frac{T}{R}$ is a constant

2. Two satellites A and B have masses m and $2m$ respectively. A is in a circular orbit of radius R and B is in a circular orbit of radius $2R$ around the earth. The ratio of their kinetic energies, T_A / T_B is **(2019 Main, 12 Jan II)**

(a) $\frac{1}{2}$ (b) 2 (c) $\sqrt{\frac{1}{2}}$ (d) 1

3. A satellite is revolving in a circular orbit at a height h from the earth surface such that $h << R$, where R is the radius of the earth. Assuming that the effect of earth's atmosphere can be neglected the minimum increase in the speed required so that the satellite could escape from the gravitational field of earth is **(2019 Main, 11 Jan II)**

(a) $\sqrt{\frac{gR}{2}}$ (b) $\sqrt{gR}$ (c) $\sqrt{2gR}$ (d) $\sqrt{gR}\,(\sqrt{2} - 1)$

4. A satellite is moving with a constant speed v in a circular orbit about the earth. An object of mass m is ejected from the satellite such that it just escapes from the gravitational pull of the earth. At the time of its ejection, the kinetic energy of the object is **(2019 Main, 10 Jan I)**

(a) $\frac{1}{2}mv^2$ (b) mv^2

(c) $\frac{3}{2}mv^2$ (d) $2mv^2$

5. The energy required to take a satellite to a height 'h' above earth surface (where, radius of earth $= 6.4 \times 10^3$ km) is E_1 and kinetic energy required for the satellite to be in a circular orbit at this height is E_2. The value of h for which E_1 and E_2 are equal is **(2019 Main, 9 Jan II)**

(a) 3.2×10^3 km (b) 1.28×10^4 km

(c) 6.4×10^3 km (d) 1.6×10^3 km

6. What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of $2R$? **(2013 Main)**

(a) $\frac{5GmM}{6R}$ (b) $\frac{2GmM}{3R}$ (c) $\frac{GmM}{2R}$ (d) $\frac{GmM}{3R}$

7. A geostationary satellite orbits around the earth in a circular orbit of radius 36,000 km. Then, the time period of a spy satellite orbiting a few hundred km above the earth's surface (R_e = 6400km) will approximately be **(2002)**

(a) 1/2 h (b) 1 h (c) 2 h (d) 4 h

8. A satellite S is moving in an elliptical orbit around the earth. The mass of the satellite is very small compared to the mass of the earth **(1998, 2M)**

(a) the acceleration of S is always directed towards the centre of the earth
(b) the angular momentum of S about the centre of the earth changes in direction, but its magnitude remains constant
(c) the total mechanical energy of S varies periodically with time
(d) the linear momentum of S remains constant in magnitude

Assertion and Reason

Mark your answer as

(*a*) *If Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I*
(*b*) *If Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I*
(*c*) *If Statement I is true; Statement II is false*
(*d*) *If Statement I is false; Statement II is true*

9. **Statement I** An astronaut in an orbiting space station above the earth experiences weightlessness.

Statement II An object moving around the earth under the influence of earth's gravitational force is in a state of 'free-fall'. **(2008, 3M)**

Fill in the Blank

10. A geostationary satellite is orbiting the earth at a height of $6R$ above the surface of the earth where R is the radius of earth. The time period of another satellite at a height of 3.5 R from the surface of the earth is ……… hours. **(1987, 2M)**

True / False

11. It is possible to put an artificial satellite into orbit in such a way that it will always remain directly over New Delhi. **(1984, 2M)**

Analytical & Descriptive Questions

12. An artificial satellite is moving in a circular orbit around the Earth with a speed equal to half the magnitude of escape velocity from the Earth. **(1990, 8M)**

(a) Determine the height of the satellite above the earth's surface.
(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the Earth, find the speed with which it hits the surface of the Earth.

13. Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 h and 8 h, respectively. The radius of the orbit of S_1 is 10^4 km when S_2 is closest to S_1. Find **(1986, 6M)**

(a) the speed of S_2 relative to S_1,
(b) the angular speed of S_2 as actually observed by an astronaut in S_1.

Topic 4 Kepler's Laws and Motion of Planets

Objective Questions I (Only one correct option)

1 If the angular momentum of a planet of mass m, moving around sun in a circular orbit is L about the centre of the sun, its areal velocity is **(Main 2019, 9 Jan I)**

(a) $\frac{4L}{m}$ (b) $\frac{2L}{m}$ (c) $\frac{L}{2m}$ (d) $\frac{L}{m}$

2. A particle is moving with a uniform speed in a circular orbit of radius R in a central force inversely proportional to the nth power of R. If the period of rotation of the particle is T, then **(2018 Main)**

(a) $T \propto R^{n/2}$ (b) $T \propto R^{3/2}$ for any value of n
(c) $T \propto R^{\frac{n}{2}+1}$ (d) $T \propto R^{\frac{n+1}{2}}$

3. A double star system consists of two stars A and B which have time periods T_A and T_B. Radius R_A and R_B and mass M_A and M_B. Choose the correct option. **(2006, 3M)**

(a) If $T_A > T_B$ then $R_A > R_B$ (b) If $T_A > T_B$ then $M_A > M_B$
(c) $\left(\frac{T_A}{T_B}\right)^2 = \left(\frac{R_A}{R_B}\right)^3$ (d) $T_A = T_B$

4. If the distance between the earth and the sun were half its present value, the number of days in a year would have been **(1996, 2M)**

(a) 64.5 (b) 129 (c) 182.5 (d) 730

5. Imagine a light planet revolving around a very massive star in a circular orbit of radius R with a period of revolution T. If the gravitational force of attraction between the planet and the star is proportional to $R^{-5/2}$, then **(1989, 2M)**

(a) T^2 is proportional to R^2
(b) T^2 is proportional to $R^{7/2}$
(c) $T^2$is proportional to $R^{3/2}$
(d) $T^2$is proportional to $R^{3.75}$

Fill in the Blanks

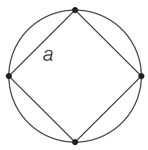
6. The ratio of earth's orbital angular momentum (about the sun) to its mass is 4.4×10^{15} m^2/s. The area enclosed by earth's orbit is approximately ……… m^2. **(1997C, 1M)**

7. According to Kepler's second law, the radius vector to a planet from the sun sweeps out equal areas in equal intervals of time. This law is a consequence of the conservation of ……… . **(1985, 2M)**

Topic 5 Miscellaneous Problems

Objective Questions I (Only one correct option)

1. Four identical particles of mass M are located at the corners of a square of side a. What should be their speed, if each of them revolves under the influence of other's gravitational field in a circular orbit circumscribing the square ? **(Main 2019, 8 April I)**



(a) $1.35\sqrt{\dfrac{GM}{a}}$ (b) $1.16\sqrt{\dfrac{GM}{a}}$

(c) $1.21\sqrt{\dfrac{GM}{a}}$ (d) $1.41\sqrt{\dfrac{GM}{a}}$

2. The mass and the diameter of a planet are three times the respective values for the earth. The period of oscillation of a simple pendulum on the earth is 2 s. The period of oscillation of the same pendulum on the planet would be **(Main 2019, 11 Jan II)**

(a) $\dfrac{2}{\sqrt{3}}$ s (b) $\dfrac{3}{2}$ s (c) $2\sqrt{3}$ s (d) $\dfrac{\sqrt{3}}{2}$ s

3. Two stars of masses 3×10^{31} kg each and at distance 2×10^{11} m rotate in a plane about their common centre of mass O. A meteorite passes through O moving perpendicular to the star's rotation plane. In order to escape from the gravitational field of this double star, the minimum speed that meteorite should have at O is (Take, gravitational constant, $G = 6.67 \times 10^{-11}$ N-m^{2}kg^{-2}) **(Main 2019, 10 Jan II)**

(a) 2.8×10^5 m/s (b) 3.8×10^4 m/s

(c) 2.4×10^4 m/s (d) 1.4×10^5 m/s

4. Four particles, each of mass M and equidistant from each other, move along a circle of radius R under the action of their mutual gravitational attraction, the speed of each particle is **(2014 Main)**

(a) $\sqrt{\dfrac{GM}{R}}$ (b) $\sqrt{2\sqrt{2}\dfrac{GM}{R}}$

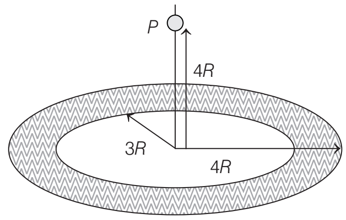
(c) $\sqrt{\dfrac{GM}{R}(1+2\sqrt{2})}$ (d) $\dfrac{1}{2}\sqrt{\dfrac{GM}{R}(1+2\sqrt{2})}$

5. A simple pendulum has a time period T_1 when on the earth's surface and T_2 when taken to a height R above the earth's surface, where R is the radius of the earth. The value of T_2/T_1 is **(2001)**

(a) 1 (b) $\sqrt{2}$

(c) 4 (d) 2

6. A thin uniform annular disc (*see figure*) of mass M has outer radius $4R$ and inner radius $3R$. The work required to take a unit mass from point P on its axis to infinity is **(2010)**



(a) $\dfrac{2GM}{7R}(4\sqrt{2}-5)$ (b) $-\dfrac{2GM}{7R}(4\sqrt{2}-5)$

(c) $\dfrac{GM}{4R}$ (d) $\dfrac{2GM}{5R}(\sqrt{2}-1)$

Objective Questions II (One or more correct option)

7. A spherical body of radius R consists of a fluid of constant density and is in equilibrium under its own gravity. If $P(r)$ is the pressure at $r(r < R)$, then the correct options is/are **(2015 Adv.)**

(a) $P\ (r = 0) = 0$ (b) $\dfrac{P\left(r=\dfrac{3R}{4}\right)}{P\left(r=\dfrac{2R}{3}\right)} = \dfrac{63}{80}$

(c) $\dfrac{P\left(r=\dfrac{3R}{5}\right)}{P\left(r=\dfrac{2R}{5}\right)} = \dfrac{16}{21}$ (d) $\dfrac{P\left(r=\dfrac{R}{2}\right)}{P\left(r=\dfrac{R}{3}\right)} = \dfrac{20}{27}$

8. Two bodies, each of mass M, are kept fixed with a separation $2L$. A particle of mass m is projected from the mid-point of the line joining their centres, perpendicular to the line. The gravitational constant is G. The correct statement(s) is (are) **(2013 Adv.)**

(a) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $4\sqrt{\dfrac{GM}{L}}$

(b) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $2\sqrt{\dfrac{GM}{L}}$

(c) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $\sqrt{\dfrac{2GM}{L}}$

(d) The energy of the mass m remains constant

Analytical & Descriptive Questions

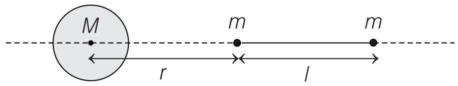
9. Distance between the centres of two stars is $10a$. The masses of these stars are M and $16M$ and their radii a and $2a$ respectively. A body of mass m is fired straight from the surface of the larger star towards the surface of the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a. **(1996, 5M)**

10. Three particles, each of mass m, are situated at the vertices of an equilateral triangle of side length a. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining the original mutual separation a. Find the initial velocity that should be given to each particle and also the time period of the circular motion. **(1988, 5M)**

Integer Answer Type Question

11. A large spherical mass M is fixed at one position and two identical masses m are kept on a line passing through the centre of M (see figure). The point masses are connected by a rigid massless rod of length l and this assembly is free to move along the line connecting them.

All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance $r = 3l$ from M the tension in the rod is zero for $m = k\left(\frac{M}{288}\right)$. The value of k is **(2015 Adv.)**



Match the following

12. A planet of mass M, has two natural satellites with masses m_1 and m_2. The radii of their circular orbits are R_1 and R_2, respectively.

Ignore the gravitational force between the satellites. Define v_1, L_1, K_1 and T_1 to be respectively, the orbital speed, angular momentum, kinetic energy and time period of revolution of satellite 1; and v_2, L_2, K_2 and T_2 to be the corresponding quantities of satellite 2. Given, $m_1 / m_2 = 2$ and $R_1 / R_2 = 1/4$, match the ratios in List-I to the numbers in List-II. **(2018 Adv.)**

	List-I		List-II
P.	v_1 / v_2	1.	1/8
Q.	L_1 / L_2	2.	1
R.	K_1 / K_2	3.	2
S.	T_1 / T_2	4.	8

(a) P → 4; Q → 2; R → 1; S → 3
(b) P → 3; Q → 2; R → 4; S → 1
(c) P → 2; Q → 3; R → 1; S → 4
(d) P → 2; Q → 3; R → 4; S → 1

Answers

Topic 1

1. (d) **2.** (c) **3.** (b)
4. (c) **5.** (b) **6.** (c) **7.** (a, b)
8. 1.24×10^{-3}

Topic 2

1. (a) **2.** (c) **3.** (b)
4. (c) **5.** (d) **6.** (a) **7.** (c)
8. (a) **9.** (b, d) **10.** (a, c, d) **11.** $h = R$
12. $v = 2\sqrt{\frac{G}{d}(M_1 + M_2)}$ **13.** 2 **14.** 3
15. $99.5\, R$

Topic 3

1. (d) **2.** (d) **3.** (d) **4.** (b)
5. (a)
6. (a) **7.** (c) **8.** (a)
9. (a) **10.** 8.48 **11.** F
12. (a) 6400 km (b) 7.9 km/s
13. (a) $-\pi \times 10^4$ km/h (b) 3×10^{-4} rad/s

Topic 4

1. (c) **2.** (d)
3. (d) **4.** (b) **5.** (b) **6.** 6.94×10^{22}
7. angular momentum

Topic 5

1. (b) **2.** (c) **3.** (a)
4. (d) **5.** (d) **6.** (a) **7.** (b, c)
8. (b, d) **9.** $\frac{3\sqrt{5}}{2}\sqrt{\frac{GM}{a}}$ **10.** $v = \sqrt{\frac{Gm}{a}},\ T = 2\pi\sqrt{\frac{a^3}{3Gm}}$
11. (7) **12.** (b)

Hints & Solutions

Topic 1 Gravitational Force and Acceleration due to Gravity

1. Let mass of given body is m. Then, it's weight on earth's surface $= mg_e$

where, g_e = acceleration due to gravity on earth's surface and weight on the surface of planet $= mg_p$

g_p = acceleration due to gravity on planet's surface.

Given,

$$\frac{mg_e}{mg_p} = \frac{9}{4} \Rightarrow \frac{g_e}{g_p} = \frac{9}{4}$$

But $g = \frac{GM}{R^2}$, so we have

$$\frac{\left(\frac{GM}{R^2}\right)}{\left(\frac{GM_p}{R_p^2}\right)} = \frac{9}{4}$$

where, M = mass of earth,

R = radius of earth,

M_p = mass of plane $= \frac{M}{9}$ (given)

and R_p = radius of planet.

$$\Rightarrow \quad \frac{M}{M_p} \cdot \frac{R_p^2}{R^2} = \frac{9}{4} \Rightarrow 9 \cdot \left(\frac{R_p}{R}\right)^2 = \frac{9}{4}$$

$$\Rightarrow \quad \frac{R_p}{R} = \frac{1}{2} \Rightarrow R_p = \frac{R}{2}$$

2. Position of particle is, $\mathbf{r} = 2t\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}}$

where, t is instantaneous time.

Velocity of particle is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 2\hat{\mathbf{i}} - 6t\hat{\mathbf{j}}$$

Now, angular momentum of particle with respect to origin is given by

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$$
$$= m\{(2t\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}}) \times (2\hat{\mathbf{i}} - 6t\hat{\mathbf{j}})\}$$
$$= m(-12t^2(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) - 6t^2(\hat{\mathbf{j}} \times \hat{\mathbf{i}}))$$

As, $\quad \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = 0$

$$\Rightarrow \quad \mathbf{L} = m(-12t^2\hat{\mathbf{k}} + 6t^2\hat{\mathbf{k}})$$

As, $\quad \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$ and $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$

$$\Rightarrow \quad \mathbf{L} = -m(6t^2)\hat{\mathbf{k}}$$

So, angular momentum of particle of mass 2 kg at time $t = 2$ s is

$$\mathbf{L} = (-2 \times 6 \times 2^2)\hat{\mathbf{k}} = -48\,\hat{\mathbf{k}}$$

3. **Key Idea** The relation of gravitational acceleration at an altitude h above the earth's surface is given as

$$g_h = g\left(1 + \frac{h}{R_e}\right)^{-2}$$

where, g is the acceleration due to gravity at earth's surface and R_e is the radius of the earth.

Given that at some height h, acceleration due to gravity,

$$g_h = 4.9\,\text{m/s}^2 \approx \frac{g}{2} \qquad \ldots \text{(i)}$$

$\therefore$ The ratio of acceleration due to gravity at earth's surface and at some altitude h is

$$\left(1 + \frac{h}{R_e}\right) = \sqrt{\frac{g}{g_h}} = \sqrt{2} \qquad \text{[From Eq. (i)]}$$

$$\therefore \quad \frac{h}{R_e} = \sqrt{2} - 1$$

or $\quad h = 0.414 \times R_e$

$\quad h = 0.414 \times 6400\,\text{km}$

($\because$ given, radius of earth, $R_e = 6400\,\text{km} = 6.4 \times 10^6$ m)

or $\quad h = 2649.6\,\text{km} = 2.6 \times 10^6$ m

Thus, at 2.6×10^6 m above the earth's surface, the acceleration due to gravity decreases to 4.9 m/s^2.

4. Inside the earth surface

$$g = \frac{GM}{R^3}\,r$$

i.e. $\quad g \propto r$

Out the earth surface $\quad g = \frac{Gm}{r^2}$

i.e. $\quad g \propto \frac{1}{r^2}$

So, till earth surface g increases linearly with distance r, shown only in graph (c).

5. Given, $R_{\text{planet}} = \frac{R_{\text{earth}}}{10}$ and density, $\rho = \frac{M_{\text{earth}}}{\frac{4}{3}\pi R_{\text{earth}}^3}$

$$= \frac{M_{\text{planet}}}{\frac{4}{3}\pi R_{\text{planet}}^3} \Rightarrow M_{\text{planet}} = \frac{M_{\text{earth}}}{10^3}$$

$$g_{\text{surface of planet}} = \frac{GM_{\text{planet}}}{R_{\text{planet}}^2} = \frac{GM_e \cdot 10^2}{10^3 \cdot R_e^2} = \frac{GM_e}{10R_e^2}$$

$$= \frac{g_{\text{surface of earth}}}{10}$$

$$g_{\text{depth of planet}} = g_{\text{surface of planet}}\left(\frac{x}{R}\right)$$

where, x = distance from centre of planet.

$\therefore$ Total force on wire

$$F = \int_{4R/5}^{R} \lambda \, dx \, g\left(\frac{x}{R}\right) = \frac{\lambda g}{R}\left[\frac{x^2}{2}\right]_{4R/5}^{R}$$

Here, $g = g_{\text{surface of planet}}, R = R_{\text{planet}}$

Substituting the given values, we get $F = 108$ N

6. $g = \dfrac{GM}{R^2}$ or $g \propto \dfrac{1}{R^2}$

g will increase if R decreases.

7. For $r \le R, F = \dfrac{GM}{R^3}.r$

or $F \propto r$

$$\frac{F_1}{F_2} = \frac{r_1}{r_2} \quad \text{for } r_1 < R$$

and $r_2 < R$

and for $r \ge R, F = \dfrac{GM}{r^2}$

or $F \propto \dfrac{1}{r^2}$

i.e. $\dfrac{F_1}{F_2} = \dfrac{r_2^2}{r_1^2}$ for $r_1 > R$

and $r_2 > R$

8. $g' = g - R\omega^2 \cos^2 \phi$

At equator $\phi = 0$

$\therefore$ $g' = g - R\omega^2$

$\Rightarrow$ $0 = g - R\omega^2$

$\therefore$ $\omega = \sqrt{\dfrac{g}{R}} = \sqrt{\dfrac{9.8}{6400 \times 10^3}}$

$= 1.24 \times 10^{-3}$ rad/s

Topic 2 Field Strength, Potential, Potential Energy and Escape Velocity

1. A satellite or spaceship in a circular orbit at a distance $(R + h)$ from centre of a planet experiences a gravitational force given by

$$F_g = \frac{GmM}{(R+h)^2}$$

where, M = mass of planet,

m = mass of spaceship,

R = radius of planet

and h = height of spaceship above surface.

This gravitational pull provides necessary centripetal pull for orbital motion of spaceship.

So, $F_g = F_{\text{centripetal}}$

$\Rightarrow$ $\dfrac{GmM}{(R+h)^2} = \dfrac{mv^2}{(R+h)}$

where, v = orbital speed of spaceship.

Orbital speed of spaceship is

$$v = \sqrt{\frac{GM}{(R+h)}} \quad \ldots\text{(i)}$$

Here, $G = 6.67 \times 10^{-11}$ N-m^2/kg^2,

$M = 8 \times 10^{22}$ kg

$R + h = (2 \times 10^6 + 20 \times 10^3)$ m

$= 2.02 \times 10^6$ m

So, substituting these values in Eq. (i), we get

$$v = \sqrt{\frac{6.67 \times 10^{-11} \times 8 \times 10^{22}}{2.02 \times 10^6}}$$

$= 1.6 \times 10^3$ ms^{-1}

Time period of rotation of spaceship will be

$$T = \frac{2\pi(R+h)}{v}$$

$\Rightarrow$ $T = \dfrac{2\pi \times 2.02 \times 10^6}{1.6 \times 10^3}$

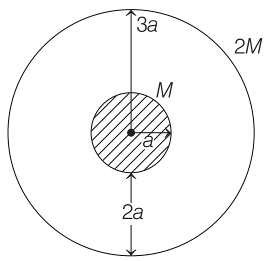
$\approx 8 \times 10^3 \text{ s} = \dfrac{8 \times 10^3}{60 \times 60}$ (h)

$= 2.2$ h

So, number of revolutions made by spaceship in 24 h,

$$n = \frac{24}{T} = \frac{24}{2.2} \approx 11 \text{ rev}$$

2. According to question, diagram will be as follows



Gravitational field due to solid sphere of radius a at a distance, $r = 3a$, i.e. $(r > R)$ is

$$E_1 = \frac{GM}{r^2} = \frac{GM}{(3a)^2} = \frac{GM}{9a^2}$$

Similarly, gravitational field due to spherical shell at a distance, $r = 3a$,

i.e. $E_2 = \dfrac{GM}{R^2}$

$= \dfrac{G(2M)}{(3a)^2} = \dfrac{2GM}{9a^2}$

Both fields are attractive in nature, so direction will be same.

Net field, $E_{\text{net}} = \dfrac{GM}{9a^2} + \dfrac{2GM}{9a^2}$

$$E_{\text{net}} = \frac{GM}{3a^2}$$

3. Given, volume of earth (V_e) is 64 times of volume of moon (V_m), i.e.

$$\frac{V_e}{V_m} = 64 = \frac{\frac{4}{3}\pi R_e^3}{\frac{4}{3}\pi R_m^3}$$

where, R_e and R_m are the radius of earth and moon, respectively.

Then, $$\frac{R_e}{R_m} = 4 \quad \ldots\text{(i)}$$

Also, since the density of moon and earth are equal, i.e.

$$\rho_m = \rho_e$$

$\Rightarrow \frac{M_e}{V_e} = \frac{M_m}{V_m}$, where M_e and M_m are the mass of the earth and moon, respectively.

$$\Rightarrow \quad \frac{M_e}{M_m} = \frac{V_e}{V_m} = 64 \quad \ldots\text{(ii)}$$

The minimum energy or escape energy delivered by the rocket launcher, so that the rocket never returns to earth is

$$E_e = \frac{GM_e m}{R_e} = E$$

where, m is the mass of the rocket.

Similarly, minimum energy that a launcher should have to escape or to never return, if rocket is launched from surface of the moon is

$$E_m = \frac{GM_m m}{R_m}$$

$\therefore$ Ratio of escape energies E_e and E_m is

$$\frac{E_e}{E_m} = \frac{\left(\frac{GM_e m}{R_e}\right)}{\left(\frac{GM_m m}{R_m}\right)} = \frac{M_e}{M_m}\cdot\frac{R_m}{R_e}$$

$$= 64 \times \frac{1}{4} = 16 \quad \text{[using Eqs. (i) and (ii)]}$$

$$\therefore \quad E_m = \frac{E_e}{16} = \frac{E}{16}$$

4. Given, $v_e = 11.2 \text{ km/s} = \sqrt{\frac{2GM_e}{R_e}}$

From energy conservation,

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_s^2 - \frac{GM_s m}{r} - \frac{GM_e m}{R_e} = 0 + 0$$

Here, r = distane of rocket from sun

$$\Rightarrow \quad v_s = \sqrt{\frac{2GM_e}{R_e} + \frac{2GM_s}{r}}$$

Given, $M_s = 3\times 10^5\ M_e$ and $r = 2.5\times 10^4\ R_e$

$$\Rightarrow \quad v_s = \sqrt{\frac{2GM_e}{R_e} + \frac{2G\ 3\times 10^5 M_e}{2.5\times 10^4 R_e}}$$

$$= \sqrt{\frac{2GM_e}{R_e}\left(1 + \frac{3\times 10^5}{2.5\times 10^4}\right)}$$

$$= \sqrt{\frac{2GM_e}{R_e}\times 13}$$

$$\Rightarrow \quad v_s \simeq 42 \text{ km/s}$$

5. $v_{\text{orbital}} = \sqrt{\frac{GM}{R}} = \sqrt{gR}$

$v_{\text{escape}} = \sqrt{2gR}$

$\therefore$ Extra velocity required $= v_{\text{escape}} - v_{\text{orbital}} = \sqrt{gR}\ (\sqrt{2}-1)$

6. $V_R = V_T - V_C$

V_R = Potential due to remaining portion

V_T = Potential due to total sphere

V_C = Potential due to cavity

Radius of cavity is $\frac{R}{2}$. Hence, volume and mass is $\frac{M}{8}$.

$$\therefore \quad V_R = -\frac{GM}{R^3}\left[1.5R^2 - 0.5\left(\frac{R}{2}\right)^2\right] + \frac{G\left(\frac{M}{8}\right)}{\left(\frac{R}{2}\right)}\left(\frac{3}{2}\right) = -\frac{GM}{R}$$

7. For $\quad r \le R, \quad \frac{mv^2}{r} = \frac{GmM}{r^2} \quad \ldots\text{(i)}$

Here, $$M = \left(\frac{4}{3}\pi r^3\right)\rho_0$$

Substituting in Eq. (i), we get $v \propto r$

i.e. v-r graph is a straight line passing through origin.

For $r > R$,

$$\frac{mv^2}{r} = \frac{Gm\left(\frac{4}{3}\pi R^3\right)\rho_0}{r^2} \quad \text{or} \quad v \propto \frac{1}{\sqrt{r}}$$

The corresponding v-r graph will be as shown in option (c).

8. $\Delta U = \frac{mgh}{1 + h/R}$

Given, $h = R$

$$\Delta U = \frac{mgR}{1 + R/R} = \frac{1}{2}mgR$$

9. Surface area of Q is four times. Therefore, radius of Q is two times. Volume is eight times. Therefore, mass of Q is also eight times.

So, let $\quad M_P = M$ and $R_P = r$

Then, $\quad M_Q = 8\text{M}$ and $R_Q = 2r$

Now, mass of R is $(M_P + M_Q)$ or 9 M. Therefore, radius of R is $(9)^{1/3} r$. Now, escape velocity from the surface of a planet is given by

$$v = \sqrt{\frac{2GM}{r}} \quad (r = \text{radius of that planet})$$

$$\therefore \quad v_P = \sqrt{\frac{2GM}{r}}$$

$$v_Q = \sqrt{\frac{2G(8M)}{(2r)}} \Rightarrow v_R = \sqrt{\frac{2G(9M)}{(9)^{1/3} r}}$$

From here we can see that,

$$\frac{v_P}{v_Q} = \frac{1}{2} \text{ and } v_R > v_Q > v_P$$

10. The gravitational field is zero at the centre of a solid sphere. The small spheres can be considered as negative mass m located at A and B. The gravitational field due to these masses at O is equal and opposite. Hence, the resultant field at O is zero, options c and d are correct because plane of these circles is y-z, i.e. perpendicular to X-axis i.e. potential at any point on these two circles will be equal due to the positive mass M and negative masses $-m$ and $-m$.

11. Kinetic energy needed to escape $= \dfrac{GM_e m}{R}$

Therefore, energy given to the particle $= \dfrac{GM_e m}{2R}$

Now, from conservation of mechanical energy. Kinetic energy at the surface of earth = Difference in potential energy at a height h and on the surface of earth

$$\therefore \quad \frac{GM_e m}{2R} = \frac{-GM_e m}{(R+h)} - \left(\frac{-GM_e m}{R}\right)$$

$$\frac{1}{2R} = \frac{1}{R} - \frac{1}{R+h}$$

$$\frac{1}{2R} = \frac{1}{R+h}$$

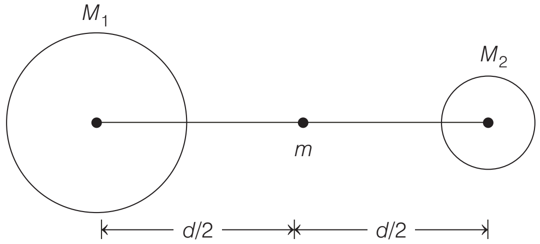
$$h + R = 2R$$

$$\Rightarrow \quad h = R$$

12. Total mechanical energy of mass m at a point midway between two centres is

$$E = -\frac{GM_1 m}{d/2} - \frac{GM_2 m}{d/2} = -\frac{2Gm}{d}(M_1 + M_2)$$

Binding energy $= \dfrac{2Gm}{d}(M_1 + M_2)$

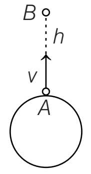


Kinetic energy required to escape the mass to infinity is,

$$\frac{1}{2} m v_e^2 = \frac{2Gm}{d}(M_1 + M_2)$$

$$\therefore \quad v_e = 2\sqrt{\frac{G(M_1 + M_2)}{d}}$$

13. At height h



$$g' = \frac{g}{\left(1 + \frac{h}{R}\right)^2} \quad \text{...(i)}$$

Given, $g' = \dfrac{g}{4}$

Substituting in equation (i) we get,

$$h = R$$

Now, from A to B,

decrease in kinetic energy = increase in potential energy

$$\Rightarrow \quad \frac{1}{2} mv^2 = \frac{mgh}{1 + \frac{h}{R}} \Rightarrow \frac{v^2}{2} = \frac{gh}{1 + \frac{h}{R}} = \frac{1}{2} gR \quad (h = R)$$

$$\Rightarrow \quad v^2 = gR \text{ or } v = \sqrt{gR}$$

Now, $\quad v_{\text{esc}} = \sqrt{2gR} = v\sqrt{2}$

$$\Rightarrow \quad N = 2$$

14. $g = \dfrac{GM}{R^2} = \dfrac{G\left(\frac{4}{3}\pi R^3\right)\rho}{R^2}$ or $g \propto \rho R$ or $R \propto \dfrac{g}{\rho}$

Now escape velocity, $v_e = \sqrt{2gR}$

or $\quad v_e \propto \sqrt{gR}$ or $v_e \propto \sqrt{g \times \dfrac{g}{\rho}} \propto \sqrt{\dfrac{g^2}{\rho}}$

$$\therefore \quad (v_e)_{\text{planet}} = (11 \text{ km s}^{-1})\sqrt{\frac{6}{121} \times \frac{3}{2}} = 3 \text{ km s}^{-1}$$

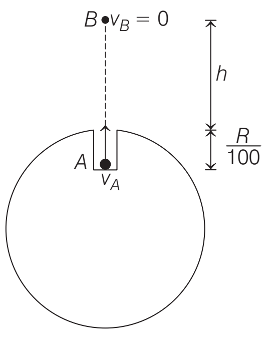
$\therefore$ The correct answer is 3.

15. Speed of particle at A, v_A

$= $ escape velocity on the surface of earth $= \sqrt{\dfrac{2GM}{R}}$

At highest point $B, v_B = 0$

Applying conservation of mechanical energy, decrease in kinetic energy = increase in gravitational potential energy



or $\quad \frac{1}{2}mv_A^2 = U_B - U_A = m(v_B - v_A)$

or $\quad \frac{v_A^2}{2} = v_B - v_A$

$$\therefore \quad \frac{GM}{R} = -\frac{GM}{R+h} - \left[-\frac{GM}{R^3}\left(1.5R^2 - 0.5\left(R - \frac{R}{100}\right)^2\right)\right]$$

or $\quad \frac{1}{R} = -\frac{1}{R+h} + \frac{3}{2R} - \left(\frac{1}{2}\right)\left(\frac{99}{100}\right)^2 \cdot \frac{1}{R}$

Solving this equation, we get

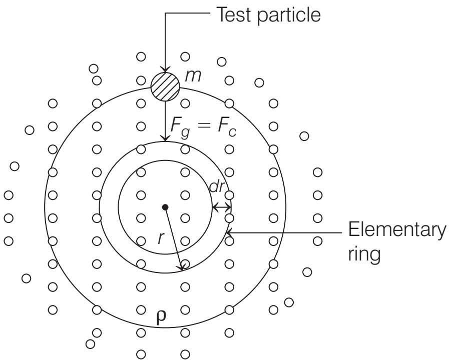
$$h = 99.5R$$

Topic 3 Motion of Satellites

1. Let the mass of the test particle be m and its orbital linear speed be v. Force of gravity of the mass-density would provide the necessary centripetal pull on test particle.

$\Rightarrow$ Gravitational force (F_g) = Centripetal force (F_c)

Let us now assume an elementary ring at a distance r from the centre of the mass density, such that mass of this elementary ring



$$\Rightarrow \quad \frac{GMm}{R^2} = \frac{mv^2}{R}$$

where, $\quad M = \int_0^R \rho \cdot 4\pi r^2 \cdot dr$

$$\Rightarrow \quad \frac{G\left(\int_0^R \frac{K}{r^2} \cdot 4\pi r^2 \cdot dr\right)}{R^2} = \frac{v^2}{R}$$

$$\left[\because \text{ given, } \rho(r) = \frac{K}{r^2}\right]$$

$$\Rightarrow \quad G \cdot 4\pi K = v^2$$

$\therefore$ Orbital speed of mass, $m = v = \sqrt{G 4\pi K}$ ... (i)

Time period of rotation of the test particle is

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{\sqrt{G \cdot 4\pi K}} \quad [\because \text{ using Eq. (i)}]$$

Hence, $\quad \frac{T}{R} = \sqrt{\frac{\pi}{GK}}$ = a constant

Alternate Solution

Since, $\quad F_g = F_c$

This relation can also be written directly as,

$$\frac{G\int_0^R \left(\frac{K}{r^2} \cdot 4\pi r^2 \cdot dr\right)}{R^2} = m\omega^2 R = m\left(2\frac{\pi}{T}\right)^2 R$$

$$\Rightarrow \quad \frac{G \cdot 4\pi KR}{R^2} = \frac{4\pi^2}{T^2} R$$

or $\quad \frac{T^2}{R^2} = \frac{\pi}{GK}$

$$\Rightarrow \quad \frac{T}{R} = \sqrt{\frac{\pi}{GK}} = \text{a constant.}$$

2. Orbital speed of a satellite in a circular orbit is

$$v_0 = \sqrt{\left(\frac{GM}{r_0}\right)}$$

where r_0 is the radius of the circular orbit.

So, kinetic energies of satellites A and B are

$$T_A = \frac{1}{2}m_A v_{OA}^2 = \frac{GMm}{2R}$$

$$T_B = \frac{1}{2}m_B v_{OB}^2 = \frac{GM(2m)}{2(2R)} = \frac{GMm}{2R}$$

So, ratio of their kinetic energies is

$$\frac{T_A}{T_B} = 1$$

3. Orbital velocity of the satellite is given as,

$$v_O = \sqrt{\frac{GM}{R+h}}$$

Since, $R >> h$

$$\therefore \quad v_O = \sqrt{\frac{GM}{R}} = \sqrt{gR} \quad \left[\because g = \frac{GM}{R^2}\right]$$

Escape velocity of the satellite,

$$v_e = \sqrt{\frac{2GM}{R+h}} = \sqrt{\frac{2GM}{R}} = \sqrt{2gR}$$

Since, we know that in order to escape the earth's gravitational field a satellite must get escape velocity.

$\therefore$ Change in velocity,

$$\Delta v = v_e - v_O = \sqrt{gR}\,(\sqrt{2} - 1)$$

4. In circular orbit of a satellite, potential energy

$$= -2 \times (\text{kinetic energy}) = -2 \times \frac{1}{2}mv^2 = -mv^2$$

Just to escape from the gravitational pull, its total mechanical energy should be zero. Therefore, its kinetic energy should be $+mv^2$.

5. The energy required for taking a satellite upto a height h from earth's surface is the difference between the energy at h height and energy at surface, then

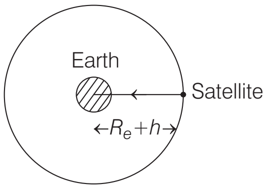
$$\Rightarrow \quad E_1 = U_f - U_i$$

$$E_1 = -\frac{GM_e m}{R_e + h} + \frac{GM_e m}{R_e} \quad \ldots\text{(i)}$$

(where, U = potential energy)

$\therefore$ Orbital velocity of satellite,

$$v_o = \sqrt{\frac{GM_e}{(R_e + h)}} \quad \text{(where, } M_e = \text{mass of earth)}$$



So energy required to perform circular motion

$$\Rightarrow \quad E_2 = \frac{1}{2}mv_o^2 = \frac{GM_e m}{2(R_e + h)}$$

$$E_2 = \frac{GM_e m}{2(R_e + h)} \quad \ldots\text{(ii)}$$

According to the question,

$$E_1 = E_2$$

$$\therefore \quad \frac{-GM_e m}{R_e + h} + \frac{GM_e m}{R_e} = \frac{GM_e m}{2(R_e + h)}$$

$$\Rightarrow \quad 3R_e = 2R_e + 2h$$

$$h = \frac{R_e}{2}$$

As radius of earth, $R_e \approx 6.4 \times 10^3$ km

Hence, $h = \frac{6.4 \times 10^3}{2}$ km or $= 3.2 \times 10^3$ km

6. E = Energy of satellite − energy of mass on the surface of planet

$$= -\frac{GMm}{2r} - \left(-\frac{GMm}{R}\right)$$

Here, $r = R + 2R = 3R$

Substituting in about equation we get, $E = \frac{5GMm}{6R}$

7. Time period of a satellite very close to earth's surface is 84.6 min. Time period increases as the distance of the satellite from the surface of earth increases. So, time period of spy satellite orbiting a few 100 km above the earth's surface should be slightly greater than 84.6 min. Therefore, the most appropriate option is (c) or 2 h.

8. Force on satellite is always towards earth, therefore, acceleration of satellite S is always directed towards centre of the earth. Net torque of this gravitational force F about centre of earth is zero. Therefore, angular momentum (both in magnitude and direction) of S about centre of earth is constant throughout. Since, the force F is conservative in nature, therefore mechanical energy of satellite remains constant. Speed of S is maximum when it is nearest to earth and minimum when it is farthest.

9. Force acting on astronaut is utilised in providing necessary centripetal force, thus he feels weightlessness, as he is in a state of free fall.

10.

$$T \propto r^{3/2}$$

$$\therefore \quad \frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2}$$

$$\text{or} \quad T_2 = \left(\frac{r_2}{r_1}\right)^{3/2}$$

$$T_1 = \left(\frac{3.5R}{7R}\right)^{3/2}(24)\text{ h} = 8.48 \text{ h}$$

(T_1 = 24 h for geostationary satellite)

11. New Delhi is not on the equatorial plane.

12. (a) Orbital speed of a satellite at distance r from centre of earth,

$$v_o = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R + h}} \quad \ldots\text{(i)}$$

Given, $$v_o = \frac{v_e}{2} = \frac{\sqrt{2GM/R}}{2} = \sqrt{\frac{GM}{2R}} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$h = R = 6400 \text{ km}$$

(b) Decrease in potential energy

= increase in kinetic energy

or $\frac{1}{2}mv^2 = \Delta U$

$$\therefore \quad v = \sqrt{\frac{2(\Delta U)}{m}}$$

$$= \sqrt{\frac{2\left(\frac{mgh}{1 + h/R}\right)}{m}} = \sqrt{gR} \quad (h = R)$$

$$= \sqrt{9.8 \times 6400 \times 10^3} = 7919 \text{ m/s} = 7.9 \text{ km/s}$$

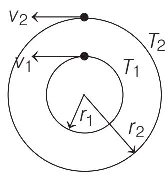
13. $T \propto r^{3/2}$ or $r \propto T^{2/3}$ $\Rightarrow$ $\frac{r_2}{r_1} = \left(\frac{T_2}{T_1}\right)^{2/3}$

$$r_2 = \left(\frac{T_2}{T_1}\right)^{2/3} r_1 = \left(\frac{8}{1}\right)^{2/3}(10^4)$$

$$= 4 \times 10^4 \text{ km}$$

Now, $$v_1 = \frac{2\pi r_1}{T_1} = \frac{(2\pi)(10^4)}{1} = 2\pi \times 10^4 \text{ km/h}$$

$$v_2 = \frac{2\pi r_2}{T_2} = \frac{(2\pi)(4 \times 10^4)}{8} = (\pi \times 10^4) \text{ km/h}$$



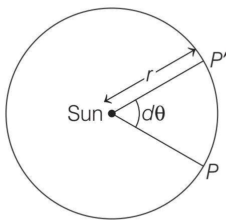
(a) Speed of S_2 relative to $S_1 = v_2 - v_1 = -\pi \times 10^4$ km/h

(b) Angular speed of S_2 as observed by S_1

$$\omega_r = \frac{|v_2 - v_1|}{|r_2 - r_1|} = \frac{\left(\pi \times 10^4 \times \frac{5}{18} \text{ m/s}\right)}{(3 \times 10^7 \text{ m})}$$
$$= 0.3 \times 10^{-3} \text{ rad/s}$$
$$= 3.0 \times 10^{-4} \text{ rad/s}$$

Topic 4 Kepler's Laws and Motion of Planets

1. According to Kepler's second law, "the line joining the planet to sun sweeps out equal areas in equal interval of time". This means the rate of change of area with time is constant.



The area covered from P to P' is dA, which is given by

$$dA = \frac{d\theta}{2\pi} \times \pi r^2$$

$$\Rightarrow \quad dA = \frac{1}{2} r^2 d\theta \text{ or } \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

where, $\frac{dA}{dt}$ = a real velocity.

$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega \text{ or } \frac{dA}{dt} = \frac{1}{2} r^2 \cdot \frac{L}{mr^2}$$

(because angular momentum, $L = mr^2\omega$)

$$\frac{dA}{dt} = \frac{L}{2m}$$

2. $$F \propto \frac{1}{R^n}$$

$$\frac{mv^2}{R} \propto \frac{1}{R^n} \quad \text{or} \quad v \propto \frac{I}{R^{\frac{n-1}{2}}}$$

$$T = \frac{2\pi R}{v} \Rightarrow T \propto \frac{R}{v}$$

$$\Rightarrow \quad T \propto R^{1 + \frac{n-1}{2}} \quad \text{or} \quad T \propto R^{\frac{n+1}{2}}$$

3. In case of binary star system, angular velocity and hence, the time period of both the stars are equal.

4. From Kepler's third law $T^2 \propto r^3$ or $T \propto (r)^{3/2}$

$$\therefore \quad \frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2} \quad \text{or} \quad T_2 = T_1 \left(\frac{r_2}{r_1}\right)^{3/2} = (365)\left(\frac{1}{2}\right)^{3/2}$$

$$\Rightarrow \quad T_2 \approx 129 \text{ days}$$

5. $$\frac{mv^2}{R} \propto R^{-5/2}$$

$$\therefore \quad v \propto R^{-3/4}$$

Now, $$T = \frac{2\pi R}{v} \quad \text{or } T^2 \propto \left(\frac{R}{v}\right)^2$$

or $$T^2 \propto \left(\frac{R}{R^{-3/4}}\right)^2 \quad \text{or } T^2 \propto R^{7/2}$$

6. Areal velocity of a planet round the sun is constant and is given by

$$\frac{dA}{dt} = \frac{L}{2m}$$

where, L = angular momentum of planet (earth) about sun
and m = mass of planet (earth).

Given, $$\frac{L}{m} = 4.4 \times 10^{15} \text{ m}^2/\text{s}$$

$\therefore$ Area enclosed by earth in time T (365 days) will be

$$\text{Area} = \frac{dA}{dt} \cdot T = \frac{L}{2m} \cdot T$$
$$= \frac{1}{2} \times 4.4 \times 10^{15} \times 365 \times 24 \times 3600 \text{ m}^2$$

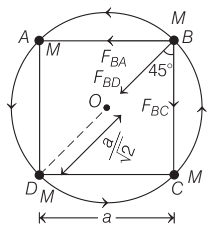
$$\text{Area} \approx 6.94 \times 10^{22} \text{ m}^2$$

7. $\frac{dA}{dt} = \frac{L}{2m}$ = constant, because angular momentum of planet (L) about the centre of sun is constant.

Thus, this law comes from law of conservation of angular momentum.

Topic 5 Miscellaneous Problems

1. In given configuration of masses, net gravitational force provides the necessary centripetal force for rotation.



Net force on mass M at position B towards centre of circle is

$$F_{BO\text{ net}} = F_{BD} + F_{BA} \sin 45° + F_{BC} \cos 45°$$
$$= \frac{GM^2}{(\sqrt{2}a)^2} + \frac{GM^2}{a^2}\left(\frac{1}{\sqrt{2}}\right) + \frac{GM^2}{a^2}\left(\frac{1}{\sqrt{2}}\right)$$

[where, diagonal length BD is $\sqrt{2}a$]

$$= \frac{GM^2}{2a^2} + \frac{GM^2}{a^2}\left(\frac{2}{\sqrt{2}}\right) = \frac{GM^2}{a^2}\left(\frac{1}{2} + \sqrt{2}\right)$$

This force will act as centripetal force.

Distance of particle from centre of circle is $\frac{a}{\sqrt{2}}$.

Here, $F_{\text{centripetal}} = \dfrac{Mv^2}{r} = \dfrac{Mv^2}{\dfrac{a}{\sqrt{2}}} = \dfrac{\sqrt{2}Mv^2}{a}$ $\left(\because r = \dfrac{a}{\sqrt{2}}\right)$

So, for rotation about the centre,

$F_{\text{centripetal}} = F_{BO\,(\text{net})}$

$\Rightarrow \quad \sqrt{2}\dfrac{Mv^2}{a} = \dfrac{GM^2}{a^2}\left(\dfrac{1}{2} + \sqrt{2}\right)$

$\Rightarrow \quad v^2 = \dfrac{GM}{a}\left(1 + \dfrac{1}{2\sqrt{2}}\right) = \dfrac{GM}{a}(1.35) \Rightarrow v = 1.16\sqrt{\dfrac{GM}{a}}$

2. Period of motion of a pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \ldots\text{(i)}$$

On the surface of earth, let period of motion is T_e and acceleration due to gravity is g_e

$$\therefore \qquad T_e = 2\pi\sqrt{\frac{l}{g_e}} \qquad \ldots\text{(ii)}$$

On the another planet, let period of motion is T_p and gravitational acceleration is g_p

$$\therefore \qquad T_p = 2\pi\sqrt{\frac{l}{g_p}} \qquad \ldots\text{(iii)}$$

($\because$ Pendulum is same, so l will be same)

From Eqs. (ii) and (iii),

$$\frac{T_e}{T_p} = \frac{2\pi\sqrt{\dfrac{l}{g_e}}}{2\pi\sqrt{\dfrac{l}{g_p}}} = \sqrt{\frac{g_p}{g_e}} \qquad \ldots\text{(iv)}$$

Now, $\quad g_e = \dfrac{GM_e}{R_e^2}$ and $g_p = \dfrac{GM_p}{R_p^2}$

Given, $\quad M_p = 3M_e$ and $R_p = 3R_e$

$\therefore \quad g_p = \dfrac{G \times 3M_e}{9R_e^2} = \dfrac{1}{3}\cdot\dfrac{GM_e}{R_e^2} = \dfrac{1}{3}g_e$

$$\Rightarrow \qquad \frac{g_p}{g_e} = \frac{1}{3} \text{ or } \sqrt{\frac{g_p}{g_e}} = \frac{1}{\sqrt{3}} \qquad \ldots\text{(v)}$$

From Eqs. (iv) and (v), $T_p = \sqrt{3}\,T_e$

or $\quad T_p = 2\sqrt{3}$ s $\qquad (\because T_e = 2\text{ s})$

3. Let us assume that stars are moving in $x\,y$-plane with origin as their centre of mass as shown in the figure below

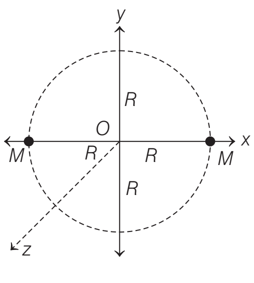
According to question,

mass of each star, $M = 3 \times 10^{31}$ kg

and diameter of circle,

$2R = 2 \times 10^{11}$ m

$\Rightarrow \quad R = 10^{11}$ m



Potential energy of meteorite at O, origin $\hat{\mathbf{j}}$ is, $\quad U_{\text{total}} = -\dfrac{2\,GMm}{r}$

If v is the velocity of meteorite at O then Kinetic energy K of the meteorite is

$$K = \frac{1}{2}mv^2$$

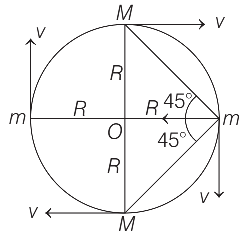
To escape from this dual star system, total mechanical energy of the meteorite at infinite distance from stars must be at least zero. By conservation of energy, we have

$$\frac{1}{2}mv^2 - \frac{2\,GMm}{R} = 0$$

$\Rightarrow v^2 = \dfrac{4\,GM}{R} = \dfrac{4 \times 6.67 \times 10^{-11} \times 3 \times 10^{31}}{10^{11}}$

$\Rightarrow \quad v = 2.8 \times 10^5$ m/s

4. Net force acting on any one particle M,



$= \dfrac{GM^2}{(2R)^2} + \dfrac{GM^2}{(R\sqrt{2})^2}\cos 45° + \dfrac{GM^2}{(R\sqrt{2})^2}\cos 45°$

$= \dfrac{GM^2}{R^2}\left(\dfrac{1}{4} + \dfrac{1}{\sqrt{2}}\right)$

This force will equal to centripetal force.

So, $\dfrac{Mv^2}{R} = \dfrac{GM^2}{R^2}\left(\dfrac{1}{4} + \dfrac{1}{\sqrt{2}}\right)$

$$v = \sqrt{\frac{GM}{4R}(1 + 2\sqrt{2})} = \frac{1}{2}\sqrt{\frac{GM}{R}(2\sqrt{2} + 1)}$$

Hence, speed of each particle in a circular motion is

$$\frac{1}{2}\sqrt{\frac{GM}{R}(2\sqrt{2} + 1)}.$$

5. $\quad T \propto \dfrac{1}{\sqrt{g}}$, i.e. $\dfrac{T_2}{T_1} = \sqrt{\dfrac{g_1}{g_2}}$

where, g_1 = acceleration due to gravity on earth's surface = g

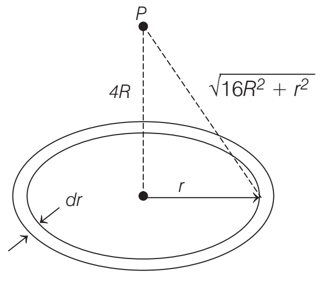
g_2 = acceleration due to gravity at a height $h = R$ from earth's surface $= g/4$

$$\left[\text{Using } g(h) = \frac{g}{\left(1 + \dfrac{h}{R}\right)^2}\right]$$

$$\Rightarrow \qquad \frac{T_2}{T_1} = \sqrt{\frac{g}{g/4}} = 2$$

6. $W = \Delta U = U_f - U_i = U_\infty - U_P = -U_P = -mV_P$

$= -V_P \qquad$ (as $m = 1$)



Potential at point P will be obtained by integration as given below.

Let dM be the mass of small ring as shown

$$dM = \frac{M}{\pi(4R)^2 - \pi(3R)^2}(2\pi r)\,dr = \frac{2Mr\,dr}{7R^2}$$

$$dV_P = -\frac{G\,dM}{\sqrt{16R^2 + r^2}}$$

$$= -\frac{2GM}{7R^2}\int_{3R}^{4R}\frac{r}{\sqrt{16R^2+r^2}}\,dr$$

$$= -\frac{2GM}{7R}(4\sqrt{2} - 5)$$

$$\therefore \quad W = +\frac{2GM}{7R}(4\sqrt{2} - 5)$$

7. Gravitational field at a distance r due to mass 'm'

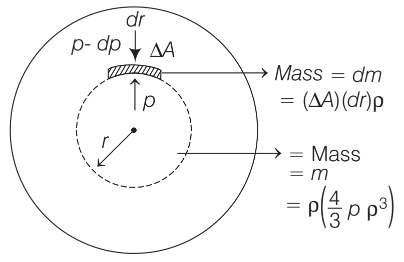
$$E = \frac{G\rho\frac{4}{3}\pi r^3}{r^2} = \frac{4G\rho\pi r}{3}$$

Consider a small element of width dr and area ΔA at a distance r.

Pressure force on this element outwards = gravitational force on 'dm' from 'm' inwards

$$\Rightarrow \quad (dp)\Delta A = E(dm)$$

$$\Rightarrow \quad -dp\cdot\Delta A = \left(\frac{4}{3}G\pi\rho r\right)(\Delta A\,dr\cdot\rho)$$



$$-\int_O^P dp = \int_R^r\left(\frac{4G\rho^2\pi}{3}\right)r\,dr$$

$$-p = \frac{4G\rho^2\pi}{3\times 2}[r^2 - R^2] \quad\Rightarrow\quad p = c(R^2 - r^2)$$

$$r = \frac{3R}{4},\ p_1 = c\left(R^2 - \frac{9R^2}{16}\right) = c\left(\frac{7R^2}{16}\right)$$

$$r = \frac{2R}{3},\ p_2 = c\left(R^2 - \frac{4R^2}{9}\right) = c\left(\frac{5R^2}{9}\right)$$

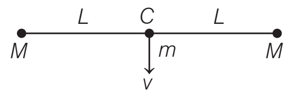
$$\frac{p_1}{p_2} = \frac{63}{80}$$

$$r = \frac{3R}{5},\ p_3 = c\left(R^2 - \frac{9}{25}R^2\right) = c\left(\frac{16R^2}{25}\right)$$

$$r = \frac{2R}{5},\ p_4 = c\left(R^2 - \frac{4R^2}{25}\right) = c\left(\frac{21R^2}{25}\right)$$

$$\frac{p_3}{p_4} = \frac{16}{21}$$

8.



Let v is the minimum velocity. From energy conservation,

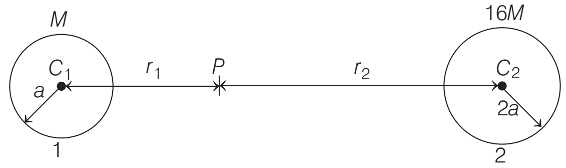
$$U_c + K_c = U_\infty + K_\infty$$

$$\therefore \quad mV_c + \frac{1}{2}mv^2 = 0 + 0$$

$$\therefore \quad v = \sqrt{-2V_c} = \sqrt{(-2)\left(\frac{-2GM}{L}\right)}$$

$$= 2\sqrt{\frac{GM}{L}}$$

9. Let there are two stars 1 and 2 as shown below.



Let P is a point between C_1 and C_2, where gravitational field strength is zero or at P field strength due to star 1 is equal and opposite to the field strength due to star 2. Hence,

$$\frac{GM}{r_1^2} = \frac{G(16M)}{r_2^2} \text{ or } \frac{r_2}{r_1} = 4 \quad \text{also} \quad r_1 + r_2 = 10a$$

$$\therefore \quad r_2 = \left(\frac{4}{4+1}\right)(10a) = 8a \quad\text{and}\quad r_1 = 2a$$

Now, the body of mass m is projected from the surface of larger star towards the smaller one. Between C_2 and P it is attracted towards 2 and between C_1 and P it will be attracted towards 1. Therefore, the body should be projected to just cross point P because beyond that the particle is attracted towards the smaller star itself.

From conservation of mechanical energy $\frac{1}{2}mv^2_{\min}$

= Potential energy of the body at P

– Potential energy at the surface of larger star.