12

Definite Integration

Topic 1 Properties of Definite Integral

Objective Questions I (Only one correct option)

1. A value of α such that

$$\int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)} = \log_e\left(\frac{9}{8}\right)$$ is **(2019 Main, 12 April II)**

(a) -2 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 2

2. If $\int_0^{\pi/2} \frac{\cot x}{\cot x + \operatorname{cosec} x} dx = m(\pi + n)$, then $m \cdot n$ is equal to **(2019 Main, 12 April I)**

(a) $-\frac{1}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) -1

3. The integral $\int_{\pi/6}^{\pi/3} \sec^{2/3} x \operatorname{cosec}^{4/3} x \, dx$ is equal to **(2019 Main, 10 April II)**

(a) $3^{5/6} - 3^{2/3}$ (b) $3^{7/6} - 3^{5/6}$
(c) $3^{5/3} - 3^{1/3}$ (d) $3^{4/3} - 3^{1/3}$

4. The value of $\int_0^{2\pi} [\sin 2x (1 + \cos 3x)] \, dx$, where $[t]$ denotes the greatest integer function, is **(2019 Main, 10 April I)**

(a) $-\pi$ (b) 2π (c) -2π (d) π

5. The value of the integral $\int_0^1 x \cot^{-1}(1 - x^2 + x^4) dx$ is **(2019 Main, 9 April II)**

(a) $\frac{\pi}{4} - \frac{1}{2}\log_e 2$ (b) $\frac{\pi}{2} - \frac{1}{2}\log_e 2$
(c) $\frac{\pi}{4} - \log_e 2$ (d) $\frac{\pi}{2} - \log_e 2$

6. The value of $\int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x} dx$ is **(2019 Main, 9 April I)**

(a) $\frac{\pi - 1}{2}$ (b) $\frac{\pi - 2}{8}$ (c) $\frac{\pi - 1}{4}$ (d) $\frac{\pi - 2}{4}$

7. Let $f(x) = \int_0^x g(t) dt$, where g is a non-zero even function. If $f(x+5) = g(x)$, then $\int_0^x f(t) dt$ equals **(2019 Main, 8 April II)**

(a) $5 \int_{x+5}^{5} g(t) dt$ (b) $\int_{5}^{x+5} g(t) dt$
(c) $2 \int_{5}^{x+5} g(t) dt$ (d) $\int_{x+5}^{5} g(t) dt$

8. If $f(x) = \frac{2 - x\cos x}{2 + x\cos x}$ and $g(x) = \log_e x$, $(x > 0)$ then the value of the integral $\int_{-\pi/4}^{\pi/4} g(f(x)) dx$ is **(2019 Main, 8 April I)**

(a) $\log_e 3$ (b) $\log_e e$
(c) $\log_e 2$ (d) $\log_e 1$

9. Let f and g be continuous functions on $[0, a]$ such that $f(x) = f(a - x)$ and $g(x) + g(a - x) = 4$, then $\int_0^a f(x) g(x) \, dx$ is equal to **(2019 Main, 12 Jan I)**

(a) $4\int_0^a f(x) \, dx$ (b) $\int_0^a f(x) \, dx$
(c) $2\int_0^a f(x) \, dx$ (d) $-3\int_0^a f(x) \, dx$

10. The integral $\int_1^e \left\{\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x\right\} \log_e x \, dx$ is equal to **(2019 Main, 12 Jan II)**

(a) $\frac{3}{2} - e - \frac{1}{2e^2}$ (b) $-\frac{1}{2} + \frac{1}{e} - \frac{1}{2e^2}$
(c) $\frac{1}{2} - e - \frac{1}{e^2}$ (d) $\frac{3}{2} - \frac{1}{e} - \frac{1}{2e^2}$

11. The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$ equals **(2019 Main, 11 Jan II)**

(a) $\frac{1}{5}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)\right)$ (b) $\frac{1}{20}\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)$
(c) $\frac{1}{10}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$ (d) $\frac{\pi}{40}$

12. The value of the integral $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx$ (where, $[x]$ denotes the greatest integer less than or equal to x) is **(2019 Main, 11 Jan I)**

(a) $4 - \sin 4$ (b) 4
(c) $\sin 4$ (d) 0

13. Let $I = \int_a^b (x^4 - 2x^2) \, dx$. If I is minimum, then the ordered pair (a, b) is **(2019 Main, 10 Jan I)**

(a) $(-\sqrt{2}, 0)$ (b) $(0, \sqrt{2})$
(c) $(\sqrt{2}, -\sqrt{2})$ (d) $(-\sqrt{2}, \sqrt{2})$

14. If $\int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2k\sec\theta}}\,d\theta = 1 - \frac{1}{\sqrt{2}}$, $(k > 0)$, then the value of k is **(2019 Main, 9 Jan II)**

(a) 1 (b) $\frac{1}{2}$

(c) 2 (d) 4

15. The value of $\int_0^{\pi} |\cos x|^3\,dx$ is **(2019 Main, 9 Jan I)**

(a) $\frac{2}{3}$ (b) $-\frac{4}{3}$ (c) 0 (d) $\frac{4}{3}$

16. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x}\,dx$ is **(2018 Main)**

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{2}$ (c) 4π (d) $\frac{\pi}{4}$

17. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to **(2017 Main)**

(a) -2 (b) 2 (c) 4 (d) -1

18. The value of $\int_{-\pi/2}^{\pi/2} \frac{x^2\cos x}{1+e^x}\,dx$ is equal to **(2016 Adv.)**

(a) $\frac{\pi^2}{4} - 2$ (b) $\frac{\pi^2}{4} + 2$

(c) $\pi^2 - e^{-\pi/2}$ (d) $\pi^2 + e^{\pi/2}$

19. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)}\,dx$ is equal to **(2015, Main)**

(a) 2 (b) 4 (c) 1 (d) 6

20. The integral $\int_{\pi/4}^{\pi/2} (2\ \text{cosec}\ x)^{17}\,dx$ is equal to **(2014 Adv.)**

(a) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16}\,du$

(b) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17}\,du$

(c) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17}\,du$

(d) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16}\,du$

21. The integral $\int_0^{\pi} \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\,dx$ is equal to **(2014 Main)**

(a) $\pi - 4$ (b) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$

(c) $4\sqrt{3} - 4$ (d) $4\sqrt{3} - 4 - \pi/3$

22. The value of the integral $\int_{-\pi/2}^{\pi/2} \left(x^2 + \log\frac{\pi - x}{\pi + x}\right)\cos x\,dx$ is **(2012)**

(a) 0 (b) $\frac{\pi^2}{2} - 4$ (c) $\frac{\pi^2}{2} + 4$ (d) $\frac{\pi^2}{2}$

23. The value of $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x\sin x^2}{\sin x^2 + \sin(\log 6 - x^2)}\,dx$ is **(2011)**

(a) $\frac{1}{4}\log\frac{3}{2}$ (b) $\frac{1}{2}\log\frac{3}{2}$ (c) $\log\frac{3}{2}$ (d) $\frac{1}{6}\log\frac{3}{2}$

24. The value of

$\int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]\,dx$ is **(2005, 1M)**

(a) 0 (b) 3 (c) 4 (d) 1

25. The value of the integral $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$ is **(2004, 1M)**

(a) $\frac{\pi}{2} + 1$ (b) $\frac{\pi}{2} - 1$ (c) -1 (d) 1

26. The integral $\int_{-1/2}^{1/2} \left([x] + \log\left(\frac{1+x}{1-x}\right)\right)dx$ equals **(2002, 1M)**

(a) $-\frac{1}{2}$ (b) 0 (c) 1 (d) $\log\left(\frac{1}{2}\right)$

27. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x}\,dx, a > 0$, is **(2001, 1M)**

(a) π (b) $a\pi$

(c) $\frac{\pi}{2}$ (d) 2π

28. If $f(x) = \begin{cases} e^{\cos x}\sin x, & \text{for } |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$

then $\int_{-2}^{3} f(x)\,dx$ is equal to **(2000, 2M)**

(a) 0 (b) 1 (c) 2 (d) 3

29. The value of the integral $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$ is **(2000, 2M)**

(a) 3/2 (b) 5/2 (c) 3 (d) 5

30. If for a real number y, $[y]$ is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\,dx$ is **(1999, 2M)**

(a) $-\pi$ (b) 0

(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

31. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to **(1999, 2M)**

(a) 2 (b) -2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

32. Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x. Then, $\int_{-1}^{1} f(x)\,dx$ is **(1998, 2M)**

(a) 1 (b) 2

(c) 0 (d) $-\frac{1}{2}$

33. If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x + \pi)$ equals **(1997, 2M)**

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$

(c) $g(x)g(\pi)$ (d) $\frac{g(x)}{g(\pi)}$

34. Let f be a positive function.

If $I_1 = \int_{1-k}^{k} x f[x(1-x)]\,dx$ and

$I_2 = \int_{1-k}^{k} f[x(1-x)]\,dx$, where $2k - 1 > 0$.

Then, $\frac{I_1}{I_2}$ is **(1997C, 2M)**

(a) 2 (b) k

(c) 1/2 (d) 1

35. The value of $\int_0^{2\pi} [2\sin x]\, dx$, where [.] represents the greatest integral functions, is **(1995, 2M)**

(a) $-\frac{5\pi}{3}$ (b) $-\pi$ (c) $\frac{5\pi}{3}$ (d) -2π

36. If $f(x) = A\sin\left(\frac{\pi x}{2}\right) + B$,

$f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)\, dx = \frac{2A}{\pi}$, then constants A and B are **(1995, 2M)**

(a) $\frac{\pi}{2}$ and $\frac{\pi}{2}$ (b) $\frac{2}{\pi}$ and $\frac{3}{\pi}$ (c) 0 and $-\frac{4}{\pi}$ (d) $\frac{4}{\pi}$ and 0

37. The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is **(1993, 1M)**

(a) 0 (b) 1
(c) $\pi/2$ (d) $\pi/4$

38. Let $f: R \to R$ and $g: R \to R$ be continuous functions. Then, the value of the integral $\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)]\,[g(x) - g(-x)]\, dx$ is **(1990, 2M)**

(a) π (b) 1 (c) -1 (d) 0

39. For any integer n, the integral $\int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x\, dx$ has the value **(1985, 2M)**

(a) π (b) 1
(c) 0 (d) None of these

40. The value of the integral $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$ is **(1983, 1M)**

(a) $\pi/4$ (b) $\pi/2$
(c) π (d) None of these

Assertion and Reason

41. Statement I The value of the integral **(2013 Main)**

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} \text{ is equal to } \pi/6.$$

Statement II $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

(a) Statement I is correct; Statement II is correct; Statement II is a correct explanation for Statement I
(b) Statement I is correct; Statement II is correct; Statement II is not a correct explanation for Statement I
(c) Statement I is correct; Statement II is false
(d) Statement I is incorrect; Statement II is correct

Passage Based Questions

Passage I

Let $F: R \to R$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1, 3)$. Let $f(x) = xF(x)$ for all $x \in R$. **(2015 Adv.)**

42. The correct statement(s) is/are

(a) $f'(1) < 0$ (b) $f(2) < 0$
(c) $f'(x) \neq 0$ for any $x \in (1, 3)$ (d) $f'(x) = 0$ for some $x \in (1, 3)$

43. If $\int_1^3 x^2 F'(x)\, dx = -12$ and $\int_1^3 x^3 F''(x)\, dx = 40$, then the correct expression(s) is/are

(a) $9f'(3) + f'(1) - 32 = 0$ (b) $\int_1^3 f(x)\, dx = 12$
(c) $9f'(3) - f'(1) + 32 = 0$ (d) $\int_1^3 f(x)\, dx = -12$

Passage II

For every function $f(x)$ which is twice differentiable, these will be good approximation of

$$\int_a^b f(x)\, dx = \left(\frac{b-a}{2}\right)\{f(a) + f(b)\},$$

for more acurate results for $c \in (a, b)$,

$$F(c) = \frac{c-a}{2}[f(a) - f(c)] + \frac{b-c}{2}[f(b) - f(c)]$$

When $c = \frac{a+b}{2}$

$$\int_a^b f(x)\, dx = \frac{b-a}{4}\{f(a) + f(b) + 2f(c)\}\, dx$$

(2006, 6M)

44. If $\lim_{t \to a} \frac{\int_a^t f(x)\, dx - \frac{(t-a)}{2}\{f(t) + f(a)\}}{(t-a)^3} = 0$,

then degree of polynomial function $f(x)$ atmost is

(a) 0 (b) 1
(c) 3 (d) 2

45. If $f''(x) < 0, \forall x \in (a, b)$, and $(c, f(c))$ is point of maxima, where $c \in (a, b)$, then $f'(c)$ is

(a) $\frac{f(b) - f(a)}{b-a}$ (b) $3\left[\frac{f(b) - f(a)}{b-a}\right]$
(c) $2\left[\frac{f(b) - f(a)}{b-a}\right]$ (d) 0

46. Good approximation of $\int_0^{\pi/2} \sin x\, dx$, is

(a) $\pi/4$ (b) $\pi(\sqrt{2}+1)/4$
(c) $\pi(\sqrt{2}+1)/8$ (d) $\frac{\pi}{8}$

Objective Questions II

(One or more than one correct option)

47. Let $f: R \to (0, 1)$ be a continuous function. Then, which of the following function(s) has (have) the value zero at some point in the interval (0, 1)? **(2017 Adv.)**

(a) $e^x - \int_0^x f(t) \sin t\, dt$ (b) $f(x) + \int_0^{\frac{\pi}{2}} f(t) \sin t\, dt$
(c) $x - \int_0^{\frac{\pi}{2} - x} f(t) \cos t\, dt$ (d) $x^9 - f(x)$

48. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)}\, dx$, then **(2017 Adv.)**

(a) $I > \log_e 99$ (b) $I < \log_e 99$
(c) $I < \frac{49}{50}$ (d) $I > \frac{49}{50}$

49. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then, the correct expression(s) is/are

(a) $\int_0^{\pi/4} x f(x)\, dx = \frac{1}{12}$ (b) $\int_0^{\pi/4} f(x)\, dx = 0$ **(2015 Adv.)**

(c) $\int_0^{\pi/4} x f(x)\, dx = \frac{1}{6}$ (d) $\int_0^{\pi/4} f(x)\, dx = 1$

50. Let $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)\, dx \le M$, then the possible values of m and M are **(2015 Adv.)**

(a) $m = 13, M = 24$

(b) $m = \frac{1}{4}, M = \frac{1}{2}$

(c) $m = -11, M = 0$

(d) $m = 1, M = 12$

51. The option(s) with the values of a and L that satisfy the equation $\frac{\int_0^{4\pi} e^t (\sin^6 at + \cos^4 at) dt}{\int_0^{\pi} e^t (\sin^6 at + \cos^4 at) dt} = L$, is/are **(2015 Adv.)**

(a) $a = 2, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$ (b) $a = 2, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

(c) $a = 4, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$ (d) $a = 4, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

52. The value(s) of $\int_0^1 \frac{x^4 (1 - x)^4}{1 + x^2}\, dx$ is (are) **(2010)**

(a) $\frac{22}{7} - \pi$ (b) $\frac{2}{105}$ (c) 0 (d) $\frac{71}{15} - \frac{3\pi}{2}$

53. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x}\, dx,\ n = 0, 1, 2, \ldots$, then **(2009)**

(a) $I_n = I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(c) $\sum_{m=1}^{10} I_{2m} = 0$ (d) $I_n = I_{n+1}$

Numerical Value

54. The value of the integral $\int_0^{1/2} \frac{1 + \sqrt{3}}{((x + 1)^2 (1 - x)^6)^{1/4}}\, dx$ is .... **(2018 Adv.)**

Fill in the Blanks

55. Let $f : [1, \infty] \to [2, \infty]$ be differentiable function such that $f(1) = 2$. If $6\int_1^x f(t) = dt = 3x f(x) - x^3,\ \forall\ x \ge 1$ then the value of $f(2)$ is .... **(2011)**

56. Let $\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}, x > 0$.

If $\int_1^4 \frac{2e^{\sin x^2}}{x}\, dx = F(k) - F(1)$, then one of the possible values of k is ..... . **(1997, 2M)**

57. The value of $\int_1^{37\pi} \frac{\pi \sin (\pi \log x)}{x}\, dx$ is .... **(1997, 2M)**

58. For $n > 0, \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx =$ .... **(1996, 2M)**

59. If for non-zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \ne b$, then $\int_1^2 f(x)\, dx =$ .... **(1996, 2M)**

60. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5 - x} + \sqrt{x}}\, dx$ is .... **(1994, 2M)**

61. The value of $\int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x}\, dx$ .... **(1993, 2M)**

62. The value of $\int_{-2}^{2} |1 - x^2|\, dx$ is ... . **(1989, 2M)**

63. The integral $\int_0^{1.5} [x^2]\, dx$, where [.] denotes the greatest function, equals .... **(1988, 2M)**

Match the Columns

64. Match the conditions/expressions in Column I with statement in Column II.

Column I		Column II	
A.	$\int_{-1}^{1} \frac{dx}{1 + x^2}$	P.	$\frac{1}{2}\log\left(\frac{2}{3}\right)$
B.	$\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$	Q.	$2\log\left(\frac{2}{3}\right)$
C.	$\int_2^3 \frac{dx}{1 - x^2}$	R.	$\frac{\pi}{3}$
D.	$\int_1^2 \frac{dx}{x\sqrt{x^2 - 1}}$	S.	$\frac{\pi}{2}$

65. Match List I with List II and select the correct answer using codes given below the lists. **(2014)**

	List I		List II
P.	The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)\, dx = 1$, is	(i)	8
Q.	The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(ii)	2
R.	$\int_{-2}^{2} \frac{3x^2}{1 + e^x} dx$ equals	(iii)	4
S.	$\frac{\left(\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1 + x}{1 - x}\right) dx\right)}{\left(\int_0^{1/2} \cos 2x \log\left(\frac{1 + x}{1 - x}\right) dx\right)}$ equals	(iv)	0

Codes

	P	Q	R	S		P	Q	R	S
(a)	(iii)	(ii)	(iv)	(i)	(b)	(ii)	(iii)	(iv)	(i)
(c)	(iii)	(ii)	(i)	(iv)	(d)	(ii)	(iii)	(i)	(iv)

Analytical & Descriptive Questions

66. The value of $\dfrac{(5050)\int_0^1 (1-x^{50})^{100}\, dx}{\int_0^1 (1-x^{50})^{101}\, dx}$ is **(2006, 6M)**

67. Evaluate
$\int_0^{\pi} e^{|\cos x|}\left[2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\, dx.$ **(2005, 2M)**

68. Evaluate $\int_{-\pi/3}^{\pi/3} \dfrac{\pi+4x^3}{2-\cos\left(|x|+\dfrac{\pi}{3}\right)}\, dx.$ **(2004, 4M)**

69. If f is an even function, then prove that $\int_0^{\pi/2} f(\cos 2x)\cos x\, dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx.$ **(2003, 2M)**

70. Evaluate $\int_0^{\pi} \dfrac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\, dx.$ **(1999, 3M)**

71. Prove that $\int_0^1 \tan^{-1}\left(\dfrac{1}{1-x+x^2}\right) dx = 2\int_0^1 \tan^{-1} x\, dx.$

Hence or otherwise, evaluate the integral
$\int_0^1 \tan^{-1}(1-x+x^2)\, dx.$ **(1998, 8M)**

72. Integrate $\int_0^{\pi/4} \log(1+\tan x)\, dx.$ **(1997C, 2M)**

73. Determine the value of $\int_{-\pi}^{\pi} \dfrac{2x(1+\sin x)}{1+\cos^2 x}\, dx.$ **(1995, 5M)**

74. Evaluate the definite integral
$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right) dx.$$ **(1995, 5M)**

75. Evaluate $\int_2^3 \dfrac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}\, dx.$ **(1993, 5M)**

76. A cubic $f(x)$ vanishes at $x=-2$ and has relative minimum / maximum at $x=-1$ and $x=1/3$.
If $\int_{-1}^1 f(x)\, dx = 14/3$, find the cubic $f(x)$. **(1992, 4M)**

77. Evaluate $\int_0^{\pi} \dfrac{x\sin(2x)\sin\left(\dfrac{\pi}{2}\cos x\right)}{2x-\pi}\, dx.$ **(1991, 4M)**

78. Show that ,
$\int_0^{\pi/2} f(\sin 2x)\sin x\, dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx.$ **(1990, 4M)**

79. Prove that for any positive integer k,
$\dfrac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + \ldots + \cos(2k-1)x]$

Hence, prove that $\int_0^{\pi/2} \sin 2kx \cdot \cot x\, dx = \pi/2.$ **(1990, 4M)**

80. If f and g are continuous functions on $[0, a]$ satisfying $f(x)=f(a-x)$ and $g(x)+g(a-x)=2$, then show that
$\int_0^a f(x)g(x)\, dx = \int_0^a f(x)\, dx.$ **(1989, 4M)**

81. Prove that the value of the integral, $\int_0^{2a} [f(x)/\{f(x)+f(2a-x)\}]\, dx$ is equal to a. **(1988, 4M)**

82. Evaluate $\int_0^1 \log[\sqrt{(1-x)}+\sqrt{(1+x)}]\, dx.$ **(1988, 5M)**

83. Evaluate $\int_0^{\pi} \dfrac{x\, dx}{1+\cos\alpha\sin x}$, $0<\alpha<\pi$. **(1986, $2\frac{1}{2}$M)**

84. Evaluate $\int_0^{\pi/2} \dfrac{x\sin x\cos x}{\cos^4 x+\sin^4 x}\, dx.$ **(1985, $2\frac{1}{2}$M)**

85. Evaluate $\int_0^{1/2} \dfrac{x\sin^{-1} x}{\sqrt{1-x^2}}\, dx.$ **(1984, 2M)**

86. Evaluate $\int_0^{\pi/4} \dfrac{\sin x+\cos x}{9+16\sin 2x}\, dx.$ **(1983, 3M)**

87. (i) Show that $\int_0^{\pi} x f(\sin x)\, dx = \dfrac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx.$ **(1982, 2M)**

(ii) Find the value of $\int_{-1}^{3/2} |x\sin\pi x|\, dx.$ **(1982, 3M)**

88. Evaluate $\int_0^1 (tx+1-x)^n dx$,

where n is a positive integer and t is a parameter independent of x. Hence, show that
$\int_0^1 x^k(1-x)^{n-k}dx = \dfrac{1}{{}^nC_k(n+1)}$, for $k=0,1,\ldots,n$. **(1981, 4M)**

Integer Answer Type Questions

89. Let $f: R \to R$ be a function defined by $f(x)=\begin{cases}[x], & x\le 2\\ 0, & x>2\end{cases}$,
where $[x]$ denotes the greatest integer less than or equal to x. If $I=\int_{-1}^2 \dfrac{xf(x^2)}{2+f(x+1)}\, dx$, then the value of $(4I-1)$ is **(2015 Adv.)**

90. If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x})\left(\dfrac{12+9x^2}{1+x^2}\right) dx$,
where $\tan^{-1} x$ takes only principal values, then the value of $\left(\log_e|1+\alpha|-\dfrac{3\pi}{4}\right)$ is **(2015 Adv.)**

91. The value of $\int_0^1 4x^3\left\{\dfrac{d^2}{dx^2}(1-x^2)^5\right\} dx$ is **(2014 Adv.)**

Topic 2 Periodicity of Integral Functions

Objective Questions I (Only one correct option)

1. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x] + [\sin x] + 4}$, where $[t]$ denotes the greatest integer less than or equal to t, is **(2019 Main, 10 Jan II)**

(a) $\frac{1}{12}(7\pi - 5)$ (b) $\frac{1}{12}(7\pi + 5)$

(c) $\frac{3}{10}(4\pi - 3)$ (d) $\frac{3}{20}(4\pi - 3)$

2. Let $T > 0$ be a fixed real number. Suppose, f is a continuous function such that for all $x \in R$. $f(x + T) = f(x)$. If $I = \int_0^T f(x)\,dx$, then the value of $\int_3^{3+3T} f(2x)\,dx$ is **(2002,1M)**

(a) $\frac{3}{2}I$ (b) I

(c) $3I$ (d) $6I$

3. Let $g(x) = \int_0^x f(t)\,dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$ for $t \in [0,1]$ and $0 \le f(t) \le \frac{1}{2}$ for $t \in [1,2]$. Then, $g(2)$ satisfies the inequality **(2000, 2M)**

(a) $-\frac{3}{2} \le g(2) < \frac{1}{2}$ (b) $0 \le g(2) < 2$

(c) $\frac{3}{2} < g(2) \le 5/2$ (d) $2 < g(2) < 4$

Analytical & Descriptive Questions

4.. Show that $\int_0^{n\pi + v} |\sin x|\,dx = 2n + 1 - \cos v$, where n is a positive integer and $0 \le v < \pi$. **(1994, 4M)**

5. Given a function $f(x)$ such that it is integrable over every interval on the real line and $f(t + x) = f(x)$, for every x and a real t, then show that the integral $\int_a^{a+t} f(x)\,dx$ is independent of a. **(1984, 4M)**

Integer Answer Type Question

6. For any real number x, let $[x]$ denotes the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x[, & \text{if } f(x) \text{ is odd} \\ 1 + [x[- x, & \text{if } f(x) \text{ is even} \end{cases}$$

Then, the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\,dx$ is...... **(2010)**

Topic 3 Estimation, Gamma Function and Derivative of Definite Integration

Objective Questions I (Only one correct option)

1. If $\int_0^x f(t)\,dt = x^2 + \int_x^1 t^2 f(t)\,dt$, then $f'\left(\frac{1}{2}\right)$ is **(2019 Main, 10 Jan II)**

(a) $\frac{24}{25}$ (b) $\frac{18}{25}$ (c) $\frac{6}{25}$ (d) $\frac{4}{5}$

2. Let $f : [0,2] \to R$ be a function which is continuous on $[0,2]$ and is differentiable on $(0,2)$ with $f(0) = 1$.

Let $F(x) = \int_0^{x^2} f(\sqrt{t})\,dt$, for $x \in [0,2]$. If $F'(x) = f'(x)$, $\forall$ $x \in (0,2)$, then $F(2)$ equals **(2014 Adv)**

(a) $e^2 - 1$ (b) $e^4 - 1$ (c) $e - 1$ (d) e^4

3. The intercepts on X-axis made by tangents to the curve, $y = \int_0^x |t|\,dt, x \in R$, which are parallel to the line $y = 2x$, are equal to **(2013 Main)**

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

4. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1 - (f'(t))^2}\,dt = \int_0^x f(t)\,dt$, $0 \le x \le 1$ and $f(0) = 0$, then **(2009)**

(a) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(b) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(c) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(d) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

5. If $\int_{\sin x}^1 t^2 f(t)\,dt = 1 - \sin x$, $\forall x \in (0, \pi/2)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

(a) 3 (b) $\sqrt{3}$ **(2005, 1M)**

(c) 1/3 (d) None of these

6. If $f(x)$ is differentiable and $\int_0^{t^2} x f(x)\,dx = \frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ equals **(2004, 1M)**

(a) $\frac{2}{5}$ (b) $-\frac{5}{2}$

(c) 1 (d) $\frac{5}{2}$

7. If $f(x)=\int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in **(2003, 1M)**

(a) (2, 2) (b) no value of x
(c) $(0,\infty)$ (d) $(-\infty,0)$

8. If $I(m,n)=\int_0^1 t^m(1+t)^n\,dt$, then the expression for $I(m,n)$ in terms of $I(m+1,n-1)$ is **(2003, 1M)**

(a) $\frac{2^n}{m+1}-\frac{n}{m+1}I(m+1,n-1)$

(b) $\frac{n}{m+1}I(m+1,n-1)$

(c) $\frac{2^n}{m+1}+\frac{n}{m+1}I(m+1,n-1)$

(d) $\frac{m}{m+1}I(m+1,n-1)$

9. Let $f(x)=\int_1^x\sqrt{2-t^2}\,dt$. Then, the real roots of the equation $x^2-f'(x)=0$ are **(2002, 1M)**

(a) ±1 (b) $\pm\frac{1}{\sqrt2}$ (c) $\pm\frac{1}{2}$ (d) 0 and 1

10. Let $f:(0,\infty)\to R$ and $F(x)=\int_0^x f(t)\,dt$.
If $F(x^2)=x^2(1+x)$, then $f(4)$ equals **(2001, 1M)**

(a) $\frac{5}{4}$ (b) 7 (c) 4 (d) 2

11. $\int_0^x f(t)\,dt=x+\int_x^1 t\,f(t)\,dt$, then the value of $f(1)$ is **(1998, 2M)**

(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) $-\frac{1}{2}$

12. Let $f:R\to R$ be a differentiable function and $f(1)=4$. Then, the value of $\lim_{x\to1}\int_4^{f(x)}\frac{2t}{x-1}dt$ is **(1990, 2M)**

(a) $8f'(1)$ (b) $4f'(1)$
(c) $2f'(1)$ (d) $f'(1)$

13. The value of the definite integral $\int_0^1(1+e^{-x^2})\,dx$ is **(1981, 2M)**

(a) -1
(b) 2
(c) $1+e^{-1}$
(d) None of the above

Objective Question II

(One or more than one correct option)

14. If $g(x)=\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)\,dt$, then **(2017 Adv.)**

(a) $g'\left(-\frac{\pi}{2}\right)=2\pi$ (b) $g'\left(-\frac{\pi}{2}\right)=-2\pi$
(c) $g'\left(\frac{\pi}{2}\right)=2\pi$ (d) $g'\left(\frac{\pi}{2}\right)=-2\pi$

Passage Based Questions

Let $f(x)=(1-x)^2\sin^2 x+x^2,\ \forall\, x\in R$ and
$g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)\,dt\ \forall\, x\in(1,\infty)$.

15. Consider the statements

P : There exists some $x\in R$ such that, $f(x)+2x=2(1+x^2)$.

Q : There exists some $x\in R$ such that, $2f(x)+1=2x(1+x)$.

Then,

(a) both P and Q are true (b) P is true and Q is false
(c) P is false and Q is true (d) both P and Q are false

16. Which of the following is true?

(a) g is increasing on $(1,\infty)$
(b) g is decreasing on $(1,\infty)$
(c) g is increasing on (1, 2) and decreasing on $(2,\infty)$
(d) g is decreasing on (1, 2) and increasing on $(2,\infty)$

Fill in the Blank

17. $f(x)=\begin{vmatrix}\sec x & \cos x & \sec^2 x+\cot x\operatorname{cosec} x\\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x\\ 1 & \cos^2 x & \cos^2 x\end{vmatrix}$.

Then, $\int_0^{\pi/2} f(x)\,dx=\ldots$. **(1987, 2M)**

Analytical & Descriptive Questions

18. For $x>0$, let $f(x)=\int_1^x\frac{\ln t}{1+t}dt$. Find the function $f(x)+f(1/x)$ and show that $f(e)+f(1/e)=1/2$, where $\ln t=\log_e t$. **(2000, 5M)**

19. Let $a+b=4$, where $a<2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx}>0,\ \forall\, x$ prove that $\int_0^a g(x)\,dx+\int_0^b g(x)\,dx$ increases as $(b-a)$ increases. **(1997, 5M)**

20. Determine a positive integer $n\le5$, such that $\int_0^1 e^x(x-1)^n dx=16-6e$ **(1992, 4M)**

21. If 'f' is a continuous function with $\int_0^x f(t)\,dt\to\infty$ as $|x|\to\infty$, then show that every line $y=mx$ intersects the curve $y^2+\int_0^x f(t)\,dt=2$ **(1991, 2M)**

22. Investigate for maxima and minima the function,
$f(x)=\int_1^x[2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]\,dt$. **(1988, 5M)**

Topic 4 Limits as the Sum

Objective Question I (Only one correct option)

1. $\lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}}+\frac{(n+2)^{1/3}}{n^{4/3}}+\ldots\ldots+\frac{(2n)^{1/3}}{n^{4/3}}\right)$ is equal to **(2019 Main, 10 April I)**

(a) $\frac{4}{3}(2)^{4/3}$ (b) $\frac{3}{4}(2)^{4/3}-\frac{4}{3}$

(c) $\frac{3}{4}(2)^{4/3}-\frac{3}{4}$ (d) $\frac{4}{3}(2)^{3/4}$

2. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\ldots+\frac{1}{5n}\right)$ is equal to **(2019 Main, 12 Jan II)**

(a) $\tan^{-1}(3)$ (b) $\tan^{-1}(2)$

(c) $\pi/4$ (d) $\pi/2$

3. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right)^{1/n}$ is equal to **(2016 Main)**

(a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$

(c) $\frac{9}{e^2}$ (d) $3\log 3-2$

Objective Questions II

(One or more than one correct option)

4. For $a\in R$ (the set of all real numbers), $a\neq -1$,

$$\lim_{n\to\infty}\frac{(1^a+2^a+\ldots+n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+\ldots+(na+n)]}=\frac{1}{60}.$$

Then, a is equal to

(a) 5 (b) 7 (c) $\frac{-15}{2}$ (d) $\frac{-17}{2}$

5. Let $S_n=\sum_{k=0}^{n}\frac{n}{n^2+kn+k^2}$ and $T_n=\sum_{k=0}^{n-1}\frac{n}{n^2+kn+k^2}$, for $n=1,2,3,\ldots$, then **(2008, 4M)**

(a) $S_n<\frac{\pi}{3\sqrt3}$ (b) $S_n>\frac{\pi}{3\sqrt3}$

(c) $T_n<\frac{\pi}{3\sqrt3}$ (d) $T_n>\frac{\pi}{3\sqrt3}$

Analytical & Descriptive Question

6. Show that, $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{6n}\right)=\log 6.$ **(1981, 2M)**

Answers

Topic 1

1. (a)	**2.** (d)	**3.** (b)	**4.** (a)
5. (a)	**6.** (c)	**7.** (d)	**8.** (d)
9. (c)	**10.** (a)	**11.** (c)	**12.** (d)
13. (d)	**14.** (c)	**15.** (d)	**16.** (d)
17. (b)	**18.** (a)	**19.** (c)	**20.** (a)
21. (d)	**22.** (b)	**23.** (a)	**24.** (c)
25. (b)	**26.** (a)	**27.** (c)	**28.** (c)
29. (b)	**30.** (c)	**31.** (a)	**32.** (a)
33. (a)	**34.** (c)	**35.** (a)	**36.** (d)
37. (d)	**38.** (d)	**39.** (c)	**40.** (a)
41. (d)	**42.** (a, b, c)	**43.** (c, d)	**44.** (b)
45. (a)	**46.** (c)	**47.** (c, d)	**48.** (b, d)
49. (a, b)	**50.** (*)	**51.** (a, c)	**52.** (a)
53. (a, b, c)	**54.** (2)	**55.** (8/3)	**56.** ($k=16$)
57. (2)	**58.** π^2	**59.** $\frac{1}{a^2-b^2}\left[a\log 2-5a+\frac{7}{2}b\right]$	
60. $\left(\frac{1}{2}\right)$	**61.** $\pi(\sqrt2-1)$	**62.** (4)	**63.** $(2-\sqrt2)$
64. $A\to S; B\to S; C\to P; D\to R$		**65.** (d)	
66. 5051	**67.** $\frac{24}{5}\left[e\cos\left(\frac{1}{2}\right)+\frac{e}{2}\sin\left(\frac{1}{2}\right)-1\right]$		
68. $\left[\frac{4\pi}{\sqrt3}\tan^{-1}\left(\frac{1}{2}\right)\right]$		**70.** $\frac{\pi}{2}$	**71.** log 2
72. $\frac{\pi}{8}(\log 2)$	**73.** π^2	**74.** $\frac{\pi}{12}[\pi+3\log(2+\sqrt3)-4\sqrt3]$	
75. $\frac{1}{2}\log 6-\frac{1}{10}$	**76.** $f(x)=x^3+x^2-x+2$		**77.** $\frac{8}{\pi^2}$
82. $\frac{1}{2}\log 2-\frac{1}{2}+\frac{\pi}{4}$		**83.** $\frac{\alpha\pi}{\sin\alpha}$	**84.** $\frac{\pi^2}{16}$
85. $\left(-\frac{\sqrt3}{12}\pi+\frac{1}{2}\right)$	**86.** $\left[\frac{1}{20}(\log 3)\right]$	**87.** (ii) $\left[\frac{3\pi+1}{\pi^2}\right]$	
88. $\left[\frac{t^{n+1}-1}{(t-1)(n+1)}\right]$		**89.** (0)	**90.** (9)
91. (2)			

Topic 2

1. (d)	**2.** (c)	**3.** (b)	**6.** (4)

Topic 3

1. (a)	**2.** (b)	**3.** (a)	**4.** (c)
5. (a)	**6.** (a)	**7.** (d)	**8.** (a)
9. (a)	**10.** (c)	**11.** (a)	**12.** (a)
13. (d)	**14.** (*)	**15.** (c)	**16.** (b)
17. $-\left(\frac{15\pi+32}{60}\right)$		**18.** $\left[\frac{1}{2}(\ln x)^2\right]$	**20.** ($n=3$)

22. At $x=1$ and $\frac{7}{5}$, $f(x)$ is maximum and minimum respectively.

Topic 4

1. (c)	**2.** (b)	**3.** (b)	**4.** (b, d)
5. (a, d)			

Hints & Solutions

Topic 1 Properties of Definite Integral

1. Let $I = \int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)}$

$= \int_{\alpha}^{\alpha+1} \frac{(x+\alpha+1)-(x+\alpha)}{(x+\alpha)(x+\alpha+1)}\,dx$

$= \int_{\alpha}^{\alpha+1} \left(\frac{1}{x+\alpha} - \frac{1}{x+\alpha+1}\right) dx$

$= \left[\log_e(x+\alpha) - \log_e(x+\alpha+1)\right]_{\alpha}^{\alpha+1}$

$= \left[\log_e\left(\frac{x+\alpha}{x+\alpha+1}\right)\right]_{\alpha}^{\alpha+1}$

$= \log_e \frac{2\alpha+1}{2\alpha+2} - \log_e \frac{2\alpha}{2\alpha+1}$

$= \log_e\left(\frac{2\alpha+1}{2\alpha+2} \times \frac{2\alpha+1}{2\alpha}\right) = \log_e\left(\frac{9}{8}\right)$ (given)

$\Rightarrow \frac{(2\alpha+1)^2}{4\alpha(\alpha+1)} = \frac{9}{8} \Rightarrow 8[4\alpha^2+4\alpha+1] = 36(\alpha^2+\alpha)$

$\Rightarrow \quad 8\alpha^2+8\alpha+2 = 9\alpha^2+9\alpha$

$\Rightarrow \quad \alpha^2+\alpha-2=0$

$\Rightarrow \quad (\alpha+2)(\alpha-1)=0$

$\Rightarrow \quad \alpha = 1, -2$

From the options we get $\alpha = -2$

2. Let $I = \int_0^{\pi/2} \frac{\cot x}{\cot x + \operatorname{cosec} x}\,dx$

$= \int_0^{\pi/2} \frac{\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x} + \frac{1}{\sin x}}\,dx = \int_0^{\pi/2} \frac{\cos x}{1+\cos x}\,dx$

$= \int_0^{\pi/2} \frac{\cos x(1-\cos x)}{1-\cos^2 x}\,dx$

$= \int_0^{\pi/2} \frac{\cos x - \cos^2 x}{\sin^2 x}\,dx$

$= \int_0^{\pi/2} (\operatorname{cosec} x \cot x - \cot^2 x)\,dx$

$= \int_0^{\pi/2} (\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x + 1)\,dx$

$= [-\operatorname{cosec} x + \cot x + x]_0^{\pi/2}$

$= \left[x + \frac{\cos x - 1}{\sin x}\right]_0^{\pi/2} = \left[x + \frac{\left(-2\sin^2\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right]_0^{\pi/2}$

$= \left[x - \tan\frac{x}{2}\right]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{1}{2}[\pi - 2]$

$= m[\pi + n]$ [given]

On comparing, we get $m = \frac{1}{2}$ and $n = -2$

$\therefore \quad m \cdot n = -1$

Alternate Solution

Let $I = \int_0^{\pi/2} \frac{\cot x}{\cot x + \operatorname{cosec} x}\,dx$

$= \int_0^{\pi/2} \frac{\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x} + \frac{1}{\sin x}}\,dx$

$= \int_0^{\pi/2} \frac{\cos x}{\cos x + 1}\,dx$

$= \int_0^{\pi/2} \frac{2\cos^2\frac{x}{2} - 1}{2\cos^2\frac{x}{2}}\,dx$

$\left[\because \cos\theta = 2\cos^2\frac{\theta}{2} - 1 \text{ and } \cos\theta + 1 = 2\cos^2\frac{\theta}{2}\right]$

$= \int_0^{\pi/2} \left(1 - \frac{1}{2}\sec^2\frac{x}{2}\right) dx$

$= \left[x - \tan\frac{x}{2}\right]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{1}{2}(\pi - 2)$

Since, $I = m(\pi - n)$

$\therefore m(\pi - n) = \frac{1}{2}(\pi - 2)$

On comparing both sides, we get

$m = \frac{1}{2}$ and $n = -2$

Now, $mn = \frac{1}{2} \times -2 = -1$

3. Let $I = \int_{\pi/6}^{\pi/3} \sec^{2/3} x \operatorname{cosec}^{4/3} x\,dx$

$= \int_{\pi/6}^{\pi/3} \frac{1}{\cos^{2/3} x \sin^{4/3} x}\,dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sec^2 x}{(\tan x)^{4/3}}\,dx$

[multiplying and dividing the denominator by $\cos^{4/3} x$]

Put, $\tan x = t$, upper limit, at $x = \pi/3 \Rightarrow t = \sqrt{3}$ and lower limit, at $x = \pi/6 \Rightarrow t = 1/\sqrt{3}$

and $\sec^2 x\,dx = dt$

So, $I = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dt}{t^{4/3}} = \left[\frac{t^{-1/3}}{-1/3}\right]_{1/\sqrt{3}}^{\sqrt{3}}$

$= -3\left(\frac{1}{3^{1/6}} - 3^{1/6}\right)$

$= 3 \cdot 3^{1/6} - 3 \cdot 3^{-1/6}$

$= 3^{7/6} - 3^{5/6}$

4. Given integral

$$I = \int_0^{2\pi} [\sin 2x \cdot (1 + \cos 3x)] dx$$

$$= \int_0^{\pi} [\sin 2x \cdot (1 + \cos 3x)] dx + \int_{\pi}^{2\pi} [\sin 2x \cdot (1 + \cos 3x)] dx$$

$$= I_1 + I_2 \text{ (let)} \qquad \text{... (i)}$$

Now, $I_2 = \int_{\pi}^{2\pi} [\sin 2x \cdot (1 + \cos 3x)] dx$

let $2\pi - x = t$, upper limit $t = 0$ and lower limit $t = \pi$

and $dx = -dt$

So, $I_2 = -\int_{\pi}^{0} [-\sin 2x \cdot (1 + \cos 3x)] dx$

$$= \int_0^{\pi} [-\sin 2x \cdot (1 + \cos 3x)] dx \qquad \text{...(ii)}$$

$\therefore$ $I = \int_0^{\pi} [\sin 2x \cdot (1 + \cos 3x)] dx + \int_0^{\pi} [-\sin 2x \cdot (1 + \cos 3x)] dx$

[from Eqs. (i) and (ii)]

$$= \int_0^{\pi} (-1) dx \quad [\because [x] + [-x] = -1, x \notin \text{Integer}]$$

$$= -\pi$$

5. Let $I = \int_0^1 x \cot^{-1}(1 - x^2 + x^4) dx$

Now, put $x^2 = t \Rightarrow 2x dx = dt$

Lower limit at $x = 0$, $t = 0$

Upper limit at $x = 1$, $t = 1$

$\therefore I = \frac{1}{2} \int_0^1 \cot^{-1}(1 - t + t^2) dt$

$$= \frac{1}{2} \int_0^1 \tan^{-1}\left(\frac{1}{1 - t + t^2}\right) dt \qquad \left[\because \cot^{-1} x = \tan^{-1} \frac{1}{x}\right]$$

$$= \frac{1}{2} \int_0^1 \tan^{-1}\left(\frac{t - (t - 1)}{1 + t(t - 1)}\right) dt$$

$$= \frac{1}{2} \left[\int_0^1 (\tan^{-1} t - \tan^{-1}(t - 1) dt\right]$$

$$\left[\because \tan^{-1} \frac{x - y}{1 + xy} = \tan^{-1} x - \tan^{-1} y\right]$$

$$\because \int_0^1 \tan^{-1}(t - 1) dt = \int_0^1 \tan^{-1}(1 - t - 1) dt = -\int_0^1 \tan^{-1}(t) dt$$

because $\int_0^a f(x) dx = \int_0^a f(a - x) dx$

So, $I = \frac{1}{2} \int_0^1 (\tan^{-1} t + \tan^{-1} t) dt$

$$= \int_0^1 \tan^{-1} t dt = [t \tan^{-1} t]_0^1 - \int_0^1 \frac{t}{1 + t^2} dt$$

[by integration by parts method]

$$= \frac{\pi}{4} - \frac{1}{2} [\log_e(1 + t^2)]_0^1 = \frac{\pi}{4} - \frac{1}{2} \log_e 2$$

6.

> **Key Idea** Use property of definite integral.
> $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x} dx \qquad \text{...(i)}$

On applying the property, $\int_a^b f(x) dx = \int_a^b f(a + b - x)\, dx$, we get

$$I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos x + \sin x} dx \qquad \text{...(ii)}$$

On adding integrals (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x)}{\sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \left[1 - \frac{1}{2}(2 \sin x \cos x)\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{2} \sin 2x\right) dx$$

$$= \left[x + \frac{1}{4} \cos 2x\right]_0^{\pi/2} = \left(\frac{\pi}{2} - 0\right) + \frac{1}{4}(-1 - 1) = \frac{\pi}{2} - \frac{1}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4} - \frac{1}{4} = \frac{\pi - 1}{4}$$

7. Given, $f(x) = \int_0^x g(t) dt$

On replacing x by $(-x)$, we get

$$f(-x) = \int_0^{-x} g(t) dt$$

Now, put $t = -u$, so

$$f(-x) = -\int_0^x g(-u) du = -\int_0^x g(u) du = -f(x)$$

[$\because g$ is an even function]

$\Rightarrow$ $f(-x) = -f(x) \Rightarrow f$ is an odd function.

Now, it is given that $f(x + 5) = g(x)$

$\therefore$ $f(5 - x) = g(-x) = g(x) = f(x + 5)$

[$\because g$ is an even function]

$$\Rightarrow \quad f(5 - x) = f(x + 5) \qquad \text{...(i)}$$

Let $I = \int_0^x f(t) dt$

Put $t = u + 5 \Rightarrow t - 5 = u \Rightarrow dt = du$

$\therefore$ $I = \int_{-5}^{x-5} f(u + 5) du = \int_{-5}^{x-5} g(u) du$

Put $u = -t \Rightarrow du = -dt$, we get

$$I = -\int_5^{5-x} g(-t) dt = \int_{5-x}^{5} g(t) dt$$

$\left[\because -\int_a^b f(x)dx = \int_b^a f(x)dx \text{ and } g \text{ is an even function}\right]$

$$I = \int_{5-x}^{5} f'(t)dt \qquad \text{[by Leibnitz rule } f'(x) = g(x)]$$

$$= f(5) - f(5-x) = f(5) - f(5+x) \qquad \text{[from Eq. (i)]}$$

$$= \int_{5+x}^{5} f'(t)dt = \int_{5+x}^{5} g(t)dt$$

8. The given functions are

$$g(x) = \log_e x,\ x > 0 \text{ and } f(x) = \frac{2 - x\cos x}{2 + x\cos x}$$

Let $\quad I = \int_{-\pi/4}^{\pi/4} g(f(x))dx$

Then, $\quad I = \int_{-\pi/4}^{\pi/4} \log_e \left(\frac{2 - x\cos x}{2 + x\cos x}\right)dx \qquad \text{...(i)}$

Now, by using the property

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx, \text{ we get}$$

$$I = \int_{-\pi/4}^{\pi/4} \log_e \left(\frac{2 + x\cos x}{2 - x\cos x}\right)dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{-\pi/4}^{\pi/4}\left[\log_e\left(\frac{2 - x\cos x}{2 + x\cos x}\right) + \log_e\left(\frac{2 + \cos x}{2\ \ x\cos x}\right)\right]dx$$

$$= \int_{-\pi/4}^{\pi/4} \log_e \left(\frac{2 - x\cos x}{2 + x\cos x} \times \frac{2 + x\cos x}{2 - x\cos x}\right)dx$$

$$[\because \log_e A + \log_e B = \log_e AB]$$

$$\Rightarrow 2I = \int_{-\pi/4}^{\pi/4} \log_e (1)dx = 0 \Rightarrow \quad I = 0 = \log_e(1)$$

9. Let $I = \int_0^a f(x)\, g(x)\, dx \qquad \text{... (i)}$

$$= \int_0^a f(a-x)\, g(a-x)\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^a f(x)\, [4 - g(x)]\, dx$$

$$[\because f(x) = f(a-x) \text{ and } g(x) + g(a-x) = 4]$$

$$= \int_0^a 4f(x)\, dx - \int_0^a f(x)\, g(x)\, dx$$

$$\Rightarrow \quad I = 4\int_0^a f(x)\, dx - I \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 2I = 4\int_0^a f(x)\, dx \quad \Rightarrow \quad I = 2\int_0^a f(x)\, dx.$$

10. Let $I = \int_1^e \left\{\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x\right\} \log_e x\, dx$

Now, put $\left(\frac{x}{e}\right)^x = t \Rightarrow \quad x\log_e\left(\frac{x}{e}\right) = \log t$

$$\Rightarrow \quad x(\log_e x - \log_e e) = \log t$$

$$\Rightarrow \quad \left[x\left(\frac{1}{x}\right) + (\log_e x - \log_e e)\right]dx = \frac{1}{t}\, dt$$

$$\Rightarrow (1 + \log_e x - 1)\, dx = \frac{1}{t}\, dt \Rightarrow (\log_e x)\, dx = \frac{1}{t}\, dt$$

Also, upper limit $x = e \Rightarrow t = 1$ and lower limit $x = 1 \Rightarrow t = \frac{1}{e}$

$$\therefore I = \int_{1/e}^{1}\left(t^2 - \frac{1}{t}\right)\cdot\frac{1}{t}\, dt \Rightarrow \quad I = \int_{1/e}^{1}(t - t^{-2})\, dt$$

$$I = \left[\left(\frac{t^2}{2} + \frac{1}{t}\right)\right]_{\frac{1}{e}}^{1} = \left\{\left(\frac{1}{2} + 1\right) - \left(\frac{1}{2e^2} + e\right)\right\} = \frac{3}{2} - e - \frac{1}{2e^2}$$

11. Let $I = \int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$

$$= \int_{\pi/6}^{\pi/4} \frac{(1 + \tan^2 x)\tan^5 x}{2\tan x\,(\tan^{10} x + 1)}\, dx \qquad \left[\because \sin 2x = \frac{2\tan x}{1 + \tan^2 x}\right]$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/4} \frac{\tan^4 x\sec^2 x}{(\tan^{10} x + 1)}\, dx$$

Put $\tan^5 x = t$ $[\because \sec^2 x = 1 + \tan^2 x]$

$\Rightarrow 5\tan^4 x\sec^2 x\, dx = dt$

x	$\frac{\pi}{6}$	$\frac{\pi}{4}$
t	$\left(\frac{1}{\sqrt{3}}\right)^5$	1

$$\therefore I = \frac{1}{2}\cdot\frac{1}{5}\int_{(1/\sqrt{3})^5}^{1} \frac{dt}{t^2 + 1} = \frac{1}{10}(\tan^{-1}(t))_{(1/\sqrt{3})^5}^{1}$$

$$= \frac{1}{10}\left(\tan^{-1}(1) - \tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$$

$$= \frac{1}{10}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$$

12. Let $I = \int_{-2}^{2} \frac{\sin^2 x}{\frac{1}{2} + \left[\frac{x}{\pi}\right]}\, dx$

Also, let $f(x) = \frac{\sin^2 x}{\frac{1}{2} + \left[\frac{x}{\pi}\right]}$

Then, $f(-x) = \frac{\sin^2(-x)}{\frac{1}{2} + \left[-\frac{x}{\pi}\right]}$ (replacing x by $-x$)

$$= \frac{\sin^2 x}{\frac{1}{2} + \left(-1 - \left[\frac{x}{\pi}\right]\right)} \qquad \left[\because [-x] = \begin{cases} -[x], & \text{if } x \in I \\ -1-[x], & \text{if } x \notin I \end{cases}\right]$$

$$\Rightarrow \qquad f(-x) = -\frac{\sin^2 x}{\frac{1}{2} + \left[\frac{x}{\pi}\right]} = -f(x)$$

i.e. $f(x)$ is odd function

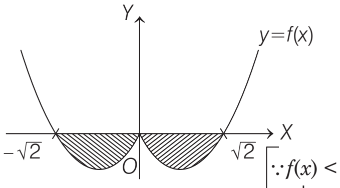
$$\therefore I = 0 \quad \left[\because \int_{-a}^{a} f(x)\, dx = \begin{cases} 0, \text{ if } f(x) \text{ is odd function} \\ 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even function} \end{cases}\right]$$

13. We have, $I = \int_a^b (x^4 - 2x^2)dx$

Let $\quad f(x) = x^4 - 2x^2 = x^2(x^2 - 2)$

$$= x^2(x - \sqrt{2})\,(x + \sqrt{2})$$

Graph of $y = f(x) = x^4 - 2x^2$ is



$\left[\because f(x) < 0 \text{ for } -\sqrt{2} < x < \sqrt{2}\right]$

$+ \quad - \quad - \quad +$

$-\sqrt{2} \quad 0 \quad \sqrt{2}$

Note that the definite integral $\int_a^b (x^4 - 2x^2)dx$ represent the area bounded by $y = f(x)$, $x = a, b$ and the X-axis.

But between $x = -\sqrt{2}$ and $x = \sqrt{2}$, $f(x)$ lies below the X-axis and so value definite integral will be negative.

Also, as long as $f(x)$ lie below the X-axis, the value of definite integral will be minimum.

$\therefore (a, b) = (-\sqrt{2}, \sqrt{2})$ for minimum of I.

14. We have, $\int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2k\sec\theta}}\, d\theta = 1 - \frac{1}{\sqrt{2}}$, $(k > 0)$

Let $I = \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2k\sec\theta}}\, d\theta = \frac{1}{\sqrt{2k}} \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{\sec\theta}}\, d\theta$

$$= \frac{1}{\sqrt{2k}} \int_0^{\pi/3} \frac{(\sin\theta)}{(\cos\theta)\sqrt{\frac{1}{\cos\theta}}}\, d\theta = \frac{1}{\sqrt{2k}} \int_0^{\pi/3} \frac{\sin\theta}{\sqrt{\cos\theta}}\, d\theta$$

Let $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt \Rightarrow \sin\theta\, d\theta = -dt$

for lower limit, $\theta = 0 \Rightarrow t = \cos 0 = 1$

for upper limit, $\theta = \frac{\pi}{3} \Rightarrow t = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2k}} \int_1^{1/2} \frac{-dt}{\sqrt{t}} = \frac{-1}{\sqrt{2k}} \int_1^{1/2} t^{-\frac{1}{2}} dt$$

$$= -\frac{1}{\sqrt{2k}} \left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right)_1^{\frac{1}{2}} = -\frac{1}{\sqrt{2k}} \left[2\sqrt{t}\right]_1^{\frac{1}{2}}$$

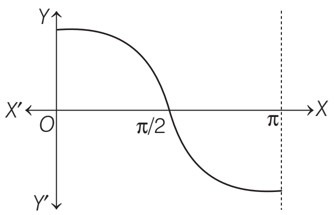
$$= -\frac{2}{\sqrt{2k}} \left[\sqrt{\frac{1}{2}} - \sqrt{1}\right] = \frac{2}{\sqrt{2k}} \left(1 - \frac{1}{\sqrt{2}}\right)$$

$\because \quad I = 1 - \frac{1}{\sqrt{2}}$ (given)

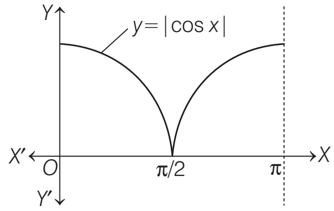
$\therefore \quad \frac{2}{\sqrt{2k}} \left(1 - \frac{1}{\sqrt{2}}\right) = 1 - \frac{1}{\sqrt{2}} \Rightarrow \frac{2}{\sqrt{2k}} = 1$

$\Rightarrow \quad 2 = \sqrt{2k} \Rightarrow 2k = 4 \Rightarrow k = 2$

15. We know, graph of $y = \cos x$ is



$\therefore$ The graph of $y = |\cos x|$ is



$I = \int_0^{\pi} |\cos x|^3 = 2\int_0^{\frac{\pi}{2}} |\cos x|^3 dx$

$\left(\because y = |\cos x| \text{ is symmetric about } x = \frac{\pi}{2}\right)$

$= 2\int_0^{\frac{\pi}{2}} \cos^3 x\, dx \qquad \left[\because \cos x \geq 0 \text{ for } x \in \left[0, \frac{\pi}{2}\right]\right]$

Now, as $\cos 3x = 4\cos^3 x - 3\cos x$

$\therefore \cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

$\therefore I = \frac{2}{4}\int_0^{\frac{\pi}{2}} (\cos 3x + 3\cos x)\, dx$

$$= \frac{1}{2}\left[\frac{\sin 3x}{3} + 3\sin x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{3}\sin\frac{3\pi}{2} + 3\sin\frac{\pi}{2}\right] - \left[\frac{1}{3}\sin 0 + 3\sin 0\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{3}(-1) + 3\right] - [0 + 0]\right\}$$

$$\left[\because \sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1\right]$$

$$= \frac{1}{2}\left[-\frac{1}{3} + 3\right] = \frac{4}{3}$$

16. Key idea Use property $= \int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Let $\qquad I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x}\, dx$

$$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2\left(-\frac{\pi}{2} + \frac{\pi}{2} - x\right)}{1 + 2^{-\frac{\pi}{2} + \frac{\pi}{2} - x}}\, dx$$

$$\left[\because \int_a^b f(x)dx = \int_a^b f(a + b - x)dx\right]$$

$$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^{-x}}\, dx$$

$$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} \frac{2^x \sin^2 x}{2^x + 1}\, dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \sin^2 x \left(\frac{2^x + 1}{2^x + 1}\right) dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$$

$\Rightarrow \quad 2I = 2\int_0^{\pi/2} \sin^2 x\, dx$ [$\because \sin^2 x$ is an even function]

$\Rightarrow \quad I = \int_0^{\pi/2} \sin^2 x dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \cos^2 x dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\Rightarrow \quad 2I = \int_0^{\pi/2} dx$

$\Rightarrow \quad 2I = [x]_0^{\pi/2} \Rightarrow I = \frac{\pi}{4}$

17. Let $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x} = \int_{\pi/4}^{3\pi/4} \frac{1-\cos x}{1-\cos^2 x} dx$

$= \int_{\pi/4}^{3\pi/4} \frac{1-\cos x}{\sin^2 x} dx$

$= \int_{\pi/4}^{3\pi/4} (\text{cosec}^2 x - \text{cosec } x \cot x) dx$

$= [-\cot x + \text{cosec } x]_{\pi/4}^{3\pi/4}$

$= [(1+\sqrt{2}) - (-1+\sqrt{2})] = 2$

18. Let $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx \quad$ ...(i)

$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos(-x)}{1+e^{-x}} dx \quad$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_{-\pi/2}^{\pi/2} x^2 \cos x \left[\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right] dx$$

$$= \int_{-\pi/2}^{\pi/2} x^2 \cos x \cdot (1)\, dx$$

$\left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ when } f(-x) = f(x)\right]$

$\Rightarrow \quad 2I = 2\int_0^{\pi/2} x^2 \cos x\, dx$

Using integration by parts, we get

$$2I = 2\,[x^2(\sin x) - (2x)(-\cos x) + (2)(-\sin x)]_0^{\pi/2}$$

$\Rightarrow \quad 2I = 2\left[\frac{\pi^2}{4} - 2\right]$

$\therefore \quad I = \frac{\pi^2}{4} - 2$

19. **PLAN** Apply the property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ and then add.

Let $\quad I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)} dx$

$= \int_2^4 \frac{2\log x}{2\log x + \log(6-x)^2} dx$

$= \int_2^4 \frac{2\log x\, dx}{2[\log x + \log(6-x)]}$

$\Rightarrow \quad I = \int_2^4 \frac{\log x\, dx}{[\log x + \log(6-x)]} \quad$ ...(i)

$\Rightarrow \quad I = \int_2^4 \frac{\log(6-x)}{\log(6-x) + \log x} dx \quad$ ...(ii)

$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

On adding Eqs. (i) and (ii), we get

$$2I = \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)} dx$$

$\Rightarrow \quad 2I = \int_2^4 dx = [x]_2^4 \Rightarrow 2I = 2$

$\Rightarrow \quad 2I = 2 \Rightarrow I = 1$

20. **PLAN** This type of question can be done using appropriate substitution.

Given, $I = \int_{\pi/4}^{\pi/2} (2\,\text{cosec}\, x)^{17}\, dx$

$$= \int_{\pi/4}^{\pi/2} \frac{2^{17} (\text{cosec}\, x)^{16}\, \text{cosec}\, x\, (\text{cosec}\, x + \cot x)}{(\text{cosec}\, x + \cot x)} dx$$

Let $\quad \text{cosec}\, x + \cot x = t$

$\Rightarrow \quad (-\text{cosec}\, x \cdot \cot x - \text{cosec}^2 x)\, dx = dt$

and $\quad \text{cosec}\, x - \cot x = 1/t$

$\Rightarrow \quad 2\,\text{cosec}\, x = t + \frac{1}{t}$

$\therefore \quad I = -\int_{\sqrt{2}+1}^{1} 2^{17} \left(\frac{t+\frac{1}{t}}{2}\right)^{16} \frac{dt}{t}$

Let $t = e^u \Rightarrow dt = e^u du.$

When $t = 1, e^u = 1 \Rightarrow u = 0$

and when $t = \sqrt{2}+1, e^u = \sqrt{2}+1$

$\Rightarrow \quad u = \ln(\sqrt{2}+1)$

$\Rightarrow \quad I = -\int_{\ln(\sqrt{2}+1)}^{0} 2(e^u + e^{-u})^{16} \frac{e^u du}{e^u}$

$$= 2\int_0^{\ln(\sqrt{2}+1)} (e^u + e^{-u})^{16}\, du$$

21. **PLAN** Use the formula, $|x-a| = \begin{cases} x-a, & x \ge a \\ -(x-a), & x < a \end{cases}$ to break given integral in two parts and then integrate separately.

$$\int_0^{\pi} \sqrt{\left(1 - 2\sin\frac{x}{2}\right)^2} dx = \int_0^{\pi} \left|1 - 2\sin\frac{x}{2}\right| dx$$

$$= \int_0^{\frac{\pi}{3}} \left(1 - 2\sin\frac{x}{2}\right) dx - \int_{\frac{\pi}{3}}^{\pi} \left(1 - 2\sin\frac{x}{2}\right) dx$$

$$= \left(x + 4\cos\frac{x}{2}\right)_0^{\frac{\pi}{3}} - \left(x + 4\cos\frac{x}{2}\right)_{\frac{\pi}{3}}^{\pi}$$

$$= 4\sqrt{3} - 4 - \frac{\pi}{3}$$

22. $I = \int_{-\pi/2}^{\pi/2} \left[x^2 + \log\left(\frac{\pi - x}{\pi + x}\right)\right] \cos x\, dx$

As, $\int_{-a}^{a} f(x)\, dx = 0$, when $f(-x) = -f(x)$

$\therefore \quad I = \int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx + 0 = 2\int_0^{\pi/2} (x^2 \cos x) \, dx$

$= 2\{(x^2 \sin x)_0^{\pi/2} - \int_0^{\pi/2} 2x \cdot \sin x \, dx\}$

$= 2\left[\frac{\pi^2}{4} - 2\{(-x \cdot \cos x)_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot (-\cos x) \, dx\}\right]$

$= 2\left[\frac{\pi^2}{4} - 2(\sin x)_0^{\pi/2}\right] = 2\left[\frac{\pi^2}{4} - 2\right] = \left(\frac{\pi^2}{2} - 4\right)$

23. Put $x^2 = t \Rightarrow x\,dx = dt/2$

$\therefore \quad I = \int_{\log 2}^{\log 3} \frac{\sin t \cdot \frac{dt}{2}}{\sin t + \sin(\log 6 - t)}$ ...(i)

Using, $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 2 + \log 3 - t)}{\sin(\log 2 + \log 3 - t) + \sin(\log 6 - (\log 2 + \log 3 - t))} dt$

$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin(t)} dt$

$\therefore \quad I = \int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t} dt$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t + \sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t} dt$

$\Rightarrow \quad 2I = \frac{1}{2}(t)_{\log 2}^{\log 3} = \frac{1}{2}(\log 3 - \log 2)$

$\therefore \quad I = \frac{1}{4}\log\left(\frac{3}{2}\right)$

24. Let $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]\,dx$

$= \int_{-2}^{0} [(x+1)^3 + 2 + (x+1)\cos(x+1)]\,dx$

Put $x + 1 = t$

$\Rightarrow \quad dx = dt$

$\therefore \quad I = \int_{-1}^{1} (t^3 + 2 + t\cos t)\,dt$

$= \int_{-1}^{1} t^3\,dt + 2\int_{-1}^{1} dt + \int_{-1}^{1} t\cos t\,dt$

$= 0 + 2 \cdot 2\,[x]_0^1 + 0$

$= 4$

[since, t^3 and $t\cos t$ are odd functions]

25. $I = \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

$= [\sin^{-1} x]_0^1 + \int_1^0 \frac{t}{t}\,dt$

[where, $t^2 = 1 - x^2 \Rightarrow t\,dt = -x\,dx$]

$= (\sin^{-1} 1 - \sin^{-1} 0) + [t]_1^0 = \pi/2 - 1$

26. $\int_{-1/2}^{1/2}\left[[x] + \log\left(\frac{1+x}{1-x}\right)\right] dx$

$= \int_{-1/2}^{1/2} [x]\,dx + \int_{-1/2}^{1/2} \log\left(\frac{1+x}{1-x}\right) dx$

$= \int_{-1/2}^{1/2} [x]\,dx + 0 \quad \left[\because \log\left(\frac{1+x}{1-x}\right) \text{ is an odd function}\right]$

$= \int_{-1/2}^{0} [x]\,dx + \int_0^{1/2} [x]\,dx = \int_{-1/2}^{0} (-1)\,dx + \int_0^{1/2} (0)\,dx$

$= -[x]_{-1/2}^{0} = -\left(0 + \frac{1}{2}\right) = -\frac{1}{2}$

27. Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x}\,dx$ ...(i)

$= \int_{\pi}^{-\pi} \frac{\cos^2(-x)}{1 + a^{-x}}\,d(-x)$

$\Rightarrow \quad I = \int_{-\pi}^{\pi} a^x \frac{\cos^2 x}{1 + a^x}\,dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_{-\pi}^{\pi} \left(\frac{1 + a^x}{1 + a^x}\right) \cos^2 x\,dx$

$= \int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_0^{\pi} \frac{1 + \cos 2x}{2}\,dx$

$= \int_0^{\pi} (1 + \cos 2x)\,dx = \int_0^{\pi} 1\,dx + \int_0^{\pi} \cos 2x\,dx$

$= [x]_0^{\pi} + 2\int_0^{\pi/2} \cos 2x\,dx = \pi + 0$

$\Rightarrow \quad 2I = \pi \quad \Rightarrow \quad I = \pi/2$

28. Given, $f(x) = \begin{cases} e^{\cos x} \sin x, & \text{for } |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$

$\therefore \int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_2^3 f(x)\,dx$

$= \int_{-2}^{2} e^{\cos x} \sin x\,dx + \int_2^3 2\,dx = 0 + 2\,[x]_2^3$

[$\because e^{\cos x} \sin x$ is an odd function]

$= 2\,[3 - 2] = 2 \quad \left[\because \int_{-2}^{3} f(x)\,dx = 2\right]$

29. $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx = \int_{e^{-1}}^{1} \left|\frac{\log_e x}{x}\right| dx - \int_1^{e^2} \left|\frac{\log_e x}{x}\right| dx$

$\left[\text{since, 1 is turning point for } \left|\frac{\log_e x}{x}\right| \text{ for + ve and − ve values}\right]$

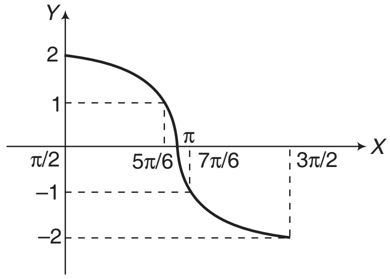
$= -\int_{e^{-1}}^{1} \frac{\log_e x}{x}\,dx + \int_1^{e^2} \left|\frac{\log_e x}{x}\right| dx$

$= -\frac{1}{2}[(\log_e x)^2]_{e^{-1}}^{1} + \frac{1}{2}[(\log_e x)^2]_1^{e^2}$

$= -\frac{1}{2}\{0 - (-1)^2\} + \frac{1}{2}(2^2 - 0) = \frac{5}{2}$

30. The graph of $y=2\sin x$ for $\pi/2 \le x \le 3\pi/2$ is given in figure. From the graph, it is clear that

$$[2\sin x] = \begin{cases} 2, & \text{if } x=\pi/2 \\ 1, & \text{if } \pi/2 < x \le 5\pi/6 \\ 0, & \text{if } 5\pi/6 < x \le \pi \\ -1, & \text{if } \pi < x \le 7\pi/6 \\ -2, & \text{if } 7\pi/6 < x \le 3\pi/2 \end{cases}$$



Therefore, $\int_{\pi/2}^{3\pi/2} [2\sin x]dx$

$$= \int_{\pi/2}^{5\pi/6} dx + \int_{5\pi/6}^{\pi} 0\,dx + \int_{\pi}^{7\pi/6} (-1)\,dx + \int_{7\pi/6}^{3\pi/2} (-2)\,dx$$

$$= [x]_{\pi/2}^{5\pi/6} + [-x]_{\pi}^{7\pi/6} + [-2x]_{7\pi/6}^{3\pi/2}$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) + \left(-\frac{7\pi}{6} + \pi\right) + \left(\frac{-2\cdot 3\pi}{2} + \frac{2\cdot 7\pi}{6}\right)$$

$$= \pi\left(\frac{5}{6} - \frac{1}{2}\right) + \pi\left(1 - \frac{7}{6}\right) + \pi\left(\frac{7}{3} - 3\right)$$

$$= \pi\left(\frac{5-3}{6}\right) + \pi\left(-\frac{1}{6}\right) + \pi\left(\frac{7-9}{3}\right) = -\pi/2$$

31. Let $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ ...(i)

$\Rightarrow$ $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(\pi - x)}$

$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\right) dx$$

$\Rightarrow$ $2I = \int_{\pi/4}^{3\pi/4} \left(\frac{2}{1-\cos^2 x}\right) dx$

$\Rightarrow$ $I = \int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x\,dx = [-\cot x]_{\pi/4}^{3\pi/4}$

$$= \left[-\cot\frac{3\pi}{4} + \cot\frac{\pi}{4}\right] = -(-1) + 1 = 2$$

32. Let $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (x - [x])\,dx = \int_{-1}^{1} x\,dx - \int_{-1}^{1} [x]\,dx$

$$= 0 - \int_{-1}^{1} [x]\,dx \quad [\because x \text{ is an odd function}]$$

But $[x] = \begin{cases} -1, & \text{if } -1 \le x < 0 \\ 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}$

$\therefore$ $\int_{-1}^{1} [x]\,dx = \int_{-1}^{0} [x]\,dx + \int_{0}^{1} [x]\,dx$

$$= \int_{-1}^{0} (-1)\,dx + \int_{0}^{1} 0\,dx$$

$$= -[x]_{-1}^{0} + 0 = -1; \ \therefore \int_{-1}^{1} f(x)\,dx = 1$$

33. Given, $g(x) = \int_0^x \cos^4 t\,dt$

$\Rightarrow$ $g(x+\pi) = \int_0^{\pi+x} \cos^4 t\,dt$

$$= \int_0^{\pi} \cos^4 t\,dt + \int_{\pi}^{\pi+x} \cos^4 t\,dt = I_1 + I_2$$

where, $I_1 = \int_0^{\pi} \cos^4 t\,dt = g(\pi)$

and $I_2 = \int_{\pi}^{\pi+x} \cos^4 t\,dt$

Put $t = \pi + y$

$\Rightarrow$ $dt = dy$

$I_2 = \int_0^x \cos^4 (y+\pi)\,dy$

$$= \int_0^x (-\cos y)^4 dy = \int_0^x \cos^4 y\,dy = g(x)$$

$\therefore$ $g(x+\pi) = g(\pi) + g(x)$

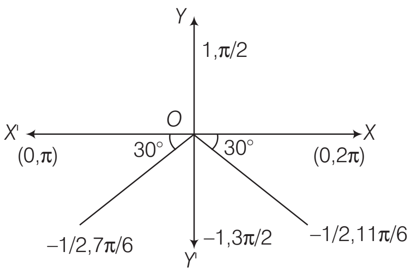
34. Given, $I_1 = \int_{1-k}^{k} x f[x(1-x)]\,dx$

$\Rightarrow$ $I_1 = \int_{1-k}^{k} (1-x) f[(1-x)x]\,dx$

$$= \int_{1-k}^{k} f[(1-x)]dx] - \int_{1-k}^{k} xf(1-x)]dx$$

$\Rightarrow$ $I_1 = I_2 - I_1 \Rightarrow \frac{I_1}{I_2} = \frac{1}{2}$

35. It is a question of greatest integer function. We have, subdivide the interval π to 2π as under keeping in view that we have to evaluate $[2\sin x]$



We know that, $\sin\frac{\pi}{6} = \frac{1}{2}$

$\therefore$ $\sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6} = -\frac{1}{2}$

$\Rightarrow$ $\sin\frac{11\pi}{6} = \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

$\Rightarrow$ $\sin\frac{9\pi}{6} = \sin\frac{3\pi}{6} = -1$

Hence, we divide the interval π to 2π as

$$\left(\pi, \frac{7\pi}{6}\right), \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right), \left(\frac{11\pi}{6}, 2\pi\right)$$

$$\sin x = \left(0, -\frac{1}{2}\right), \left(-1, -\frac{1}{2}\right), \left(-\frac{1}{2}, 0\right)$$

$$\Rightarrow \quad 2\sin x = (0, -1), (-2, -1), (-1, 0)$$

$$\Rightarrow \quad [2\sin x] = -1$$

$$= \int_{\pi}^{7\pi/6} [2\sin x]\, dx + \int_{7\pi/6}^{11\pi/6} [2\sin x]\, dx + \int_{11\pi/6}^{2\pi} [2\sin x]\, dx$$

$$= \int_{\pi}^{7\pi/6} (-1)\, dx + \int_{7\pi/6}^{11\pi/6} (-2)\, dx + \int_{11\pi/6}^{2\pi} (-1)\, dx$$

$$= -\frac{\pi}{6} - 2\left(\frac{4\pi}{6}\right) - \frac{\pi}{6} = -\frac{10\pi}{6} = -\frac{5\pi}{3}$$

36. Given, $f(x) = A\sin\left(\frac{\pi x}{2}\right) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$

and $\int_0^1 f(x)\, dx = \frac{2A}{\pi}$

$$f'(x) = \frac{A\pi}{2}\cos\frac{\pi x}{2} \Rightarrow f'\left(\frac{1}{2}\right) = \frac{A\pi}{2}\cos\frac{\pi}{4} = \frac{A\pi}{2\sqrt{2}}$$

But $f'\left(\frac{1}{2}\right) = \sqrt{2}$ $\therefore$ $\frac{A\pi}{2\sqrt{2}} = \sqrt{2} \Rightarrow A = \frac{4}{\pi}$

Now, $\int_0^1 f(x)\, dx = \frac{2A}{\pi} \Rightarrow \int_0^1 \left\{A\sin\left(\frac{\pi x}{2}\right) + B\right\} dx = \frac{2A}{\pi}$

$$\Rightarrow \left[-\frac{2A}{\pi}\cos\frac{\pi x}{2} + Bx\right]_0^1 = \frac{2A}{\pi} \Rightarrow B + \frac{2A}{\pi} = \frac{2A}{\pi}$$

$$\Rightarrow \quad B = 0$$

37. Let $I = \int_0^{\pi/2} \frac{1}{1 + \tan^3 x}\, dx = \int_0^{\pi/2} \frac{1}{1 + \frac{\sin^3 x}{\cos^3 x}}\, dx$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\, dx \quad \text{...(i)}$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\cos^3\left(\frac{\pi}{2} - x\right) + \sin^3\left(\frac{\pi}{2} - x\right)}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} 1\, dx \Rightarrow 2I = [x]_0^{\pi/2} = \pi/2 \Rightarrow I = \pi/4$$

38. Let $I = \int_{-\pi/2}^{\pi/2} [f(x) + f(-x)]\, [g(x) - g(-x)]\, dx$

Let $\phi(x) = [f(x) + f(-x)]\, [g(x) - g(-x)]$

$\Rightarrow$ $\phi(-x) = [f(-x) + f(x)]\, [g(-x) - g(x)]$

$\Rightarrow$ $\phi(-x) = -\phi(x)$

$\Rightarrow$ $\phi(x)$ is an odd function.

$\therefore$ $\int_{-\pi/2}^{\pi/2} \phi(x)\, dx = 0$

39. Let $I = \int_0^{\pi} e^{\cos^2 x} \cdot \cos^3\{(2n+1)x\}\, dx$

Using $\int_0^a f(x)\, dx = \begin{cases} 0, & f(a - x) = -f(x) \\ 2\int_0^{a/2} f(x)\, dx, & f(a - x) = f(x) \end{cases}$

Again, let $f(x) = e^{\cos^2 x} \cdot \cos^3\{(2n+1)x\}$

$\therefore$ $f(\pi - x) = (e^{\cos^2 x})\{-\cos^3(2n+1)x\} = -f(x)$

$\therefore$ $I = 0$

40. Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} 1\, dx$$

$$\therefore \quad I = \frac{\pi}{4}$$

41. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$ ...(i)

$$\therefore \quad I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot x}}$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\, dx}{1 + \sqrt{\tan x}} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} dx$$

$$\Rightarrow \quad 2I = [x]_{\pi/6}^{\pi/3}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{12}$$

Statement I is false.

But $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$ is a true statement by property of definite integrals.

42. According to the given data,

$$F'(x) < 0, \ \forall x \in (1, 3)$$

We have, $f(x) = xF(x)$

$\Rightarrow$ $f'(x) = F(x) + xF'(x)$ ...(i)

$\Rightarrow$ $f'(1) = F(1) + F'(1) < 0$

[given $F(1) = 0$ and $F'(x) < 0$]

Also, $f(2) = 2F(2) < 0$ [using $F(x) < 0, \forall x \in (1, 3)$]

Now, $f'(x) = F(x) + xF'(x) < 0$

[using $F(x) < 0, \ \forall x \in (1, 3)$]

$\Rightarrow$ $f'(x) < 0$

43. Given, $\int_1^3 x^2F'(x)\,dx = -12$

$\Rightarrow [x^2F(x)]_1^3 - \int_1^3 2x\cdot F(x)\,dx = -12$

$\Rightarrow 9F(3) - F(1) - 2\int_1^3 f(x)\,dx = -12$

[$\because xF(x) = f(x)$, given]

$\Rightarrow -36 - 0 - 2\int_1^3 f(x)\,dx = -12$

$\therefore \int_1^3 f(x)\,dx = -12$ and $\int_1^3 x^3F''(x)dx = 40$

$\Rightarrow [x^3F'(x)]_1^3 - \int_1^3 3x^2F'(x)\,dx = 40$

$\Rightarrow [x^2(xF'(x)]_1^3 - 3\times(-12) = 40$

$\Rightarrow \{x^2\cdot[f'(x) - F(x)]\}_1^3 = 4$

$\Rightarrow 9[f'(3) - F(3)] - [f'(1) - F(1)] = 4$

$\Rightarrow 9[f'(3) + 4] - [f'(1) - 0] = 4$

$\Rightarrow 9f'(3) - f'(1) = -32$

44. Given, $\lim_{t\to a}\dfrac{\int_a^t f(x)\,dx - \dfrac{(t-a)}{2}\{f(t)+f(a)\}}{(t-a)^3} = 0$

Using L'Hospital's rule, put $t - a = h$

$\Rightarrow \lim_{h\to 0}\dfrac{\int_a^{a+h} f(x)\,dx - \dfrac{h}{2}\{f(a+h)+f(a)\}}{h^3} = 0$

$\Rightarrow \lim_{h\to 0}\dfrac{f(a+h) - \dfrac{1}{2}\{f(a+h)+f(a)\} - \dfrac{h}{2}\{f'(a+h)\}}{3h^2} = 0$

Again, using L' Hospital's rule,

$\lim_{h\to 0}\dfrac{f'(a+h) - \dfrac{1}{2}f'(a+h) - \dfrac{1}{2}f'(a+h) - \dfrac{h}{2}f''(a+h)}{6h} = 0$

$\Rightarrow \lim_{h\to 0}\dfrac{-\dfrac{h}{2}f''(a+h)}{6h} = 0$

$\Rightarrow f''(a) = 0, \forall a \in R$

$\Rightarrow f(x)$ must have maximum degree 1.

45. $F'(c) = (b-a)f'(c) + f(a) - f(b)$

$F''(c) = f''(c)(b-a) < 0$

$\Rightarrow F'(c) = 0 \Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b-a}$

46. $\int_0^{\pi/2}\sin x\,dx = \dfrac{\dfrac{\pi}{2} - 0}{4}\left[\sin 0 + \sin\left(\dfrac{\pi}{2}\right) + 2\sin\left(\dfrac{0 + \dfrac{\pi}{2}}{2}\right)\right]$

$= \dfrac{\pi}{8}(1 + \sqrt{2})$

47. $\because e^x \in (1, e)$ in $(0, 1)$ and $\int_0^x f(t)\sin t\,dt \in (0, 1)$ in $(0, 1)$

$\therefore e^x - \int_0^x f(t)\sin t\,dt$ cannot be zero.

So, option (a) is incorrect.

(b) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$ always positive

$\therefore$ Option (b) is incorrect.

(c) Let $h(x) = x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$,

$h(0) = -\int_0^{\frac{\pi}{2}} f(t)\cos t\,dt < 0$

$h(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t)\cos t\,dt > 0$

$\therefore$ Option (c) is correct.

(d) Let $g(x) = x^9 - f(x)$

$g(0) = -f(0) < 0$

$g(1) = 1 - f(1) > 0$

$\therefore$ Option (d) is correct.

48. $I = \sum_{k=1}^{98}\int_k^{k+1}\dfrac{(k+1)}{x(x+1)}dx$

Clearly, $I > \sum_{k=1}^{98}\int_k^{k+1}\dfrac{(k+1)}{(x+1)^2}dx$

$\Rightarrow I > \sum_{k=1}^{98}(k+1)\int_k^{k+1}\dfrac{1}{(x+1)^2}dx$

$\Rightarrow I > \sum_{k=1}^{98}(-(k+1))\left[\dfrac{1}{k+2} - \dfrac{1}{k+1}\right] \Rightarrow I > \sum_{k=1}^{98}\dfrac{1}{k+2}$

$\Rightarrow I > \dfrac{1}{3} + \ldots + \dfrac{1}{100} > \dfrac{98}{100} \Rightarrow I > \dfrac{49}{50}$

Also, $I < \sum_{k=1}^{98}\int_k^{k+1}\dfrac{k+1}{x(k+1)}dx = \sum_{k=1}^{98}[\log_e(k+1) - \log_e k]$

$I < \log_e 99$

49. Here, $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$

for all $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

$\therefore f(x) = 7\tan^6 x\sec^2 x - 3\tan^2 x\sec^2 x$

$= (7\tan^6 x - 3\tan^2 x)\sec^2 x$

Now, $\int_0^{\pi/4} x f(x)dx = \int_0^{\pi/4} \underset{\text{I}}{x}\,\underset{\text{II}}{(7\tan^6 x - 3\tan^2 x)\sec^2 x}\,dx$

$= [x(\tan^7 x - \tan^3 x)]_0^{\pi/4}$

$- \int_0^{\pi/4} 1\,(\tan^7 x - \tan^3 x)\,dx$

$= 0 - \int_0^{\pi/4}\tan^3 x(\tan^4 x - 1)dx$

$= -\int_0^{\pi/4}\tan^3 x\,(\tan^2 x - 1)\sec^2 x\,dx$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore \int_0^{\pi/4} x f(x)dx = -\int_0^1 t^3(t^2 - 1)\,dt$

$$= \int_0^1 (t^3 - t^5)\,dt = \left[\frac{t^4}{4} - \frac{t^5}{5}\right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Also, $\int_0^{\pi/4} f(x)\,dx = \int_0^{\pi/4} (7\tan^6 x - 3\tan^2 x)\sec^2 x\,dx$

$$= \int_0^1 (7t^6 - 3t^2)\,dt = [t^7 - t^3]_0^1 = 0$$

50. Here, $f'(x) = \frac{192\,x^3}{2 + \sin^4 \pi x}$ $\therefore \frac{192x^3}{3} \le f'(x) \le \frac{192x^3}{2}$

On integrating between the limits $\frac{1}{2}$ to x, we get

$$\int_{1/2}^{x} \frac{192x^3}{3}\,dx \le \int_{1/2}^{x} f'(x)\,dx \le \int_{1/2}^{x} \frac{192x^3}{2}\,dx$$

$\Rightarrow \frac{192}{12}\left(x^4 - \frac{1}{16}\right) \le f(x) - f(0) \le 24x^4 - \frac{3}{2}$

$\Rightarrow 16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}$

Again integrating between the limits $\frac{1}{2}$ to 1, we get

$$\int_{1/2}^{1} (16x^4 - 1)\,dx \le \int_{1/2}^{1} f(x)\,dx \le \int_{1/2}^{1}\left(24x^4 - \frac{3}{2}\right)dx$$

$\Rightarrow \left[\frac{16x^5}{5} - x\right]_{1/2}^{1} \le \int_{1/2}^{1} f(x)\,dx \le \left[\frac{24x^5}{5} - \frac{3}{2}x\right]_{1/2}^{1}$

$\Rightarrow \left(\frac{11}{5} + \frac{2}{5}\right) \le \int_{1/2}^{1} f(x)\,dx \le \left(\frac{33}{10} + \frac{6}{10}\right)$

$\Rightarrow 2.6 \le \int_{1/2}^{1} f(x)\,dx \le 3.9$

(*) None of the option is correct.

51. Let $I_1 = \int_0^{4\pi} e^t (\sin^6 at + \cos^4 at)\,dt$

$$= \int_0^{\pi} e^t (\sin^6 at + \cos^4 at)\,dt + \int_{\pi}^{2\pi} e^t (\sin^6 at + \cos^4 at)\,dt + \int_{2\pi}^{3\pi} e^t (\sin^6 at + \cos^4 at)\,dt + \int_{3\pi}^{4\pi} e^t (\sin^6 at + \cos^4 at)\,dt$$

$\therefore \quad I_1 = I_2 + I_3 + I_4 + I_5$ ...(i)

Now, $I_3 = \int_{\pi}^{2\pi} e^t (\sin^6 at + \cos^4 at)\,dt$

Put $t = \pi + x \Rightarrow dt = dx$

$\therefore \quad I_3 = \int_0^{\pi} e^{\pi + x} \cdot (\sin^6 at + \cos^4 at)\,dt = e^{\pi} \cdot I_2$ ...(ii)

Now, $I_4 = \int_{2\pi}^{3\pi} e^t (\sin^6 at + \cos^4 at)\,dt$

Put $t = 2\pi + x \Rightarrow dt = dx$

$\therefore \quad I_4 = \int_0^{\pi} e^{x + 2\pi} (\sin^6 at + \cos^4 at)\,dt = e^{2\pi} \cdot I_2$ ...(iii)

and $I_5 = \int_{3\pi}^{4\pi} e^t (\sin^6 at + \cos^4 at)\,dt$

Put $t = 3\pi + x$

$\therefore \quad I_5 = \int_0^{\pi} e^{3\pi + x} (\sin^6 at + \cos^4 at)\,dt = e^{3\pi} \cdot I_2$ ...(iv)

From Eqs. (i), (ii), (iii) and (iv), we get

$I_1 = I_2 + e^{\pi} \cdot I_2 + e^{2\pi} \cdot I_2 + e^{3\pi} \cdot I_2 = (1 + e^{\pi} + e^{2\pi} + e^{3\pi})\,I_2$

$$\therefore \quad L = \frac{\int_0^{4\pi} e^t (\sin^6 at + \cos^4 at)\,dt}{\int_0^{\pi} e^t (\sin^6 at + \cos^4 at)\,dt}$$

$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi})$$

$$= \frac{1 \cdot (e^{4\pi} - 1)}{e^{\pi} - 1} \text{ for } a \in R$$

52. Let $I = \int_0^1 \frac{x^4 (1-x)^4}{1 + x^2}\,dx = \int_0^1 \frac{(x^4 - 1)(1-x)^4 + (1-x)^4}{(1 + x^2)}\,dx$

$$= \int_0^1 (x^2 - 1)(1-x)^4\,dx + \int_0^1 \frac{(1 + x^2 - 2x)^2}{(1 + x^2)}\,dx$$

$$= \int_0^1 \left\{(x^2 - 1)(1-x)^4 + (1 + x^2) - 4x + \frac{4x^2}{(1 + x^2)}\right\} dx$$

$$= \int_0^1 \left((x^2 - 1)(1-x)^4 + (1 + x^2) - 4x + 4 - \frac{4}{1 + x^2}\right) dx$$

$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1 + x^2}\right) dx$$

$$= \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x - 4\tan^{-1} x\right]_0^1$$

$$= \frac{1}{7} - \frac{4}{6} + \frac{5}{5} - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4} - 0\right) = \frac{22}{7} - \pi$$

53. Given $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x)\sin x}\,dx$ ...(i)

Using $\int_a^b f(x)\,dx = \int_a^b f(b + a - x)\,dx$, we get

$I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1 + \pi^x)\sin x}\,dx$ ...(ii)

On adding Eqs. (i) and (ii), we have

$$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x}\,dx = 2\int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$$

$\left[\because f(x) = \frac{\sin nx}{\sin x} \text{ is an even function}\right]$

$\Rightarrow \quad I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$

Now, $I_{n+2} - I_n = \int_0^{\pi} \frac{\sin (n+2)x - \sin nx}{\sin x}\,dx$

$$= \int_0^{\pi} \frac{2\cos (n+1)x \cdot \sin x}{\sin x}\,dx$$

$$= 2\int_0^{\pi} \cos (n+1)x\,dx = 2\left[\frac{\sin (n+1)x}{(n+1)}\right]_0^{\pi} = 0$$

$\therefore \quad I_{n+2} = I_n$ ...(iii)

Since, $I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$

$\Rightarrow \quad I_1 = \pi$ and $I_2 = 0$

From Eq. (iii) $\quad I_1 = I_3 = I_5 = \ldots = \pi$

and $\quad I_2 = I_4 = I_6 = \ldots = 0$

$\Rightarrow \quad \sum_{m=1}^{10} I_{2m+1} = 10\pi$ and $\sum_{m=1}^{10} I_{2m} = 0$

$\therefore$ Correct options are (a), (b), (c).

54. (2) Let $I = \int_0^{1/2} \frac{1+\sqrt{3}}{[(x+1)^2(1-x)^6]^{1/4}} dx$

$$\Rightarrow \quad I = \int_0^{1/2} \frac{1+\sqrt{3}}{(1-x)^2\left[\left(\frac{1-x}{1+x}\right)^6\right]^{1/4}} dx$$

Put $\frac{1-x}{1+x} = t \Rightarrow \frac{-2\,dx}{(1+x)^2} = dt$

when $x = 0, t = 1, x = \frac{1}{2}, t = \frac{1}{3}$

$\therefore$

$$I = \int_1^{1/3} \frac{(1+\sqrt{3})\,dt}{-2(t)^{6/4}}$$

$$\Rightarrow \quad I = \frac{-(1+\sqrt{3})}{2}\left[\frac{-2}{\sqrt{t}}\right]_1^{1/3}$$

$$\Rightarrow \quad I = (1+\sqrt{3})(\sqrt{3}-1) \Rightarrow I = 3 - 1 = 2$$

55. Given, $f(1) = \frac{1}{3}$ and $6\int_1^x f(t)dt = 3x\,f(x) - x^3, \forall\, x \geq 1$

Using Newton-Leibnitz formula.
Differentiating both sides

$\Rightarrow \quad 6f(x)\cdot 1 - 0 = 3f(x) + 3xf'(x) - 3x^2$

$\Rightarrow \quad 3xf'(x) - 3f(x) = 3x^2 \Rightarrow f'(x) - \frac{1}{x}f(x) = x$

$\Rightarrow \quad \frac{xf'(x) - f'(x)}{x^2} = 1 \Rightarrow \frac{d}{dx}\left\{\frac{x}{x}\right\} = 1$

On integrating both sides, we get

$\Rightarrow \quad \frac{f(x)}{x} = x + c \qquad \left[\because f(1) = \frac{1}{3}\right]$

$\frac{1}{3} = 1 + c \Rightarrow c = \frac{2}{3}$ and $f(x) = x^2 - \frac{2}{3}x$

$\therefore \quad f(2) = 4 - \frac{4}{3} = \frac{8}{3}$

NOTE Here, $f(1) = 2$, does not satisfy given function.

$\therefore \quad f(1) = \frac{1}{3}$

For that $f(x) = x^2 - \frac{2}{3}x$ and $f(2) = 4 - \frac{4}{3} = \frac{8}{3}$

56. Given, $\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1)$

Put $x^2 = t$

$\Rightarrow \quad 2x\,dx = dt$

$$\Rightarrow \quad \int_1^{16} 2\frac{e^{\sin t}}{t}\cdot\frac{dt}{2} = F(k) - F(1)$$

$$\Rightarrow \quad \int_1^{16} \frac{e^{\sin t}}{t} dt = F(k) - F(1)$$

$$\Rightarrow \quad [F(t)]_1^{16} = F(k) - F(1)$$

$$\left[\because \frac{d}{dx}\{F(x)\} = \frac{e^{\sin x}}{x}, \text{ given}\right]$$

$\Rightarrow \quad F(16) - F(1) = F(k) - F(1)$

$\therefore \quad k = 16$

57. Let $I = \int_1^{37\pi} \frac{\pi \sin(\pi \log x)}{x} dx$

Put $\pi \log x = t$

$\Rightarrow \quad \frac{\pi}{x} dx = dt$

$$\therefore \quad I = \int_0^{37\pi} \sin(t)\,dt = -[\cos t]_0^{37\pi} = -[\cos 37\pi - \cos 0]$$

$$= -[(-1) - 1] = 2$$

58. Let $I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \qquad \text{...(i)}$

$$I = \int_0^{2\pi} \frac{(2\pi - x)[\sin(2\pi - x)]^{2n}}{[\sin(2\pi - x)]^{2n} + [\cos(2\pi - x)]^{2n}} dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{2\pi} \frac{(2\pi - x)\cdot\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{2\pi\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx - \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{2\pi\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx - I \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{\pi\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow \quad I = \pi\left[\int_0^{\pi} \frac{\pi\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx + \int_0^{\pi} \frac{\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)} dx\right]$$

$$\left[\begin{array}{l}\text{using property} \\ \int_0^{2a} f(x)\,dx = \int_0^a [f(x) + f(2a - x)]dx\end{array}\right]$$

$$I = \pi\left[\int_0^{\pi} \frac{\sin^{2n} x\,dx}{\sin^{2n} x + \cos^{2n} x} dx + \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx\right]$$

$$\Rightarrow \quad I = 2\pi\int_0^{\pi} \frac{\sin^{2n} x\,dx}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow \quad I = 4\pi\left[\int_0^{\pi/2} \frac{\sin^{2n} x\,dx}{\sin^{2n} x + \cos^{2n} x} dx\right] \qquad \text{...(ii)}$$

$$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\sin^{2n}(\pi/2 - x)}{\sin^{2n}(\pi/2 - x) + \cos^{2n}(\pi/2 - x)} dx$$

$$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} dx \qquad \text{...(iii)}$$

On adding Eqs. (ii) and (iii), we get

$$2I = 4\pi\int_0^{\pi/2} \frac{\sin^{2n} x + \cos^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$$

$$\Rightarrow \quad 2I = 4\pi\int_0^{\pi/2} 1\, dx = 4\pi\, [x]_0^{\pi/2} = 4\pi \cdot \frac{\pi}{2}$$

$$\Rightarrow \quad I = \pi^2$$

59. Given, $af(x) + bf(1/x) = \dfrac{1}{x} - 5$ ...(i)

Replacing x by $1/x$ in Eq. (i), we get

$$af(1/x) + bf(x) = x - 5 \quad \text{...(ii)}$$

On multiplying Eq. (i) by a and Eq. (ii) by b, we get

$$a^2 f(x) + abf(1/x) = a\left(\frac{1}{x} - 5\right) \quad \text{...(iii)}$$

$$abf(1/x) + b^2 f(x) = b\,(x - 5) \quad \text{...(iv)}$$

On subtracting Eq. (iv) from Eq. (iii), we get

$$(a^2 - b^2)\, f(x) = \frac{a}{x} - bx - 5a + 5b$$

$$\Rightarrow \quad f(x) = \frac{1}{(a^2 - b^2)}\left(\frac{a}{x} - bx - 5a + 5b\right)$$

$$\Rightarrow \quad \int_1^2 f(x)\, dx = \frac{1}{(a^2 - b^2)}\int_1^2 \left(\frac{a}{x} - bx - 5a + 5b\right) dx$$

$$= \frac{1}{(a^2 - b^2)}\left[a \log|x| - \frac{b}{2}x^2 - 5(a - b)x\right]_1^2$$

$$= \frac{1}{(a^2 - b^2)}\left[a \log 2 - 2b - 10\,(a - b) - a \log 1 + \frac{b}{2} + 5\,(a - b)\right]$$

$$= \frac{1}{(a^2 - b^2)}\left[a \log 2 - 5a + \frac{7}{2}b\right]$$

60. Let $\quad I = \displaystyle\int_2^3 \frac{\sqrt{x}}{\sqrt{5 - x} + \sqrt{x}}\, dx$ ...(i)

$$\Rightarrow \quad I = \int_2^3 \frac{\sqrt{2 + 3 - x}}{\sqrt{(2 + 3) - (5 - x)} + \sqrt{2 + 3 - x}}\, dx$$

$$\Rightarrow \quad I = \int_2^3 \frac{\sqrt{5 - x}}{\sqrt{x} + \sqrt{5 - x}}\, dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_2^3 \frac{\sqrt{x} + \sqrt{5 - x}}{\sqrt{5 - x} + \sqrt{x}}\, dx \Rightarrow 2I = \int_2^3 1\, dx = 1 \quad \Rightarrow I = \frac{1}{2}$$

61. Let $\quad I = \displaystyle\int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x}\, dx$ ...(i)

$$\Rightarrow \quad I = \int_{\pi/4}^{3\pi/4} \frac{\left(\frac{\pi}{4} + \frac{3\pi}{4} - x\right)}{1 + \sin\left(\frac{\pi}{4} + \frac{3\pi}{4} - x\right)}\, dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx\right]$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\pi - x}{1 + \sin(\pi - x)}\, dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\pi}{1 + \sin x}\, dx - \int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x}\, dx$$

$$= \pi\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \sin x} - I \quad \text{[from Eq. (i)]}$$

$$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \frac{dx}{(1 + \sin x)}$$

$$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \frac{(1 - \sin x)}{(1 + \sin x)(1 - \sin x)}\, dx$$

$$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \frac{(1 - \sin x)}{1 - \sin^2 x}\, dx$$

$$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} (\sec^2 x - \sec x \cdot \tan x)\, dx$$

$$= \frac{\pi}{2}\,[\tan x - \sec x]_{\pi/4}^{3\pi/4}$$

$$= \frac{\pi}{2}\,[-1 - 1 - (-\sqrt{2} - \sqrt{2})]$$

$$= \frac{\pi}{2}\,(-2 + 2\sqrt{2}) = \pi\,(\sqrt{2} - 1)$$

62. $\displaystyle\int_{-2}^{2} |1 - x^2|\, dx$

$$= \int_{-2}^{-1} (x^2 - 1)\, dx + \int_{-1}^{1} (1 - x^2)\, dx + \int_1^2 (x^2 - 1)\, dx$$

$$= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_1^2$$

$$= \left(-\frac{1}{3} + 1 + \frac{8}{3} - 2\right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) + \left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right)$$

$$= 4$$

63. $\displaystyle\int_0^{1.5} [x^2]\, dx = \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{1.5} 2\, dx$

$$= 0 + [x]_1^{\sqrt{2}} + 2\,[x]_{\sqrt{2}}^{1.5}$$

$$= (\sqrt{2} - 1) + 2\,(1.5 - \sqrt{2})$$

$$= \sqrt{2} - 1 + 3 - 2\sqrt{2}$$

$$= 2 - \sqrt{2}$$

64. (A) Let $I = \displaystyle\int_{-1}^{1} \frac{dx}{1 + x^2}$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore \quad I = 2\int_0^{\pi/4} d\theta = \frac{\pi}{2}$$

(B) Let $\quad I = \displaystyle\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

Put $\quad x = \sin\theta$

$$\Rightarrow \quad dx = \cos\theta\, d\theta$$

$$\therefore \quad I = \int_0^{\pi/2} 1\, d\theta = \frac{\pi}{2}$$

(C) $\int_2^3 \frac{dx}{1-x^2} = \frac{1}{2}\left[\log\left(\frac{1+x}{1-x}\right)\right]_2^3$

$= \frac{1}{2}\left[\log\left(\frac{4}{-2}\right) - \log\left(\frac{3}{-1}\right)\right] = \frac{1}{2}\left[\log\left(\frac{2}{3}\right)\right]$

(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

65. (P) **PLAN** (i) A polynomial satisfying the given conditions is taken.
(ii) The other conditions are also applied and the number of polynomial is taken out.

Let $f(x) = ax^2 + bx + c$

$f(0) = 0 \Rightarrow c = 0$

Now, $\int_0^1 f(x)\,dx = 1$

$\Rightarrow \left(\frac{ax^3}{3} + \frac{bx^2}{2}\right)_0^1 = 1 \Rightarrow \frac{\alpha}{3} + \frac{\beta}{2} = 1$

$\Rightarrow 2a + 3b = 6$

As a, b are non-negative integers.

So, $a = 0, b = 2$ or $a = 3, b = 0$

$\therefore f(x) = 2x$ or $f(x) = 3x^2$

(Q) **PLAN** Such type of questions are converted into only sine or cosine expression and then the number of points of maxima in given interval are obtained.

$f(x) = \sin(x^2) + \cos(x^2)$

$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos(x^2) + \frac{1}{\sqrt{2}}\sin(x^2)\right]$

$= \sqrt{2}\left[\cos x^2 \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\sin(x^2)\right]$

$= \sqrt{2}\cos\left(x^2 - \frac{\pi}{4}\right)$

For maximum value, $x^2 - \frac{\pi}{4} = 2n\pi \Rightarrow x^2 = 2n\pi + \frac{\pi}{4}$

$\Rightarrow x = \pm\sqrt{\frac{\pi}{4}}$, for $n = 0 \Rightarrow x = \pm\sqrt{\frac{9\pi}{4}}$, for $n = 1$

So, $f(x)$ attains maximum at 4 points in $[-\sqrt{13}, \sqrt{13}]$.

(R) **PLAN**

(i) $\int_{-a}^a f(x)\,dx = \int_{-a}^a f(-x)\,dx$

(ii) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(-x) = f(x)$, i.e. f is an even function.

$I = \int_{-2}^2 \frac{3x^2}{1+e^x}\,dx$

and $I = \int_{-2}^2 \frac{3x^2}{1+e^{-x}}\,dx$

$\Rightarrow 2I = \int_{-2}^2 \left(\frac{3x^2}{1+e^x} + \frac{3x^2(e^x)}{e^x+1}\right)dx$

$2I = \int_{-2}^2 3x^2\,dx \Rightarrow 2I = 2\int_0^2 3x^2\,dx$

$I = [x^3]_0^2 = 8$

(S) **PLAN** $\int_{-a}^a f(x)\,dx = 0$

If $f(-x) = -f(x)$, i.e. $f(x)$ is an odd function.

Let $f(x) = \cos 2x \log\left(\frac{1+x}{1-x}\right)$

$f(-x) = \cos 2x \log\left(\frac{1-x}{1+x}\right) = -f(x)$

Hence, $f(x)$ is an odd function.

So, $\int_{-1/2}^{1/2} f(x)\,dx = 0$

(P) → (ii); (Q) → (iii); (R) → (i); (S) → (iv)

66. Let $I_2 = \int_0^1 (1-x^{50})^{101}\,dx$,

$= [(1-x^{50})^{101}\cdot x]_0^1 + \int_0^1 (1-x^{50})^{100}\,50\cdot x^{49}\cdot x\,dx$

[using integration by parts]

$= 0 - \int_0^1 (50)(101)(1-x^{50})^{100}(-x^{50})\,dx$

$= -(50)(101)\int_0^1 (1-x^{50})^{101}\,dx$

$+ (50)(101)\int_0^1 (1-x^{50})^{100}\,dx = 5050 I_2 + 5050 I_1$

$\therefore I_2 + 5050 I_2 = 5050 I_1$

$\Rightarrow \frac{(5050) I_1}{I_2} = 5051$

67. Let

$I = \int_0^\pi e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$

$\Rightarrow I = \int_0^\pi e^{|\cos x|}\cdot \sin x \cdot 2\sin\left(\frac{1}{2}\cos x\right)dx$

$+ \int_0^\pi e^{|\cos x|}\cdot 3\cos\left(\frac{1}{2}\cos x\right)\cdot \sin x\,dx$

$\Rightarrow I = I_1 + I_2$ ...(i)

$\left[\text{using} \int_0^{2a} f(x)\,dx = \begin{cases} 0, & f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & f(2a-x) = +f(x)\end{cases}\right]$

where, $I_1 = 0$ $[\because f(\pi - x) = -f(x)]$ ...(ii)

and $I_2 = 6\int_0^{\pi/2} e^{\cos x}\cdot \sin x \cdot \cos\left(\frac{1}{2}\cos x\right)dx$

Now, $I_2 = 6\int_0^1 e^t \cdot \cos\left(\frac{t}{2}\right)dt$

[put $\cos x = t \Rightarrow -\sin x\,dx = dt$]

$= 6\left[e^t \cos\left(\frac{t}{2}\right) + \frac{1}{2}\int e^t \sin\frac{t}{2}\,dt\right]_0^1$

$= 6\left[e^t \cos\left(\frac{t}{2}\right) + \frac{1}{2}\left(e^t \sin\frac{t}{2} - \int \frac{e^t}{2}\cos\frac{t}{2}\,dt\right)\right]_0^1$

$= 6\left[e^t \cos\frac{t}{2} + \frac{1}{2}e^t \sin\frac{t}{2}\right]_0^1 - \frac{I_2}{4}$

$$= \frac{24}{5}\left[e\cos\left(\frac{1}{2}\right) + \frac{e}{2}\sin\left(\frac{1}{2}\right) - 1\right] \quad \ldots\text{(iii)}$$

From Eq. (i), we get

$$I = \frac{24}{5}\left[e\cos\left(\frac{1}{2}\right) + \frac{e}{2}\sin\left(\frac{1}{2}\right) - 1\right]$$

68. Let $I = \int_{-\pi/3}^{\pi/3} \frac{\pi\, dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} + 4\int_{-\pi/3}^{\pi/3} \frac{x^3\, dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}$

Using $\int_{-a}^{a} f(x)\, dx = \begin{cases} 0, & f(-x) = -f(x) \\ 2\int_0^a f(x)\, dx, & f(-x) = f(x) \end{cases}$

$$\therefore \quad I = 2\int_0^{\pi/3} \frac{\pi\, dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} + 0$$

$$\left[\because \frac{x^3\, dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} \text{ is odd}\right]$$

$$I = 2\pi \int_0^{\pi/3} \frac{dx}{2 - \cos(x + \pi/3)}$$

Put $x + \frac{\pi}{3} = t \Rightarrow dx = dt$

$$\therefore \quad I = 2\pi\int_{\pi/3}^{2\pi/3} \frac{dt}{2 - \cos t} = 2\pi\int_{\pi/3}^{2\pi/3} \frac{\sec^2 \frac{t}{2}\, dt}{1 + 3\tan^2 \frac{t}{2}}$$

Put $\tan\frac{t}{2} = u \Rightarrow \sec^2\frac{t}{2}\, dt = 2\, du$

$$\Rightarrow \quad I = 2\pi\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2\, du}{1 + 3u^2} = \frac{4\pi}{3}\left[\sqrt{3}\tan^{-1}\sqrt{3}u\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}}(\tan^{-1} 3 - \tan^{-1} 1) = \frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \quad \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\, dx = \frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$$

69. Let $I = \int_0^{\pi/2} f(\cos 2x)\cos x\, dx \quad \ldots\text{(i)}$

$$\Rightarrow \quad I = \int_0^{\pi/2} f\left(\cos 2\left(\frac{\pi}{2} - x\right)\right)\cdot\cos\left(\frac{\pi}{2} - x\right)\, dx$$

$$\left[\text{using} \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} f(\cos 2x)\sin x\, dx \quad \ldots\text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\cos 2x)(\sin x + \cos x)\, dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\cos 2x)[\cos(x - \pi/4)]\, dx$$

Put $-x + \frac{\pi}{4} = t \Rightarrow -dx = dt$

$$\therefore \quad 2I = -\sqrt{2}\int_{\pi/4}^{-\pi/4} f\left[\cos\left(\frac{\pi}{2} - 2t\right)\right]\cos t\, dt$$

$$\Rightarrow \quad 2I = \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\sin 2t)\cos t\, dt$$

$$\therefore \quad I = \sqrt{2}\int_0^{\pi/4} f(\sin 2t)\cos t\, dt$$

70. Let $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx \quad \ldots\text{(i)}$

$$= \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi - x)} + e^{-\cos(\pi - x)}}\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\, dx \quad \ldots\text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$= \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx = \int_0^{\pi} 1\, dx = [x]_0^{\pi} = \pi$$

$$\Rightarrow \quad I = \pi/2$$

71. $\int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right)dx = \int_0^1 \tan^{-1}\left[\frac{1 - x + x}{1 - x(1 - x)}\right]dx$

$$= \int_0^1 [\tan^{-1}(1 - x) - \tan^{-1} x]\, dx$$

$$= \int_0^1 \tan^{-1}[1 - (1 - x)]\, dx + \int_0^1 \tan^{-1} x\, dx$$

$$= 2\int_0^1 \tan^{-1} x\, dx \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right] \quad \ldots\text{(i)}$$

Now, $\int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right)dx$

$$= \int_0^1 \left[\frac{\pi}{2} - \cot^{-1}\left(\frac{1}{1 - x + x^2}\right)\right]dx$$

$$= \frac{\pi}{2} - \int_0^1 \tan^{-1}(1 - x + x^2)\, dx$$

$$\therefore \int_0^1 \tan^{-1}(1 - x + x^2)\, dx = \frac{\pi}{2} - \int_0^1 \tan^{-1}\frac{1}{(1 - x + x^2)}\, dx$$

$$= \frac{\pi}{2} - 2I_1$$

where, $I_1 = \int_0^1 \tan^{-1} x\, dx = [x\tan^{-1} x]_0^1 - \int_0^1 \frac{x\, dx}{1 + x^2}$

$$= \frac{\pi}{4} - \frac{1}{2}[\log(1 + x^2)]_0^1 = \frac{\pi}{4} - \frac{1}{2}\log 2$$

$$\therefore \int_0^1 \tan^{-1}(1 - x + x^2)\, dx = \frac{\pi}{2} - 2\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) = \log 2$$

72. Let $I = \int_0^{\pi/4} \log(1 + \tan x)\, dx \quad \ldots\text{(i)}$

$$I = \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right)dx$$

$$\therefore \quad I = \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right) dx$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right) dx \Rightarrow I = \int_0^{\pi/4} \log 2\, dx - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2 \quad \Rightarrow \quad I = \frac{\pi}{8}(\log 2)$$

73. Let $\quad I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$

$$I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x} dx$$

$$\Rightarrow \quad I = I_1 + I_2$$

Now, $\quad I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx$

Let $\quad f(x) = \frac{2x}{1+\cos^2 x}$

$$\Rightarrow \quad f(-x) = \frac{-2x}{1+\cos^2(-x)} = \frac{-2x}{1+\cos^2 x} = -f(x)$$

$\Rightarrow \quad f(-x) = -f(x)$ which shows that $f(x)$ is an odd function.

$\therefore \qquad I_1 = 0$

Again, let $\qquad g(x) = \frac{2x\sin x}{1+\cos^2 x}$

$$\Rightarrow g(-x) = \frac{2(-x)\sin(-x)}{1+\cos^2(-x)} = \frac{2x\sin x}{1+\cos^2 x} = g(x)$$

$\Rightarrow g(-x) = g(x)$ which shows that $g(x)$ is an even function.

$$\therefore \quad I_2 = \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x} dx = 2\cdot 2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$$

$$= 4\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+[\cos(\pi-x)]^2} dx = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x} dx$$

$$= 4\int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x} dx - 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$$

$$\Rightarrow I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - I_2$$

$$\Rightarrow 2I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Put $\qquad \cos x = t \;\Rightarrow\; -\sin x\, dx = dt$

$$\therefore \quad I_2 = -2\pi\int_1^{-1} \frac{dt}{1+t^2} = 2\pi\int_{-1}^{1} \frac{dt}{1+t^2} = 4\pi\int_0^{1} \frac{dt}{1+t^2}$$

$$= 4\pi[\tan^{-1} t]_0^1 = 4\pi[\tan^{-1} 1 - \tan^{-1} 0]$$

$$= 4\pi(\pi/4 - 0) = \pi^2$$

$$\therefore \quad I = I_1 + I_2 = 0 + \pi^2 = \pi^2$$

74. Let $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right) dx \qquad \ldots$(i)

Put $x = -y \Rightarrow dx = -dy$

$$\therefore \qquad I = \int_{1/\sqrt{3}}^{-1/\sqrt{3}} \frac{y^4}{1-y^4}\cos^{-1}\left(\frac{-2y}{1+y^2}\right)(-1)\, dy$$

Now, $\cos^{-1}(-x) = \pi - \cos^{-1} x$ for $-1 \le x \le 1$.

$$\therefore \quad I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{y^4}{1-y^4}\left[\pi - \cos^{-1}\left(\frac{2y}{1+y^2}\right)\right] dy$$

$$= \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{y^4}{1-y^4} dy - \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{y^4}{1-y^4}\cos^{-1}\left(\frac{2y}{1+y^2}\right) dy$$

$$= \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx - \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

$$\Rightarrow \quad I = \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx - I \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow 2I = \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx = \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left[-1 + \frac{1}{1-x^4}\right] dx$$

$$= -\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} 1\, dx + \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{1-x^4}$$

$$= -\pi[x]_{-1/\sqrt{3}}^{1/\sqrt{3}} + \pi I_1, \text{ where } I_1 = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{1-x^4}$$

$$\Rightarrow \qquad 2I = -\pi\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) + \pi I_1 = -\frac{2\pi}{\sqrt{3}} + \pi I_1$$

Now, $I_1 = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{1-x^4} = 2\int_0^{1/\sqrt{3}} \frac{dx}{1-x^4}$

[since, the integral is an even function]

$$= \int_0^{1/\sqrt{3}} \frac{1+1+x^2-x^2}{(1-x^2)(1+x^2)} dx$$

$$= \int_0^{1/\sqrt{3}} \frac{1}{1-x^2} dx + \int_0^{1/\sqrt{3}} \frac{1}{1+x^2} dx$$

$$= \int_0^{1/\sqrt{3}} \frac{1}{(1-x)(1+x)} dx + \int_0^{1/\sqrt{3}} \frac{1}{(1+x^2)} dx$$

$$= \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{1}{1-x} dx + \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{1}{1+x} dx + \int_0^{1/\sqrt{3}} \frac{1}{1+x^2} dx$$

$$= \left[-\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + \tan^{-1} x\right]_0^{1/\sqrt{3}}$$

$$= \frac{1}{2}\left[\ln\left|\frac{1+x}{1-x}\right|\right]_0^{1/\sqrt{3}} + [\tan^{-1} x]_0^{1/\sqrt{3}}$$

$$= \frac{1}{2}\ln\left|\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}\right| + \tan^{-1}\frac{1}{\sqrt{3}}$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right| + \frac{\pi}{6} = \frac{1}{2}\ln\left|\frac{(\sqrt{3}+1)^2}{3-1}\right| + \frac{\pi}{6}$$

$$= \frac{1}{2}\ln(2+\sqrt{3}) + \frac{\pi}{6}$$

$$\therefore 2I = \frac{-2\pi}{\sqrt{3}} + \frac{\pi}{2}\ln(2+\sqrt{3}) + \frac{\pi^2}{6}$$

$$= \frac{\pi}{6}[\pi + 3\ln(2+\sqrt{3}) - 4\sqrt{3}]$$

$$\Rightarrow I = \frac{\pi}{12}[\pi + 3\ln(2+\sqrt{3}) - 4\sqrt{3}]$$

Alternate Solution

Since, $\cos^{-1} y = \frac{\pi}{2} - \sin^{-1} y$

$\therefore \quad \cos^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{\pi}{2} - \sin^{-1}\frac{2x}{1+x^2} = \frac{\pi}{2} - 2\tan^{-1} x$

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left[\frac{\pi}{2}\cdot\frac{x^4}{1-x^4} - \frac{x^4}{1-x^4} 2\tan^{-1} x\right] dx$$

$$\left[\because \frac{x^4}{1-x^4} 2\tan^{-1} x \text{ is an odd function}\right]$$

$$\therefore \quad I = 2\cdot\frac{\pi}{2}\int_0^{\frac{1}{\sqrt{3}}} \left(-1 + \frac{1}{1-x^4}\right) dx + 0$$

$$= \frac{\pi}{2}\int_0^{1/\sqrt{3}} \left(-2 + \frac{1}{1-x^2} + \frac{1}{1+x^2}\right) dx$$

$$= \frac{\pi}{2}\left[-2x + \frac{1}{2\cdot 1}\log\frac{1+x}{1-x} + \tan^{-1} x\right]_0^{1/\sqrt{3}}$$

$$= \frac{\pi}{2}\left[-\frac{2}{\sqrt{3}} + \frac{1}{2}\log\frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\pi}{6}\right]$$

$$= \frac{\pi}{12}\ [\pi + 3\ \log(2+\sqrt{3}) - 4\sqrt{3}]$$

75. Let $I = \int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} dx$

$$= \int_2^3 \frac{2x^5 - 2x^3 + x^4 + 1 + 2x^2}{(x^2+1)(x^2-1)(x^2+1)} dx$$

$$= \int_2^3 \frac{2x^3(x^2-1) + (x^2+1)^2}{(x^2+1)^2(x^2-1)} dx$$

$$= \int_2^3 \frac{2x^3(x^2-1)}{(x^2+1)^2(x^2-1)} dx + \int_2^3 \frac{(x^2+1)^2}{(x^2+1)^2(x^2-1)} dx$$

$$= \int_2^3 \frac{2x^3}{(x^2+1)^2} dx + \int_2^3 \frac{1}{(x^2-1)} dx$$

$\Rightarrow \quad I = I_1 + I_2$

Now, $\quad I_1 = \int_2^3 \frac{2x^3}{(x^2+1)^2} dx$

Put $x^2 + 1 = t \ \Rightarrow\ 2x\,dx = dt$

$$\therefore \quad I_1 = \int_5^{10} \frac{(t-1)}{t^2} dt = \int_5^{10} \frac{1}{t} dt - \int_5^{10} \frac{1}{t^2} dt$$

$$= [\log t]_5^{10} + \left[\frac{1}{t}\right]_5^{10}$$

$$= \log 10 - \log 5 + \frac{1}{10} - \frac{1}{5}$$

$$= \log 2 - \frac{1}{10}$$

Again, $\quad I_2 = \int_2^3 \frac{1}{(x^2-1)} dx = \int_2^3 \frac{1}{(x-1)(x+1)} dx$

$$= \frac{1}{2}\int_2^3 \frac{1}{(x-1)} dx - \frac{1}{2}\int_2^3 \frac{1}{(x+1)} dx$$

$$= \left[\frac{1}{2}\log(x-1)\right]_2^3 - \frac{1}{2}\left[\ \log(x+1)\right]_2^3$$

$$= \frac{1}{2}\log\frac{2}{1} - \frac{1}{2}\log\frac{4}{3}$$

From Eq. (i), $I = I_1 + I_2$

$$= \log 2 - \frac{1}{10} + \frac{1}{2}\log 2 - \frac{1}{2}\log\frac{4}{3}$$

$$= \log\,[2\cdot 2^{1/2}\left(\frac{4}{3}\right)^{-1/2}] - \frac{1}{10} = \frac{1}{2}\log 6 - \frac{1}{10}$$

76. Since, $f(x)$ is a cubic polynomial. Therefore, $f'(x)$ is a quadratic polynomial and $f(x)$ has relative maximum and minimum at $x = \frac{1}{3}$ and $x = -1$ respectively, therefore, -1 and $1/3$ are the roots of $f'(x) = 0$.

$$\therefore \quad f'(x) = a(x+1)\left(x - \frac{1}{3}\right) = a\left(x^2 - \frac{1}{3}x + x - \frac{1}{3}\right)$$

$$= a\left(x^2 + \frac{2}{3}x - \frac{1}{3}\right)$$

Now, integrating w.r. t. x, we get

$$f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3}\right) + c$$

where, c is constant of integration.

Again, $f(-2) = 0$

$$\therefore \quad f(-2) = a\left(-\frac{8}{3} + \frac{4}{3} + \frac{2}{3}\right) + c$$

$$\Rightarrow \quad 0 = a\left(\frac{-8+4+2}{3}\right) + c$$

$$\Rightarrow \quad 0 = \frac{-2a}{3} + c \ \Rightarrow\ c = \frac{2a}{3}$$

$$\therefore \quad f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3}\right) + \frac{2a}{3} = \frac{a}{3}(x^3 + x^2 - x + 2)$$

Again, $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$ [given]

$$\Rightarrow \int_{-1}^{1} \frac{a}{3}(x^3 + x^2 - x + 2)\,dx = \frac{14}{3}$$

$$\Rightarrow \int_{-1}^{1} \frac{a}{3}(0 + x^2 + 0 + 2)\,dx = \frac{14}{3}$$

[$\because$ $y = x^3$ and $y = -x$ are odd functions]

$$\Rightarrow \frac{a}{3}\left[2\int_0^1 x^2\,dx + 4\int_0^1 1\,dx\right] = \frac{14}{3}$$

$$\Rightarrow \quad \frac{a}{3}\left[\left(\frac{2x^3}{3} + 4x\right)\right]_0^1 = \frac{14}{3}$$

$$\Rightarrow \quad \frac{a}{3}\left(\frac{2}{3} + 4\right) = \frac{14}{3} \ \Rightarrow\ \frac{a}{3}\left(\frac{14}{3}\right) = \frac{14}{3}$$

$$\Rightarrow \quad a = 3$$

Hence, $\quad f(x) = x^3 + x^2 - x + 2$

77. Let $I=\int_0^{\pi}\frac{x\sin(2x)\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}\,dx$ ...(i)

Then $I=\int_0^{\pi}\frac{(\pi-x)\cdot\sin 2(\pi-x)\cdot\sin\left[\frac{\pi}{2}\cos(\pi-x)\right]}{2(\pi-x)-\pi}\,dx$ ...(ii)

$\Rightarrow\quad I=\int_0^{\pi}\frac{(\pi-x)\cdot\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{\pi-2x}\,dx$

$\Rightarrow\quad I=\int_0^{\pi}\frac{(x-\pi)\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}\,dx$ ... (iii)

On adding Eqs. (i) and (iii), we get

$$2I=\int_0^{\pi}\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$\Rightarrow\quad 2I=2\int_0^{\pi}\sin x\cos x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$\Rightarrow\quad I=\int_0^{\pi}\sin x\cos x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$\left[\text{put } \frac{\pi}{2}\cos x=t\ \Rightarrow\ -\frac{\pi}{2}\sin x\,dx=dt\ \Rightarrow\sin x\,dx=-\frac{2}{\pi}\,dt\right]$

$$\therefore\quad I=-\frac{2}{\pi}\int_{\pi/2}^{-\pi/2}\frac{2t}{\pi}\cdot\sin t\,dt$$

$$=\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2}t\sin t\,dt$$

$$\Rightarrow\quad I=\frac{4}{\pi^2}\left[-t\cos t+\sin t\right]_{-\pi/2}^{\pi/2}=\frac{4}{\pi^2}\times 2=\frac{8}{\pi^2}$$

78. Let $I=\int_0^{\pi/2}f(\sin 2x)\sin x\,dx$ ...(i)

Then, $I=\int_0^{\pi/2}f\left[\sin 2\left(\frac{\pi}{2}-x\right)\right]\sin\left(\frac{\pi}{2}-x\right)dx$

$=\int_0^{\pi/2}f[\sin 2x]\cdot\cos x\,dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi/2}f(\sin 2x)(\sin x+\cos x)\,dx$$

$$=2\int_0^{\pi/4}f(\sin 2x)(\sin x+\cos x)\,dx$$

$$=2\sqrt{2}\int_0^{\pi/4}f(\sin 2x)\sin\left(x+\frac{\pi}{4}\right)dx$$

$$=2\sqrt{2}\int_0^{\pi/4}f\left(\sin 2\left(\frac{\pi}{4}-x\right)\right)\sin\left(\frac{\pi}{4}-x+\frac{\pi}{4}\right)dx$$

$$=2\sqrt{2}\int_0^{\pi/4}f(\cos 2x)\cos x\,dx$$

$$\therefore\quad I=\sqrt{2}\int_0^{\pi/4}f(\cos 2x)\cos x\,dx$$

Hence, $\int_0^{\pi/2}f(\sin 2x)\cdot\sin x\,dx=\sqrt{2}\int_0^{\pi/4}f(\cos 2x)\cos x\,dx$

79. We know that,

$2\sin x\,[\cos x+\cos 3x+\cos 5x+\ldots+\cos(2k-1)x]$

$=2\sin x\cos x+2\sin x\cos 3x+2\sin x\cos 5x+\ldots+2\sin x\cos(2k-1)x$

$=\sin 2x+(\sin 4x-\sin 2x)+(\sin 6x-\sin 4x)+\ldots+\{\sin 2kx-\sin(2k-2)x\}$

$=\sin 2kx$

$\therefore\ 2\,[\cos x+\cos 3x+\cos 5x+\ldots+\cos(2k-1)x]=\frac{\sin 2kx}{\sin x}$ ...(i)

Now, $\sin 2kx\cdot\cot x=\frac{\sin 2kx}{\sin x}\cdot\cos x$

$=2\cos x\,[\cos x+\cos 3x+\cos 5x+\ldots+\cos(2k-1)x]$ [from Eq. (i)]

$=[2\cos^2 x+2\cos x\cos 3x+2\cos x\cos 5x+\ldots+2\cos x\cos(2k-1)x]$

$=(1+\cos 2x)+(\cos 4x+\cos 2x)+(\cos 6x+\cos 4x)+\ldots+\{\cos 2kx+\cos(2k-2)x\}$

$=1+2\,[\cos 2x+\cos 4x+\cos 6x+\ldots+\cos(2k-2)x]+\cos 2kx$

$\therefore\ \int_0^{\pi/2}(\sin 2kx)\cdot\cot x\,dx$

$=\int_0^{\pi/2}1\cdot dx+2\int_0^{\pi/2}(\cos 2x+\cos 4x\ldots\cos(2k-2)x)\,dx+\int_0^{\pi/2}\cos(2k)x\,dx$

$$=\frac{\pi}{2}+2\left[\frac{\sin 2x}{2}+\frac{\sin 4x}{4}+\ldots+\frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2}+\left[\frac{\sin(2k)x}{2k}\right]_0^{\pi/2}=\frac{\pi}{2}$$

80. Let $I=\int_0^a f(x)\cdot g(x)\,dx$

$I=\int_0^a f(a-x)\cdot g(a-x)\,dx=\int_0^a f(x)\cdot\{2-g(x)\}\,dx$

$[\because f(a-x)=f(x)$ and $g(x)+g(a-x)=2]$

$=2\int_0^a f(x)\,dx-\int_0^a f(x)\,g(x)\,dx$

$$\Rightarrow\quad I=2\int_0^a f(x)\,dx-I$$

$$\Rightarrow\quad 2I=2\int_0^a f(x)\,dx$$

$$\therefore\quad \int_0^a f(x)\,g(x)\,dx=\int_0^a f(x)\,dx$$

81. Let $I=\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}\,dx$ ...(i)

$I=\int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f(x)}\,dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{2a}1\,dx=2a\ \Rightarrow\ I=a$$

82. Let $I = \int_0^1 \log(\sqrt{1-x} + \sqrt{1+x})\,dx$

Put $x = \cos 2\theta$

$\Rightarrow dx = -2\sin 2\theta\,d\theta$

$\therefore\; I = -2\int_{\pi/4}^{0} \log[\sqrt{1-\cos 2\theta} + \sqrt{1+\cos 2\theta}]\,(\sin 2\theta)\,d\theta$

$= -2\int_{\pi/4}^{0} \log[\sqrt{2}\,(\sin\theta + \cos\theta)]\sin 2\theta\,d\theta$

$= -2\int_{\pi/4}^{0} [(\log\sqrt{2})\sin 2\theta + \log(\sin\theta + \cos\theta)\cdot\sin 2\theta]\,d\theta$

$= -2\log\sqrt{2}\left[\frac{-\cos 2\theta}{2}\right]_{\pi/4}^{0} - 2\int_{\pi/4}^{0} \underset{\text{I}}{\log(\sin\theta+\cos\theta)}\cdot\underset{\text{II}}{\sin 2\theta}\,d\theta$

$= \log\sqrt{2} - 2\left[-\left\{\log(\sin\theta+\cos\theta)\cdot\frac{\cos 2\theta}{2}\right\}_{\pi/4}^{0} - \int_{\pi/4}^{0}\left(\frac{\cos\theta - \sin\theta}{\cos\theta+\sin\theta}\times\frac{-\cos 2\theta}{2}\right)d\theta\right]$

$= \log(\sqrt{2}) - 2\left[0 + \frac{1}{2}\int_{\pi/4}^{0}(\cos\theta - \sin\theta)^2\,d\theta\right]$

$= \frac{1}{2}\log 2 - \int_{\pi/4}^{0}(1-\sin 2\theta)\,d\theta$

$= \frac{1}{2}\log 2 - \left[\theta + \frac{\cos 2\theta}{2}\right]_{\pi/4}^{0}$

$= \frac{1}{2}\log 2 - \left(\frac{1}{2} - \frac{\pi}{4}\right) = \frac{1}{2}\log 2 - \frac{1}{2} + \frac{\pi}{4}$

83. Let $\quad I = \int_0^{\pi}\frac{x}{1+\cos\alpha\sin x}\,dx \qquad \ldots\text{(i)}$

$\Rightarrow \quad I = \int_0^{\pi}\frac{(\pi - x)}{1+\cos\alpha\sin(\pi - x)}\,dx$

$\Rightarrow \quad I = \int_0^{\pi}\frac{(\pi - x)}{1+\cos\alpha\sin x}\,dx \qquad \ldots\text{(ii)}$

On adding Eqs. (i) and (ii), we get

$2I = \pi\int_0^{\pi}\frac{dx}{1+\cos\alpha\sin x}$

$\Rightarrow \quad 2I = \pi\int_0^{\pi}\frac{\sec^2\frac{x}{2}\,dx}{(1+\tan^2\frac{x}{2}) + 2\cos\alpha\tan\frac{x}{2}}$

Put $\tan\frac{x}{2} = t \;\Rightarrow\; \sec^2\frac{x}{2}\,dx = 2\,dt$

$\therefore \quad 2I = \pi\int_0^{\infty}\frac{2\,dt}{1+t^2+2t\cos\alpha}$

$\Rightarrow \quad 2I = 2\pi\int_0^{\infty}\frac{dt}{(t+\cos\alpha)^2 + \sin^2\alpha}$

$I = \frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$

$= \frac{\pi}{\sin\alpha}[\tan^{-1}(\infty) - \tan^{-1}(\cot\alpha)]$

$= \frac{\pi}{\sin\alpha}\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right) = \frac{\alpha\pi}{\sin\alpha}$

$\therefore \quad I = \frac{\alpha\pi}{\sin\alpha}$

84. Let $I = \int_0^{\pi/2}\frac{x\sin x\cdot\cos x}{\cos^4 x + \sin^4 x}\,dx$

$\Rightarrow I = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cdot\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right) + \cos^4\left(\frac{\pi}{2}-x\right)}\,dx$

$\Rightarrow I = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\cdot\sin x\cos x}{\cos^4 x + \sin^4 x}\,dx$

$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi/2}\frac{\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx - \int_0^{\pi/2}\frac{x\sin x\cdot\cos x}{\sin^4 x + \cos^4 x}\,dx$

$= \frac{\pi}{2}\int_0^{\pi/2}\frac{\sin x\cdot\cos x}{\sin^4 x + \cos^4 x}\,dx - I$

$\Rightarrow 2I = \frac{\pi}{2}\int_0^{\pi/2}\frac{\tan x\cdot\sec^2 x}{\tan^4 x + 1}\,dx$

$\Rightarrow 2I = \frac{\pi}{2}\cdot\frac{1}{2}\int_0^{\pi/2}\frac{1}{1+(\tan^2 x)^2}\,d(\tan^2 x)$

$\Rightarrow 2I = \frac{\pi}{4}\cdot[\tan^{-1} t]_0^{\infty} = \frac{\pi}{4}(\tan^{-1}\infty - \tan^{-1} 0)$

[where, $t = \tan^2 x$]

$\Rightarrow \quad I = \frac{\pi^2}{16}$

85. Let $I = \int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ Put $\sin^{-1}x = \theta \;\Rightarrow\; x = \sin\theta$

$\Rightarrow \; dx = \cos\theta\,d\theta$

$\therefore \; I = \int_0^{\pi/6}\frac{\theta\sin\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta = \int_0^{\pi/6}\theta\sin\theta\,d\theta$

$= [-\theta\cos\theta]_0^{\pi/6} + \int_0^{\pi/6}\cos\theta\,d\theta$

$= \left(-\frac{\pi}{6}\cos\frac{\pi}{6} + 0\right) + \left(\sin\frac{\pi}{6} - \sin 0\right) = -\frac{\sqrt{3}\,\pi}{12} + \frac{1}{2}$

86. Let $\quad I = \int_0^{\pi/4}\frac{(\sin x + \cos x)}{9 + 16\sin 2x}\,dx$

$I = \int_0^{\pi/4}\frac{\sin x + \cos x}{25 - 16(\sin x - \cos x)^2}\,dx$

Put $\;4(\sin x - \cos x) = t \Rightarrow 4(\cos x + \sin x)\,dx = dt$

$\therefore \quad I = \frac{1}{4}\int_{-4}^{0}\frac{dt}{25 - t^2} = \frac{1}{4}\cdot\frac{1}{2\,(5)}\log\left[\left|\frac{5+t}{5-t}\right|\right]_{-4}^{0}$

$$I = \frac{1}{40}\left[\log\left|\frac{5+0}{5-0}\right| - \log\left|\frac{5-4}{5+4}\right|\right]$$

$$= \frac{1}{40}\left(\log 1 - \log\frac{1}{9}\right) = \frac{1}{40}\log 9 = \frac{1}{20}(\log 3)$$

87. (i) Let $I = \int_0^{\pi} x\, f(\sin x)\, dx$ ...(i)

$\Rightarrow \quad I = \int_0^{\pi} (\pi - x)\, f(\sin x)\, dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \pi\, f(\sin x)\, dx$$

$$\therefore \int_0^{\pi} x\, f(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx$$

(ii) Let $I = \int_{-1}^{3/2} |x \sin \pi x|\, dx$

Since, $|x\sin \pi x| = \begin{cases} x\sin\pi x, & -1 < x \le 1 \\ -x\sin\pi x, & 1 < x < \frac{3}{2} \end{cases}$

$$\therefore \quad I = \int_{-1}^{1} x\sin\pi x\, dx + \int_1^{3/2} -x\sin\pi x\, dx$$

$$= 2\left[-\frac{x\cos\pi x}{\pi}\right]_0^1 - 2\int_0^1 1\cdot\left(\frac{-\cos\pi x}{\pi}\right)dx - \left\{\left[\frac{-x\cos\pi x}{\pi}\right]_1^{3/2} - \int_1^{3/2}\left(\frac{-\cos\pi x}{\pi}\right)dx\right\}$$

$$= 2\left(\frac{1}{\pi}\right) + \frac{2}{\pi}\cdot\left[\frac{\sin\pi x}{\pi}\right]_0^1 - \left(-\frac{1}{\pi}\right) - \frac{1}{\pi}\left[\frac{\sin\pi x}{\pi}\right]_1^{3/2}$$

$$= \frac{2}{\pi} + \frac{2}{\pi^2}(0-0) + \frac{1}{\pi} + \frac{1}{\pi^2}(+1-0)$$

$$= \frac{3}{\pi} + \frac{1}{\pi^2} = \left(\frac{3\pi+1}{\pi^2}\right)$$

88. Let $I = \int_0^1 (tx + 1 - x)^n\, dx = \int_0^1 \{(t-1)x + 1\}^n\, dx$

$$= \left[\frac{((t-1)x+1)^{n+1}}{(n+1)(t-1)}\right]_0^1 = \frac{1}{n+1}\left(\frac{t^{n+1}-1}{t-1}\right)$$

$$= \frac{1}{n+1}(1 + t + t^2 + \ldots + t^n) \quad \ldots(i)$$

Again, $I = \int_0^1 (tx + 1 - x)^n\, dx = \int_0^1 [(1-x) + tx]^n\, dx$

$$= \int_0^1 [{}^nC_0(1-x)^n + {}^nC_1(1-x)^{n-1}(tx)$$
$${}^nC_2(1-x)^{n-2}(tx)^2 + \ldots + {}^nC_n(tx)^n +]\, dx$$

$$= \int_0^1 \left[\sum_{r=0}^{n} {}^nC_r(1-x)^{n-r}(tx)^r\right]dx$$

$$= \sum_{r=0}^{n} {}^nC_r\left[\int_0^1 (1-x)^{n-r}\cdot x^r\, dx\right]t^r \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\sum_{r=0}^{n} {}^nC_r\left[\int_0^1 (1-x)^{n-r}\cdot x^r\, dx\right]t^r = \frac{1}{n+1}(1 + t + \ldots + t^n)$$

On equating coefficient of t^k on both sides, we get

$${}^nC_k\left[\int_0^1 (1-x)^{n-k}\cdot x^k\, dx\right] = \frac{1}{n+1}$$

$$\Rightarrow \quad \int_0^1 (1-x)^{n-k} x^k\, dx = \frac{1}{(n+1)\,{}^nC_k}$$

89. Here, $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$

$$\therefore \quad I = \int_{-1}^{2} \frac{x\, f(x^2)}{2 + f(x+1)}\, dx$$

$$= \int_{-1}^{0} \frac{x\, f(x^2)}{2 + f(x+1)}\, dx + \int_0^1 \frac{x\, f(x^2)}{2 + f(x+1)}\, dx + \int_1^{\sqrt{2}} \frac{x\, f(x^2)}{2 + f(x+1)}\, dx + \int_{\sqrt{2}}^{\sqrt{3}} \frac{x\, f(x^2)}{2 + f(x+1)}\, dx + \int_{\sqrt{3}}^{2} \frac{x\, f(x^2)}{2 + f(x+1)}\, dx$$

$$= \int_{-1}^{0} 0\, dx + \int_0^1 0\, dx + \int_1^{\sqrt{2}} \frac{x\cdot 1}{2+0}\, dx + \int_{\sqrt{2}}^{\sqrt{3}} 0\, dx + \int_{\sqrt{3}}^{2} 0\, dx$$

$\left[\begin{array}{l} \because -1 < x < 0 \Rightarrow 0 < x^2 < 1 \Rightarrow [x^2] = 0, \\ \quad 0 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow [x^2] = 0, \\ 1 < x < \sqrt{2} \Rightarrow \begin{cases} 1 < x^2 < 2 & \Rightarrow [x^2] = 1 \\ 2 < x+1 < 1+\sqrt{2} & \Rightarrow f(x+1) = 0, \end{cases} \\ \quad \sqrt{2} < x < \sqrt{3} \Rightarrow 2 < x^2 < 3 \Rightarrow f(x^2) = 0, \\ \text{and } \sqrt{3} < x < 2 \Rightarrow 3 < x^2 < 4 \Rightarrow f(x^2) = 0 \end{array}\right]$

$$\Rightarrow \quad I = \int_1^{\sqrt{2}} \frac{x}{2}\, dx = \left[\frac{x^2}{4}\right]_1^{\sqrt{2}} = \frac{1}{4}(2-1) = \frac{1}{4}$$

$$\therefore \quad 4I = 1 \Rightarrow 4I - 1 = 0$$

90. Here, $\alpha = \int_0^1 e^{(9x + 3\tan^{-1}x)}\left(\frac{12 + 9x^2}{1+x^2}\right)dx$

Put $9x + 3\tan^{-1}x = t$

$$\Rightarrow \quad \left(9 + \frac{3}{1+x^2}\right)dx = dt$$

$$\therefore \quad \alpha = \int_0^{9+3\pi/4} e^t\, dt = [e^t]_0^{9+3\pi/4} = e^{9+3\pi/4} - 1$$

$$\Rightarrow \quad \log_e|1+\alpha| = 9 + \frac{3\pi}{4}$$

$$\Rightarrow \quad \log_e|\alpha + 1| - \frac{3\pi}{4} = 9$$

91. **PLAN** Integration by parts

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int\left(\frac{d}{dx}[f(x)]\int g(x)\, dx\right)dx$$

Given, $I = \int_0^1 \underset{\text{I}}{4x^3}\, \frac{d^2}{dx^2}\underset{\text{II}}{(1-x^2)^5}\, dx$

$$= \left[4x^3\frac{d}{dx}(1-x^2)^5\right]_0^1 - \int_0^1 12x^2\frac{d}{dx}(1-x^2)^5\, dx$$

$$= \left[4x^3 \times 5(1-x^2)^4 \ (-2x)\right]_0^1 - 12\left[[x^2(1-x^2)^5]_0^1 - \int_0^1 2x(1-x^2)^5 \, dx\right]$$

$$= 0 - 0 - 12(0-0) + 12\int_0^1 2x(1-x^2)^5 \, dx$$

$$= 12 \times \left[-\frac{(1-x^2)^6}{6}\right]_0^1 = 12\left[0 + \frac{1}{6}\right] = 2$$

Topic 2 Periodicity of Integral Functions

1. Let $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{[x] + [\sin x] + 4}$

$$= \int_{\frac{-\pi}{2}}^{-1} \frac{dx}{[x]+[\sin x]+4} + \int_{-1}^{0} \frac{dx}{[x]+[\sin x]+4} + \int_0^1 \frac{dx}{[x]+[\sin x]+4} + \int_1^{\frac{\pi}{2}} \frac{dx}{[x]+[\sin x]+4}$$

$\because$ $[x] = \begin{cases} -2, & -\pi/2 < x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < \pi/2 \end{cases}$

and $[\sin x] = \begin{cases} -1, & -\pi/2 < x < -1 \\ -1, & -1 < x < 0 \\ 0, & 0 < x < 1 \\ 0, & 1 < x < \pi/2 \end{cases}$

[$\because$ For $x < 0, -1 \le \sin x < 0$ and for $x > 0, 0 < \sin x \le 1$]

So, $I = \int_{\frac{-\pi}{2}}^{-1} \frac{dx}{-2-1+4} + \int_{-1}^{0} \frac{dx}{-1-1+4} + \int_0^1 \frac{dx}{0+0+4} + \int_1^{\frac{\pi}{2}} \frac{dx}{1+0+4}$

$$= \int_{\frac{-\pi}{2}}^{-1} \frac{dx}{1} + \int_{-1}^{0} \frac{dx}{2} + \int_0^1 \frac{dx}{4} + \int_1^{\frac{\pi}{2}} \frac{dx}{5}$$

$$= \left(-1 + \frac{\pi}{2}\right) + \frac{1}{2}(0+1) + \frac{1}{4}(1-0) + \frac{1}{5}\left(\frac{\pi}{2} - 1\right)$$

$$= \left(-1 + \frac{1}{2} + \frac{1}{4} - \frac{1}{5}\right) + \left(\frac{\pi}{2} + \frac{\pi}{10}\right)$$

$$= \frac{-20+10+5-4}{20} + \frac{5\pi + \pi}{10}$$

$$= -\frac{9}{20} + \frac{3\pi}{5} = \frac{3}{20}(4\pi - 3)$$

2. $\int_3^{3+3T} f(2x)\, dx$ Put $2x = y \Rightarrow dx = \frac{1}{2} dy$

$\therefore$ $\frac{1}{2}\int_6^{6+6T} f(y)\, dy = \frac{6I}{2} = 3I$

3. Given, $g(x) = \int_0^x f(t)\, dt$

$\Rightarrow$ $g(2) = \int_0^2 f(t)\, dt = \int_0^1 f(t)\, dt + \int_1^2 f(t) dt$

Now, $\frac{1}{2} \le f(t) \le 1$ for $t \in [0,1]$

We get $\int_0^1 \frac{1}{2}\, dt \le \int_0^1 f(t)\, dt \le \int_0^1 1\, dt$

$\Rightarrow$ $\frac{1}{2} \le \int_0^1 f(t)\, dt \le 1$ ...(i)

Again, $0 \le f(t) \le \frac{1}{2}$ for $t \in [1,2]$ ...(ii)

$\Rightarrow$ $\int_1^2 0\, dt \le \int_1^2 f(t)\, dt \le \int_1^2 dt$

$\Rightarrow$ $0 \le \int_1^2 f(t)\, dt \le \frac{1}{2}$

From Eqs. (i) and (ii), we get

$$\frac{1}{2} \le \int_0^1 f(t)\, dt + \int_1^2 f(t)\, dt \le \frac{3}{2}$$

$\Rightarrow$ $\frac{1}{2} \le g(2) \le \frac{3}{2}$

$\Rightarrow$ $0 \le g(2) < 2$

4. $\int_0^{n\pi + v} |\sin x|\, dx = \int_0^{\pi} |\sin x|\, dx + \int_{\pi}^{2\pi} |\sin x|\, dx + \ldots + \int_{(n-1)\pi}^{n\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$

$$= \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$$

Now to solve, $\int_{(r-1)\pi}^{r\pi} |\sin x|\, dx$, we have

$$x = (r-1)\pi + t$$

$\Rightarrow$ $\sin x = \sin[(r-1)\pi + t] = (-1)^{r-1} \sin t$

and when $x = (r-1)\pi$, $t = 0$ and when

$$x = r\pi, t = \pi$$

$\therefore$ $\int_{(r-1)\pi}^{r\pi} |\sin x|\, dx = \int_0^{\pi} |(-1)^{r-1} \sin t|\, dt$

$$= \int_0^{\pi} |\sin t|\, dt = \int_0^{\pi} \sin t\, dt$$

$$= [-\cos t]_0^{\pi} = -\cos\pi + \cos 0 = 2$$

Again, $\int_{n\pi}^{n\pi+v} |\sin x|\, dx$, putting $x = n\pi + t$

Then, $\int_{n\pi}^{n\pi+v} |\sin x|\, dx = \int_0^{v} |(-1)^n \sin t|\, dt = \int_0^{v} \sin t\, dt$

$$= [-\cos t]_0^{v} = -\cos v + \cos 0 = 1 - \cos v$$

$\therefore$ $\int_0^{n\pi+v} |\sin x|\, dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$

$$= \sum_{r=1}^{n} 2 + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$$

$$= 2n + 1 - \cos v$$

5. Let $\phi(a) = \int_a^{a+t} f(x)\, dx$

On differentiating w.r.t. a, we get

$\phi'(a) = f(a+t)\cdot 1 - f(a)\cdot 1 = 0$ [given, $f(x+t) = f(x)$]

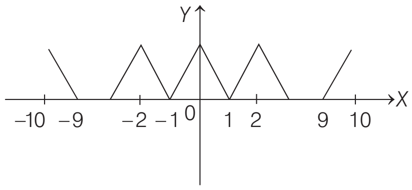
$\therefore$ $\phi(a)$ is constant.

$\Rightarrow$ $\int_a^{a+t} f(x)\, dx$ is independent of a.

6. Given, $f(x)=\begin{cases} x-[x], & \text{if } [x] \text{ is odd.} \\ 1+[x]-x, & \text{if } [x] \text{ is even.}\end{cases}$

$f(x)$ and $\cos \pi x$ both are periodic with period 2 and both are even.

$\therefore \quad \int_{-10}^{10} f(x)\cos \pi x\, dx = 2\int_{0}^{10} f(x) \cos \pi x\, dx$



$$= 10 \int_0^2 f(x) \cos \pi x \, dx$$

Now, $\int_0^1 f(x) \cos \pi x\, dx$

$$= \int_0^1 (1-x) \cos \pi x\, dx = -\int_0^1 u \cos \pi u\, du$$

and $\int_1^2 f(x) \cos \pi x\, dx = \int_1^2 (x-1) \cos \pi x\, dx$

$$= -\int_0^1 u \cos \pi u\, du$$

$\therefore \quad \int_{-10}^{10} f(x) \cos \pi x\, dx = -20 \int_0^1 u \cos \pi u\, du = \dfrac{40}{\pi^2}$

$\Rightarrow \quad \dfrac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x\, dx = 4$

Topic 3 Estimation, Gamma Function and Derivative of Definite Integration

1. Given, $\int_0^x f(t)\, dt = x^2 + \int_x^1 t^2 f(t) dt$

On differentiating both sides, w.r.t. 'x', we get

$$f(x) = 2x + 0 - x^2 f(x)$$

$$\left[\because \frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t) dt\right] = f(\psi(x)) \frac{d}{dx}\psi(x) - f(\phi(x)) \frac{d}{dx} \phi(x)\right]$$

$\Rightarrow \quad (1+x^2) f(x) = 2x \Rightarrow f(x) = \dfrac{2x}{1+x^2}$

On differentiating w.r.t. 'x' we get

$$f'(x) = \frac{(1+x^2)(2)-(2x)(0+2x)}{(1+x^2)^2}$$

$$= \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

$$\therefore f'\left(\frac{1}{2}\right) = \frac{2-2\left(\frac{1}{2}\right)^2}{\left(1+\left(\frac{1}{2}\right)^2\right)^2} = \frac{2-2\left(\frac{1}{4}\right)}{\left(1+\frac{1}{4}\right)^2} = \frac{2-\frac{1}{2}}{\left(\frac{5}{4}\right)^2} = \frac{\frac{3}{2}}{\frac{25}{16}} = \frac{24}{25}$$

2. **PLAN** Newton-Leibnitz's formula

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\, dt\right] = f\{\psi(x)\}\left\{\frac{d}{dx}\psi(x)\right\} - f\{\phi(x)\}\left\{\frac{d}{dx}\phi(x)\right\}$$

Given, $\quad F(x) = \int_0^{x^2} f(\sqrt{t})\, dt$

$\therefore \quad F'(x) = 2x\, f(x)$

Also, $\quad F'(x) = f'(x)$

$\Rightarrow \quad 2x\, f(x) = f'(x)$

$\Rightarrow \quad \dfrac{f'(x)}{f(x)} = 2x$

$\Rightarrow \quad \int \dfrac{f'(x)}{f(x)} dx = \int 2x\, dx \Rightarrow \ln f(x) = x^2 + c$

$\Rightarrow \quad f(x) = e^{x^2+c} \Rightarrow f(x) = K e^{x^2} \quad [K = e^c]$

Now, $\quad f(0) = 1$

$\therefore \quad 1 = K$

Hence, $\quad f(x) = e^{x^2}$

$$F(2) = \int_0^4 e^t dt = [e^t]_0^4 = e^4 - 1$$

3. Given, $y = \int_0^x |t|\, dt$

$\therefore \quad \dfrac{dy}{dx} = |x| \cdot 1 - 0 = |x| \quad$ [by Leibnitz's rule]

$\because$ Tangent to the curve $y = \int_0^x |t|\, dt, x \in R$ are parallel to the line $y = 2x$

$\therefore$ Slope of both are equal $\Rightarrow x = \pm 2$

Points, $\quad y = \int_0^{\pm 2} |t|\, dt = \pm 2$

Equation of tangent is

$y-2 = 2(x-2)$ and $y+2 = 2(x+2)$

For x intercept put $y = 0$, we get

$0-2 = 2(x-2)$ and $0+2 = 2(x+2)$

$\Rightarrow \quad x = \pm 1$

4. Given $\int_0^x \sqrt{1-\{f'(t)\}^2}\, dt = \int_0^x f(t)\, dt, 0 \le x \le 1$

Differentiating both sides w.r.t. x by using Leibnitz's rule, we get

$$\sqrt{1-\{f'(x)\}^2} = f(x) \Rightarrow f'(x) = \pm\sqrt{1-\{f(x)\}^2}$$

$\Rightarrow \int \dfrac{f'(x)}{\sqrt{1-\{f(x)\}^2}} dx = \pm \int dx \Rightarrow \sin^{-1}\{f(x)\} = \pm x + c$

Put $\quad x = 0 \Rightarrow \sin^{-1}\{f(0)\} = c$

$\Rightarrow \quad c = \sin^{-1}(0) = 0 \quad [\because f(0) = 0]$

$\therefore \quad f(x) = \pm \sin x$

but $\quad f(x) \geq 0, \forall x \in [0, 1]$

$\therefore \quad f(x) = \sin x$

As we know that,

$$\sin x < x, \ \forall \ x > 0$$

$$\therefore \quad \sin\left(\frac{1}{2}\right) < \frac{1}{2} \text{ and } \sin\left(\frac{1}{3}\right) < \frac{1}{3}$$

$$\Rightarrow \quad f\left(\frac{1}{2}\right) < \frac{1}{2} \text{ and } f\left(\frac{1}{3}\right) < \frac{1}{3}$$

5. Since $\int_{\sin x}^{1} t^2 f(t)\, dt = 1 - \sin x$, thus to find $f(x)$.

On differentiating both sides using Newton Leibnitz formula

i.e. $$\frac{d}{dx}\int_{\sin x}^{1} t^2 f(t)\, dt = \frac{d}{dx}(1 - \sin x)$$

$$\Rightarrow \{1^2 f(1)\}\cdot(0) - (\sin^2 x)\cdot f(\sin x)\cdot \cos x = -\cos x$$

$$\Rightarrow \quad f(\sin x) = \frac{1}{\sin^2 x}$$

For $f\left(\frac{1}{\sqrt{3}}\right)$ is obtained when $\sin x = 1/\sqrt{3}$

i.e. $$f\left(\frac{1}{\sqrt{3}}\right) = (\sqrt{3})^2 = 3$$

6. Here, $\int_0^{t^2} x f(x)\, dx = \frac{2}{5} t^5$

Using Newton Leibnitz's formula, differentiating both sides, we get

$$t^2\{f(t^2)\}\left\{\frac{d}{dt}(t^2)\right\} - 0\cdot f(0)\left\{\frac{d}{dt}(0)\right\} = 2t^4$$

$$\Rightarrow \quad t^2 f(t^2) 2t = 2t^4 \quad \Rightarrow \quad f(t^2) = t$$

$$\therefore \quad f\left(\frac{4}{25}\right) = -\frac{2}{5} \qquad \left[\text{putting } t = \frac{2}{5}\right]$$

$$\Rightarrow \quad f\left(\frac{4}{25}\right) = \frac{2}{5}$$

7. Given, $\quad f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$

On differentiating both sides using Newton's Leibnitz's formula, we get

$$f'(x) = e^{-(x^2+1)^2}\left\{\frac{d}{dx}(x^2+1)\right\} - e^{-(x^2)^2}\left\{\frac{d}{dx}(x^2)\right\}$$

$$= e^{-(x^2+1)^2}\cdot 2x - e^{-(x^2)^2}\cdot 2x$$

$$= 2xe^{-(x^4+2x^2+1)}(1 - e^{2x^2+1})$$

[where, $e^{2x^2+1} > 1, \forall x$ and $e^{-(x^4+2x^2+1)} > 0, \forall x$]

$$\therefore \quad f'(x) > 0$$

which shows $2x < 0$ or $x < 0 \quad \Rightarrow \ x \in (-\infty, 0)$

8. Here, $I(m, n) = \int_0^1 t^m (1+t)^n\, dt$ reduce into $I(m+1, n-1)$

[we apply integration by parts taking $(1+t)^n$ as first and t^m as second function]

$$\therefore \ I(m, n) = \left[(1+t)^n \cdot \frac{t^{m+1}}{m+1}\right]_0^1 - \int_0^1 n(1+t)^{(n-1)}\cdot \frac{t^{m+1}}{m+1}\, dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1}\int_0^1 (1+t)^{(n-1)}\cdot t^{m+1}\, dt$$

$$\therefore \ I(m, n) = \frac{2^n}{m+1} - \frac{n}{m+1}\cdot I(m+1, n-1)$$

9. Given, $\quad f(x) = \int_1^x \sqrt{2 - t^2}\, dt \quad \Rightarrow \quad f'(x) = \sqrt{2 - x^2}$

Also, $x^2 - f'(x) = 0$

$$\therefore \quad x^2 = \sqrt{2 - x^2} \quad \Rightarrow \quad x^4 = 2 - x^2$$

$$\Rightarrow \quad x^4 + x^2 - 2 = 0 \quad \Rightarrow \quad x = \pm 1$$

10. Given, $\quad F(x) = \int_0^x f(t)\, dt$

By Leibnitz's rule,

$$F'(x) = f(x) \qquad \ldots\text{(i)}$$

But $\quad F(x^2) = x^2(1 + x) = x^2 + x^3 \qquad$ [given]

$$\Rightarrow \quad F(x) = x + x^{3/2} \quad \Rightarrow \quad F'(x) = 1 + \frac{3}{2}x^{1/2}$$

$$\Rightarrow \quad f(x) = F'(x) = 1 + \frac{3}{2}x^{1/2} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad f(4) = 1 + \frac{3}{2}(4)^{1/2} \ \Rightarrow \ f(4) = 1 + \frac{3}{2}\times 2 = 4$$

11. Given, $\quad \int_0^x f(t) dt = x + \int_x^1 t\, f(t)\, dt$

On differentiating both sides w.r.t. x, we get

$$f(x)\, 1 = 1 - x f(x)\cdot 1 \quad \Rightarrow \quad (1 + x) f(x) = 1$$

$$\Rightarrow \quad f(x) = \frac{1}{1+x} \quad \Rightarrow \quad f(1) = \frac{1}{1+1} = \frac{1}{2}$$

12. $$\lim_{x\to 1}\int_4^{f(x)} \frac{2t}{x-1}\, dt = \lim_{x\to 1}\frac{\int_4^{f(x)} 2t\, dt}{x-1}$$

[using L' Hospital's rule]

$$= \lim_{x\to 1}\frac{2f(x)\cdot f'(x)}{1} = 2f(1)\cdot f'(1)$$

$$= 8f'(1) \qquad [\because f(1) = 4]$$

13. If $f(x)$ is a continuous function defined on $[a, b]$, then

$$m\,(b - a) \leq \int_a^b f(x)\, dx \leq M(b - a)$$

where, M and m are maximum and minimum values respectively of $f(x)$ in $[a, b]$.

Here, $f(x) = 1 + e^{-x^2}$ is continuous in $[0, 1]$.

Now, $\quad 0 < x < 1 \Rightarrow x^2 < x \Rightarrow e^{x^2} < e^x \Rightarrow e^{-x^2} > e^{-x}$

Again, $0 < x < 1 \Rightarrow x^2 > 0 \Rightarrow e^{x^2} > e^0 \Rightarrow e^{-x^2} < 1$

$$\therefore \quad e^{-x} < e^{-x^2} < 1, \forall \ x \in [0, 1]$$

$$\Rightarrow \quad 1 + e^{-x} < 1 + e^{-x^2} < 2, \forall x \in [0, 1]$$

$$\Rightarrow \quad \int_0^1 (1 + e^{-x})\, dx < \int_0^1 (1 + e^{-x^2})\, dx < \int_0^1 2\, dx$$

$$\Rightarrow \quad 2 - \frac{1}{e} < \int_0^1 (1 + e^{-x^2})\, dx < 2$$

14. $g(x) = \int_{\sin x}^{\sin 2x} \sin^{-1}(t)dt$

$$g'(x) = 2\cos 2x \sin^{-1}(\sin 2x) - \cos x \sin^{-1}(\sin x)$$

$$g'\left(\frac{\pi}{2}\right) = -2\sin^{-1}(0) = 0$$

$$g'\left(-\frac{\pi}{2}\right) = -2\sin^{-1}(0) = 0$$

No option is matching.

15. Here, $f(x) + 2x = (1-x)^2 \cdot \sin^2 x + x^2 + 2x$ ...(i)

where, $P : f(x) + 2x = 2(1+x)^2$ ...(ii)

$\therefore \quad 2(1+x^2) = (1-x)^2 \sin^2 x + x^2 + 2x$

$\Rightarrow \quad (1-x)^2 \sin^2 x = x^2 - 2x + 2$

$\Rightarrow \quad (1-x)^2 \sin^2 x = (1-x)^2 + 1$

$\Rightarrow \quad (1-x)^2 \cos^2 x = -1$

which is never possible.

$\therefore P$ is false.

Again, let $Q : h(x) = 2f(x) + 1 - 2x(1+x)$

where, $\quad h(0) = 2f(0) + 1 - 0 = 1$

$h(1) = 2f(1) + 1 - 4 = -3$, as $h(0)\,h(1) < 0$

$\Rightarrow h(x)$ must have a solution.

$\therefore \quad Q$ is true.

16. Here, $f(x) = (1-x)^2 \cdot \sin^2 x + x^2 \geq 0, \forall x$.

and $\quad g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \log t\right) f(t)\, dt$

$\Rightarrow \quad g'(x) = \left\{\frac{2(x-1)}{(x+1)} - \log x\right\} \cdot \underbrace{f(x)}_{+\text{ve}}$ ...(i)

For $g'(x)$ to be increasing or decreasing,

let $\quad \phi(x) = \frac{2(x-1)}{(x+1)} - \log x$

$$\phi'(x) = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-(x-1)^2}{x(x+1)^2}$$

$\phi'(x) < 0$, for $x > 1 \Rightarrow \phi(x) < \phi(1) \Rightarrow \phi(x) < 0$ ...(ii)

From Eqs. (i) and (ii), we get

$$g'(x) < 0 \text{ for } x \in (1, \infty)$$

$\therefore g(x)$ is decreasing for $x \in (1, \infty)$.

17. Given, $f(x) = \begin{vmatrix} \sec x & \cos x & \operatorname{cosec} x \cdot \cot x + \sec^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$

Applying $R_3 \to \frac{1}{\cos x} R_3$,

$$f(x) = \cos x \begin{vmatrix} \sec x & \cos x & \operatorname{cosec} x \cdot \cot x + \sec^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ \sec x & \cos x & \cos x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3 \quad \Rightarrow \quad f(x)$

$$= \cos x \begin{vmatrix} 0 & 0 & \operatorname{cosec} x \cdot \cot x + \sec^2 x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ \sec x & \cos x & \cos x \end{vmatrix}$$

$= (\operatorname{cosec} x \cdot \cot x + \sec^2 x - \cos x) \cdot (\cos^3 x - \cos x) \cdot \cos x$

$= -\left[\frac{\sin^2 x + \cos^3 x - \cos^3 x \cdot \sin^2 x}{\sin^2 x \cdot \cos^2 x}\right] \cdot \cos^2 x \cdot \sin^2 x$

$= -\sin^2 x - \cos^3 x(1 - \sin^2 x) = -\sin^2 x - \cos^5 x$

$\therefore \quad \int_0^{\pi/2} f(x)\, dx = -\int_0^{\pi/2} (\sin^2 x + \cos^5 x)\, dx$

$$\left[\because \int_0^{\pi/2} \sin^m x \cdot \cos^n x\, dx = \frac{\sqrt{\frac{m+1}{2}}\sqrt{\frac{n+2}{2}}}{2\sqrt{\frac{m+n+2}{2}}}\right]$$

$$\int_0^{\pi/2} f(x)\, dx = -\left\{\frac{\sqrt{\frac{3}{2}} \cdot \sqrt{\frac{1}{2}}}{2\sqrt{2}} + \frac{\sqrt{\frac{6}{2}} \cdot \sqrt{\frac{1}{2}}}{2\sqrt{\frac{7}{2}}}\right\}$$

$$= -\left\{\frac{1/2 \cdot \pi}{2} + \frac{2\sqrt{\pi}}{2 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}\right\} = -\left\{\frac{\pi}{4} + \frac{8}{15}\right\} = -\left(\frac{15\pi + 32}{60}\right)$$

18. $f(x) = \int_1^x \frac{\ln t}{1+t}\, dt$ for $x > 0$ [given]

Now, $\quad f(1/x) = \int_1^{1/x} \frac{\ln t}{1+t}\, dt$

Put $t = 1/u \quad \Rightarrow \quad dt = (-1/u^2)\, du$

$\therefore \quad f(1/x) = \int_1^x \frac{\ln(1/u)}{1+1/u} \cdot \frac{(-1)}{u^2}\, du$

$$= \int_1^x \frac{\ln u}{u(u+1)}\, du = \int_1^x \frac{\ln t}{t(1+t)}\, dt$$

Now, $\quad f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln t}{(1+t)}\, dt + \int_1^x \frac{\ln t}{t(1+t)}\, dt$

$$= \int_1^x \frac{(1+t)\ln t}{t(1+t)}\, dt = \int_1^x \frac{\ln t}{t}\, dt = \frac{1}{2}[(\ln t)^2]_1^x = \frac{1}{2}(\ln x)^2$$

Put $x = e$,

$$f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}(\ln e)^2 = \frac{1}{2}$$ **Hence proved.**

19. Let $t = b - a$ and $a + b = 4$ [given]

$\Rightarrow \quad t = 4 - a - a$

$\Rightarrow \quad t = 4 - 2a$

$\Rightarrow \quad a = 2 - \frac{t}{2}$

and $\quad t = b - (4 - b)$

$\Rightarrow \quad t = 2b - 4$

$\Rightarrow \quad \frac{t}{2} = b - 2$

$\Rightarrow \quad b = 2 + \frac{t}{2}$

Again, $\quad a < 2$ [given]

$\Rightarrow \quad 2 - \frac{\pi}{2} < 2$

$\Rightarrow \quad \frac{\pi}{2} > 0 \quad \Rightarrow \quad t > 0$

Now, $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

$$= \int_0^{2-t/2} g(x)\,dx + \int_0^{2+t/2} g(x)\,dx$$

Let $\quad F(x) = \int_0^{2-t/2} g(x)\,dx + \int_0^{2+t/2} g(x)\,dx$

For $t > 0$, $F'(t) = -\frac{1}{2} g\left(2 - \frac{t}{2}\right) + \frac{1}{2} g\left(2 + \frac{t}{2}\right)$

[using Leibnitz's rule]

$$= \frac{1}{2} g\left(2 + \frac{t}{2}\right) - \frac{1}{2} g\left(2 - \frac{t}{2}\right)$$

Again, $\quad \frac{dg}{dx} > 0, \forall\, x \in R$ [given]

Now, $2 - t/2 < 2 + t/2 \;\therefore\; t > 0$

We get $g(2 + t/2) - g(2 - t/2) > 0, \forall\, t > 0$

So, $\quad F'(t) > 0, \forall\, t > 0$

Hence, $F(t)$ increases with t, therefore $F(t)$ increases as $(b - a)$ increases.

20. Let $I_n = \int_0^1 e^x (x-1)^n\,dx$

Put $x - 1 = t \;\Rightarrow\; dx = dt$

$\therefore \quad I_n = \int_{-1}^0 e^{t+1} \cdot t^n dt = e\int_{-1}^0 t^n e^t dt$

$$= e\left([t^n e^t]_{-1}^0 - n\int_{-1}^0 t^{n-1} e^t dt\right)$$

$$= e\left(0 - (-1)^n e^{-1} - n\int_{-1}^0 t^{n-1} e^t dt\right)$$

$$= (-1)^{n+1} - ne\int_{-1}^0 t^{n-1} e^t\,dt$$

$\Rightarrow \quad I_n = (-1)^{n+1} - nI_{n-1}$ ...(i)

For $n = 1$, $I_1 = \int_0^1 e^x (x-1)\,dx = [e^x (x-1)]_0^1 - \int_0^1 e^x\,dx$

$$= e^1 (1-1) - e^0 (0-1) - [e^x]_0^1 = 1 - (e-1) = 2 - e$$

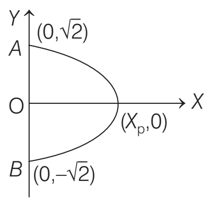
Therefore, from Eq. (i), we get

$$I_2 = (-1)^{2+1} - 2I_1 = -1 - 2(2-e) = 2e - 5$$

and $\quad I_3 = (-1)^{3+1} - 3I_2 = 1 - 3(2e-5) = 16 - 6e$

Hence, $n = 3$ is the answer.

21. Since, f is continuous function and $\int_0^x f(t)\,dt \to \infty$, as $|x| \to \infty$. To show that every line $y = mx$ intersects the curve $\quad y^2 + \int_0^x f(t)\,dt = 2$



At $x = 0$, $y = \pm\sqrt{2}$

Hence, $(0, \sqrt{2}), (0, -\sqrt{2})$ are the point of intersection of the curve with the Y-axis.

As $x \to \infty$, $\int_0^x f(t)\,dt \to \infty$ for a particular x (say x_n), then $\int_0^x f(t)\,dt = 2$ and for this value of x, $y = 0$

The curve is symmetrical about X-axis.

Thus, we have that there must be some x, such that $f(x_n) = 2$.

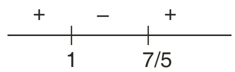
Thus, $y = mx$ intersects this closed curve for all values of m.

22. Given, $f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]\,dt$

$\therefore \quad f'(x) = [2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2] \cdot 1 - 0$

$$= (x-1)(x-2)^2 [2(x-2) + 3(x-1)]$$

$$= (x-1)(x-2)^2 (5x - 7)$$



$\therefore f(x)$ attains maximum at $x = 1$ and $f(x)$ attains minimum at $x = \frac{7}{5}$.

Topic 4 Limits as the Sum

1. Let $\quad p = \lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}} + \frac{(n+2)^{1/3}}{n^{4/3}} + \ldots + \frac{(2n)^{1/3}}{n^{4/3}}\right)$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{(n+r)^{1/3}}{n^{4/3}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\left(1 + \frac{r}{n}\right)^{1/3} n^{1/3}}{n^{4/3}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(1 + \frac{r}{n}\right)^{1/3}$$

Now, as per integration as limit of sum.

Let $\frac{r}{n} = x$ and $\frac{1}{n} = dx$ $\quad [\because n \to \infty]$

Then, upper limit of integral is 1 and lower limit of integral is 0.

So, $p = \int_0^1 (1+x)^{1/3}\,dx \quad \left[\because \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx\right]$

$$= \left[\frac{3}{4}(1+x)^{4/3}\right]_0^1 = \frac{3}{4}(2^{4/3} - 1) = \frac{3}{4}(2)^{4/3} - \frac{3}{4}$$

2. Clearly,

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots + \frac{1}{5n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots + \frac{n}{n^2+(2n)^2}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{2n} \frac{n}{n^2+r^2}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{2n} \frac{1}{1+\left(\frac{r}{n}\right)^2} \cdot \frac{1}{n} = \int_0^2 \frac{dx}{1+x^2}$$

$$\left[\because \lim_{n\to\infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^p f(x)dx\right]$$

$$= [\tan^{-1} x]_0^2 = \tan^{-1} 2$$

3. Let $l = \lim_{n \to \infty} \left(\frac{(n+1)\cdot(n+2)\dots(3n)}{n^{2n}}\right)^{\frac{1}{n}}$

$$= \lim_{n \to \infty} \left(\frac{(n+1)\cdot(n+2)\dots(n+2n)}{n^{2n}}\right)^{\frac{1}{n}}$$

$$= \lim_{n \to \infty} \left(\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\dots\left(\frac{n+2n}{n}\right)\right)^{\frac{1}{n}}$$

Taking log on both sides, we get

$$\log l = \lim_{n \to \infty} \frac{1}{n}\left[\log\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{2n}{n}\right)\right\}\right]$$

$$\Rightarrow \quad \log l = \lim_{n \to \infty} \frac{1}{n}$$

$$\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \dots + \log\left(1+\frac{2n}{n}\right)\right]$$

$$\Rightarrow \quad \log l = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} \log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \quad \log l = \int_0^2 \log(1+x)\,dx$$

$$\Rightarrow \quad \log l = \left[\log(1+x)\cdot x - \int \frac{1}{1+x}\cdot x\,dx\right]_0^2$$

$$\Rightarrow \quad \log l = [\log(1+x)\cdot x]_0^2 - \int_0^2 \frac{x+1-1}{1+x}\,dx$$

$$\Rightarrow \quad \log l = 2\cdot\log 3 - \int_0^2 \left(1 - \frac{1}{1+x}\right) dx$$

$$\Rightarrow \quad \log l = 2\cdot\log 3 - [x - \log|1+x|]_0^2$$

$$\Rightarrow \quad \log l = 2\cdot\log 3 - [2 - \log 3]$$

$$\Rightarrow \quad \log l = 3\cdot\log 3 - 2$$

$$\Rightarrow \quad \log l = \log 27 - 2$$

$$\therefore \quad l = e^{\log 27 - 2} = 27\cdot e^{-2} = \frac{27}{e^2}$$

4. **PLAN** Converting Infinite series into definite Integral

i.e. $\lim_{n\to\infty} \frac{h(n)}{n}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=g(n)}^{h(n)} f\left(\frac{r}{n}\right) = \int f(x)dx$$

$\lim_{n\to\infty} \frac{g(n)}{n}$

where, $\frac{r}{n}$ is replaced with x.

Σ is replaced with integral.

Here,

$$\lim_{n\to\infty} \frac{1^a + 2^a + \dots + n^a}{(n+1)^{a-1}\{(na+1)+(na+2)+\dots+(na+n)\}} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\cdot\left[n^2a + \frac{n(n+1)}{2}\right]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{\left(1+\frac{1}{n}\right)^{a-1}\cdot(2na+n+1)} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\left(2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a\right)\cdot \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{a-1}\cdot\left(2a+1+\frac{1}{n}\right)} = \frac{1}{60}$$

$$\Rightarrow 2\int_0^1 (x^a)\,dx \cdot \frac{1}{1\cdot(2a+1)} = \frac{1}{60}$$

$$\Rightarrow \quad \frac{2\cdot[x^{a+1}]_0^1}{(2a+1)\cdot(a+1)} = \frac{1}{60}$$

$$\therefore \quad \frac{2}{(2a+1)(a+1)} = \frac{1}{60} \Rightarrow (2a+1)(a+1) = 120$$

$$\Rightarrow \quad 2a^2 + 3a + 1 - 120 = 0 \quad \Rightarrow \quad 2a^2 + 3a - 119 = 0$$

$$\Rightarrow \quad (2a+17)(a-7) = 0 \quad \Rightarrow \quad a = 7, \frac{-17}{2}$$

5. Given, $S_n = \sum_{k=0}^{n} \frac{n}{n^2 + kn + k^2}$

$$= \sum_{k=0}^{n} \frac{1}{n}\cdot\left(\frac{1}{1+\frac{k}{n}+\frac{k^2}{n^2}}\right) < \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n}\left(\frac{1}{1+\frac{k}{n}+\left(\frac{k}{n}\right)^2}\right)$$

$$= \int_0^1 \frac{1}{1+x+x^2}\,dx$$

$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\cdot\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}} \quad \text{i.e.} \quad S_n < \frac{\pi}{3\sqrt{3}}$$

Similarly, $T_n > \frac{\pi}{3\sqrt{3}}$

6. $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{6n}\right) = \sum_{r=1}^{5n} \frac{1}{n+r}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{5n} \frac{1}{\left(1+\frac{r}{n}\right)}$$

$$= \int_0^5 \frac{dx}{1+x} = [\log(1+x)]_0^5 = \log 6 - \log 1 = \log 6$$