8

Functions

Topic 1 Classification of Functions, Domain and Range and Even, Odd Functions

Objective Questions I (Only one correct option)

1. The domain of the definition of the function $f(x)=\frac{1}{4-x^2}+\log_{10}(x^3-x)$ is **(2019 Main, 9 April II)**

(a) $(-1,0)\cup(1,2)\cup(3,\infty)$ (b) $(-2,-1)\cup(-1,0)\cup(2,\infty)$
(c) $(-1,0)\cup(1,2)\cup(2,\infty)$ (d) $(1,2)\cup(2,\infty)$

2. Let $f(x)=a^x\,(a>0)$ be written as $f(x)=f_1(x)+f_2(x)$, where $f_1(x)$ is an even function and $f_2(x)$ is an odd function. Then $f_1(x+y)+f_1(x-y)$ equals **(2019 Main, 8 April II)**

(a) $2f_1(x+y)\cdot f_2(x-y)$ (b) $2f_1(x+y)\cdot f_1(x-y)$
(c) $2f_1(x)\cdot f_2(y)$ (d) $2f_1(x)\cdot f_1(y)$

3. Domain of definition of the function $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ for real valued x, is **(2003, 2M)**

(a) $\left[-\frac{1}{4},\frac{1}{2}\right]$ (b) $\left[-\frac{1}{2},\frac{1}{2}\right]$
(c) $\left(-\frac{1}{2},\frac{1}{9}\right)$ (d) $\left[-\frac{1}{4},\frac{1}{4}\right]$

4. Range of the function $f(x)=\frac{x^2+x+2}{x^2+x+1};x\in R$ is **(2003, 2M)**

(a) $(1,\infty)$ (b) $(1, 11/7)$
(c) $(1, 7/3]$ (d) $(1, 7/5)$

5. Let $f(x)=(1+b^2)x^2+2bx+1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is **(2001, 1M)**

(a) $[0,1]$ (b) $\left[0,\frac{1}{2}\right]$
(c) $\left[\frac{1}{2},1\right]$ (d) $(0,1]$

6. The domain of definition of $f(x)=\frac{\log_2(x+3)}{x^2+3x+2}$ is **(2001, 1M)**

(a) $R/\{-1,-2\}$
(b) $(-2,\infty)$
(c) $R/\{-1,-2,-3\}$
(d) $(-3,\infty)/\{-1,-2\}$

7. The domain of definition of the function $y(x)$ is given by the equation $2^x+2^y=2$, is **(2000, 1M)**

(a) $0<x\le 1$ (b) $0\le x\le 1$
(c) $-\infty<x\le 0$ (d) $-\infty<x<1$

8. Let $f(\theta)=\sin\theta\,(\sin\theta+\sin 3\theta)$. Then, $f(\theta)$ **(2000, 1M)**

(a) ≥ 0, only when $\theta\ge 0$ (b) ≤ 0, for all real θ
(c) ≥ 0, for all real θ (d) ≤ 0, only when $\theta\le 0$

9. The domain of definition of the function $y=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$ is **(1983, 1M)**

(a) $(-3,-2)$ excluding -2.5 (b) $[0,1]$ excluding 0.5
(c) $(-2,1)$ excluding 0 (d) None of these

Match the Columns

Match the conditions/expressions in Column I with statement in Column II.

10. Let $f(x)=\frac{x^2-6x+5}{x^2-5x+6}$. **(2007, 6M)**

	Column I		Column II
A.	If $-1<x<1$, then $f(x)$ satisfies	p.	$0<f(x)<1$
B.	If $1<x<2$, then $f(x)$ satisfies	q.	$f(x)<0$
C.	If $3<x<5$, then $f(x)$ satisfies	r.	$f(x)>0$
D.	If $x>5$, then $f(x)$ satisfies	s.	$f(x)<1$

Objective Question II
(One or more than one correct option)

11. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains **(1986, 2M)**

(a) $\left(-\infty,-\frac{3}{2}\right)$ (b) $\left(-\frac{3}{2},-\frac{1}{4}\right)$
(c) $\left(-\frac{1}{4},\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},3\right)$

Fill in the Blanks

12. If $f(x) = \sin \log \left(\frac{\sqrt{4 - x^2}}{1 - x} \right)$, then the domain of $f(x)$ is… . **(1985, 2M)**

13. The domain of the function $f(x) = \sin^{-1} \left(\log_2 \frac{x^2}{2} \right)$ is given by ... **(1984, 2M)**

14. The values of $f(x) = 3 \sin \left(\sqrt{\frac{\pi^2}{16} - x^2} \right)$ lie in the interval… **(1983, 2M)**

True/False

15. If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively, then $f_1(x) + f_2(x)$ is defined on $D_1 \cap D_2$. **(1988, 1M)**

Analytical & Descriptive Questions

16. Find the range of values of t for which

$$2 \sin t = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}, \; t \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

(2005, 2M)

17. Let $y = \sqrt{\frac{(x+1)(x-3)}{(x-2)}}$.

Find all the real values of x for which y takes real values. **(1980, 2M)**

Topic 2 Composite of Functions

Objective Questions I (Only one correct option)

1. For $x \in \left(0, \frac{3}{2} \right)$, let $f(x) = \sqrt{x}$, $g(x) = \tan x$ and $h(x) = \frac{1 - x^2}{1 + x^2}$. If $\phi(x) = ((hof)og)(x)$, then $\phi\left(\frac{\pi}{3}\right)$ is equal to **(2019 Main, 12 April I)**

(a) $\tan \frac{\pi}{12}$ (b) $\tan \frac{11\pi}{12}$

(c) $\tan \frac{7\pi}{12}$ (d) $\tan \frac{5\pi}{12}$

2. Let $f(x) = x^2, x \in R$. For any $A \subseteq R$, define $g(A) = \{x \in R : f(x) \in A\}$. If $S = [0, 4]$, then which one of the following statements is not true? **(2019 Main, 10 April I)**

(a) $f(g(S)) = S$ (b) $g(f(S)) \neq S$

(c) $g(f(S)) = g(S)$ (d) $f(g(S)) \neq f(S)$

3. Let $\sum_{k=1}^{10} f(a + k) = 16(2^{10} - 1)$, where the function f satisfies $f(x + y) = f(x) f(y)$ for all natural numbers x, y and $f(1) = 2$. Then, the natural number 'a' is **(2019 Main, 9 April I)**

(a) 2 (b) 4

(c) 3 (d) 16

4. If $f(x) = \log_e \left(\frac{1 - x}{1 + x} \right), |x| < 1$, then $f\left(\frac{2x}{1 + x^2} \right)$ is equal to **(2019 Main, 8 April I)**

(a) $2f(x)$ (b) $2f(x^2)$

(c) $(f(x))^2$ (d) $-2f(x)$

5. For $x \in R - \{0, 1\}$, let $f_1(x) = \frac{1}{x}$, $f_2(x) = 1 - x$ and $f_3(x) = \frac{1}{1 - x}$ be three given functions. If a function, $J(x)$ satisfies $(f_2 \circ J \circ f_1)(x) = f_3(x)$, then $J(x)$ is equal to **(2019 Main, 9 Jan I)**

(a) $f_2(x)$ (b) $f_3(x)$

(c) $f_1(x)$ (d) $\frac{1}{x} f_3(x)$

6. Let $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ be such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy$, $\forall x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to **(2017 Main)**

(a) 330 (b) 165 (c) 190 (d) 255

7. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then, the set of all x satisfying $(fogogof)(x) = (gogof)(x)$, where $(fog)(x) = f(g(x))$, is **(2011)**

(a) $\pm \sqrt{n\pi}, n \in \{0, 1, 2, \ldots\}$

(b) $\pm \sqrt{n\pi}, n \in \{1, 2, \ldots\}$

(c) $\pi/2 + 2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

(d) $2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

8. Let $f(x) = \frac{\alpha x}{x + 1}, x \neq -1$. Then, for what value of α is $f[f(x)] = x$? **(2001, 1M)**

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 1 (d) -1

9. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, then for all x, $f[g(x)]$ is equal to **(2001, 1M)**

(a) x (b) 1

(c) $f(x)$ (d) $g(x)$

10. If $g\{f(x)\} = |\sin x|$ and $f\{g(x)\} = (\sin \sqrt{x})^2$, then **(1998, 2M)**

(a) $f(x) = \sin^2 x, g(x) = \sqrt{x}$

(b) $f(x) = \sin x, g(x) = |x|$

(c) $f(x) = x^2, g(x) = \sin \sqrt{x}$

(d) f and g cannot be determined

11. If $f(x) = \cos(\log x)$, then $f(x) \cdot f(y) - \frac{1}{2} \left[f\left(\frac{x}{y}\right) + f(xy) \right]$ has the value **(1983, 1M)**

(a) -1 (b) $\frac{1}{2}$

(c) -2 (d) None of these

12. Let $f(x)=|x-1|$. Then, **(1983, 1M)**

(a) $f(x^2)=\{f(x)\}^2$
(b) $f(x+y)=f(x)+f(y)$
(c) $f(|x|)=|f(x)|$
(d) None of the above

Objective Questions II

(One or more than one correct option)

13. Let $f(x)=\sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right]$ for all $x\in R$ and $g(x)=\frac{\pi}{2}\sin x$ for all $x\in R$. Let $(fog)(x)$ denotes $f\{g(x)\}$ and $(gof)(x)$ denotes $g\{f(x)\}$. Then, which of the following is/are true? **(2015 Adv.)**

(a) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$ (b) Range of fog is $\left[-\frac{1}{2},\frac{1}{2}\right]$
(c) $\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$
(d) There is an $x\in R$ such that $(gof)(x)=1$

14. If $f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$, where $[x]$ stands for the greatest integer function, then **(1991, 2M)**

(a) $f(\pi/2)=-1$ (b) $f(\pi)=1$
(c) $f(-\pi)=0$ (d) $f(\pi/4)=1$

15. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at (0, 0) and $[x, g(x)]$ is $\sqrt{3}/4$, then the function $g(x)$ is **(1989, 2M)**

(a) $g(x)=\pm\sqrt{1-x^2}$ (b) $g(x)=\sqrt{1-x^2}$
(c) $g(x)=-\sqrt{1-x^2}$ (d) $g(x)=\sqrt{1+x^2}$

16. If $y=f(x)=\frac{x+2}{x-1}$, then **(1984, 3M)**

(a) $x=f(y)$ (b) $f(1)=3$
(c) y increases with x for $x<1$
(d) f is a rational function of x

Fill in the Blanks

17. If $f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then $(g\circ f)(x)=\ldots$. **(1996, 2M)**

True/False

18. If $f(x)=(a-x^n)^{1/n}$, where $a>0$ and n is a positive integer, then $f[f(x)]=x$. **(1983, 1M)**

Analytical & Descriptive Questions

19. Find the natural number a for which

$$\sum_{k=1}^{n} f(a+k)=16(2^n-1),$$

where the function f satisfies the relation $f(x+y)=f(x)f(y)$ for all natural numbers x, y and further $f(1)=2$. **(1992, 6M)**

Topic 3 Types of Functions

Objective Questions I (Only one correct option)

1. If the function $f:\mathbf{R}-\{1,-1\}\to A$ defined by $f(x)=\frac{x^2}{1-x^2}$, is surjective, then A is equal to **(2019 Main, 9 April I)**

(a) $\mathbf{R}-\{-1\}$
(b) $[0,\infty)$
(c) $\mathbf{R}-[-1,0)$
(d) $\mathbf{R}-(-1,0)$

2. Let a function $f:(0,\infty)\longrightarrow(0,\infty)$ be defined by $f(x)=\left|1-\frac{1}{x}\right|$. Then, f is **(2019 Main, 11 Jan II)**

(a) injective only
(b) both injective as well as surjective
(c) not injective but it is surjective
(d) neither injective nor surjective

3. The number of functions f from $\{1, 2, 3, \ldots, 20\}$ onto $\{1, 2, 3, \ldots, 20\}$ such that $f(k)$ is a multiple of 3, whenever k is a multiple of 4, is **(2019 Main, 11 Jan II)**

(a) $(15)!\times 6!$
(b) $5^6\times 15$
(c) $5!\times 6!$
(d) $6^5\times(15)!$

4. Let $f: R\to R$ be defined by $f(x)=\frac{x}{1+x^2}$, $x\in R$. Then, the range of f is **(2019 Main, 11 Jan I)**

(a) $\left[-\frac{1}{2},\frac{1}{2}\right]$ (b) $(-1,1)-\{0\}$
(c) $R-\left[-\frac{1}{2},\frac{1}{2}\right]$ (d) $R-[-1,1]$

5. Let N be the set of natural numbers and two functions f and g be defined as $f, g: N\longrightarrow N$ such that

$$f(n)=\begin{cases}\frac{n+1}{2}; & \text{if } n \text{ is odd}\\ \frac{n}{2}; & \text{if } n \text{ is even}\end{cases}$$

and $g(n)=n-(-1)^n$. Then, fog is **(2019 Main, 10 Jan II)**

(a) one-one but not onto (b) onto but not one-one
(c) both one-one and onto (d) neither one-one nor onto

6. Let $A=\{x\in R: x$ is not a positive integer$\}$. Define a function $f: A\to R$ as $f(x)=\frac{2x}{x-1}$, then f is **(2019 Main, 9 Jan II)**

(a) injective but not surjective
(b) not injective
(c) surjective but not injective
(d) neither injective nor surjective

7. The function $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$ is **(2017 Main)**

(a) invertible
(b) injective but not surjective
(c) surjective but not injective
(d) neither injective nor surjective

8. The function $f: [0, 3] \to [1, 29]$, defined by

$f(x) = 2x^3 - 15x^2 + 36x + 1$, is **(2012)**

(a) one-one and onto (b) onto but not one-one
(c) one-one but not onto (d) neither one-one nor onto

9. $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$, $g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$

Then, $f - g$ is **(2005, 1M)**

(a) one-one and into (b) neither one-one nor onto
(c) many one and onto (d) one-one and onto

10. If $f : [0, \infty) \to [0, \infty)$ and $f(x) = \frac{x}{1+x}$, then f is **(2003, 2M)**

(a) one-one and onto (b) one-one but not onto
(c) onto but not one-one (d) neither one-one nor onto

11. Let function $f : R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$. Then, f is **(2002, 1M)**

(a) one-to-one and onto (b) one-to-one but not onto
(c) onto but not one-to-one (d) neither one-to-one nor onto

12. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then, the number of onto functions from E to F is **(2001, 1M)**

(a) 14 (b) 16
(c) 12 (d) 8

Match the Columns

Match the conditions/expressions in Column I with statement in Column II.

13. Let $f_1 : R \to R$, $f_2 : [0, \infty] \to R$, $f_3 : R \to R$ and $f_4 : R \to [0, \infty)$ be defined by **(2014 Adv.)**

$$f_1(x) = \begin{cases} |x|, & \text{if } x < 0 \\ e^x, & \text{if } x \geq 0 \end{cases}; \; f_2(x) = x^2; \; f_3(x) = \begin{cases} \sin x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

and $$f_4(x) = \begin{cases} f_2[f_1(x)], & \text{if } x < 0 \\ f_2[f_1(x)] - 1, & \text{if } x \geq 0 \end{cases}$$

	Column I		Column II
A.	f_4 is	p.	onto but not one-one
B.	f_3 is	q.	neither continuous nor one-one
C.	$f_2 o f_1$ is	r.	differentiable but not one-one
D.	f_2 is	s.	continuous and one-one

Codes

	A	B	C	D
(a)	r	p	s	q
(b)	p	r	s	q
(c)	r	p	q	s
(d)	p	r	q	s

14. Let the functions defined in Column I have domain $(-\pi/2, \pi/2)$ and range $(-\infty, \infty)$ **(1992, 2M)**

	Column I		Column II
A.	$1 + 2x$	p.	onto but not one-one
B.	$\tan x$	q.	one-one but not onto
		r.	one-one and onto
		s.	neither one-one nor onto

Objective Question II

(One or more than one correct option)

15. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be given by

$f(x) = [\log(\sec x + \tan x)]^3$. Then,

(a) $f(x)$ is an odd function
(b) $f(x)$ is a one-one function
(c) $f(x)$ is an onto function
(d) $f(x)$ is an even function

Fill in the Blanks

16. There are exactly two distinct linear functions, ..., and... which map $\{-1, 1\}$ onto $\{0, 2\}$. **(1989, 2M)**

True/False

17. The function $f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$ is not one-to-one. **(1983, 1M)**

Analytical & Descriptive Question

18. A function $f : IR \to IR$, where IR, is the set of real numbers, is defined by $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$.

Find the interval of values of α for which is onto. Is the functions one-to-one for $\alpha = 3$? Justify your answer. **(1996, 5M)**

19. Let A and B be two sets each with a finite number of elements. Assume that there is an injective mapping from A to B and that there is an injective mapping from B to A. Prove that there is a bijective mapping from A to B. **(1981, 2M)**

Topic 4 Inverse and Periodic Functions

Objective Questions I (Only one correct option)

1. If X and Y are two non-empty sets where $f : X \to Y$, is function is defined such that

$$f(C) = \{f(x) : x \in C\} \text{ for } C \subseteq X \text{ and}$$
$$f^{-1}(D) = \{x : f(x) \in D\} \text{ for } D \subseteq Y,$$

for any $A \subseteq Y$ and $B \subseteq Y$, then **(2005, 1M)**

(a) $f^{-1}\{f(A)\} = A$
(b) $f^{-1}\{f(A)\} = A$, only if $f(X) = Y$
(c) $f\{f^{-1}(B)\} = B$, only if $B \subseteq f(x)$
(d) $f\{f^{-1}(B)\} = B$

2. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $g\{f(x)\}$ is invertible in the domain **(2004, 1M)**

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[0, \pi]$

3. Suppose $f(x) = (x+1)^2$ for $x \geq -1$. If $g(x)$ is the function whose graph is reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals **(2002, 1M)**

(a) $-\sqrt{x} - 1, x \geq 0$ (b) $\frac{1}{(x+1)^2}, x > -1$
(c) $\sqrt{x+1}, x \geq -1$ (d) $\sqrt{x} - 1, x \geq 0$

4. If $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals **(2001, 1M)**

(a) $\frac{x + \sqrt{x^2 - 4}}{2}$ (b) $\frac{x}{1 + x^2}$
(c) $\frac{x - \sqrt{x^2 - 4}}{2}$ (d) $1 + \sqrt{x^2 - 4}$

5. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is **(1999, 2M)**

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1 + \sqrt{1 + 4\log_2 x})$
(c) $\frac{1}{2}(1 - \sqrt{1 + 4\log_2 x})$ (d) not defined

6. If $f(x) = 3x - 5$, then $f^{-1}(x)$ **(1998, 2M)**

(a) is given by $\frac{1}{3x - 5}$
(b) is given by $\frac{x + 5}{3}$
(c) does not exist because f is not one-one
(d) does not exist because f is not onto

7. Which of the following functions is periodic? **(1983, 1M)**

(a) $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to the real number x
(b) $f(x) = \sin(1/x)$ for $x \neq 0$, $f(0) = 0$
(c) $f(x) = x \cos x$
(d) None of the above

Objective Question II
(One or more than one correct option)

8. Let $f : (0, 1) \to R$ be defined by $f(x) = \frac{b - x}{1 - bx}$, where b is a constant such that $0 < b < 1$. Then, **(2011)**

(a) f is not invertible on (0, 1)
(b) $f \neq f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$
(c) $f = f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$
(d) f^{-1} is differentiable on (0, 1)

Assertion and Reason

For the following questions, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows.

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

9. Let $F(x)$ be an indefinite integral of $\sin^2 x$.

Statement I The function $F(x)$ satisfies $F(x + \pi) = F(x)$ for all real x.
Because
Statement II $\sin^2(x + \pi) = \sin^2 x$, for all real x. **(2007, 3M)**

Analytical & Descriptive Question

10. Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false $f(x) = 1$, $f(y) \neq 1$, $f(z) \neq 2$ determine $f^{-1}(1)$. **(1982, 2M)**

11. If f is an even function defined on the interval $(-5, 5)$, then four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are .... **(1996, 1M)**

Answers

Topic 1

1. (c) **2.** (d) **3.** (a) **4.** (c)
5. (d) **6.** (d) **7.** (d) **8.** (c)
9. (c) **10.** A → p; B → q; C → q; D → p
11. (a,d) **12.** (−2,1)
13. Domain ∈ [−2,−1] ∪ [1, 2] **14.** $\left[0, \frac{3}{\sqrt{2}}\right]$ **15.** True
16. $t \in \left[-\frac{\pi}{2}, \frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$ **17.** $x \in [-1, 2) \cup [3, \infty)$

Topic 2

1. (b) **2.** (c) **3.** (c) **4.** (a)
5. (b) **6.** (a) **7.** (b) **8.** (d)
9. (b) **10.** (a) **11.** (d) **12.** (d)
13. (a,b,c) **14.** (a, c) **15.** (b, c) **16.** (a, d)
17. 1 **18.** True **19.** ($a = 3$)

Topic 3

1. (c) **2.** (d) **3.** (a) **4.** (a)
5. (b) **6.** (a) **7.** (c) **8.** (b)
9. (d) **10.** (b) **11.** (a) **12.** (a)
13. (d) **14.** A → q; B → r
15. (a, b, c) **16.** $y = x + 1$ and $y = -x + 1$
17. True **18.** $2 \le \alpha \le 14$, No

Topic 4

1. (c) **2.** (b) **3.** (d) **4.** (a)
5. (b) **6.** (b) **7.** (a) **8.** (b)
9. (d)
10. $f^{-1}(1) = y$ **11.** $\left(\frac{\pm 3 \pm \sqrt{5}}{2}\right)$

Hints & Solutions

Topic 1 Classification of Functions, Domain and Range

1. Given function $f(x) = \frac{1}{4 - x^2} + \log_{10}(x^3 - x)$

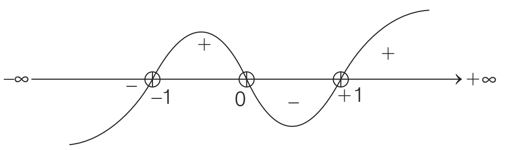
For domain of $f(x)$

$$4 - x^2 \ne 0 \Rightarrow x \ne \pm 2 \quad \text{...(i)}$$

and $x^3 - x > 0$

$\Rightarrow$ $x(x-1)(x+1) > 0$

From Wavy curve method,



$$x \in (-1, 0) \cup (1, \infty) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get the domain of $f(x)$ as $(-1, 0) \cup (1, 2) \cup (2, \infty)$.

2. Given, function $f(x) = a^x$, $a > 0$ is written as sum of an even and odd functions $f_1(x)$ and $f_2(x)$ respectively.

Clearly, $f_1(x) = \frac{a^x + a^{-x}}{2}$ and $f_2(x) = \frac{a^x - a^{-x}}{2}$

So, $f_1(x + y) + f_1(x - y)$

$$= \frac{1}{2}[a^{x+y} + a^{-(x+y)}] + \frac{1}{2}[a^{x-y} + a^{-(x-y)}]$$

$$= \frac{1}{2}\left[a^x a^y + \frac{1}{a^x a^y} + \frac{a^x}{a^y} + \frac{a^y}{a^x}\right]$$

$$= \frac{1}{2}\left[a^x\left(a^y + \frac{1}{a^y}\right) + \frac{1}{a^x}\left(\frac{1}{a^y} + a^y\right)\right]$$

$$= \frac{1}{2}\left(a^x + \frac{1}{a^x}\right)\left(a^y + \frac{1}{a^y}\right)$$

$$= 2\left(\frac{a^x + a^{-x}}{2}\right)\left(\frac{a^y + a^{-y}}{2}\right) = 2 f_1(x) \cdot f_1(y)$$

3. Here, $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$, to find domain we must have,

$$\sin^{-1}(2x) + \frac{\pi}{6} \ge 0 \qquad \left[\text{but } -\frac{\pi}{2} \le \sin^{-1}\theta \le \frac{\pi}{2}\right]$$

$$-\frac{\pi}{6} \le \sin^{-1}(2x) \le \frac{\pi}{2}$$

$$\sin\left(\frac{-\pi}{6}\right) \le 2x \le \sin\frac{\pi}{2} \quad \Rightarrow \quad \frac{-1}{2} \le 2x \le \frac{1}{2}$$

$$\frac{-1}{4} \le x \le \frac{1}{2}$$

$\because$ $x \in \left[\frac{-1}{4}, \frac{1}{2}\right]$

4. Let $y = f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}$, $x \in R$

$\therefore$ $y = \frac{x^2 + x + 2}{x^2 + x + 1}$

$$y = 1 + \frac{1}{x^2 + x + 1} \qquad \text{[i.e. } y > 1\text{]} \quad \text{...(i)}$$

$\Rightarrow$ $yx^2 + yx + y = x^2 + x + 2$

$\Rightarrow$ $x^2(y - 1) + x(y - 1) + (y - 2) = 0, \forall x \in R$

Since, x is real, $D \ge 0$

$\Rightarrow$ $(y - 1)^2 - 4(y - 1)(y - 2) \ge 0$

$\Rightarrow$ $(y - 1)\{(y - 1) - 4(y - 2)\} \ge 0$

$\Rightarrow \quad (y-1)(-3y+7) \geq 0$

$\Rightarrow \quad 1 \leq y \leq \frac{7}{3} \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii), Range $\in \left(1, \frac{7}{3}\right]$

5. Given, $f(x) = (1+b^2)x^2 + 2bx + 1$

$$= (1+b^2)\left(x + \frac{b}{1+b^2}\right)^2 + 1 - \frac{b^2}{1+b^2}$$

$m(b) =$ minimum value of $f(x) = \frac{1}{1+b^2}$ is positive

and $m(b)$ varies from 1 to 0, so range $= (0, 1]$

6. Given, $f(x) = \frac{\log_2(x+3)}{(x^2+3x+2)} = \frac{\log_2(x+3)}{(x+1)(x+2)}$

For numerator, $x + 3 > 0$

$\Rightarrow \quad x > -3 \quad \ldots\text{(i)}$

and for denominator, $(x+1)(x+2) \neq 0$

$\Rightarrow \quad x \neq -1, -2 \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii),

Domain is $(-3, \infty)/\{-1, -2\}$

7. Given, $2^x + 2^y = 2, \ \forall\, x, y \in R$

But $\quad 2^x, 2^y > 0, \ \forall\, x, y \in R$

Therefore, $2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2$

Taking log on both sides with base 2, we get

$\log_2 0 < \log_2 2^x < \log_2 2 \quad \Rightarrow \quad -\infty < x < 1$

8. It is given,

$f(\theta) = \sin\theta\,(\sin\theta + \sin 3\theta)$

$= (\sin\theta + 3\sin\theta - 4\sin^3\theta)\sin\theta$

$= (4\sin\theta - 4\sin^3\theta)\sin\theta \ = \sin^2\theta\,(4 - 4\sin^2\theta)$

$= 4\sin^2\theta\cos^2\theta = (2\sin\theta\cos\theta)^2$

$= (\sin 2\theta)^2 \geq 0$

which is true for all θ.

9. For domain of y,

$1 - x > 0, 1 - x \neq 1 \quad \text{and} \quad x + 2 > 0$

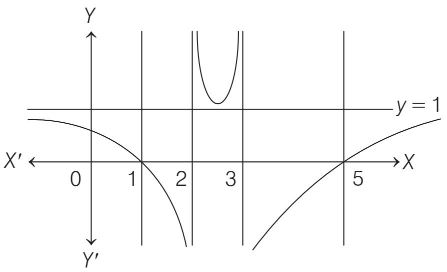
$\Rightarrow \quad x < 1, x \neq 0 \quad \text{and} \quad x > -2$

$\Rightarrow \quad -2 < x < 1$ excluding 0

$\Rightarrow \quad x \in (-2, 1) - \{0\}$

10. Given, $f(x) = \frac{(x-1)(x-5)}{(x-2)(x-3)}$

The graph of $f(x)$ is shown as :



A. If $-1 < x < 1 \Rightarrow \ 0 < f(x) < 1$

B. If $1 < x < 2 \ \Rightarrow \ f(x) < 0$

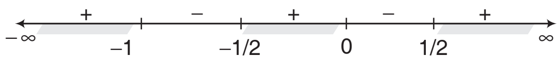
C. If $3 < x < 5 \ \Rightarrow \ f(x) < 0$

D. If $x > 5 \ \Rightarrow \ 0 < f(x) < 1$

11. Since, $\quad \frac{2x-1}{2x^3+3x^2+x} > 0$

$\Rightarrow \quad \frac{(2x-1)}{x(2x^2+3x+1)} > 0$

$\Rightarrow \quad \frac{(2x-1)}{x(2x+1)(x+1)} > 0$



Hence, the solution set is,

$x \in (-\infty, -1) \cup (-1/2, 0) \cup (1/2, \infty)$

Hence, (a) and (d) are the correct options.

12. Given, $f(x) = \sin\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)$

For domain, $\frac{\sqrt{4-x^2}}{1-x} > 0$, $4 - x^2 > 0$ and $1 - x \neq 0$

$\Rightarrow \quad (1-x) > 0 \quad \text{and} \quad 4 - x^2 > 0$

$\Rightarrow \quad x < 1 \quad \text{and} \quad |x| < 2 \ \Rightarrow -2 < x < 1$

Thus, domain $\in (-2, 1)$.

13. Given, $f(x) = \sin^{-1}\left(\log_2 \frac{x^2}{2}\right)$

For domain, $\quad -1 \leq \log_2 \frac{x^2}{2} \leq 1$

$\Rightarrow \quad \frac{1}{2} \leq \frac{x^2}{2} \leq 2$

$\Rightarrow \quad 1 \leq x^2 \leq 4$

$\Rightarrow \quad 1 \leq |x| \leq 2$

$\Rightarrow \quad$ Domain $\in [-2, -1] \cup [1, 2]$

14. Given, $f(x) = 3\sin\sqrt{\frac{\pi^2}{16} - x^2}$

$\Rightarrow$ Domain $\in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$\therefore$ For range, $f'(x) = 3\cos\left(\sqrt{\frac{\pi^2}{16} - x^2}\right) \cdot \frac{1(-2x)}{2\sqrt{\frac{\pi^2}{16} - x^2}} = 0$

Where, $\quad \cos\left(\sqrt{\frac{\pi^2}{16} - x^2}\right) = 0 \quad \text{or} \quad x = 0$

$$\left[\begin{array}{l}\text{neglecting } \cos\left(\sqrt{\frac{\pi^2}{16} - x^2}\right) = 0 \ \Rightarrow \ \frac{\pi^2}{16} - x^2 = \frac{\pi^2}{4} \\ \Rightarrow x^2 = -\frac{3\pi^2}{16}, \text{ never possible}\end{array}\right]$$

$\Rightarrow \quad x = 0$

Thus, $f(0) = 3\sin\frac{\pi}{4} = \frac{3}{\sqrt{2}}$

and $f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = 0$

Hence, range $\in \left[0, \frac{3}{\sqrt{2}}\right]$

15. Since, domains of $f_1(x)$ and $f_2(x)$ are D_1 and D_2.
Thus, domain of $[f_1(x) + f_2(x)]$ is $D_1 \cap D_2$.
Hence, given statement is true.

16. Given, $2\sin t = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Put $2\sin t = y \Rightarrow -2 \le y \le 2$

$\therefore \quad y = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$

$\Rightarrow (3y - 5)x^2 - 2x(y - 1) - (y + 1) = 0$

Since, $x \in R - \{1, -1/3\}$

[as, $3x^2 - 2x - 1 \ne 0 \Rightarrow (x - 1)(x + 1/3) \ne 0$]

$\therefore \quad D \ge 0$

$\Rightarrow \quad 4(y - 1)^2 + 4(3y - 5)(y + 1) \ge 0$

$\Rightarrow \quad y^2 - y - 1 \ge 0$

$\Rightarrow \quad \left(y - \frac{1}{2}\right)^2 - \frac{5}{4} \ge 0$

$\Rightarrow \quad \left(y - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(y - \frac{1}{2} + \frac{\sqrt{5}}{2}\right) \ge 0$

$\Rightarrow \quad y \le \frac{1 - \sqrt{5}}{2}$

or $\quad y \ge \frac{1 + \sqrt{5}}{2}$

$\Rightarrow \quad 2\sin t \le \frac{1 - \sqrt{5}}{2}$

or $\quad 2\sin t \ge \frac{1 + \sqrt{5}}{2}$

$\Rightarrow \quad \sin t \le \sin\left(-\frac{\pi}{10}\right)$

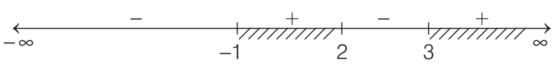
or $\quad \sin t \ge \sin\left(\frac{3\pi}{10}\right)$

$\Rightarrow \quad t \le -\frac{\pi}{10}$ or $t \ge \frac{3\pi}{10}$

Hence, range of t is $\left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$.

17. Since, $y = \sqrt{\frac{(x + 1)(x - 3)}{(x - 2)}}$ takes all real values only

when $\frac{(x + 1)(x - 3)}{(x - 2)} \ge 0$



$\Rightarrow \quad -1 \le x < 2$ or $x \ge 3$

$\therefore \quad x \in [-1, 2) \cup [3, \infty)$.

Topic 2 Composite of Functions and Even, Odd Functions

1. Given, for $x \in (0, 3/2)$, functions

$$f(x) = \sqrt{x} \quad \ldots \text{(i)}$$

$$g(x) = \tan x \quad \ldots \text{(ii)}$$

and $$h(x) = \frac{1 - x^2}{1 + x^2} \quad \ldots \text{(iii)}$$

Also given, $\phi(x) = ((hof)og)(x) = (hof)(g(x))$

$= h(f(g(x)))$

$= h(f(\tan x))$

$= h(\sqrt{\tan x}) = \frac{1 - (\sqrt{\tan x})^2}{1 + (\sqrt{\tan x})^2}$

$= \frac{1 - \tan x}{1 + \tan x} = \tan\left(\frac{\pi}{4} - x\right)$

Now, $\phi\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$

$= \tan\left(\frac{3\pi - 4\pi}{12}\right) = \tan\left(-\frac{\pi}{12}\right)$

$= -\tan\left(\frac{\pi}{12}\right) = \tan\left(\pi - \frac{\pi}{12}\right)$

$= \tan\left(\frac{11\pi}{12}\right)$

2. Given, functions $f(x) = x^2, x \in R$
and $g(A) = \{x \in R : f(x) \in A\}; A \subseteq R$
Now, for $S = [0, 4]$

$g(S) = \{x \in R : f(x) \in S = [0, 4]\}$

$= \{x \in R : x^2 \in [0, 4]\}$

$= \{x \in R : x \in [-2, 2]\}$

$\Rightarrow \quad g(S) = [-2, 2]$

So, $\quad f(g(S)) = [0, 4] = S$

Now, $f(S) = \{x^2 : x \in S = [0, 4]\} = [0, 16]$

and $g(f(S)) = \{x \in R : f(x) \in f(S) = [0, 16]\}$

$= \{x \in R : f(x) \in [0, 16]\}$

$= \{x \in R : x^2 \in [0, 16]\}$

$= \{x \in R : x \in [-4, 4]\} = [-4, 4]$

From above, it is clear that $g(f(S)) = g(S)$.

3. Given, $f(x + y) = f(x) \cdot f(y)$

Let $\quad f(x) = \lambda^x \quad$ [where $\lambda > 0$]

$\because \quad f(1) = 2 \quad$ (given)

$\therefore \quad \lambda = 2$

So, $\sum_{k=1}^{10} f(a + k) = \sum_{k=1}^{10} \lambda^{a+k} = \lambda^a\left(\sum_{k=1}^{10} \lambda^k\right)$

$= 2^a [2^1 + 2^2 + 2^3 + \ldots\ldots + 2^{10}]$

$= 2^a \left[\frac{2(2^{10} - 1)}{2 - 1}\right]$

[by using formula of sum of n-terms of a GP having first term 'a' and common ratio 'r', is

$S_n = \frac{a(r^n - 1)}{r - 1}$, where $r > 1$]

$\Rightarrow \quad 2^{a+1}(2^{10}-1)=16(2^{10}-1)$ (given)

$\Rightarrow \quad 2^{a+1}=16=2^4 \Rightarrow a+1=4 \Rightarrow a=3$

4. Given, $f(x)=\log_e\left(\frac{1-x}{1+x}\right), |x|<1$, then

$$f\left(\frac{2x}{1+x^2}\right)=\log_e\left(\frac{1-\frac{2x}{1+x^2}}{1+\frac{2x}{1+x^2}}\right) \qquad \left[\because \left|\frac{2x}{1+x^2}\right|<1\right]$$

$$=\log_e\left(\frac{\frac{1+x^2-2x}{1+x^2}}{\frac{1+x^2+2x}{1+x^2}}\right)=\log_e\left(\frac{(1-x)^2}{(1+x)^2}\right)=\log_e\left(\frac{1-x}{1+x}\right)^2$$

$$=2\log_e\left(\frac{1-x}{1+x}\right) \qquad [\because \log_e|A|^m=m\log_e|A|]$$

$$=2f(x) \qquad \left[\because f(x)=\log_e\left(\frac{1-x}{1+x}\right)\right]$$

5. We have,

$$f_1(x)=\frac{1}{x},\ f_2(x)=1-x \text{ and } f_3(x)=\frac{1}{1-x}$$

Also, we have $(f_2 \, o \, J \, o \, f_1)(x)=f_3(x)$

$\Rightarrow \quad f_2((J \, o \, f_1)(x))=f_3(x)$

$\Rightarrow \quad f_2(J(f_1(x)))=f_3(x)$

$\Rightarrow \quad 1-J(f_1(x))=\frac{1}{1-x}$

$\left[\because f_2(x)=1-x \text{ and } f_3(x)=\frac{1}{1-x}\right]$

$\Rightarrow \quad 1-J\left(\frac{1}{x}\right)=\frac{1}{1-x} \qquad \left[\because f_1(x)=\frac{1}{x}\right]$

$\Rightarrow \quad J\left(\frac{1}{x}\right)=1-\frac{1}{1-x}$

$$=\frac{1-x-1}{1-x}=\frac{-x}{1-x}$$

Now, put $\frac{1}{x}=X$, then

$$J(X)=\frac{\frac{-1}{X}}{1-\frac{1}{X}} \qquad \left[\because x=\frac{1}{X}\right]$$

$$=\frac{-1}{X-1}=\frac{1}{1-X}$$

$\Rightarrow \quad J(X)=f_3(X)$ or $J(x)=f_3(x)$

6. We have, $f(x)=ax^2+bx+c$

Now, $f(x+y)=f(x)+f(y)+xy$

Put $\quad y=0 \Rightarrow f(x)=f(x)+f(0)+0$

$\Rightarrow \quad f(0)=0$

$\Rightarrow \quad c=0$

Again, put $\quad y=-x$

$\therefore \quad f(0)=f(x)+f(-x)-x^2$

$\Rightarrow \quad 0=ax^2+bx+ax^2-bx-x^2$

$\Rightarrow \quad 2ax^2-x^2=0$

$\Rightarrow \quad a=\frac{1}{2}$

Also, $a+b+c=3$

$\Rightarrow \quad \frac{1}{2}+b+0=3 \Rightarrow b=\frac{5}{2}$

$\therefore \quad f(x)=\frac{x^2+5x}{2}$

Now, $\quad f(n)=\frac{n^2+5n}{2}=\frac{1}{2}n^2+\frac{5}{2}n$

$$\therefore \quad \sum_{n=1}^{10} f(n)=\frac{1}{2}\sum_{n=1}^{10} n^2+\frac{5}{2}\sum_{n=1}^{10} n$$

$$=\frac{1}{2}\cdot\frac{10\times 11\times 21}{6}+\frac{5}{2}\times\frac{10\times 11}{2}$$

$$=\frac{385}{2}+\frac{275}{2}=\frac{660}{2}=330$$

7. $f(x)=x^2,\ g(x)=\sin x$

$(gof)(x)=\sin x^2$

$go(gof)(x)=\sin(\sin x^2)$

$(fogogof)(x)=(\sin(\sin x^2))^2 \quad \ldots\text{(i)}$

Again, $\quad (gof)(x)=\sin x^2$

$(gogof)(x)=\sin(\sin x^2) \quad \ldots\text{(ii)}$

Given, $\quad (fogogof)(x)=(gogof)(x)$

$\Rightarrow \quad (\sin(\sin x^2))^2=\sin(\sin x^2)$

$\Rightarrow \quad \sin(\sin x^2)\{\sin(\sin x^2)-1\}=0$

$\Rightarrow \sin(\sin x^2)=0$ or $\sin(\sin x^2)=1$

$\Rightarrow \quad \sin x^2=0 \quad$ or $\quad \sin x^2=\frac{\pi}{2}$

$\therefore \quad x^2=n\pi$

$\left[\sin x^2=\frac{\pi}{2} \text{ is not possible as } -1\le\sin\theta\le 1\right]$

$x=\pm\sqrt{n\pi}$

8. Given, $f(x)=\frac{\alpha x}{x+1}$

$$f[f(x)]=f\left(\frac{\alpha x}{x+1}\right)=\frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1}+1}$$

$$=\frac{\frac{\alpha^2 x}{x+1}}{\frac{\alpha x+(x+1)}{x+1}}=\frac{\alpha^2 x}{(\alpha+1)x+1}=x \text{ [given]} \quad \ldots\text{(i)}$$

$\Rightarrow \quad \alpha^2 x=(\alpha+1)x^2+x$

$\Rightarrow \quad x[\alpha^2-(\alpha+1)x-1]=0$

$\Rightarrow \quad x(\alpha+1)(\alpha-1-x)=0$

$\Rightarrow \quad \alpha-1=0 \quad$ and $\quad \alpha+1=0$

$\Rightarrow \quad \alpha=-1$

But $\alpha=1$ does not satisfy the Eq. (i).

9. $g(x) = 1 + x - [x]$ is greater than 1

since $x - [x] > 0$

$f[g(x)] = 1$, since $f(x) = 1$ for all $x > 0$

10. Let $\quad f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Now, $\quad fog(x) = f[g(x)] = f(\sqrt{x}) = \sin^2 \sqrt{x}$

and $\quad gof(x) = g[f(x)] = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$

Again, let $f(x) = \sin x$, $g(x) = |x|$

$$fog(x) = f[g(x)] = f(|x|)$$
$$= \sin|x| \neq (\sin\sqrt{x})^2$$

When $\quad f(x) = x^2,\ g(x) = \sin\sqrt{x}$

$$fog(x) = f[g(x)] = f(\sin\sqrt{x}) = (\sin\sqrt{x})^2$$

and $\quad (gof)(x) = g[f(x)] = g(x^2) = \sin\sqrt{x^2}$

$$= \sin|x| \neq |\sin x|$$

11. Given, $f(x) = \cos(\log x)$

$$\therefore\ f(x)\cdot f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$$

$$= \cos(\log x)\cdot\cos(\log y) - \frac{1}{2}[\cos(\log x - \log y) + \cos(\log x + \log y)]$$

$$= \cos(\log x)\cdot\cos(\log y) - \frac{1}{2}[(2\cos(\log x)\cdot\cos(\log y)]$$

$$= \cos(\log x)\cdot\cos(\log y) - \cos(\log x)\cdot\cos(\log y) = 0$$

12. Given, $\quad f(x) = |x - 1|$

$\therefore \quad f(x^2) = |x^2 - 1|$

and $\quad \{f(x)\}^2 = (x-1)^2$

$\Rightarrow \quad f(x^2) \neq (f(x))^2$, hence (a) is false.

Also, $f(x+y) = |x+y-1|$

and $\quad f(x) = |x-1|$,

$\quad f(y) = |y-1|$

$\Rightarrow \quad f(x+y) \neq f(x) + f(y)$, hence (b) is false.

$\quad f(|x|) = ||x| - 1|$

and $\quad |f(x)| = ||x-1|| = |x-1|$

$\therefore \quad f(|x|) \neq |f(x)|$, hence (c) is false.

13. (a) $f(x) = \sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right], x \in R$

$$= \sin\left(\frac{\pi}{6}\sin\theta\right),\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ where } \theta = \frac{\pi}{2}\sin x$$

$$= \sin\alpha,\ \alpha \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right], \text{where } \alpha = \frac{\pi}{6}\sin\theta$$

$$\therefore \quad f(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Hence, range of $f(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

So, option (a) is correct.

(b) $\quad f\{g(x)\} = f(t),\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow f(t) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$\therefore$ Option (b) is correct.

(c) $$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right]}{\frac{\pi}{2}(\sin x)}$$

$$= \lim_{x\to 0}\frac{\sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right]}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}\cdot\frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\left(\frac{\pi}{2}\sin x\right)}$$

$$= 1 \times \frac{\pi}{6} \times 1 = \frac{\pi}{6}$$

$\therefore$ Option (c) is correct.

(d) $\quad g\{f(x)\} = 1$

$$\Rightarrow \quad \frac{\pi}{2}\sin\{f(x)\} = 1$$

$$\Rightarrow \quad \sin\{f(x)\} = \frac{2}{\pi} \quad \text{...(i)}$$

But $\quad f(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

$$\therefore \quad \sin\{f(x)\} \in \left[-\frac{1}{2}, \frac{1}{2}\right] \quad \text{...(ii)}$$

$\Rightarrow \sin\{f(x)\} \neq \frac{2}{\pi}$, $\quad$ [from Eqs. (i) and (ii)]

i.e. $\quad$ No solution.

$\therefore$ Option (d) is not correct.

14. Since, $\quad f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

$\Rightarrow \quad f(x) = \cos(9)x + \cos(-10)x$

[using $[\pi^2] = 9$ and $[-\pi^2] = -10$]

$$\therefore \quad f\left(\frac{\pi}{2}\right) = \cos\frac{9\pi}{2} + \cos 5\pi = -1$$

$$f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$$

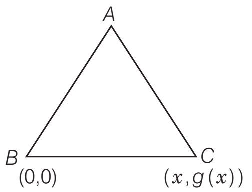
$$f(-\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$$

$$f\left(\frac{\pi}{4}\right) = \cos\frac{9\pi}{4} + \cos\frac{10\pi}{4} = \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

Hence, (a) and (c) are correct options.

15. Since, area of equilateral triangle $= \frac{\sqrt{3}}{4}(BC)^2$

$$\Rightarrow \quad \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\cdot[x^2 + g^2(x)] \Rightarrow g^2(x) = 1 - x^2$$



$$\Rightarrow \quad g(x) = \sqrt{1 - x^2} \text{ or } -\sqrt{1 - x^2}$$

Hence, (b) and (c) are the correct options.

16. Given, $\quad y = f(x) = \frac{x+2}{x-1}$

$$\Rightarrow \quad yx - y = x + 2 \quad \Rightarrow \quad x(y-1) = y + 2$$

$$\Rightarrow \quad x = \frac{y+2}{y-1} \quad \Rightarrow \quad x = f(y)$$

Here, $f(1)$ does not exist, so domain $\in R-\{1\}$

$$\frac{dy}{dx}=\frac{(x-1)\cdot 1-(x+2)\cdot 1}{(x-1)^2}$$
$$=-\frac{3}{(x-1)^2}$$

$\Rightarrow f(x)$ is decreasing for all $x \in R-\{1\}$.
Also, f is rational function of x.
Hence, (a) and (d) are correct options.

17. $f(x)=\sin^2 x+\sin^2(x+\pi/3)+\cos x\cos(x+\pi/3)$

$$\Rightarrow f(x)=\sin^2 x+(\sin x\cos\pi/3+\cos x\sin\pi/3)^2+\cos x\cos(x+\pi/3)$$

$$\Rightarrow f(x)=\sin^2 x+\left(\frac{\sin x\cdot 1}{2}+\frac{\cos x\cdot\sqrt{3}}{2}\right)^2+\cos x(\cos x\cos\pi/3-\sin x\sin\pi/3)$$

$$\Rightarrow f(x)=\sin^2 x+\frac{\sin^2 x}{4}+\frac{3\cos^2 x}{4}+\frac{2\cdot\sqrt{3}}{4}\sin x\cos x+\frac{\cos^2 x}{2}-\cos x\sin x\cdot\frac{\sqrt{3}}{2}$$

$$=\sin^2 x+\frac{\sin^2 x}{4}+\frac{3\cos^2 x}{4}+\frac{\cos^2 x}{2}$$
$$=\frac{5}{4}\sin^2 x+\frac{5}{4}\cos^2 x=\frac{5}{4}$$

and $\quad gof(x)=g\{f(x)\}=g(5/4)=1$

Alternate Solution

$$f(x)=\sin^2 x+\sin^2(x+\pi/3)+\cos x\cos(x+\pi/3)$$
$$\Rightarrow f'(x)=2\sin x\cos x+2\sin(x+\pi/3)\cos(x+\pi/3)-\sin x\cos(x+\pi/3)-\cos x\sin(x+\pi/3)$$
$$=\sin 2x+\sin(2x+2\pi/3)-[\sin(x+x+\pi/3)]$$
$$=2\sin\left(\frac{2x+2x+2\pi/3}{2}\right)\cdot\cos\left(\frac{2x-2x-2\pi/3}{2}\right)-\sin(2x+\pi/3)$$
$$=2[\sin(2x+\pi/3)\cdot\cos\pi/3]-\sin(2x+\pi/3)$$
$$=2\left[\sin(2x+\pi/3)\cdot\frac{1}{2}\right]-\sin\left(2x+\frac{\pi}{3}\right)=0$$

$\Rightarrow f(x)=c$, where c is a constant.
But $f(0)=\sin^2 0+\sin^2(\pi/3)+\cos 0\cos\pi/3$

$$=\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{2}=\frac{3}{4}+\frac{1}{2}=\frac{5}{4}$$

Therefore, $(gof)(x)=g[f(x)]=g(5/4)=1$

18. Given, $\quad f(x)=(a-x^n)^{1/n}$

$\Rightarrow \quad f[f(x)]=[a-\{(a-x^n)^{1/n}\}^n]^{1/n}=(x^n)^{1/n}=x$

$\therefore \quad f[f(x)]=x$

Hence, given statement is true.

19. Let $f(n)=2^n$ for all positive integers n.

Now, for $n=1$, $f(1)=2=2!$

$\Rightarrow$ It is true for $n=1$.

Again, let $f(k)$ is true.

$\Rightarrow \quad f(k)=2^k$, for some $k\in N$.

Again, $f(k+1)=f(k)\cdot f(1)$ [by definition]

$=2^k\cdot 2$ [from induction assumption]

$=2^{k+1}$

Therefore, the result is true for $n=k+1$. Hence, by principle of mathematical induction,

$$f(n)=2^n,\ \forall\ n\in N$$

Now, $\quad \sum_{k=1}^{n} f(a+k)=\sum_{k=1}^{n} f(a)f(k)=f(a)\sum_{k=1}^{n}2^k$

$$=f(a)\cdot\frac{2(2^n-1)}{2-1}$$
$$=2^a\cdot 2(2^n-1)=2^{a+1}(2^n-1)$$

But $\quad \sum_{k=1}^{n} f(a+k)=16(2^n-1)=2^4(2^n-1)$

Therefore, $\quad a+1=4 \Rightarrow a=3$

Topic 3 Types of Functions

1. Given, function $f:\mathbf{R}-\{1,-1\}\to A$ defined as

$f(x)=\dfrac{x^2}{1-x^2}=y$ (let)

$\Rightarrow \quad x^2=y(1-x^2)$ $[\because x^2\neq 1]$

$\Rightarrow \quad x^2(1+y)=y$

$\Rightarrow \quad x^2=\dfrac{y}{1+y}$ [provided $y\neq -1$]

$\because \quad x^2\geq 0$

$\Rightarrow \quad \dfrac{y}{1+y}\geq 0\Rightarrow y\in(-\infty,-1)\cup[0,\infty)$

Since, for surjective function, range of $f=$ codomain

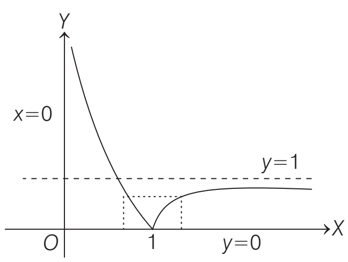
$\therefore$ Set A should be $\mathbf{R}-[-1,0)$.

2. We have, $f(x)=\dfrac{|x-1|}{x}=\begin{cases}-\dfrac{(x-1)}{x}, & \text{if } 0<x\leq 1\\ \dfrac{x-1}{x}, & \text{if } x>1\end{cases}$

$$=\begin{cases}\dfrac{1}{x}-1, & \text{if } 0<x\leq 1\\ 1-\dfrac{1}{x}, & \text{if } x>1\end{cases}$$

Now, let us draw the graph of $y=f(x)$

Note that when $x\to 0$, then $f(x)\to\infty$, when $x=1$, then $f(x)=0$, and when $x\to\infty$, then $f(x)\to 1$



Clearly, $f(x)$ is not injective because if $f(x) < 1$, then f is many one, as shown in figure.

Also, $f(x)$ is not surjective because range of $f(x)$ is $[0, \infty[$ and but in problem co-domain is $(0, \infty)$, which is wrong.

$\therefore f(x)$ is neither injective nor surjective

3. According to given information, we have if

$k \in \{4, 8, 12, 16, 20\}$

Then, $f(k) \in \{3, 6, 9, 12, 15, 18\}$

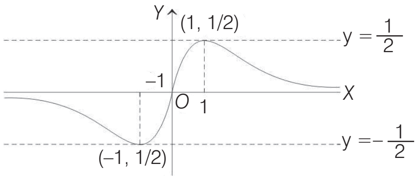
$[\because \text{Codomain } (f) = \{1, 2, 3, \ldots, 20\}]$

Now, we need to assign the value of $f(k)$ for $k \in \{4, 8, 12, 16, 20\}$ this can be done in ${}^6C_5 \cdot 5!$ ways $= 6 \cdot 5! = 6!$ and remaining 15 element can be associated by $15!$ ways.

$\therefore$ Total number of onto functions $= = 15!\,6!$

4. We have, $f(x) = \dfrac{x}{1 + x^2}, x \in R$

Ist Method $f(x)$ is an odd function and maximum occur at $x = 1$



From the graph it is clear that range of $f(x)$ is

$$\left[-\frac{1}{2}, \frac{1}{2}\right]$$

IInd Method $f(x) = \dfrac{1}{x + \dfrac{1}{x}}$

If $x > 0$, then by AM $\geq$ GM, we get $x + \dfrac{1}{x} \geq 2$

$\Rightarrow \quad \dfrac{1}{x + \dfrac{1}{x}} \leq \dfrac{1}{2} \quad \Rightarrow \quad 0 < f(x) \leq \dfrac{1}{2}$

If $x < 0$, then by AM $\geq$ GM, we get $x + \dfrac{1}{x} \leq -2$

$\Rightarrow \quad \dfrac{1}{x + \dfrac{1}{x}} \geq -\dfrac{1}{2} \quad \Rightarrow -\dfrac{1}{2} \leq f(x) < 0$

If $x = 0$, then $f(x) = \dfrac{0}{1 + 0} = 0$

Thus, $\quad -\dfrac{1}{2} \leq f(x) \leq \dfrac{1}{2}$

Hence, $f(x) \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

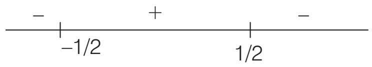
IIIrd Method

Let $y = \dfrac{x}{1 + x^2} \Rightarrow yx^2 - x + y = 0$

$\because \; x \in R$, so $D \geq 0$

$\Rightarrow \quad 1 - 4y^2 \geq 0$

$\Rightarrow (1 - 2y)(1 + 2y) \geq 0 \Rightarrow y \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$



So, range is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

5. Given, $f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even,} \end{cases}$

and $\quad g(n) = n - (-1)^n = \begin{cases} n + 1, & \text{if } n \text{ is odd} \\ n - 1, & \text{if } n \text{ is even} \end{cases}$

Now, $f(g(n)) = \begin{cases} f(n+1), & \text{if } n \text{ is odd} \\ f(n-1), & \text{if } n \text{ is even} \end{cases}$

$= \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n-1+1}{2} = \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

$= f(x)$

[$\because$ if n is odd, then $(n+1)$ is even and if n is even, then $(n-1)$ is odd]

Clearly, function is not one-one as $f(2) = f(1) = 1$

But it is onto function.

[$\because$ If $m \in N$ (codomain) is odd, then $2m \in N$ (domain) such that $f(2m) = m$ and

if $m \in N$ codomain is even, then

$2m - 1 \in N$ (domain) such that $f(2m - 1) = m$]

$\therefore$ Function is onto but not one-one

6. We have a function $f : A \to R$ defined as, $f(x) = \dfrac{2x}{x - 1}$

One-one Let $x_1, x_2 \in A$ such that

$$f(x_1) = f(x_2)$$

$\Rightarrow \quad \dfrac{2x_1}{x_1 - 1} = \dfrac{2x_2}{x_2 - 1}$

$\Rightarrow \quad 2x_1x_2 - 2x_1 = 2x_1x_2 - 2x_2$

$\Rightarrow \quad x_1 = x_2$

Thus, $f(x_1) = f(x_2)$ has only one solution, $x_1 = x_2$

$\therefore \; f(x)$ is one-one (injective)

Onto Let $x = 2$, then $f(2) = \dfrac{2 \times 2}{2 - 1} = 4$

But $x = 2$ is not in the domain, and $f(x)$ is one-one function

$\therefore f(x)$ can never be 4.

Similarly, $f(x)$ can not take many values.

Hence, $f(x)$ is into (not surjective).

$\therefore f(x)$ is injective but not surjective.

7. We have, $f(x) = \dfrac{x}{1 + x^2}$

$\therefore \quad f\left(\dfrac{1}{x}\right) = \dfrac{\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} = \dfrac{x}{1 + x^2} = f(x)$

$\therefore \quad f\left(\frac{1}{2}\right) = f(2) \text{ or } f\left(\frac{1}{3}\right) = f(3) \text{ and so on.}$

So, $f(x)$ is many-one function.

Again, let $\quad y = f(x) \Rightarrow y = \frac{x}{1+x^2}$

$\Rightarrow \quad y + x^2 y = x \Rightarrow yx^2 - x + y = 0$

As, $\quad x \in R$

$\therefore \quad (-1)^2 - 4\,(y)(y) \ge 0$

$\Rightarrow \quad 1 - 4y^2 \ge 0$

$\Rightarrow \quad y \in \left[\frac{-1}{2}, \frac{1}{2}\right]$

$\therefore$ Range = Codomain $= \left[\frac{-1}{2}, \frac{1}{2}\right]$

So, $f(x)$ is surjective.

Hence, $f(x)$ is surjective but not injective.

8. **PLAN** To check nature of function.

(i) One-one To check one-one, we must check whether $f'(x) > 0$ or $f'(x) < 0$ in given domain.

(ii) Onto To check onto, we must check Range = Codomain

Description of Situation To find range in given domain $[a, b]$, put $f'(x) = 0$ and find $x = \alpha_1, \alpha_2, \ldots, \alpha_n \in [a, b]$

Now, find $\{f(a), f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n), f(b)\}$

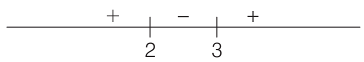
its greatest and least values gives you range.

Now, $f : [0, 3] \to [1, 29]$

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

$\therefore \quad f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$$= 6(x-2)(x-3)$$



For given domain [0, 3], $f(x)$ is increasing as well as decreasing $\Rightarrow$ many-one

Now, put $\quad f'(x) = 0$

$\Rightarrow \quad x = 2, 3$

Thus, for range $\quad f(0) = 1,\ f(2) = 29,\ f(3) = 28$

$\Rightarrow \quad$ Range $\in [1, 29]$

$\therefore$ Onto but not one-one.

9. Let $\phi(x) = f(x) - g(x) = \begin{cases} x, & x \in Q \\ -x, & x \notin Q \end{cases}$

Now, to check one-one.

Take any straight line parallel to X-axis which will intersect $\phi(x)$ only at one point.

$\Rightarrow \phi(x)$ is one-one.

To check onto

As $\quad f(x) = \begin{cases} x, & x \in Q \\ -x, & x \notin Q \end{cases}$, which shows

$y = x$ and $y = -x$ for rational and irrational values

$\Rightarrow y \in$ real numbers.

$\therefore \quad$ Range = Codomain $\Rightarrow$ onto

Thus, $f - g$ is one-one and onto.

10. Given, $f : [0, \infty) \to [0, \infty)$

Here, domain is $[0, \infty)$ and codomain is $[0, \infty)$. Thus, to check one-one

Since, $f(x) = \frac{x}{1+x} \Rightarrow f'(x) = \frac{1}{(1+x)^2} > 0, \forall x \in [0, \infty)$

$\therefore f(x)$ is increasing in its domain. Thus, $f(x)$ is one-one in its domain. To check onto (we find range)

Again, $\quad y = f(x) = \frac{x}{1+x}$

$\Rightarrow \quad y + yx = x$

$\Rightarrow \quad x = \frac{y}{1-y} \Rightarrow \frac{y}{1-y} \ge 0$

Since, $x \ge 0$, therefore $0 \le y < 1$

i.e. Range $\ne$ Codomain

$\therefore f(x)$ is one-one but not onto.

11. Given, $\quad f(x) = 2x + \sin x$

$\Rightarrow \quad f'(x) = 2 + \cos x \Rightarrow f'(x) > 0, \forall x \in R$

which shows $f(x)$ is one-one, as $f(x)$ is strictly increasing.

Since, $f(x)$ is increasing for every $x \in R$,

$\therefore f(x)$ takes all intermediate values between $(-\infty, \infty)$.

Range of $f(x) \in R$.

Hence, $f(x)$ is one-to-one and onto.

12. The number of onto functions from

$E = \{1, 2, 3, 4\}$ to $F = \{1, 2\}$

= Total number of functions which map E to F

− Number of functions for which map $f(x) = 1$ and $f(x) = 2$ for all $x \in E = 2^4 - 2 = 14$

13. **PLAN**

(i) For such questions, we need to properly define the functions and then we draw their graphs.

(ii) From the graphs, we can examine the function for continuity, differentiability, one-one and onto.

$$f_1(x) = \begin{cases} -x, & x < 0 \\ e^x, & x \ge 0 \end{cases}$$

$$f_2(x) = x^2, x \ge 0$$

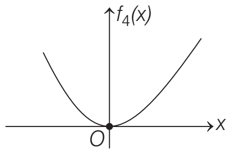
$$f_3(x) = \begin{cases} \sin x, & x < 0 \\ x, & x \ge 0 \end{cases}$$

$$f_4(x) = \begin{cases} f_2(f_1(x)), & x < 0 \\ f_2(f_1(x)) - 1, & x \ge 0 \end{cases}$$

Now, $\quad f_2(f_1(x)) = \begin{cases} x^2, & x < 0 \\ e^{2x}, & x \ge 0 \end{cases}$

$\Rightarrow \quad f_4 = \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \ge 0 \end{cases}$

As $f_4(x)$ is continuous, $f'_4(x) = \begin{cases} 2x, & x < 0 \\ 2e^{2x}, & x > 0 \end{cases}$



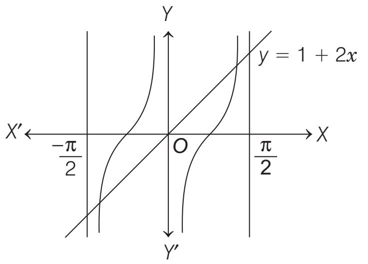
Graph for $f_4(x)$

$f_4'(0)$ is not defined. Its range is $[0, \infty)$.

Thus, range = codomain = $[0, \infty)$, thus f_4 is onto.

Also, horizontal line (drawn parallel to X-axis) meets the curve more than once, thus function is not one-one.

14. $y = 1 + 2x$ is linear function, therefore it is one-one and its range is $(-\pi + 1, \pi + 1)$. Therefore, $(1 + 2x)$ is one-one but not onto so (A) $\to$ (q). Again, see the figure.



It is clear from the graph that $y = \tan x$ is one-one and onto, therefore (B) $\to$ (r).

15. PLAN

(i) If $f'(x) > 0, \forall x \in (a, b)$, then $f(x)$ is an increasing function in (a, b) and thus $f(x)$ is one-one function in (a, b).

(ii) If range of $f(x)$ = codomain of $f(x)$, then $f(x)$ is an onto function.

(iii) A function $f(x)$ is said to be an odd function, if $f(-x) = -f(x), \forall x \in R$, i.e.
$f(-x) + f(x) = 0, \forall x \in R$

$$f(x) = [\ln(\sec x + \tan x)]^3$$

$$f'(x) = \frac{3[\ln(\sec x + \tan x)]^2 (\sec x \tan x + \sec^2 x)}{(\sec x + \tan x)}$$

$$f'(x) = 3 \sec x \, [\ln(\sec x + \tan x)]^2 > 0, \forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$f(x)$ is an increasing function.

$\therefore$ $f(x)$ is an one-one function.

$$(\sec x + \tan x) = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right), \text{ as } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ then}$$

$$0 < \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) < \infty$$

$$0 < \sec x + \tan x < \infty$$

$$\Rightarrow \quad -\infty < \ln(\sec x + \tan x) < \infty$$

$$-\infty < [\ln(\sec x + \tan x)]^3 < \infty$$

$$\Rightarrow \quad -\infty < f(x) < \infty$$

Range of $f(x)$ is R and thus $f(x)$ is an ont function.

$$f(-x) = [\ln(\sec x - \tan x)]^3 = \left[\ln\left(\frac{1}{\sec x + \tan x}\right)\right]^3$$

$$f(-x) = -[\ln(\sec x + \tan x)]^3$$

$$f(x) + f(-x) = 0$$

$\Rightarrow$ $f(x)$ is an odd function.

16. Let $y = ax + b$ and $y = cx + d$ be two linear functions.

When $x = -1, y = 0$ and $x = 1, y = 2$, then

$$0 = -a + b \quad \text{and} \quad a + b = 2 \;\Rightarrow\; a = b = 1$$

$\therefore$ $y = x + 1$ ...(i)

Again, when $x = -1, y = 2$ and $x = 1, y = 0$, then

$$-c + d = 2 \quad \text{and} \quad c + d = 0$$

$\Rightarrow$ $d = 1$ and $c = -1$

$\therefore$ $y = -x + 1$ ...(ii)

Hence, two linear functions are $y = x + 1$ and $y = -x + 1$

17. Given, $f(x) = \dfrac{x^2 + 4x + 30}{x^2 - 8x + 18}$

$$\Rightarrow \quad f'(x) = \frac{\begin{bmatrix} (x^2 - 8x + 18)(2x + 4) \\ -(x^2 + 4x + 30)(2x - 8) \end{bmatrix}}{(x^2 - 8x + 18)^2}$$

$$= \frac{2(-6x^2 - 12x + 156)}{(x^2 - 8x + 18)^2} = \frac{-12(x^2 + 2x - 26)}{(x^2 - 8x + 18)^2}$$

which shows $f'(x)$ is positive and negative both.

$\therefore$ $f(x)$ is many one.

Hence, given statement is true.

18. Let $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$$\Rightarrow \quad \alpha y + 6xy - 8x^2 y = \alpha x^2 + 6x - 8$$

$$\Rightarrow \quad -\alpha x^2 - 8x^2 y + 6xy - 6x + \alpha y + 8 = 0$$

$$\Rightarrow \quad \alpha x^2 + 8x^2 y - 6xy + 6x - \alpha y - 8 = 0$$

$$\Rightarrow \quad x^2(\alpha + 8y) + 6x(1 - y) - (8 + \alpha y) = 0$$

Since, x is real.

$$\Rightarrow \quad B^2 - 4AC \geq 0$$

$$\Rightarrow \quad 36(1 - y)^2 + 4(\alpha + 8y)(8 + \alpha y) \geq 0$$

$$\Rightarrow \quad 9(1 - 2y + y^2) + [8\alpha + (64 + \alpha^2)y + 8\alpha y^2] \geq 0$$

$$\Rightarrow \quad y^2(9 + 8\alpha) + y(46 + \alpha^2) + 9 + 8\alpha \geq 0 \quad \text{...(i)}$$

$$\Rightarrow \quad A > 0, D \leq 0, \;\Rightarrow\; 9 + 8\alpha > 0$$

and $(46 + \alpha^2)^2 - 4(9 + 8\alpha)^2 \leq 0$

$$\Rightarrow \quad \alpha > -9/8$$

and $[46 + \alpha^2 - 2(9 + 8\alpha)][46 + \alpha^2 + 2(9 + 8\alpha)] \leq 0$

$$\Rightarrow \quad \alpha > -9/8$$

and $(\alpha^2 - 16\alpha + 28)(\alpha^2 + 16\alpha + 64) \leq 0$

$$\Rightarrow \quad \alpha > -9/8$$

and $[(\alpha - 2)(\alpha - 14)](\alpha + 8)^2 \leq 0$

$$\Rightarrow \quad \alpha > -9/8$$

and $(\alpha - 2)(\alpha - 14) \leq 0$ $[\because (\alpha + 8)^2 \geq 0]$

$$\Rightarrow \quad \alpha > -9/8$$

and $2 \leq \alpha \leq 14$

$$\Rightarrow \quad 2 \leq \alpha \leq 14$$

Thus, $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ will be onto, if $2 \le \alpha \le 14$

Again, when $\alpha = 3$

$$f(x) = \frac{3x^2 + 6x - 8}{3 + 6x - 8x^2}, \text{ in this case } f(x) = 0$$

$$\Rightarrow \quad 3x^2 + 6x - 8 = 0$$

$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{36 + 96}}{6} = \frac{-6 \pm \sqrt{132}}{6} = \frac{1}{3}(-3 \pm \sqrt{33})$$

This shows that

$$f\left[\frac{1}{3}(-3 + \sqrt{33})\right] = f\left[\frac{1}{3}(-3 - \sqrt{33})\right] = 0$$

Therefore, f is not one-to-one.

19. Since, there is an injective mapping from A to B, each element of A has unique image in B.

Similarly, there is also an injective mapping from B to A, each element of B has unique image in A or in other words there is one to one onto mapping from A to B.

Thus, there is bijective mapping from A to B.

Topic 4 Inverse and Periodic Functions

1. Since, only (c) satisfy given definition

i.e. $\quad f\{f^{-1}(B)\} = B$

Only, if $\quad B \subseteq f(x)$

2. By definition of composition of function,

$g(f(x)) = (\sin x + \cos x)^2 - 1$, is invertible (i.e. bijective)

$\Rightarrow \quad g\{f(x)\} = \sin 2x$ is bijective.

We know, $\sin x$ is bijective, only when $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Thus, $g\{f(x)\}$ is bijective, if $-\dfrac{\pi}{2} \le 2x \le \dfrac{\pi}{2}$

$$\Rightarrow \quad -\frac{\pi}{4} \le x \le \frac{\pi}{4}$$

3. It is only to find the inverse.

Let $\quad y = f(x) = (x+1)^2$, for $x \ge -1$

$$\pm\sqrt{y} = x + 1, \quad x \ge -1$$

$$\Rightarrow \quad \sqrt{y} = x + 1 \quad \Rightarrow \quad y \ge 0, x + 1 \ge 0$$

$$\Rightarrow \quad x = \sqrt{y} - 1$$

$$\Rightarrow \quad f^{-1}(y) = \sqrt{y} - 1$$

$$\Rightarrow \quad f^{-1}(x) = \sqrt{x} - 1 \quad \Rightarrow \quad x \ge 0$$

4. Let $y = x + \dfrac{1}{x} \quad \Rightarrow \quad y = \dfrac{x^2 + 1}{x}$

$\Rightarrow \quad xy = x^2 + 1$

$\Rightarrow \quad x^2 - xy + 1 = 0 \quad \Rightarrow \quad x = \dfrac{y \pm \sqrt{y^2 - 4}}{2}$

$$\Rightarrow \quad f^{-1}(y) = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x \pm \sqrt{x^2 - 4}}{2}$$

Since, the range of the inverse function is $[1, \infty)$, then

we take $\quad f^{-1}(x) = \dfrac{x + \sqrt{x^2 - 4}}{2}$

If we consider $\quad f^{-1}(x) = \dfrac{x - \sqrt{x^2 - 4}}{2}$, then $f^{-1}(x) > 1$

This is possible only if $(x-2)^2 > x^2 - 4$

$$\Rightarrow \quad x^2 + 4 - 4x > x^2 - 4$$

$$\Rightarrow \quad 8 > 4x$$

$$\Rightarrow \quad x < 2, \text{ where } x > 2$$

Therefore, (a) is the answer.

5. Let $y = 2^{x(x-1)}$, where $y \ge 1$ as $x \ge 1$

Taking $\log_2$ on both sides, we get

$$\log_2 y = \log_2 2^{x(x-1)}$$

$$\Rightarrow \quad \log_2 y = x(x-1)$$

$$\Rightarrow \quad x^2 - x - \log_2 y = 0$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}$$

For $y \ge 1$, $\log_2 y \ge 0 \Rightarrow 4\log_2 y \ge 0 \Rightarrow 1 + 4\log_2 y \ge 1$

$$\Rightarrow \quad \sqrt{1 + 4\log_2 y} \ge 1$$

$$\Rightarrow \quad -\sqrt{1 + 4\log_2 y} \le -1$$

$$\Rightarrow \quad 1 - \sqrt{1 + 4\log_2 y} \le 0$$

But $\quad x \ge 1$

So, $x = 1 - \sqrt{1 + 4\log_2 y}$ is not possible.

Therefore, we take $x = \dfrac{1}{2}(1 + \sqrt{1 + 4\log_2 y})$

$$\Rightarrow \quad f^{-1}(y) = \frac{1}{2}(1 + \sqrt{1 + 4\log_2 y})$$

$$\Rightarrow \quad f^{-1}(x) = \frac{1}{2}(1 + \sqrt{1 + 4\log_2 x})$$

6. Given, $f(x) = 3x - 5$ [given]

Let $\quad y = f(x) = 3x - 5 \quad \Rightarrow \quad y + 5 = 3x$

$$\Rightarrow \quad x = \frac{y + 5}{3}$$

$$f^{-1}(y) = \frac{y + 5}{3}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x + 5}{3}$$

7. Clearly, $f(x) = x - [x] = \{x\}$

which has period 1.

And $\sin \dfrac{1}{x}$, $x \cos x$ are non-periodic functions.

8. Here, $f(x) = \dfrac{b - x}{1 - bx}$, where $0 < b < 1, 0 < x < 1$

For function to be invertible, it should be one-one onto.

$\therefore$ Check Range :

Let $f(x) = y \Rightarrow y = \dfrac{b-x}{1-bx}$

$\Rightarrow y - bxy = b - x \Rightarrow x(1-by) = b - y$

$\Rightarrow x = \dfrac{b-y}{1-by}$, where $0 < x < 1$

$\therefore \quad 0 < \dfrac{b-y}{1-by} < 1 \Rightarrow \dfrac{b-y}{1-by} > 0$ and $\dfrac{b-y}{1-by} < 1$

$\Rightarrow \quad y < b$ or $y > \dfrac{1}{b}$ ...(i)

$\dfrac{(b-1)(y+1)}{1-by} < 0 - 1 < y < \dfrac{1}{b}$ ...(ii)

From Eqs. (i) and (ii), we get

$$y \in \left(-1, \frac{1}{b}\right) \subset \text{Codomain}$$

9. Given, $F(x) = \int \sin^2 x\, dx = \int \dfrac{1-\cos 2x}{2} dx$

$F(x) = \dfrac{1}{4}(2x - \sin 2x) + C$

Since, $F(x+\pi) \neq F(x)$

Hence, Statement I is false.

But Statement II is true as $\sin^2 x$ is periodic with period π.

10. It gives three cases

***Case* I** When $f(x) = 1$ is true.

In this case, remaining two are false.

$\therefore \quad f(y) = 1$ and $f(z) = 2$

This means x and y have the same image, so $f(x)$ is not an injective, which is a contradiction.

***Case* II** When $f(y) \neq 1$ is true.

If $f(y) \neq 1$ is true, then the remaining statements are false.

$\therefore \quad f(x) \neq 1$ and $f(z) = 2$

i.e. both x and y are not mapped to 1. So, either both associate to 2 or 3. Thus, it is not injective.

***Case* III** When $f(z) \neq 2$ is true.

If $f(z) \neq 2$ is true, then remaining statements are false.

$\therefore$ If $\quad f(x) \neq 1$ and $f(y) = 1$

But f is injective.

Thus, we have $\quad f(x) = 2, f(y) = 1$ and $f(z) = 3$

Hence, $\quad f^{-1}(1) = y$

11. Since, f is an even function,

then $f(-x) = f(x), \forall x \in (-5, 5)$

Given, $\quad f(x) = f\left(\dfrac{x+1}{x+2}\right)$ ...(i)

$\Rightarrow \quad f(-x) = f\left(\dfrac{-x+1}{-x+2}\right)$

$\Rightarrow \quad f(x) = f\left(\dfrac{-x+1}{-x+2}\right)$ $[\because f(-x) = f(x)]$

Taking f^{-1} on both sides, we get

$x = \dfrac{-x+1}{-x+2}$

$\Rightarrow \quad -x^2 + 2x = -x + 1$

$\Rightarrow \quad x^2 - 3x + 1 = 0$

$\Rightarrow \quad x = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

Again, $\quad f(x) = f\left(\dfrac{x+1}{x+2}\right)$

$\Rightarrow \quad f(-x) = f\left(\dfrac{x+1}{x+2}\right)$ $[\because f(-x) = f(x)]$

Taking f^{-1} on both sides, we get

$-x = \dfrac{x+1}{x+2}$

$\Rightarrow \quad x^2 + 3x + 1 = 0$

$\Rightarrow \quad x = \dfrac{-3 \pm \sqrt{9-4}}{2} = \dfrac{-3 \pm \sqrt{5}}{2}$

Therefore, four values of x are $\dfrac{\pm 3 \pm \sqrt{5}}{2}$.