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Complex Numbers

Topic 1 Complex Number in Iota Form

Objective Questions I (Only one correct option)

1 Let $z \in C$ with Im $(z) = 10$ and it satisfies $\dfrac{2z - n}{2z + n} = 2i - 1$ for some natural number n, then **(2019 Main, 12 April II)**

(a) $n = 20$ and $\text{Re}(z) = -10$ (b) $n = 40$ and $\text{Re}(z) = 10$
(c) $n = 40$ and $\text{Re}(z) = -10$ (d) $n = 20$ and $\text{Re}(z) = 10$

2 All the points in the set $S = \left\{\dfrac{\alpha + i}{\alpha - i} : \alpha \in \mathbf{R}\right\} (i = \sqrt{-1})$ lie on a **(2019 Main, 9 April I)**

(a) circle whose radius is $\sqrt{2}$.
(b) straight line whose slope is -1.
(c) circle whose radius is 1.
(d) straight line whose slope is 1.

3 Let $z \in C$ be such that $|z| < 1$. If $\omega = \dfrac{5 + 3z}{5(1 - z)}$, then **(2019 Main, 9 April II)**

(a) $4 \text{ Im}(\omega) > 5$ (b) $5\text{Re}(\omega) > 1$
(c) $5 \text{ Im}(\omega) < 1$ (d) $5\text{Re}(\omega) > 4$

4 Let $\left(-2 - \dfrac{1}{3}i\right)^3 = \dfrac{x + iy}{27}$ $(i = \sqrt{-1})$, where x and y are real numbers, then $y - x$ equals **(2019 Main, 11 Jan I)**

(a) 91 (b) 85 (c) -85 (d) -91

5. Let $A = \left\{\theta \in \left(-\dfrac{\pi}{2}, \pi\right) : \dfrac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \text{ is purely imaginary}\right\}$

Then, the sum of the elements in A is **(2019 Main, 9 Jan I)**

(a) $\dfrac{3\pi}{4}$ (b) $\dfrac{5\pi}{6}$ (c) π (d) $\dfrac{2\pi}{3}$

6. A value of θ for which $\dfrac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary, is **(2016 Main)**

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{6}$ (c) $\sin^{-1}\left(\dfrac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

7. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then **(1998, 2M)**

(a) $x = 3, y = 1$ (b) $x = 1, y = 1$ (c) $x = 0, y = 3$ (d) $x = 0, y = 0$

8. The value of sum $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals **(1998, 2M)**

(a) i (b) $i - 1$ (c) $-i$ (d) 0

9. The smallest positive integer n for which $\left(\dfrac{1 + i}{1 - i}\right)^n = 1$, is **(1980, 2M)**

(a) 8 (b) 16
(c) 12 (d) None of these

Objective Question II

(One or more than one correct option)

10. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\dfrac{az + b}{z + 1}\right) = y$, then which of the following is(are) possible value(s) of x? **(2017 Adv.)**

(a) $1 - \sqrt{1 + y^2}$ (b) $-1 - \sqrt{1 - y^2}$
(c) $1 + \sqrt{1 + y^2}$ (d) $-1 + \sqrt{1 - y^2}$

Topic 2 Conjugate and Modulus of a Complex Number

Objective Questions I (Only one correct option)

1 The equation $|z - i| = |z - 1|$, $i = \sqrt{-1}$, represents **(2019 Main, 12 April I)**

(a) a circle of radius $\dfrac{1}{2}$
(b) the line passing through the origin with slope 1
(c) a circle of radius 1
(d) the line passing through the origin with slope -1

2 If $a > 0$ and $z = \dfrac{(1 + i)^2}{a - i}$, has magnitude $\sqrt{\dfrac{2}{5}}$, then $\bar{z}$ is equal to **(2019 Main, 10 April I)**

(a) $\dfrac{1}{5} - \dfrac{3}{5}i$ (b) $-\dfrac{1}{5} - \dfrac{3}{5}i$
(c) $-\dfrac{1}{5} + \dfrac{3}{5}i$ (d) $-\dfrac{3}{5} - \dfrac{1}{5}i$

3 Let z_1 and z_2 be two complex numbers satisfying $|z_1|=9$ and $|z_2-3-4i|=4$. Then, the minimum value of $|z_1-z_2|$ is **(2019 Main, 12 Jan II)**

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) 0

4 If $\frac{z-\alpha}{z+\alpha}$ $(\alpha \in R)$ is a purely imaginary number and $|z|=2$, then a value of α is **(2019 Main, 12 Jan I)**

(a) $\sqrt{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

5 Let z be a complex number such that $|z|+z=3+i$ (where $i=\sqrt{-1}$).
Then, $|z|$ is equal to **(2019 Main, 11 Jan II)**

(a) $\frac{\sqrt{34}}{3}$ (b) $\frac{5}{3}$ (c) $\frac{\sqrt{41}}{4}$ (d) $\frac{5}{4}$

6. A complex number z is said to be unimodular, if $|z|\neq 1$. If z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular.
Then, the point z_1 lies on a **(2015 Main)**

(a) straight line parallel to X-axis
(b) straight line parallel to Y-axis
(c) circle of radius 2
(d) circle of radius $\sqrt{2}$

7. If z is a complex number such that $|z|\geq 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$ **(2014 Main)**

(a) is equal to 5/2
(b) lies in the interval (1, 2)
(c) is strictly greater than 5/2
(d) is strictly greater than 3/2 but less than 5/2

8. Let complex numbers α and $1/\bar{\alpha}$ lies on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$, respectively.
If $z_0=x_0+iy_0$ satisfies the equation $2|z_0|^2=r^2+2$, then $|\alpha|$ is equal to **(2013 Adv.)**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

9. Let z be a complex number such that the imaginary part of z is non-zero and $a=z^2+z+1$ is real. Then, a cannot take the value **(2012)**

(a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

10. Let $z=x+iy$ be a complex number where, x and y are integers. Then, the area of the rectangle whose vertices are the root of the equation $z\bar{z}^3+\bar{z}z^3=350$, is **(2009)**

(a) 48 (b) 32 (c) 40 (d) 80

11. If $|z|=1$ and $z\neq\pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on **(2007, 3M)**

(a) a line not passing through the origin
(b) $|z|=\sqrt{2}$
(c) the X-axis
(d) the Y-axis

12. If $w=\alpha+i\beta$, where $\beta\neq 0$ and $z\neq 1$, satisfies the condition that $\left(\frac{w-\bar{w}z}{1-z}\right)$ is purely real, then the set of values of z is **(2006, 3M)**

(a) $|z|=1, z\neq 2$ (b) $|z|=1$ and $z\neq 1$
(c) $z=\bar{z}$ (d) None of these

13. If $|z|=1$ and $w=\frac{z-1}{z+1}$ (where, $z\neq -1$), then Re (w) is **(2003, 1M)**

(a) 0 (b) $\frac{1}{|z+1|^2}$ (c) $\left|\frac{1}{z+1}\right|\cdot\frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$

14. For all complex numbers z_1, z_2 satisfying $|z_1|=12$ and $|z_2-3-4i|=5$, the minimum value of $|z_1-z_2|$ is **(2002, 1M)**

(a) 0 (b) 2
(c) 7 (d) 17

15. If z_1, z_2 and z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$, then $|z_1+z_2+z_3|$ is **(2000, 2M)**

(a) equal to 1 (b) less than 1
(c) greater than 3 (d) equal to 3

16. For positive integers n_1, n_2 the value of expression $(1+i)^{n_1}+(1+i^3)^{n_1}+(1+i^5)^{n_2}+(1+i^7)^{n_2}$, here $i=\sqrt{-1}$ is a real number, if and only if **(1996, 2M)**

(a) $n_1=n_2+1$ (b) $n_1=n_2-1$
(c) $n_1=n_2$ (d) $n_1>0, n_2>0$

17. The complex numbers $\sin x+i\cos 2x$ and $\cos x-i\sin 2x$ are conjugate to each other, for **(1988, 2M)**

(a) $x=n\pi$ (b) $x=0$
(c) $x=(n+1/2)\pi$ (d) no value of x

18. The points z_1, z_2, z_3 and z_4 in the complex plane are the vertices of a parallelogram taken in order, if and only if **(1983, 1M)**

(a) $z_1+z_4=z_2+z_3$ (b) $z_1+z_3=z_2+z_4$
(c) $z_1+z_2=z_3+z_4$ (d) None of these

19. If $z=x+iy$ and $w=(1-iz)/(z-i)$, then $|w|=1$ implies that, in the complex plane **(1983, 1M)**

(a) z lies on the imaginary axis (b) z lies on the real axis
(c) z lies on the unit circle (d) None of these

20. The inequality $|z-4|<|z-2|$ represents the region given by **(1982, 2M)**

(a) Re $(z)\geq 0$ (b) Re $(z)<0$
(c) Re $(z)>0$ (d) None of these

21. If $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$, then **(1982, 2M)**

(a) Re $(z)=0$ (b) Im $(z)=0$
(c) Re $(z)>0$, Im $(z)>0$ (d) Re $(z)>0$, Im $(z)<0$

22. The complex numbers $z=x+iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right|=1$, lie on **(1981, 2M)**

(a) the X-axis
(b) the straight line $y=5$
(c) a circle passing through the origin
(d) None of the above

Objective Questions II

(One or more than one correct option)

23. Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y \in R, i = \sqrt{-1})$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE? **(2018 Adv.)**

(a) If L has exactly one element, then $|s| \neq |t|$

(b) If $|s| = |t|$, then L has infinitely many elements

(c) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(d) If L has more than one element, then L has infinitely many elements

24. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\dfrac{z_1 + z_2}{z_1 - z_2}$ may be **(1986, 2M)**

(a) zero

(b) real and positive

(c) real and negative

(d) purely imaginary

25. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1 \bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies **(1985, 2M)**

(a) $|w_1| = 1$ (b) $|w_2| = 1$

(c) $\text{Re}(w_1 \bar{w}_2) = 0$ (d) None of these

Passage Based Problems

Read the following passages and answer the questions that follow.

Passage I

Let A, B, C be three sets of complex number as defined below

$$A = \{z : \text{Im}(z) \geq 1\}$$
$$B = \{z : |z - 2 - i| = 3\}$$
$$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$$

(2008, 12M)

26. $\min_{z \in s} |1 - 3i - z|$ is equal to

(a) $\dfrac{2 - \sqrt{3}}{2}$ (b) $\dfrac{2 + \sqrt{3}}{2}$

(c) $\dfrac{3 - \sqrt{3}}{2}$ (d) $\dfrac{3 + \sqrt{3}}{2}$

27. Area of S is equal to

(a) $\dfrac{10\pi}{3}$ (b) $\dfrac{20\pi}{3}$ (c) $\dfrac{16\pi}{3}$ (d) $\dfrac{32\pi}{3}$

28. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(a) -6 and 3 (b) -3 and 6

(c) -6 and 6 (d) -3 and 9

Passage II

Let $S = S_1 \cap S_2 \cap S_3$, where

$$S_1 = \{z \in C : |z| < 4\}, S_2 = \left\{z \in C : \text{Im}\left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\}$$

and $S_3 : \{z \in C : \text{Re}\, z > 0\}$ **(2008)**

29. Let z be any point in $A \cap B \cap C$.

The $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(a) 25 and 29 (b) 30 and 34

(c) 35 and 39 (d) 40 and 44

30. The number of elements in the set $A \cap B \cap C$ is

(a) 0 (b) 1

(c) 2 (d) ∞

Match the Columns

31. Match the statements of Column I with those of Column II.

Here, z takes values in the complex plane and Im (z) and Re (z) denote respectively, the imaginary part and the real part of z **(2010)**

	Column I		Column II
A.	The set of points z satisfying $\lvert z - i\lvert z\rvert\rvert = \lvert z + i\lvert z\rvert\rvert$ is contained in or equal to	p.	an ellipse with eccentricity 4/5
B.	The set of points z satisfying $\lvert z + 4\rvert + \lvert z - 4\rvert = 0$ is contained in or equal to	q.	the set of points z satisfying Im $(z) = 0$
C.	If $\lvert w\rvert = 2$, then the set of points $z = w - \dfrac{1}{w}$ is contained in or equal to	r.	the set of points z satisfying $\lvert \text{Im}(z)\rvert \leq 1$
D.	If $\lvert w\rvert = 1$, then the set of points $z = w + \dfrac{1}{w}$ is contained in or equal to	s.	the set of points satisfying $\lvert \text{Re}(z)\rvert \leq 2$
		t.	the set of points z satisfying $\lvert z\rvert \leq 3$

Fill in the Blanks

32. If α, β, γ are the cube roots of p, $p < 0$, then for any x, y and z then $\dfrac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = \ldots$. **(1990, 2M)**

33. For any two complex numbers z_1, z_2 and any real numbers a and b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \ldots$. **(1988, 2M)**

34. If the expression $\dfrac{\left[\sin\left(\dfrac{x}{2}\right) + \cos\left(\dfrac{x}{2}\right) - i\tan(x)\right]}{\left[1 + 2i\sin\left(\dfrac{x}{2}\right)\right]}$

is real, then the set of all possible values of x is... . **(1987, 2M)**

True/False

35. If three complex numbers are in AP. Then, they lie on a circle in the complex plane **(1985 M)**

36. If the complex numbers, z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 = 0$. **(1984, 1M)**

37. For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \le x_2$ and $y_1 \le y_2$. Then, for all complex numbers z with $1 \cap z$, we have $\frac{1-z}{1+z} \cap 0$. **(1981, 2M)**

Analytical & Descriptive Questions

38. Find the centre and radius of the circle formed by all the points represented by $z = x + iy$ satisfying the relation $\left|\frac{z-\alpha}{z-\beta}\right| = k\ (k \ne 1)$, where α and β are the constant complex numbers given by $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. **(2004, 2M)**

39. Prove that there exists no complex number z such that $|z| < 1/3$ and $\sum_{r=1}^{n} a_r z^r = 1$, where $|a_r| < 2$. **(2003, 2M)**

40. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$. **(2003, 2M)**

41. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$, if and only if $z = w$ or $z\,\bar{w} = 1$. **(1999, 10M)**

42. Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$. **(1996, 2M)**

43. If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$. **(1995, 5M)**

44. A relation R on the set of complex numbers is defined by $z_1 R z_2$, if and only if $\frac{z_1 - z_2}{z_1 + z_2}$ is real.
Show that R is an equivalence relation. **(1982, 2M)**

45. Find the real values of x and y for which the following equation is satisfied
$$\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i.$$ **(1980, 2M)**

46. Express $\frac{1}{(1-\cos\theta) + 2i\sin\theta}$ in the form $A + iB$. **(1979, 3M)**

47. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$ **(1978, 2M)**

Integer Answer Type Question

48. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the maximum value of $|2z - 6 + 5i|$ is ...... **(2011)**

Topic 3 Argument of a Complex Number

Objective Questions I (Only one correct option)

1. Let z_1 and z_2 be any two non-zero complex numbers such that $3|z_1| = 4|z_2|$. If $z = \frac{3z_1}{2z_2} + \frac{2z_2}{3z_1}$, then **(2019 Main, 10 Jan I)**

(a) $|z| = \frac{1}{2}\sqrt{\frac{17}{2}}$ (b) $\text{Im}(z) = 0$
(c) $\text{Re}(z) = 0$ (d) $|z| = \sqrt{\frac{5}{2}}$

2. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ is equal to **(2013 Main)**

(a) $-\theta$ (b) $\frac{\pi}{2} - \theta$ (c) θ (d) $\pi - \theta$

3. If $\arg(z) < 0$, then $\arg(-z) - \arg(z)$ equals **(2000, 2M)**

(a) π (b) $-\pi$
(c) $-\pi/2$ (d) $\pi/2$

4. Let z and w be two complex numbers such that $|z| \le 1$, $|w| \le 1$ and $|z + iw| = |z - i\bar{w}| = 2$, then z equals **(1995, 2M)**

(a) 1 or i (b) i or $-i$
(c) 1 or -1 (d) i or -1

5. Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$, then z equals **(1995, 2M)**

(a) w (b) $-w$ (c) $\bar{w}$ (d) $-\bar{w}$

6. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to **(1987, 2M)**

(a) $-\pi$ (b) $-\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{2}$

7. If a, b, c and u, v, w are the complex numbers representing the vertices of two triangles such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where r is a complex number, then the two triangles **(1985, 2M)**

(a) have the same area (b) are similar
(c) are congruent (d) None of these

Objective Questions II
(One or more than one correct option)

8. For a non-zero complex number z, let $\arg(z)$ denote the principal argument with $-\pi < \arg(z) \le \pi$. Then, which of the following statement(s) is (are) FALSE ? **(2018 Adv.)**

(a) $\arg(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(b) The function $f: R \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in R$, is continuous at all points of R, where $i = \sqrt{-1}$.

(c) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π.

(d) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$, lies on a straight line.

9. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t) z_1 + t z_2$ for some real number t with $0 < t < 1$. If $\arg(w)$ denotes the principal argument of a non-zero complex number w, then **(2010)**

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ (b) $\arg(z - z_1) = \arg(z - z_2)$

(c) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ (d) $\arg(z - z_1) = \arg(z_2 - z_1)$

Match the Columns

10. Match the conditions/expressions in Column I with statement in Column II ($z \neq 0$ is a complex number)

	Column I		Column II
A.	$\text{Re}(z) = 0$	p.	$\text{Re}(z^2) = 0$
B.	$\arg(z) = \frac{\pi}{4}$	q.	$\text{Im}(z^2) = 0$
		r.	$\text{Re}(z^2) = \text{Im}(z^2)$

Analytical & Descriptive Questions

11. $|z| \leq 1, |w| \leq 1$, then show that
$|z - w|^2 \leq (|z| - |w|)^2 + (\arg z - \arg w)^2$ **(1995, 5M)**

12. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $(z - z_1)/(z - z_2)$ is $\pi/4$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$. **(1991, 4M)**

Topic 4 Rotation of a Complex Number

Objective Questions I (Only one correct option)

1. Let $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$. If $R(z)$ and $I(z)$ respectively denote the real and imaginary parts of z, then **(2019 Main, 10 Jan II)**

(a) $R(z) > 0$ and $I(z) > 0$ (b) $I(z) = 0$
(c) $R(z) < 0$ and $I(z) > 0$ (d) $R(z) = -3$

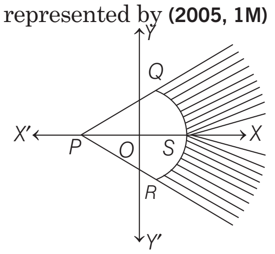
2. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and then it moves through an angle $\frac{\pi}{2}$ in anti-clockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by **(2008, 3M)**

(a) $6 + 7i$ (b) $-7 + 6i$ (c) $7 + 6i$ (d) $-6 + 7i$

3. A man walks a distance of 3 units from the origin towards the North-East (N 45° E) direction. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then, the position of P in the Argand plane is **(2007, 3M)**

(a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)e^{i\pi/4}$ (c) $(4 + 3i)e^{i\pi/4}$ (d) $(3 + 4i)e^{i\pi/4}$

4. The shaded region, where $P = (-1, 0)$, $Q = (-1 + \sqrt{2}, \sqrt{2})$ $R = (-1 + \sqrt{2}, -\sqrt{2})$, $S = (1, 0)$ is represented by **(2005, 1M)**



(a) $|z + 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$

(b) $|z + 1| < 2, |\arg(z + 1)| < \frac{\pi}{2}$

(c) $|z + 1| > 2, |\arg(z + 1)| > \frac{\pi}{4}$

(d) $|z - 1| < 2, |\arg(z + 1)| > \frac{\pi}{2}$

5. If $0 < \alpha < \frac{\pi}{2}$ is a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q = \{\cos(\alpha - \theta), \sin(\alpha - \theta)\}$, then Q is obtained from P by **(2002, 2M)**

(a) clockwise rotation around origin through an angle α
(b) anti-clockwise rotation around origin through an angle α
(c) reflection in the line through origin with slope $\tan\alpha$
(d) reflection in the line through origin with slope $\tan\frac{\alpha}{2}$

6. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is **(2001, 1M)**

(a) of area zero
(b) right angled isosceles
(c) equilateral
(d) obtuse angled isosceles

Objective Questions II
(One or more than one correct option)

7. Let $a, b \in R$ and $a^2 + b^2 \neq 0$.

Suppose $S = \left\{z \in C : z = \frac{1}{a + i\,bt}, t \in R, t \neq 0\right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on **(2016 Adv.)**

(a) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(b) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(c) the X-axis for $a \neq 0, b = 0$

(d) the Y-axis for $a = 0, b \neq 0$

8. Let $W = \frac{\sqrt{3}+i}{2}$ and $P = \{W^n : n = 1,2,3,\ldots\}$.
Further $H_1 = \left\{z \in C : \text{Re}(z) > \frac{1}{2}\right\}$
and $H_2 = \left[z \in C : \text{Re}(z) < \frac{-1}{2}\right]$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2$ is equal to **(2013 JEE Adv.)**

(a) $\frac{\pi}{2}$
(b) $\frac{\pi}{6}$
(c) $\frac{2\pi}{3}$
(d) $\frac{5\pi}{6}$

Fill in the Blanks

9. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then $z_2 = \ldots$, $z_3 = \ldots$. **(1994, 2M)**

10. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1 + i$ and $2 - i$ respectively, then A represents the complex number …or… **(1993, 2M)**

11. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a = \ldots$ and $b = \ldots$. **(1990, 2M)**

Analytical & Descriptive Questions

12. If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}i$. Find the other vertices of square. **(2005, 4M)**

13. Let $\bar{b}z + b\bar{z} = c$, $b \neq 0$, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of the point z_2 through the line, then show that $c = \bar{z}_1 b + z_2 \bar{b}$. **(1997C, 5M)**

14. Let z_1 and z_2 be the roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin prove that $p^2 = 4q \cos^2\left(\frac{\alpha}{2}\right)$. **(1997, 5M)**

15. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that
$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$. **(1986, $2\frac{1}{2}$ M)**

16. Show that the area of the triangle on the argand diagram formed by the complex number z, iz and $z + iz$ is $\frac{1}{2}|z|^2$. **(1986, $2\frac{1}{2}$ M)**

17. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$. **(1983, 2M)**

18. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then, prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$. **(1981, 4M)**

Integer Answer Type Question

19. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$$ is **(2016 Adv.)**

Topic 5 De-Moivre's Theorem, Cube Roots and *n*th Roots of Unity

Objective Questions I (Only one correct option)

1. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then **(2019 Main, 10 April II)**

(a) $\bar{z}w = -i$
(b) $z\bar{w} = \frac{1-i}{\sqrt{2}}$
(c) $\bar{z}w = i$
(d) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$

2. If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ $(i = \sqrt{-1})$, then $(1 + iz + z^5 + iz^8)^9$ is equal to **(2019 Main, 8 April II)**

(a) 1 (b) $(-1 + 2i)^9$ (c) -1 (d) 0

3. Let z_0 be a root of the quadratic equation, $x^2 + x + 1 = 0$, If $z = 3 + 6iz_0^{81} - 3iz_0^{93}$, then arg z is equal to **(2019 Main, 9 Jan II)**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) 0 (d) $\frac{\pi}{3}$

4. Let $z = \cos\theta + i \sin\theta$. Then, the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is **(2009)**

(a) $\frac{1}{\sin 2°}$
(b) $\frac{1}{3 \sin 2°}$
(c) $\frac{1}{2 \sin 2°}$
(d) $\frac{1}{4 \sin 2°}$

5. The minimum value of $|a + b\omega + c\omega^2|$, where a, b and c are all not equal integers and $\omega\ (\neq 1)$ is a cube root of unity, is **(2005, 1M)**

(a) $\sqrt{3}$ (b) $\frac{1}{2}$ (c) 1 (d) 0

6. If $\omega\ (\neq 1)$ be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n is **(2004, 1M)**

(a) 2 (b) 3 (c) 5 (d) 6

7. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$ is **(2002, 1M)**

(a) 3ω (b) $3\omega(\omega - 1)$ (c) $3\omega^2$ (d) $3\omega(1 - \omega)$

8. Let z_1 and z_2 be nth roots of unity which subtend a right angled at the origin, then n must be of the form (where, k is an integer) **(2001, 1M)**

(a) $4k + 1$ (b) $4k + 2$ (c) $4k + 3$ (d) $4k$

9. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to **(1999, 2M)**

(a) $1 - i\sqrt{3}$ (b) $-1 + i\sqrt{3}$ (c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

10. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to **(1998, 2M)**

(a) 128ω (b) -128ω (c) $128\omega^2$ (d) $-128\omega^2$

11. If $\omega\ (\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$, then A and B are respectively **(1995, 2M)**

(a) 0, 1 (b) 1, 1
(c) 1, 0 (d) –1, 1

12. The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is **(1998, 2M)**

(a) – 1 (b) 0 (c) $-i$ (d) i

Match the Columns

13. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, \ldots 9$.

	Column I		Column II
P.	For each z_k, there exists a z_j such that $z_k \cdot z_j = 1$	(i)	True
Q.	There exists a $k \in \{1, 2, \ldots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers	(ii)	False
R.	$\frac{|1 - z_1|\,|1 - z_2| \ldots |1 - z_9|}{10}$ equal	(iii)	1
S.	$1 - \sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	(iv)	2

(2011)

Codes

	P	Q	R	S
(a)	(i)	(ii)	(iv)	(iii)
(b)	(ii)	(i)	(iii)	(iv)
(c)	(i)	(ii)	(iii)	(iv)
(d)	(ii)	(i)	(iv)	(iii)

Fill in the Blanks

14. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex number z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to ... . **(2010)**

15. The value of the expression
$1(2 - \omega)(2 - \omega^2) + 2(3 - \omega)(3 - \omega^2) + \ldots + (n - 1)\cdot(n - \omega)(n - \omega^2)$,
where, ω is an imaginary cube root of unity, is…. **(1996, 2M)**

True/False

16. The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle. **(1988, 1M)**

Analytical & Descriptive Questions

17. Let a complex number α, $\alpha \neq 1$, be a root of the equation

$$z^{p+q} - z^p - z^q + 1 = 0$$

where, p and q are distinct primes. Show that either

$$1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$$

or

$$1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$$

but not both together. **(2002, 5M)**

18. If $1, a_1, a_2, \ldots, a_{n-1}$ are the n roots of unity, then show that $(1 - a_1)(1 - a_2)(1 - a_3)\ldots(1 - a_{n-1}) = n$ **(1984, 2M)**

19. It is given that n is an odd integer greater than 3, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x + 1)^n - x^n - 1$. **(1980, 3M)**

20. If $x = a + b$, $y = a\alpha + b\beta$, $z = a\beta + b\alpha$, where α, β are complex cube roots of unity, then show that $xyz = a^3 + b^3$. **(1979, 3M)**

Integer Answer Type Question

21. Let $\omega = e^{j\pi/3}$ and a, b, c, x, y, z be non-zero complex numbers such that $a + b + c = x$, $a + b\omega + c\omega^2 = y$, $a + b\omega^2 + c\omega = z$.

Then, the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is …… **(2011)**

Answers

Topic 1

1. (c) 2. (c) 3. (b) 4. (a)
5. (d) 6. (d) 7. (d) 8. (b)
9. (d) 10. (b, d)

Topic 2

1. (b) 2. (b) 3. (d) 4. (d)
5. (b) 6. (c) 7. (b) 8. (c)
9. (d) 10. (a) 11. (d) 12. (b)
13. (a) 14. (b) 15. (a) 16. (d)
17. (d) 18. (b) 19. (b) 20. (d)
21. (b) 22. (a) 23. (a, c, d) 24. (a,d)
25. (a, b, c) 26. (c) 27. (b) 28. (d)
29. (c) 30. (b)
31. A $\to$ q, r ; B $\to$ p; C $\to$ p, s, t ; D $\to$ q, r, s, t 32. ω^2
33. $(a^2+b^2)(|z_1|^2+|z_2|^2)$
34. $x=2n\pi+2\alpha$, $\alpha=\tan^{-1}k$, where $k\in(1,2)$ or $x=2n\pi$
35. False 36. True 37. True
38. Centre $=\dfrac{\alpha-k^2\beta}{1-k^2}$, Radius $=\left|\dfrac{k(\alpha-\beta)}{1-k^2}\right|$
42. $\left[z=i,\pm\dfrac{\sqrt{3}}{2}-\dfrac{i}{2}\right]$
45. $(x=3$ and $y=-1)$
46. $A+iB=\dfrac{1}{2\left(1+3\cos^2\dfrac{\theta}{2}\right)}-i\dfrac{\cot(\theta/2)}{1+3\cos^2(\theta/2)}$ 48. 5

Topic 3

1. (*) 2. (c) 3. (a)
4. (c) 5. (d) 6. (c) 7. (b)
8. (a, b, d) 9. (a, c, d) 10. A $\to$ q ; B $\to$ p

Topic 4

1. (b) 2. (d) 3. (d) 4. (a)
5. (d) 6. (c) 7. (d) 8. (c, d)
9. $z_2=-2, z_3=1-i\sqrt{3}$ 10. $3-\dfrac{i}{2}$ or $1-\dfrac{3i}{2}$
11. $a=b=2\pm\sqrt{3}$
12. $z_2=-\sqrt{3}\,i, z_3=(1-\sqrt{3})+i$ and $z_4=(1+\sqrt{3})-i$
19. (4)

Topic 5

1. (a) 2. (c) 3. (a)
4. (d) 5. (c) 6. (b)
7. (b) 8. (d) 9. (c) 10. (d)
11. (b) 12. (d) 13. (c) 14. (1)
15. $\left(\dfrac{n(n+1)}{2}\right)^2-n$ 16. True 21. (3)

Hints & Solutions

Topic 1 Complex Number in Iota Form

1. Let $z=x+10i$, as $\text{Im}(z)=10$ (given).

Since z satisfies,

$$\frac{2z-n}{2z+n}=2i-1, n\in N,$$

$\therefore\ (2x+20i-n)=(2i-1)(2x+20i+n)$

$\Rightarrow (2x-n)+20i=(-2x-n-40)+(4x+2n-20)i$

On comparing real and imaginary parts, we get

$2x-n=-2x-n-40$ and $20=4x+2n-20$

$\Rightarrow\quad 4x=-40$ and $4x+2n=40$

$\Rightarrow\quad x=-10$ and $-40+2n=40\Rightarrow n=40$

So, $\quad n=40$ and $x=\text{Re}(z)=-10$

2. Let $x+iy=\dfrac{\alpha+i}{\alpha-i}$

$$\Rightarrow x+iy=\frac{(\alpha+i)^2}{\alpha^2+1}=\frac{(\alpha^2-1)+(2\alpha)i}{\alpha^2+1}=\frac{\alpha^2-1}{\alpha^2+1}+\left(\frac{2\alpha}{\alpha^2+1}\right)i$$

On comparing real and imaginary parts, we get

$$x=\frac{\alpha^2-1}{\alpha^2+1} \text{ and } y=\frac{2\alpha}{\alpha^2+1}$$

Now, $x^2+y^2=\left(\dfrac{\alpha^2-1}{\alpha^2+1}\right)^2+\left(\dfrac{2\alpha}{\alpha^2+1}\right)^2$

$$=\frac{\alpha^4+1-2\alpha^2+4\alpha^2}{(\alpha^2+1)^2}=\frac{(\alpha^2+1)^2}{(\alpha^2+1)^2}=1$$

$\Rightarrow\quad x^2+y^2=1$

Which is an equation of circle with centre (0, 0) and radius 1 unit.

So, $S=\left\{\dfrac{\alpha+i}{\alpha-i};\alpha\in\mathbf{R}\right\}$ lies on a circle with radius 1.

3. Given complex number

$$\omega=\frac{5+3z}{5(1-z)}$$

$\Rightarrow\quad 5\omega-5\omega z=5+3z$

$\Rightarrow\quad (3+5\omega)z=5\omega-5$

$\Rightarrow\quad |3+5\omega||z|=|5\omega-5|$ ...(i)

[applying modulus both sides and $|z_1z_2|=|z_1||z_2|$]

$\because\quad |z|<1$

$\therefore\ |3+5\omega|>|5\omega-5|$ [from Eq. (i)]

$\Rightarrow \quad \left|\omega + \frac{3}{5}\right| > |\omega - 1|$

Let $\omega = x + iy$, then $\left(x + \frac{3}{5}\right)^2 + y^2 > (x-1)^2 + y^2$

$\Rightarrow \quad x^2 + \frac{9}{25} + \frac{6}{5}x > x^2 + 1 - 2x$

$\Rightarrow \quad \frac{16x}{5} > \frac{16}{25} \Rightarrow x > \frac{1}{5} \Rightarrow 5x > 1$

$\Rightarrow \quad 5\ \text{Re}(\omega) > 1$

4. We have, $\frac{x+iy}{27} = \left(-2 - \frac{1}{3}i\right)^3 = \left[\frac{-1}{3}(6+i)\right]^3$

$\Rightarrow \frac{x+iy}{27} = -\frac{1}{27}(216 + 108i + 18i^2 + i^3)$

$= -\frac{1}{27}(198 + 107i)$

$[\because (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2,\ i^2 = -1,\ i^3 = -i]$

On equating real and imaginary part, we get

$x = -198$ and $y = -107$

$\Rightarrow y - x = -107 + 198 = 91$

5. Let $z = \left(\frac{3+2i\sin\theta}{1-2i\sin\theta}\right) \times \left(\frac{1+2i\sin\theta}{1+2i\sin\theta}\right)$

(rationalising the denominator)

$= \frac{3 - 4\sin^2\theta + 8i\sin\theta}{1 + 4\sin^2\theta}$

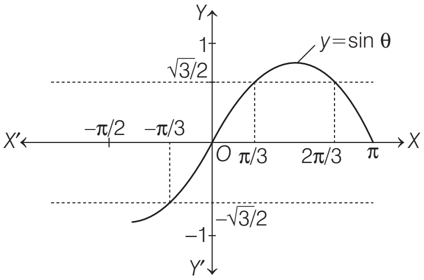
$[\because a^2 - b^2 = (a+b)(a-b)$ and $i^2 = -1]$

$= \left(\frac{3-4\sin^2\theta}{1+4\sin^2\theta}\right) + \left(\frac{8\sin\theta}{1+4\sin^2\theta}\right)i$

As z is purely imaginary, so real part of $z = 0$

$\therefore \frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0 \Rightarrow 3 - 4\sin^2\theta = 0$

$\Rightarrow \quad \sin^2\theta = \frac{3}{4} \Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2}$



$\Rightarrow \quad \theta \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}\right\}$

Sum of values of $\theta = \frac{2\pi}{3}$.

6. Let $z = \frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary. Then, we have

$\text{Re}(z) = 0$

Now, consider $z = \frac{2+3i\sin\theta}{1-2i\sin\theta}$

$= \frac{(2+3i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$

$= \frac{2 + 4i\sin\theta + 3i\sin\theta + 6i^2\sin^2\theta}{1^2 - (2i\sin\theta)^2}$

$= \frac{2 + 7i\sin\theta - 6\sin^2\theta}{1 + 4\sin^2\theta}$

$= \frac{2 - 6\sin^2\theta}{1+4\sin^2\theta} + i\,\frac{7\sin\theta}{1+4\sin^2\theta}$

$\because \quad \text{Re}(z) = 0$

$\therefore \quad \frac{2-6\sin^2\theta}{1+4\sin^2\theta} = 0 \Rightarrow 2 = 6\sin^2\theta$

$\Rightarrow \quad \sin^2\theta = \frac{1}{3}$

$\Rightarrow \quad \sin\theta = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \sin^{-1}\left(\pm\frac{1}{\sqrt{3}}\right) = \pm\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

7. Given, $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$

$\Rightarrow \quad -3i\begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = x + iy$

$\Rightarrow \quad x + iy = 0$ $[\because C_2$ and C_3 are identical$]$

$\Rightarrow \quad x = 0,\ y = 0$

8. $\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13} i^n(1+i) = (1+i)\sum_{n=1}^{13} i^n$

$= (1+i)(i + i^2 + i^3 + \dots + i^{13}) = (1+i)\left[\frac{i - (1 - i^{13})}{1-i}\right]$

$= (1+i)\left[\frac{i(1-i)}{1-i}\right] = (1+i)\,i = i - 1$

Alternate Solution

Since, sum of any four consecutive powers of iota is zero.

$\therefore \sum_{n=1}^{13}(i^n + i^{n+1}) = (i + i^2 + \dots + i^{13})$

$+ (i^2 + i^3 + \dots + i^{14}) = i + i^2 = i - 1$

9. Since, $\left(\frac{1+i}{1-i}\right)^n = 1 \quad \Rightarrow \quad \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n = 1$

$\Rightarrow \quad \left(\frac{2i}{2}\right)^n = 1$

$\Rightarrow \quad i^n = 1$

The smallest positive integer n for which $i^n = 1$ is 4.

$\therefore \quad n = 4$

10. $\frac{az+b}{z+1} = \frac{ax + b + aiy}{(x+1) + iy} = \frac{(ax + b + aiy)((x+1) - iy)}{(x+1)^2 + y^2}$

$\therefore \quad \text{Im}\left(\frac{az+b}{z+1}\right) = \frac{-(ax+b)y + ay(x+1)}{(x+1)^2 + y^2}$

$\Rightarrow \quad \dfrac{(a-b)y}{(x+1)^2+y^2} = y$

$\because \quad a-b=1$

$\therefore \quad (x+1)^2+y^2=1$

$\therefore \quad x=-1\pm\sqrt{1-y^2}$

Topic 2 Conjugate and Modulus of Complex Number

1. Let the complex number $z = x + iy$

Also given, $|z-i| = |z-1|$

$\Rightarrow |x+iy-i| = |x+iy-1|$

$\Rightarrow \sqrt{x^2+(y-1)^2} = \sqrt{(x-1)^2+y^2}$

$[\because |z| = \sqrt{(\text{Re}(z))^2+(\text{Im}(z))^2}]$

On squaring both sides, we get

$x^2+y^2-2y+1 = x^2+y^2-2x+1$

$\Rightarrow y = x$, which represents a line through the origin with slope 1.

2. The given complex number $z = \dfrac{(1+i)^2}{a-i}$

$= \dfrac{(1-1+2i)(a+i)}{a^2+1}$ $\quad [\because i^2=-1]$

$= \dfrac{2i(a+i)}{a^2+1} = \dfrac{-2+2ai}{a^2+1}$ $\quad \ldots$(i)

$\because \quad |z| = \sqrt{2/5}$ $\quad$ [given]

$\Rightarrow \quad \sqrt{\dfrac{4+4a^2}{(a^2+1)^2}} = \sqrt{\dfrac{2}{5}} \Rightarrow \dfrac{2}{\sqrt{1+a^2}} = \sqrt{\dfrac{2}{5}}$

$\Rightarrow \quad \dfrac{4}{1+a^2} = \dfrac{2}{5} \Rightarrow a^2+1=10$

$\Rightarrow \quad a^2 = 9 \Rightarrow a = 3$ $\quad [\because a>0]$

$\therefore \quad z = \dfrac{-2+6i}{10}$ $\quad$ [From Eq. (i)]

So, $\bar{z} = \left(\dfrac{-2+6i}{10}\right) = \left(-\dfrac{1}{5}+\dfrac{3}{5}i\right) \Rightarrow \bar{z} = -\dfrac{1}{5}-\dfrac{3}{5}i$

$[\because$ if $z = x+iy$, then $\bar{z} = x-iy]$

3. Clearly $|z_1| = 9$, represents a circle having centre $C_1(0,0)$ and radius $r_1 = 9$.

and $|z_2-3-4i| = 4$ represents a circle having centre $C_2(3,4)$ and radius $r_2 = 4$.

The minimum value of $|z_1-z_2|$ is equals to minimum distance between circles $|z_1| = 9$ and $|z_2-3-4i| = 4$.

$\because C_1C_2 = \sqrt{(3-0)^2+(4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$

and $|r_1-r_2| = |9-4| = 5 \Rightarrow C_1C_2 = |r_1-r_2|$

$\therefore$ Circles touches each other internally.

Hence, $\quad |z_1-z_2|_{\min} = 0$

4. Since, the complex number $\dfrac{z-\alpha}{z+\alpha}$ $(\alpha \in R)$ is purely imaginary number, therefore

$\dfrac{z-\alpha}{z+\alpha} + \dfrac{\bar{z}-\alpha}{\bar{z}+\alpha} = 0$ $\quad [\because \alpha \in R]$

$\Rightarrow \quad z\bar{z} - \alpha\bar{z} + \alpha z - \alpha^2 + z\bar{z} - \alpha z + \alpha\bar{z} - \alpha^2 = 0$

$\Rightarrow \quad 2|z|^2 - 2\alpha^2 = 0$ $\quad [\because z\bar{z} = |z|^2]$

$\Rightarrow \quad \alpha^2 = |z|^2 = 4$ $\quad [|z| = 2$ given$]$

$\Rightarrow \quad \alpha = \pm 2$

5. We have, $|z| + z = 3+i$

Let $\quad z = x+iy$

$\therefore \quad \sqrt{x^2+y^2} + x + iy = 3+i$

$\Rightarrow \quad (x+\sqrt{x^2+y^2}) + iy = 3+i$

$\Rightarrow \quad x+\sqrt{x^2+y^2} = 3$ and $y = 1$

Now, $\quad \sqrt{x^2+1} = 3-x$

$\Rightarrow \quad x^2+1 = 9-6x+x^2$

$\Rightarrow \quad 6x = 8 \Rightarrow x = \dfrac{4}{3}$

$\therefore \quad z = \dfrac{4}{3}+i$

$\Rightarrow \quad |z| = \sqrt{\dfrac{16}{9}+1} = \sqrt{\dfrac{25}{9}} \Rightarrow |z| = \dfrac{5}{3}$

6. **PLAN** If z is unimodular, then $|z| = 1$. Also, use property of modulus i.e. $z\bar{z} = |z|^2$

Given, z_2 is not unimodular i.e. $|z_2| \neq 1$

and $\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular.

$\Rightarrow \quad \left|\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}\right| = 1 \Rightarrow |z_1-2z_2|^2 = |2-z_1\bar{z}_2|^2$

$\Rightarrow (z_1-2z_2)(\bar{z}_1-2\bar{z}_2) = (2-z_1\bar{z}_2)(2-\bar{z}_1z_2)$ $\quad [z\bar{z} = |z|^2]$

$\Rightarrow |z_1|^2+4|z_2|^2-2\bar{z}_1z_2-2z_1\bar{z}_2$

$= 4+|z_1|^2|z_2|^2-2\bar{z}_1z_2-2z_1\bar{z}_2 \Rightarrow (|z_2|^2-1)(|z_1|^2-4) = 0$

$\because \quad |z_2| \neq 1$

$\therefore \quad |z_1| = 2$

Let $\quad z_1 = x+iy \Rightarrow x^2+y^2 = (2)^2$

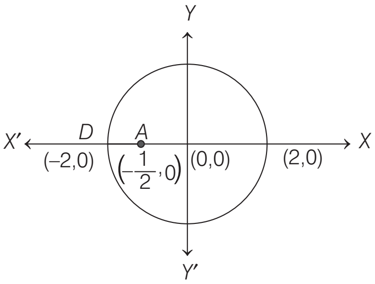
$\therefore$ Point z_1 lies on a circle of radius 2.

7. $|z| \geq 2$ is the region on or outside circle whose centre is (0, 0) and radius is 2.

Minimum $\left|z+\dfrac{1}{2}\right|$ is distance of z, which lie on circle $|z| = 2$ from $(-1/2, 0)$.

$\therefore$ Minimum $\left|z+\dfrac{1}{2}\right| =$ Distance of $\left(-\dfrac{1}{2}, 0\right)$ from $(-2, 0)$

$= \sqrt{\left(-2+\dfrac{1}{2}\right)^2+0} = \dfrac{3}{2} = \sqrt{\left(\dfrac{-1}{2}+2\right)^2+0} = \dfrac{3}{2}$



Geometrically Min $\left| z+\frac{1}{2} \right| = AD$

Hence, minimum value of $\left| z+\frac{1}{2} \right|$ lies in the interval (1, 2).

8. **PLAN** Intersection of circles, the basic concept is to solve the equations simultaneously and using properties of modulus of complex numbers.

Formula used $|z|^2 = z \cdot \bar{z}$

and $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= |z_1|^2 - z_1\bar{z}_2 - z_2\bar{z}_1 + |z_2|^2$

Here, $(x - x_0)^2 + (y - y_0)^2 = r^2$

and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$ can be written as,

$$|z - z_0|^2 = r^2 \text{ and } |z - z_0|^2 = 4r^2$$

Since, α and $\frac{1}{\bar{\alpha}}$ lies on first and second respectively.

$\therefore$ $|\alpha - z_0|^2 = r^2$ and $\left|\frac{1}{\bar{\alpha}} - z_0\right|^2 = 4r^2$

$\Rightarrow$ $(\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = r^2$

$\Rightarrow$ $|\alpha|^2 - z_0\bar{\alpha} - \bar{z}_0\alpha + |z_0|^2 = r^2$ ...(i)

and $\left|\frac{1}{\bar{\alpha}} - z_0\right|^2 = 4r^2$

$\Rightarrow$ $\left(\frac{1}{\bar{\alpha}} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 4r^2$

$\Rightarrow$ $\frac{1}{|\alpha|^2} - \frac{z_0}{\alpha} - \frac{\bar{z}_0}{\bar{\alpha}} + |z_0|^2 = 4r^2$

Since, $|\alpha|^2 = \alpha \cdot \alpha$

$\Rightarrow$ $\frac{1}{|\alpha|^2} - \frac{z_0 \cdot \bar{\alpha}}{|\alpha|^2} - \frac{\bar{z}_0}{|\alpha|^2} \cdot \alpha + |z_0|^2 = 4r^2$

$\Rightarrow$ $1 - z_0\bar{\alpha} - \bar{z}_0\alpha + |\alpha|^2 |z_0|^2 = 4r^2|\alpha|^2$ ...(ii)

On subtracting Eqs. (i) and (ii), we get

$$(|\alpha|^2 - 1) + |z_0|^2(1 - |\alpha|^2) = r^2(1 - 4|\alpha|^2)$$

$\Rightarrow$ $(|\alpha|^2 - 1)(1 - |z_0|^2) = r^2(1 - 4|\alpha|^2)$

$\Rightarrow$ $(|\alpha|^2 - 1)\left(1 - \frac{r^2+2}{2}\right) = r^2(1 - 4|\alpha|^2)$

Given, $|z_0|^2 = \frac{r^2+2}{2}$

$\Rightarrow$ $(|\alpha|^2 - 1)\cdot\left(\frac{-r^2}{2}\right) = r^2(1 - 4|\alpha|^2)$

$\Rightarrow$ $|\alpha|^2 - 1 = -2 + 8|\alpha|^2$

$\Rightarrow$ $7|\alpha|^2 = 1$

$\therefore$ $|\alpha| = 1/\sqrt{7}$

9. **PLAN** If $ax^2 + bx + c = 0$ has roots α, β, then

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For roots to be real $b^2 - 4ac \geq 0$.

Description of Situation As imaginary part of $z = x + iy$ is non-zero.

$\Rightarrow$ $y \neq 0$

Method I Let $z = x + iy$

$\therefore$ $a = (x+iy)^2 + (x+iy) + 1$

$\Rightarrow$ $(x^2 - y^2 + x + 1 - a) + i(2xy + y) = 0$

$\Rightarrow$ $(x^2 - y^2 + x + 1 - a) + iy(2x+1) = 0,$ ...(i)

It is purely real, if $y(2x+1) = 0$

but imaginary part of z, i.e. y is non-zero.

$\Rightarrow$ $2x + 1 = 0$ or $x = -1/2$

From Eq. (i), $\frac{1}{4} - y^2 - \frac{1}{2} + 1 - a = 0$

$\Rightarrow$ $a = -y^2 + \frac{3}{4} \Rightarrow a < \frac{3}{4}$

Method II Here, $z^2 + z + (1-a) = 0$

$\therefore$ $z = \frac{-1 \pm \sqrt{1 - 4(1-a)}}{2 \times 1}$

$\Rightarrow$ $z = \frac{-1 \pm \sqrt{4a - 3}}{2}$

For z do not have real roots, $4a - 3 < 0 \Rightarrow a < \frac{3}{4}$

10. Since, $z\bar{z}(z^2 + \bar{z}^2) = 350$

$\Rightarrow$ $2(x^2 + y^2)(x^2 - y^2) = 350$

$\Rightarrow$ $(x^2 + y^2)(x^2 - y^2) = 175$

Since, $x, y \in I$, the only possible case which gives integral solution, is

$x^2 + y^2 = 25$ ... (i)

$x^2 - y^2 = 7$ ... (ii)

From Eqs. (i) and (ii),

$$x^2 = 16, y^2 = 9 \Rightarrow x = \pm 4, y = \pm 3$$

$\therefore$ Area of rectangle $= 8 \times 6 = 48$

11. Let $z = \cos\theta + i\sin\theta$

$\Rightarrow$ $\frac{z}{1-z^2} = \frac{\cos\theta + i\sin\theta}{1 - (\cos 2\theta + i\sin 2\theta)}$

$= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - 2i\sin\theta\cos\theta}$

$= \frac{\cos\theta + i\sin\theta}{-2i\sin\theta(\cos\theta + i\sin\theta)} = \frac{i}{2\sin\theta}$

Hence, $\frac{z}{1-z^2}$ lies on the imaginary axis i.e. Y-axis.

Alternate Solution

Let $E = \dfrac{z}{1-z^2} = \dfrac{z}{z\bar{z}-z^2} = \dfrac{1}{\bar{z}-z}$ which is an imaginary.

12. Let $z_1 = \dfrac{w - \overline{w}z}{1-z}$ be purely real $\Rightarrow \quad z_1 = \bar{z}_1$

$\therefore \quad \dfrac{w-\overline{w}z}{1-z} = \dfrac{\overline{w}-w\bar{z}}{1-\bar{z}}$

$\Rightarrow \quad w - w\bar{z} - \overline{w}z + \overline{w}z\cdot\bar{z} = \overline{w} - z\overline{w} - w\bar{z} + wz\cdot\bar{z}$

$\Rightarrow \quad (w-\overline{w}) + (\overline{w}-w)|z|^2 = 0$

$\Rightarrow \quad (w-\overline{w})(1-|z|^2) = 0$

$\Rightarrow \quad |z|^2 = 1 \quad$ [as $w - \overline{w} \neq 0$, since $\beta \neq 0$]

$\Rightarrow \quad |z| = 1$ and $z \neq 1$

13. Since, $|z| = 1$ and $w = \dfrac{z-1}{z+1}$

$\Rightarrow \quad z - 1 = wz + w \quad \Rightarrow \quad z = \dfrac{1+w}{1-w} \quad \Rightarrow \quad |z| = \dfrac{|1+w|}{|1-w|}$

$\Rightarrow \quad |1-w| = |1+w| \qquad [\because |z| = 1]$

On squaring both sides, we get

$1 + |w|^2 - 2|w|\,\text{Re}(w) = 1 + |w|^2 + 2|w|\,\text{Re}(w)$

[using $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2|z_1||z_2|\,\text{Re}(\bar{z}_1 z_2)$]

$\Rightarrow \quad 4|w|\,\text{Re}\,|w| = 0$

$\Rightarrow \quad \text{Re}(w) = 0$

14. We know, $|z_1 - z_2| = |z_1 - (z_2 - 3 - 4i) - (3 + 4i)|$

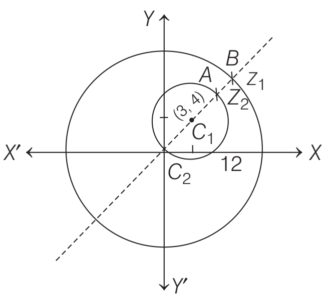
$\geq |z_1| - |z_2 - 3 - 4i| - |3 + 4i|$

$\geq 12 - 5 - 5 \quad$ [using $|z_1 - z_2| \geq |z_1| - |z_2|$]

$\therefore \quad |z_1 - z_2| \geq 2$

Alternate Solution

Clearly from the figure $|z_1 - z_2|$ is minimum when z_1, z_2 lie along the diameter.



$\therefore \quad |z_1 - z_2| \geq C_2B - C_2A \geq 12 - 10 = 2$

15. Given, $|z_1| = |z_2| = |z_3| = 1$

Now, $|z_1| = 1$

$\Rightarrow \quad |z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1$

Similarly, $z_2\bar{z}_2 = 1, z_3\bar{z}_3 = 1$

Again now, $\left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$

$\Rightarrow \quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1 \Rightarrow |\overline{z_1 + z_2 + z_3}| = 1$

$\Rightarrow \quad |z_1 + z_2 + z_3| = 1$

16. $(1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$

$= [{}^{n_1}C_0 + {}^{n_1}C_1 i + {}^{n_1}C_2 i^2 + {}^{n_1}C_3 i^3 + \dots]$
$+ [{}^{n_1}C_0 - {}^{n_1}C_1 i + {}^{n_1}C_2 i^2 - {}^{n_1}C_3 i^3 + \dots]$
$+ [{}^{n_2}C_0 + {}^{n_2}C_1 i + {}^{n_2}C_2 i^2 + {}^{n_2}C_3 i^3 + \dots]$
$+ [{}^{n_2}C_0 - {}^{n_2}C_1 i + {}^{n_2}C_2 i^2 - {}^{n_2}C_3 i^3 + ..]$

$= 2[{}^{n_1}C_0 + {}^{n_1}C_2 i^2 + {}^{n_1}C_4 i^4 + \dots]$
$+ 2[{}^{n_2}C_0 + {}^{n_2}C_2 i^2 + {}^{n_2}C_4 i^4 + \dots]$

$= 2[{}^{n_1}C_0 - {}^{n_1}C_2 + {}^{n_1}C_4 - \dots] + 2[{}^{n_2}C_0 - {}^{n_2}C_2 + {}^{n_2}C_4 - \dots]$

This is a real number irrespective of the values of n_1 and n_2.

Alternate Solution

$\{(1+i)^{n_1} + (1-i)^{n_1}\} + \{(1+i)^{n_2} + (1-i)^{n_2}\}$

$\Rightarrow$ A real number for all n_1 and $n_2 \in R$.

[$\because z + \bar{z} = 2\,\text{Re}(z) \Rightarrow (1+i)^{n_1} + (1-i)^{n_1}$ is real number for all $n \in R$]

17. Since, $\overline{(\sin x + i\cos 2x)} = \cos x - i\sin 2x$

$\Rightarrow \quad \sin x - i\cos 2x = \cos x - i\sin 2x$

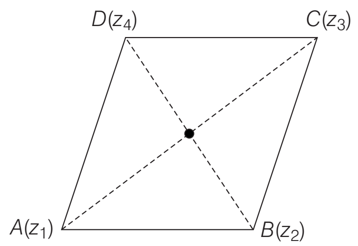
$\Rightarrow \quad \sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow \quad \tan x = 1$ and $\tan 2x = 1$

$\Rightarrow \quad x = \pi/4$ and $x = \pi/8$ which is not possible at same time.

Hence, no solution exists.

18. Since, z_1, z_2, z_3, z_4 are the vertices of parallelogram.



$\therefore$ Mid-point of AC = mid-point of BD

$\Rightarrow \quad \dfrac{z_1 + z_3}{2} = \dfrac{z_2 + z_4}{2}$

$\Rightarrow \quad z_1 + z_3 = z_2 + z_4$

19. Since, $|w| = 1 \Rightarrow \left|\dfrac{1-iz}{z-i}\right| = 1$

$\Rightarrow \quad |z - i| = |1 - iz|$

$\Rightarrow \quad |z - i| = |z + i| \qquad [\because |1 - iz| = |-i||z + i| = |z + i|]$

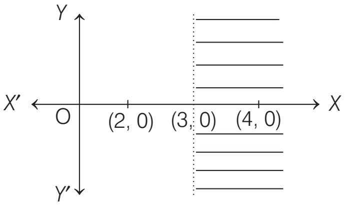
$\therefore$ It is a perpendicular bisector of $(0, 1)$ and $(0, -1)$ i.e. X-axis. Thus, z lies on the real axis.

20. Given, $|z - 4| < |z - 2|$

Since, $|z - z_1| > |z - z_2|$ represents the region on right side of perpendicular bisector of z_1 and z_2.

$\therefore \quad |z - 2| > |z - 4|$

$\Rightarrow \quad \text{Re}(z) > 3$ and $\text{Im}(z) \in R$



21. Given, $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$

$$\left[\because \omega = \frac{-1 + i\sqrt{3}}{2} \text{ and } \omega^2 = \frac{-1 - i\sqrt{3}}{2}\right]$$

Now, $\frac{\sqrt{3} + i}{2} = -i\left(\frac{-1 + i\sqrt{3}}{2}\right) = -i\omega$

and $\frac{\sqrt{3} - i}{2} = i\left(\frac{-1 - i\sqrt{3}}{2}\right) = i\omega^2$

$\therefore \quad z = (-i\omega)^5 + (i\omega^2)^5 = -i\omega^2 + i\omega$

$= i(\omega - \omega^2) = i(i\sqrt{3}) = -\sqrt{3}$

$\Rightarrow \quad \text{Re}(z) < 0$ and $\text{Im}(z) = 0$

Alternate Solution

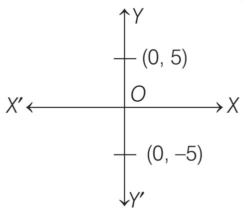
We know that, $z + \bar{z} = 2\,\text{Re}(z)$

If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then

z is purely real. i.e. Im $(z) = 0$

22. Given, $\left|\frac{z - 5i}{z + 5i}\right| = 1 \Rightarrow |z - 5i| = |z + 5i|$

[$\because$ if $|z - z_1| = |z - z_2|$, then it is a perpendicular bisector of z_1 and z_2]



$\therefore$ Perpendicular bisector of (0, 5) and (0, – 5) is X-axis.

23. We have,

$$sz + t\bar{z} + r = 0 \quad \ldots\text{(i)}$$

On taking conjugate

$$\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0 \quad \ldots\text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$z = \frac{\bar{r}t - r\bar{s}}{|s|^2 - |t|^2}$$

(a) For unique solutions of z

$|s|^2 - |t|^2 \neq 0 \Rightarrow |s| \neq |t|$

It is true

(b) If $|s| = |t|$, then $\bar{r}t - r\bar{s}$ may or may not be zero.

So, z may have no solutions.

$\therefore L$ may be an empty set.

It is false.

(c) If elements of set L represents line, then this line and given circle intersect at maximum two point. Hence, it is true.

(d) In this case locus of z is a line, so L has infinite elements. Hence, it is true.

24. Given, $|z_1| = |z_2|$

Now, $\frac{z_1 + z_2}{z_1 - z_2} \times \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_1\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_1 - z_2\bar{z}_2}{|z_1 - z_2|^2}$

$= \frac{|z_1|^2 + (z_2\bar{z}_1 - z_1\bar{z}_2) - |z_2|^2}{|z_1 - z_2|^2}$

$= \frac{z_2\bar{z}_1 - z_1\bar{z}_2}{|z_1 - z_2|^2} \quad [\because |z_1|^2 = |z_2|^2]$

As, we know $z - \bar{z} = 2i\,\text{Im}(z)$

$\therefore \quad z_2\bar{z}_1 - z_1\bar{z}_2 = 2i\,\text{Im}(z_2\bar{z}_1)$

$\therefore \quad \frac{z_1 + z_2}{z_1 - z_2} = \frac{2i\,\text{Im}(z_2\bar{z}_1)}{|z_1 - z_2|^2}$

which is purely imaginary or zero.

25. Since, $z_1 = a + ib$ and $z_2 = c + id$

$\Rightarrow \quad |z_1|^2 = a^2 + b^2 = 1$ and $|z_2|^2 = c^2 + d^2 = 1 \quad \ldots\text{(i)}$

$[\because |z_1| = |z_2| = 1]$

Also, Re $(z_1\bar{z}_2) = 0 \Rightarrow ac + bd = 0$

$\Rightarrow \quad \frac{a}{b} = -\frac{d}{c} = \lambda \quad$ [say]…(ii)

From Eqs. (i) and (ii), $b^2\lambda^2 + b^2 = c^2 + \lambda^2c^2$

$\Rightarrow \quad b^2 = c^2$ and $a^2 = d^2$

Also, given $w_1 = a + ic$ and $w_2 = b + id$

Now, $|w_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$

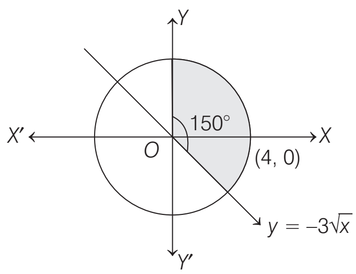
$|w_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$

and $\text{Re}(w_1\overline{w_2}) = ab + cd = (b\lambda)b + c(-\lambda c)$ [from Eq. (i)]

$= \lambda(b^2 - c^2) = 0$

26. $\min_{Z \in S} |1 - 3i - z| =$ perpendicular distance of point (1, – 3) from the line $\sqrt{3}x + y = 0 \Rightarrow \frac{|\sqrt{3} - 3|}{\sqrt{3 + 1}} = \frac{3 - \sqrt{3}}{2}$

27. Since, $S = S_1 \cap S_2 \cap S_3$



Clearly, the shaded region represents the area of sector

$\therefore \quad S = \frac{1}{2} r^2 \theta = \frac{1}{2} \times 4^2 \times \frac{5\pi}{6} = \frac{20\pi}{3}$

28. Since, $\quad |w - (2+i)| < 3 \Rightarrow |w| - |2+i| < 3$

$\Rightarrow \quad -3 + \sqrt{5} < |w| < 3 + \sqrt{5}$

$\Rightarrow \quad -3 - \sqrt{5} < -|w| < 3 - \sqrt{5} \qquad \ldots(i)$

Also, $\quad |z - (2+i)| = 3$

$\Rightarrow \quad -3 + \sqrt{5} \le |z| \le 3 + \sqrt{5} \qquad \ldots(ii)$

$\therefore \quad -3 < |z| - |w| + 3 < 9$

29. $|z + 1 - i|^2 + |z - 5 - i|^2$

$= (x+1)^2 + (y-1)^2 + (x-5)^2 + (y-1)^2$

$= 2(x^2 + y^2 - 4x - 2y) + 28$

$= 2(4) + 28 = 36 \qquad [\because x^2 + y^2 - 4x - 2y = 4]$

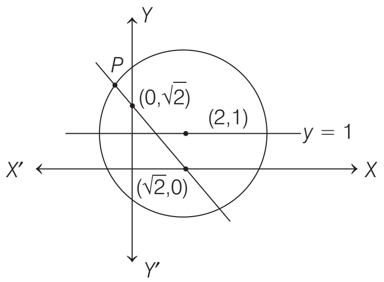
30. Let $z = x + iy$

Set A corresponds to the region $y \ge 1 \qquad \ldots(i)$

Set B consists of points lying on the circle, centre at (2, 1) and radius 3.

i.e. $\quad x^2 + y^2 - 4x - 2y = 4 \qquad \ldots(ii)$

Set C consists of points lying on the $x + y = \sqrt{2} \qquad \ldots(iii)$



Clearly, there is only one point of intersection of the line $x + y = \sqrt{2}$ and circle $x^2 + y^2 - 4x - 2y = 4$.

31. A. Let $\quad z = x + iy$

$\Rightarrow$ we get $\quad y\sqrt{x^2 + y^2} = 0$

$\Rightarrow \quad y = 0$

$\Rightarrow \quad I_m(z) = 0$

B. We have

$2ae = 8,\ 2a = 10$

$\Rightarrow \quad 10e = 8$

$\Rightarrow \quad e = \frac{4}{5}$

$\Rightarrow \quad b^2 = 25\left(1 - \frac{16}{25}\right) = 9$

$\therefore \quad \frac{x^2}{25} + \frac{y^2}{9} = 1$

C. Let $\quad w = 2(\cos\theta + i\sin\theta)$

$\therefore \quad z = 2(\cos\theta + i\sin\theta) - \frac{1}{2(\cos\theta + i\sin\theta)}$

$= 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$

$= \frac{3}{2}\cos\theta + \frac{5}{2}i\sin\theta$

Let $\quad z = x + iy$

$\Rightarrow \quad x = \frac{3}{2}\cos\theta$ and $y = \frac{5}{2}\sin\theta$

$\Rightarrow \quad \left(\frac{2x}{3}\right)^2 + \left(\frac{2y}{5}\right)^2 = 1$

$\Rightarrow \quad \frac{x^2}{9/4} + \frac{y^2}{25/4} = 1$

$\therefore \quad e = \sqrt{1 - \frac{9/4}{25/4}} = \frac{4}{5}$

D. Let $\quad w = \cos\theta + i\sin\theta$

Then, $\quad z = x + iy = \cos\theta + i\sin\theta + \frac{1}{\cos\theta + i\sin\theta}$

$= 2\cos\theta$

$\Rightarrow \quad x = 2\cos\theta,\ y = 0$

32. $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = \frac{x(p)^{1/3} + y(p)^{1/3}\omega + z(p)^{1/3}\omega^2}{x(p)^{1/3}\omega^2 + y(p)^{1/3}\omega^3 + z(p)^{1/3}\omega}$

$\frac{\omega^2(x + y\omega + z\omega^2)}{\omega^2(x\omega + y\omega^2 + z)}$

$= \frac{\omega^2(x + y\omega + z\omega^2)}{x + y\omega + z\omega^2} = \omega^2$

33. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= [a^2|z_1|^2 + b^2|z_2|^2 - 2ab\,\text{Re}(z_1\bar{z}_2)]$

$+ [b^2|z_1|^2 + a^2|z_2|^2 + 2ab\,\text{Re}(z_1\bar{z}_2)]$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

34. $\frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - i\tan x}{1 + 2i\sin\frac{x}{2}} \in R$

$= \frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2} - i\tan x\right)\left(1 - 2i\sin\frac{x}{2}\right)}{1 + 4\sin^2\frac{x}{2}}$

Since, it is real, so imaginary part will be zero.

$\therefore \quad -2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - \tan x = 0$

$\Rightarrow \quad 2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)\cos x + 2\sin\frac{x}{2}\cos\frac{x}{2} = 0$

$\Rightarrow \sin\frac{x}{2}\left[\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) + \cos\frac{x}{2}\right] = 0$

$\therefore \quad \sin\frac{x}{2} = 0$

$\Rightarrow \quad x = 2n\pi \qquad \ldots(i)$

or $\quad \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) + \cos\frac{x}{2} = 0$

On dividing by $\cos^3 \frac{x}{2}$, we get

$$\left(\tan\frac{x}{2}+1\right)\left(1-\tan^2\frac{x}{2}\right)+\left(1+\tan^2\frac{x}{2}\right)=0$$

$\Rightarrow \quad \tan^3\frac{x}{2}-\tan\frac{x}{2}-2=0$

Let $\quad \tan\frac{x}{2}=t$

and $\quad f(t)=t^3-t-2$

Then, $\quad f(1)=-2<0$

and $\quad f(2)=4>0$

Thus, $f(t)$ changes sign from negative to positive in the interval (1, 2).

$\therefore$ Let $t=k$ be the root for which

$$f(k)=0 \quad \text{and} \quad k\in(1,2)$$

$\therefore \quad t=k \quad \text{or} \quad \tan\frac{x}{2}=k=\tan\alpha$

$\Rightarrow \quad x/2=n\pi+\alpha$

$\Rightarrow \quad \begin{cases} x=2n\pi+2\alpha, \alpha=\tan^{-1}k, \text{ where } k\in(1,2) \\ \quad\quad \text{or } x=2n\pi \end{cases}$

35. Since, z_1, z_2, z_3 are in AP.

$\Rightarrow \quad 2z_2=z_1+z_3$

i.e. points are collinear, thus do not lie on circle. Hence, it is a false statement.

36. Since, z_1, z_2, z_3 are vertices of equilateral triangle and $|z_1|=|z_2|=|z_3|$

$\Rightarrow z_1, z_2, z_3$ lie on a circle with centre at origin.

$\Rightarrow$ Circumcentre = Centroid

$\Rightarrow \quad 0=\frac{z_1+z_2+z_3}{3}$

$\therefore \quad z_1+z_2+z_3=0$

37. Let $z=x+iy \Rightarrow 1\cap z$ gives $1\cap x+iy$

or $\quad 1\le x$ and $0\le y$ $\quad$ ...(i)

Given, $\quad \frac{1-z}{1+z}\cap 0 \Rightarrow \frac{1-x-iy}{1+x+iy}\cap 0$

$\Rightarrow \quad \frac{(1-x-iy)(1+x-iy)}{(1+x+iy)(1+x-iy)}\cap 0+0i$

$\Rightarrow \quad \frac{1-x^2-y^2}{(1+x)^2+y^2}-\frac{2iy}{(1+x)^2+y^2}\cap 0+0i$

$\Rightarrow \quad x^2+y^2\ge 1$

and $\quad -2y\le 0$

or $x^2+y^2\ge 1$ and $y\ge 0$ which is true by Eq. (i).

38. As we know, $|z|^2=z\cdot\bar{z}$

Given, $\quad \frac{|z-\alpha|^2}{|z-\beta|^2}=k^2$

$\Rightarrow \quad (z-\alpha)(\bar{z}-\bar{\alpha})=k^2(z-\beta)(\bar{z}-\bar{\beta})$

$\Rightarrow \quad |z|^2-\alpha\bar{z}-\bar{\alpha}z+|\alpha|^2=k^2(|z|^2-\beta\bar{z}-\bar{\beta}z+|\beta|^2)$

$\Rightarrow \quad |z|^2(1-k^2)-(\alpha-k^2\beta)\bar{z}-(\bar{\alpha}-\bar{\beta}k^2)z$

$$+(|\alpha|^2-k^2|\beta|^2)=0$$

$\Rightarrow \quad |z|^2-\frac{(\alpha-k^2\beta)}{(1-k^2)}\bar{z}-\frac{(\bar{\alpha}-\bar{\beta}k^2)}{(1-k^2)}z+\frac{|\alpha|^2-k^2|\beta|^2}{(1-k^2)}=0$ ...(i)

On comparing with equation of circle,

$$|z|^2+a\bar{z}+\bar{a}z+b=0$$

whose centre is $(-a)$ and radius $=\sqrt{|a|^2-b}$

$\therefore$ Centre for Eq. (i) $=\frac{\alpha-k^2\beta}{1-k^2}$

and radius $=\sqrt{\left(\frac{\alpha-k^2\beta}{1-k^2}\right)\left(\frac{\bar{\alpha}-k^2\bar{\beta}}{1-k^2}\right)-\frac{\alpha\bar{\alpha}-k^2\beta\bar{\beta}}{1-k^2}}$

$$=\left|\frac{k(\alpha-\beta)}{1-k^2}\right|$$

39. Given, $a_1z+a_2z^2+\ldots+a_nz^n=1$

and $\quad |z|<\frac{1}{3}$ $\quad$ ...(i)

$\therefore \quad |a_1z+a_2z^2+a_3z^3+\ldots+a_nz^n|=1$

$\Rightarrow \quad |a_1z|+|a_2z^2|+|a_3z^3|+\ldots+|a_nz^n|\ge 1$

[using $|z_1+z_2|\le|z_1|+|z_2|$]

$\Rightarrow \quad 2\{(|z|+|z|^2+|z|^3+\ldots+|z|^n\}>1$ $\quad$ [using $|a_r|<2$]

$\Rightarrow \quad \frac{2|z|(1-|z|^n)}{1-|z|}>1$ $\quad$ [using sum of n terms of GP]

$\Rightarrow \quad 2|z|-2|z|^{n+1}>1-|z|$

$\Rightarrow \quad 3|z|>1+2|z|^{n+1}$

$\Rightarrow \quad |z|>\frac{1}{3}+\frac{2}{3}|z|^{n+1}$

$\Rightarrow \quad |z|>\frac{1}{3}$, which contradicts $\quad$ ...(ii)

$\therefore$ There exists no complex number z such that

$$|z|<1/3 \text{ and } \sum_{r=1}^{n}a_rz^r=1$$

40. Given, $|z_1|<1$ and $|z_2|>1$ $\quad$ ...(i)

Then, to prove

$$\left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right|<1 \quad \left[\text{using } \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\right]$$

$\Rightarrow \quad |1-z_1\bar{z}_2|<|z_1-z_2|$ $\quad$ ...(ii)

On squaring both sides, we get,

$(1-z_1\bar{z}_2)(1-\bar{z}_1z_2)<(z_1-z_2)(\bar{z}_1-\bar{z}_2)$ $\quad$ [using $|z|^2=z\bar{z}$]

$\Rightarrow \quad 1-z_1\bar{z}_2-\bar{z}_1z_2+z_1\bar{z}_1z_2\bar{z}_2<z_1\bar{z}_1-z_1\bar{z}_2-z_2\bar{z}_1+z_2\bar{z}_2$

$\Rightarrow \quad 1+|z_1|^2|z_2|^2<|z_1|^2+|z_2|^2$

$\Rightarrow \quad 1-|z_1|^2-|z_2|^2+|z_1|^2|z_2|^2<0$

$\Rightarrow \quad (1-|z_1|^2)(1-|z_2|^2)<0$ $\quad$ ...(iii)

which is true by Eq. (i) as $|z_1|<1$ and $|z_2|>1$

$\therefore \quad (1-|z_1|^2)>0$ and $(1-|z_2|^2)<0$

$\therefore$ Eq. (iii) is true whenever Eq. (ii) is true.

$\Rightarrow \quad \left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right|<1$ $\quad$ **Hence proved.**

41. Given, $|z|^2 w - |w|^2 z = z - w$

$\Rightarrow z\bar{z}\, w - w\bar{w}\, z = z - w$ $\quad [\because\ |z|^2 = z\bar{z}\ ]$ ...(i)

Taking modulus of both sides, we get

$$|zw||\bar{z} - \bar{w}| = |z - w|$$

$$\Rightarrow |zw||\bar{z} - \bar{w}| = |\overline{z - w}| \quad [\therefore\ |z| = |\bar{z}|\ ]$$

$$\Rightarrow |zw||\bar{z} - \bar{w}| = |\bar{z} - \bar{w}|$$

$$\Rightarrow |\bar{z} - \bar{w}|(|zw| - 1) = 0$$

$$\Rightarrow |\overline{z - w}| = 0 \quad \text{or} \quad |zw| - 1 = 0$$

$$\Rightarrow |z - w| = 0 \quad \text{or} \quad |zw| = 1$$

$$\Rightarrow z - w = 0 \quad \text{or} \quad |z\,w| = 1$$

$$\Rightarrow z = w \quad \text{or} \quad |zw| = 1$$

Now, suppose $z \neq w$

Then, $|zw| = 1$ or $|z||w| = 1$

$$\Rightarrow |z| = \frac{1}{|w|} = r \quad \text{[say]}$$

Let $\quad z = re^{i\theta}$ and $w = \dfrac{1}{r}e^{i\phi}$

On putting these values in Eq. (i), we get

$$r^2\left(\frac{1}{r}e^{i\phi}\right) - \frac{1}{r^2}(re^{i\theta}) = re^{i\theta} - \frac{1}{r}e^{i\phi}$$

$$\Rightarrow re^{i\phi} - \frac{1}{r}e^{i\theta} = re^{i\theta} - \frac{1}{r}e^{i\phi}$$

$$\Rightarrow \left(r + \frac{1}{r}\right)e^{i\phi} = \left(r + \frac{1}{r}\right)e^{i\theta}$$

$$\Rightarrow e^{i\phi} = e^{i\theta} \Rightarrow \phi = \theta$$

Therefore, $\quad z = re^{i\theta}$ and $w = \dfrac{1}{r}e^{i\theta}$

$$\Rightarrow z\bar{w} = re^{i\theta}\cdot\frac{1}{r}e^{-i\theta} = 1$$

NOTE 'If and only if ' means we have to prove the relation in both directions.

Conversely

Assuming that $z = w$ or $z\bar{w} = 1$

If $\quad z = w$, then

$$\text{LHS} = z\bar{z}\,w - w\,\bar{w}z = |z|^2 \cdot z - |w|^2 \cdot z$$

$$= |z|^2 \cdot z - |z|^2 \cdot z = 0$$

and $\quad \text{RHS} = z - w = 0$

If $\quad zw = 1$, then $\bar{z}\bar{w} = 1$ and

$$\text{LHS} = z\bar{z}\,w - w\bar{w}\,z = \bar{z}\cdot 1 - \bar{w}\cdot 1$$

$$= \bar{z} - \bar{w} = z - \bar{w} = 0 = \text{RHS}$$

Hence proved.

Alternate Solution

We have, $\quad |z|^2 w - |w|^2 z = z - w$

$$\Leftrightarrow |z|^2 w - |w|^2 z - z + w = 0$$

$$\Leftrightarrow (|z|^2 + 1)w - (|w|^2 + 1)z = 0$$

$$\Leftrightarrow (|z|^2 + 1)w = (|w|^2 + 1)z$$

$$\Leftrightarrow \frac{z}{w} = \frac{|z|^2 + 1}{|w|^2 + 1}$$

$\therefore\ \dfrac{z}{w}$ is purely real.

$$\Leftrightarrow \frac{\bar{z}}{\bar{w}} = \frac{z}{w} \Rightarrow z\bar{w} = \bar{z}w \quad \text{...(i)}$$

Again, $\quad |z|^2 w - |w|^2 z = z - w$

$$\Leftrightarrow z\cdot\bar{z}w - w\cdot\bar{w}z = z - w$$

$$\Leftrightarrow z(\bar{z}w - 1) - w(z\bar{w} - 1) = 0$$

$$\Leftrightarrow (z - w)(z\bar{w} - 1) = 0 \quad \text{[from Eq. (i)]}$$

$$\Leftrightarrow z = w \quad \text{or} \quad z\bar{w} = 1$$

Therefore, $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$.

42. Let $\quad z = x + iy$.

Given, $\quad \bar{z} = iz^2$

$$\Rightarrow (\overline{x + iy}) = i(x + iy)^2$$

$$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy)$$

$$\Rightarrow x - iy = -2xy + i(x^2 - y^2)$$

NOTE It is a compound equation, therefore we can generate from it more than one primary equations.

On equating real and imaginary parts, we get

$$x = -2xy \quad \text{and} \quad -y = x^2 - y^2$$

$$\Rightarrow x + 2xy = 0 \quad \text{and} \quad x^2 - y^2 + y = 0$$

$$\Rightarrow x(1 + 2y) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad y = -1/2$$

When $\quad x = 0,\ x^2 - y^2 + y = 0 \Rightarrow 0 - y^2 + y = 0$

$$\Rightarrow y(1 - y) = 0 \Rightarrow y = 0 \quad \text{or} \quad y = 1$$

When, $\quad y = -1/2,\ x^2 - y^2 + y = 0$

$$\Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0 \Rightarrow x^2 = \frac{3}{4}$$

$$\Rightarrow x = \pm\frac{\sqrt{3}}{2}$$

Therefore, $\quad z = 0 + i\,0,\ 0 + i\,; \pm\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}$

$$\Rightarrow z = i, \pm\frac{\sqrt{3}}{2} - \frac{i}{2} \quad [\because z \neq 0]$$

43. Given, $iz^3 + z^2 - z + i = 0$

$$\Rightarrow iz^3 - i^2z^2 - z + i = 0 \quad [\because\ i^2 = -1]$$

$$\Rightarrow iz^2(z - i) - 1(z - i) = 0$$

$$\Rightarrow (iz^2 - 1)(z - i) = 0$$

$$\Rightarrow z - i = 0 \quad \text{or} \quad iz^2 - 1 = 0$$

$$\Rightarrow z = i \quad \text{or} \quad z^2 = \frac{1}{i} = -i$$

If $\quad z = i$, then $|z| = |i| = 1$

If $\ z^2 = -i$, then $|z^2| = |-i| = 1$

$$\Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$$

44. Here, $z_1Rz_2 \Leftrightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

(i) **Reflexive** $z_1Rz_1 \Leftrightarrow \dfrac{z_1 - z_1}{z_1 + z_2} = 0$ $\quad$ [purely real]

$\therefore \quad z_1Rz_1$ is reflexive.

(ii) **Symmetric** $z_1Rz_2 \Leftrightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$$\Rightarrow \frac{-(z_2 - z_1)}{z_1 + z_2} \text{ is real} \Rightarrow z_2Rz_1$$

$$\therefore \quad z_1Rz_2 \Rightarrow z_2Rz_1$$

Therefore, it is symmetric.

(iii) **Transitive** $z_1 R z_2$

$\Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

and $z_2 R z_3$

$\Rightarrow \dfrac{z_2 - z_3}{z_2 + z_3}$ is real

Here, let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$

$\therefore \dfrac{z_1 - z_2}{z_1 + z_2}$ is real $\Rightarrow \dfrac{(x_1 - x_2) + i\,(y_1 - y_2)}{(x_1 + x_2) + i\,(y_1 + y_2)}$ is real

$$\Rightarrow \frac{\{(x_1 - x_2) + i\,(y_1 - y_2)\}\{(x_1 + x_2) - i\,(y_1 + y_2)\}}{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

$$\Rightarrow (y_1 - y_2)(x_1 + x_2) - (x_1 - x_2)(y_1 + y_2) = 0$$

$$\Rightarrow 2x_2 y_1 - 2y_2 x_1 = 0$$

$$\Rightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2} \quad \text{... (i)}$$

Similarly, $z_2 R z_3$

$$\Rightarrow \frac{x_2}{y_2} = \frac{x_3}{y_3} \quad \text{... (ii)}$$

From Eqs. (i) and (ii), we have $\dfrac{x_1}{y_1} = \dfrac{x_3}{y_3} \Rightarrow z_1 R z_3$

Thus, $z_1 R z_2$ and $z_2 R z_3 \Rightarrow z_1 R z_3$. [transitive]

Hence, R is an equivalence relation.

45. $\dfrac{(1 + i)\,x - 2i}{3 + i} + \dfrac{(2 - 3i)\,y + i}{3 - i} = i$

$\Rightarrow (1 + i)(3 - i)\,x - 2i\,(3 - i) + (3 + i)(2 - 3i)y + i\,(3 + i) = 10i$

$\Rightarrow 4x + 2ix - 6i - 2 + 9y - 7iy + 3i - 1 = 10i$

$\Rightarrow 4x + 9y - 3 = 0$ and $2x - 7y - 3 = 10$

$\Rightarrow x = 3$ and $y = -1$

46. Now, $\dfrac{1}{(1 - \cos\theta) + 2i\sin\theta} = \dfrac{1}{2\sin^2\dfrac{\theta}{2} + 4i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}$

$$= \frac{1}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + 2i\cos\frac{\theta}{2}\right)} \times \frac{\sin\frac{\theta}{2} - 2i\cos\frac{\theta}{2}}{\left(\sin\frac{\theta}{2} - 2i\cos\frac{\theta}{2}\right)}$$

$$= \frac{\sin\frac{\theta}{2} - 2i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\right)}$$

$$= \frac{\sin\frac{\theta}{2} - 2i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1 + 3\cos^2\frac{\theta}{2}\right)}$$

$$\Rightarrow A + iB = \frac{1}{2\left(1 + 3\cos^2\frac{\theta}{2}\right)} - i\,\frac{\cot\frac{\theta}{2}}{1 + 3\cos^2\frac{\theta}{2}}$$

47. Since, $(x + iy)^2 = \dfrac{a + ib}{c + id}$

$\Rightarrow |x + iy|^2 = \dfrac{|a + ib|}{|c + id|}$ $\left[\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right]$

$\Rightarrow (x^2 + y^2) = \dfrac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$

$\Rightarrow (x^2 + y^2)^2 = \dfrac{a^2 + b^2}{c^2 + d^2}$

Hence proved.

48. Given, $|z - 3 - 2i| \le 2$ ...(i)

To find minimum of $|2z - 6 + 5i|$

or $2\left|z - 3 + \dfrac{5}{2}i\right|$, using triangle inequality

i.e. $||z_1| - |z_2|| \le |z_1 + z_2|$

$$\therefore \left|z - 3 + \frac{5}{2}i\right| = \left|z - 3 - 2i + 2i + \frac{5}{2}i\right|$$

$$= \left|(z - 3 - 2i) + \frac{9}{2}i\right|$$

$$\ge \left||z - 3 - 2i| - \frac{9}{2}\right| \ge \left|2 - \frac{9}{2}\right| \ge \frac{5}{2}$$

$$\Rightarrow \left|z - 3 + \frac{5}{2}i\right| \ge \frac{5}{2} \text{ or } |2z - 6 + 5i| \ge 5$$

Topic 3 Argument of a Complex Number

1. (*) Given, $3|z_1| = 4|z_2| \Rightarrow \dfrac{|z_1|}{|z_2|} = \dfrac{4}{3}$ $[\because z_2 \ne 0 \Rightarrow |z_2| \ne 0]$

$$\therefore \frac{z_1}{z_2} = \left|\frac{z_1}{z_2}\right| e^{i\theta} \text{ and } \frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{-i\theta}$$

$[\because z = |z|(\cos\theta + i\sin\theta) = |z|\,e^{i\theta}]$

$$\Rightarrow \frac{z_1}{z_2} = \frac{4}{3}e^{i\theta} \text{ and } \frac{z_2}{z_1} = \frac{3}{4}e^{-i\theta}$$

$$\Rightarrow \frac{3}{2}\frac{z_1}{z_2} = 2e^{i\theta} \text{ and } \frac{2}{3}\frac{z_2}{z_1} = \frac{1}{2}e^{-i\theta}$$

On adding these two, we get

$$z = \frac{3}{2}\frac{z_1}{z_2} + \frac{2}{3}\frac{z_2}{z_1} = 2e^{i\theta} + \frac{1}{2}e^{-i\theta}$$

$$= 2\cos\theta + 2i\sin\theta + \frac{1}{2}\cos\theta - \frac{1}{2}i\sin\theta$$

$[\because e^{\pm i\theta} = (\cos\theta \pm i\sin\theta)]$

$$= \frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta$$

$$\Rightarrow |z| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{34}{4}} = \sqrt{\frac{17}{2}}$$

Note that z is neither purely imaginary and nor purely real.

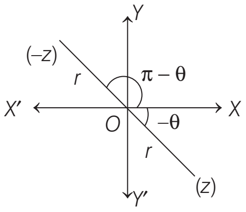
'*' *None of the options is correct.*

2. Given, $|z| = 1$, $\arg z = \theta \therefore z = e^{i\theta}$

$\therefore \quad \bar{z} = e^{-i\theta} \Rightarrow \bar{z} = \frac{1}{z}$

$\therefore \quad \arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+\frac{1}{z}}\right) = \arg(z) = \theta$

3. Since, $\arg(z) < 0 \Rightarrow \arg(z) = -\theta$



$\Rightarrow \quad z = r\cos(-\theta) + i\sin(-\theta)$

$= r(\cos\theta - i\sin\theta)$

and $\quad -z = -r[\cos\theta - i\sin\theta]$

$= r[\cos(\pi-\theta) + i\sin(\pi-\theta)]$

$\therefore \quad \arg(-z) = \pi - \theta$

Thus, $\arg(-z) - \arg(z)$

$= \pi - \theta - (-\theta) = \pi$

Alternate Solution

Reason $\arg(-z) - \arg z = \arg\left(\frac{-z}{z}\right) = \arg(-1) = \pi$

and also $\arg z - \arg(-z) = \arg\left(\frac{z}{-z}\right) = \arg(-1) = \pi$

4. Given, $\quad |z + iw| = |z - i\bar{w}| = 2$

$\Rightarrow \quad |z - (-iw)| = |z - (i\bar{w})| = 2$

$\Rightarrow \quad |z - (-iw)| = |z - (-i\bar{w})|$

$\therefore$ z lies on the perpendicular bisector of the line joining $-iw$ and $-i\bar{w}$. Since, $-i\bar{w}$ is the mirror image of $-iw$ in the X-axis, the locus of z is the X-axis.

Let $\quad z = x + iy$ and $y = 0$.

Now, $\quad |z| \le 1 \Rightarrow x^2 + 0^2 \le 1 \Rightarrow -1 \le x \le 1$.

$\therefore$ z may take values given in option (c).

Alternate Solution

$|z + iw| \le |z| + |iw| = |z| + |w|$

$\le 1 + 1 = 2$

$\therefore \quad |z + iw| \le 2$

$\Rightarrow \quad |z + iw| = 2$ holds when

$\arg z - \arg iw = 0$

$\Rightarrow \quad \arg\frac{z}{iw} = 0$

$\Rightarrow \frac{z}{iw}$ is purely real.

$\Rightarrow \frac{z}{w}$ is purely imaginary.

Similarly, when $|z - i\bar{w}| = 2$, then $\frac{z}{\bar{w}}$ is purely imaginary

Now, given relation

$|z + iw| = |z - i\bar{w}| = 2$

Put $w = i$, we get

$|z + i^2| = |z + i^2| = 2$

$\Rightarrow \quad |z - 1| = 2$

$\Rightarrow \quad z = -1 \qquad [\because |z| \le 1]$

Put $w = -i$, we get

$|z - i^2| = |z - i^2| = 2$

$\Rightarrow \quad |z + 1| = 2 \Rightarrow z = 1 \qquad [\because |z| \le 1]$

$\therefore$ $z = 1$ or -1 is the correct option.

5. Since, $|z| = |w|$ and $\arg(z) = \pi - \arg(w)$

Let $\quad w = re^{i\theta}$, then $\bar{w} = re^{-i\theta}$

$\therefore \quad z = re^{i(\pi-\theta)} = re^{i\pi} \cdot e^{-i\theta} = -re^{-i\theta} = -\bar{w}$

6. Given, $|z_1 + z_2| = |z_1| + |z_2|$

On squaring both sides, we get

$|z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\arg z_1 - \arg z_2)$

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow \quad 2|z_1||z_2|\cos(\arg z_1 - \arg z_2) = 2|z_1||z_2|$

$\Rightarrow \quad \cos(\arg z_1 - \arg z_2) = 1$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = 0$

7. Since a, b, c and u, v, w are the vertices of two triangles.

Also, $\quad c = (1-r)a + rb$

and $\quad w = (1-r)u + rv \qquad \ldots(i)$

Consider $\begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix}$

Applying $R_3 \to R_3 - \{(1-r)R_1 + rR_2\}$

$= \begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c-(1-r)a-rb & w-(1-r)u-rv & 1-(1-r)-r \end{vmatrix}$

$= \begin{vmatrix} a & u & 1 \\ b & v & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \qquad$ [from Eq. (i)]

8. (a) Let $\quad z = -1 - i$ and $\arg(z) = \theta$

Now, $\quad \tan\theta = \left|\frac{\text{im}(z)}{\text{Re}(z)}\right| = \left|\frac{-1}{-1}\right| = 1$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

Since, $x < 0$, $y < 0$

$\therefore \quad \arg(z) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$

(b) We have, $f(t) = \arg(-1 + it)$

$\arg(-1+it) = \begin{cases} \pi - \tan^{-1} t, & t \ge 0 \\ -(\pi + \tan^{-1} t), & t < 0 \end{cases}$

This function is discontinuous at $t = 0$.

(c) We have,

$$\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$$

Now, $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2n\pi$

$\therefore$ $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$

$= \arg(z_1) - \arg(z_2) + 2n\pi - \arg(z_1) + \arg(z_2)$

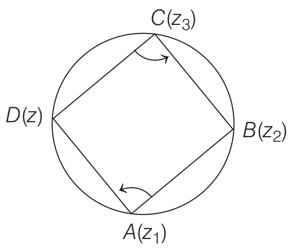
$= 2n\pi$

So, given expression is multiple of 2π.

(d) We have, $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$

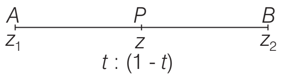
$\Rightarrow \left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right)$ is purely real

Thus, the points $A(z_1), B(z_2), C(z_3)$ and $D(z)$ taken in order would be concyclic if purely real.
Hence, it is a circle.



$\therefore$ (a), (b), (d) are false statement.

9. Given, $z = \frac{(1-t)z_1 + tz_2}{(1-t)+t}$



Clearly, z divides z_1 and z_2 in the ratio of $t:(1-t)$, $0 < t < 1$

$\Rightarrow$ $AP + BP = AB$ i.e. $|z - z_1| + |z - z_2| = |z_1 - z_2|$

$\Rightarrow$ Option (a) is true.

and $\arg(z - z_1) = \arg(z_2 - z)$

$= \arg(z_2 - z_1)$

$\Rightarrow$ Option (b) is false and option (d) is true.

Also, $\arg(z - z_1) = \arg(z_2 - z_1)$

$\Rightarrow$ $\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$

$\therefore$ $\frac{z-z_1}{z_2-z_1}$ is purely real.

$\Rightarrow$ $\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$

or $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

Option (c) is correct.

10. Let $z = a + ib$.

Given, $\text{Re}(z) = 0 \Rightarrow a = 0$

Then, $z = ib \Rightarrow z^2 = -b^2$ or $\text{Im}(z^2) = 0$

Therefore, A $\to$ q

Also, given, $\arg(z) = \frac{\pi}{4}$.

Let $z = r\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Then, $z^2 = r^2\left(\cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4}\right) + 2ir^2\cos\frac{\pi}{4}\sin\frac{\pi}{4}$

$= ir^2 \sin\pi/2 = ir^2$

Therefore, $\text{Re}(z^2) = 0 \Rightarrow$ B $\to$ p.

$\Rightarrow$ $a = b = 2 - \sqrt{3}$ $[\because a, b \leftarrow (0,1)]$

11. Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$

We have, $|z| = r_1, |w| = r_2, \arg(z) = \theta_1$ and $\arg(w) = \theta_2$

Given, $|z| \le 1, |w| < 1$

$\Rightarrow$ $r_1 \le 1$ and $r_2 \le 1$

Now,

$z - w = (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)$

$\Rightarrow |z - w|^2 = (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$= r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2$

$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1r_2[1 - \cos(\theta_1 - \theta_2)]$

$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$

$\le |r_1 - r_2|^2 + 4\left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|^2$ $[\because r_1, r_2 \le 1]$

and $|\sin\theta| \le |\theta|, \forall \theta \in R$

Therefore, $|z - w|^2 \le |r_1 - r_2|^2 + 4\left|\frac{\theta_1 - \theta_2}{2}\right|^2$

$\le |r_1 - r_2|^2 + |\theta_1 - \theta_2|^2$

$\Rightarrow$ $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$

Alternate Solution

$|z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos(\arg z - \arg w)$

$= |z|^2 + |w|^2 - 2|z||w| + 2|z||w| - 2|z||w|\cos(\arg z - \arg w)$

$= (|z| - |w|)^2 + 2|z||w| \cdot 2\sin^2\left(\frac{\arg z - \arg w}{2}\right)$ ...(i)

$\therefore$ $|z - w|^2 \le (|z| - |w|)^2 + 4 \cdot 1 \cdot 1\left(\frac{\arg z - \arg w}{2}\right)^2$

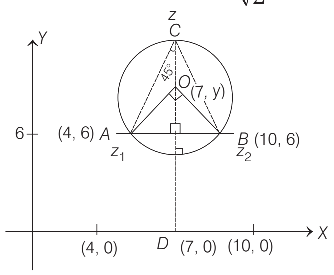
$[\because \sin\theta \le \theta]$

$\Rightarrow$ $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$

12. Since, $z_1 = 10 + 6i$, $z_2 = 4 + 6i$

and $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \dfrac{\pi}{4}$ represents locus of z is a circle shown as from the figure whose centre is $(7, y)$ and $\angle AOB = 90°$, clearly $OC = 9 \Rightarrow OD = 6 + 3 = 9$

$\therefore$ Centre = (7, 9) and radius $= \dfrac{6}{\sqrt{2}} = 3\sqrt{2}$



$\Rightarrow$ Equation of circle is $|z - (7 + 9i)| = 3\sqrt{2}$

Topic 4 Rotation of a Complex Number

1. Given, $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$

$\because$ Euler's form of

$$\frac{\sqrt{3}}{2} + \frac{i}{2} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = e^{i(\pi/6)}$$

and $\dfrac{\sqrt{3}}{2} - \dfrac{i}{2} = \cos\left(\dfrac{-\pi}{6}\right) + i\sin\left(-\dfrac{\pi}{6}\right) = e^{-i\pi/6}$

So, $z = (e^{i\pi/6})^5 + (e^{-i\pi/6})^5 = e^{i\frac{5\pi}{6}} + e^{-i\frac{5\pi}{6}}$

$$= \left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$$

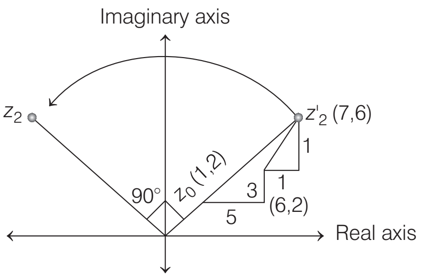
$[\because e^{i\theta} = \cos\theta + i\sin\theta]$

$$= 2\cos\frac{5\pi}{6}$$

$\therefore$ $I(z) = 0$ and $R(z) = -2\cos\dfrac{\pi}{6} = -\sqrt{3} < 0$

$\left[\because \cos\dfrac{5\pi}{6} = \cos\left(\pi - \dfrac{\pi}{6}\right) = -\cos\dfrac{\pi}{6}\right]$

2.



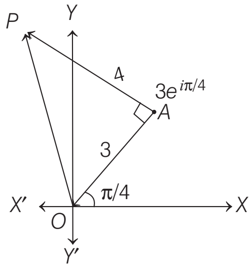
$z_2' = (6 + \sqrt{2}\cos 45°, 5 + \sqrt{2}\sin 45°) = (7, 6) = 7 + 6i$

By rotation about (0, 0),

$$\frac{z_2}{z_2'} = e^{i\pi/2} \Rightarrow z_2 = z_2'\left(e^{i\frac{\pi}{2}}\right)$$

$$= (7 + 6i)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = (7 + 6i)(i) = -6 + 7i$$

3. Let $OA = 3$, so that the complex number associated with A is $3e^{i\pi/4}$. If z is the complex number associated with P, then



$$\frac{z - 3e^{i\pi/4}}{0 - 3e^{i\pi/4}} = \frac{4}{3}e^{-i\pi/2} = -\frac{4i}{3}$$

$\Rightarrow 3z - 9e^{i\pi/4} = 12ie^{i\pi/4}$

$\Rightarrow \quad z = (3 + 4i)e^{i\pi/4}$

4. Since, $|PQ| = |PS| = |PR| = 2$

$\therefore$ Shaded part represents the external part of circle having centre (–1, 0) and radius 2.

As we know equation of circle having centre z_0 and radius r, is $|z - z_0| = r$

$\therefore \quad |z - (-1 + 0i)| > 2$

$\Rightarrow \quad |z + 1| > 2$

Also, argument of $z + 1$ with respect to positive direction of X-axis is $\pi/4$.

$\therefore \quad \arg(z + 1) \le \dfrac{\pi}{4}$ …(i)

and argument of $z + 1$ in anticlockwise direction is $-\pi/4$.

$\therefore \quad -\pi/4 \le \arg(z + 1)$ …(ii)

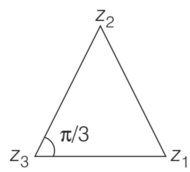
From Eqs. (i) and (ii),

$|\arg(z + 1)| \le \pi/4$

5. In the Argand plane, P is represented by e^{i0} and Q is represented by $e^{i(\alpha - \theta)}$

Now, rotation about a line with angle α is given by $e^{\theta} \to e^{(\alpha - \theta)}$. Therefore, Q is obtained from P by reflection in the line making an angle $\alpha/2$.

6. $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2} = \dfrac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})}$



$$= \frac{1 - i^2 3}{2(1 + i\sqrt{3})} = \frac{4}{2(1 + i\sqrt{3})} = \frac{2}{(1 + i\sqrt{3})}$$

$\Rightarrow \quad \dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{1 + i\sqrt{3}}{2} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$\Rightarrow \quad \left|\dfrac{z_2 - z_3}{z_1 - z_3}\right| = 1$ and $\arg\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right) = \dfrac{\pi}{3}$

Hence, the triangle is an equilateral.

Alternate Solution

$\therefore \quad \dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2}$

$\Rightarrow \quad \dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{2}{1 - i\sqrt{3}} = \dfrac{1 + i\sqrt{3}}{2} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$\Rightarrow \quad \arg\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right) = \dfrac{\pi}{3}$ and also $\left|\dfrac{z_2 - z_3}{z_1 - z_3}\right| = 1$

Therefore, triangle is equilateral.

7. Here, $x + iy = \frac{1}{a + ibt} \times \frac{a - ibt}{a - ibt}$

$\therefore \quad x + iy = \frac{a - ibt}{a^2 + b^2t^2}$

Let $\quad a \neq 0, b \neq 0$

$\therefore \quad x = \frac{a}{a^2 + b^2t^2}$ and $y = \frac{-bt}{a^2 + b^2t^2}$

$\Rightarrow \quad \frac{y}{x} = \frac{-bt}{a} \Rightarrow t = \frac{ay}{bx}$

On putting $x = \frac{a}{a^2 + b^2t^2}$, we get

$$x\left(a^2 + b^2 \cdot \frac{a^2y^2}{b^2x^2}\right) = a \Rightarrow a^2(x^2 + y^2) = ax$$

or $\quad x^2 + y^2 - \frac{x}{a} = 0 \quad \ldots \text{(i)}$

or $\quad \left(x - \frac{1}{2a}\right)^2 + y^2 = \frac{1}{4a^2}$

$\therefore$ Option (a) is correct.

For $\quad a \neq 0$ and $b = 0$,

$$x + iy = \frac{1}{a} \Rightarrow x = \frac{1}{a}, y = 0$$

$\Rightarrow z$ lies on X-axis.

$\therefore$ Option (c) is correct.

For $a = 0$ and $b \neq 0$, $x + iy = \frac{1}{ibt} \Rightarrow x = 0, y = -\frac{1}{bt}$

$\Rightarrow z$ lies on Y-axis.

$\therefore$ Option (d) is correct.

8. **PLAN** It is the simple representation of points on Argand plane and to find the angle between the points.

Here, $P = W^n = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n = \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}$

$$H_1 = \left\{z \in C : \text{Re}(z) > \frac{1}{2}\right\}$$

$\therefore P \cap H_1$ represents those points for which $\cos\frac{n\pi}{6}$ is + ve.

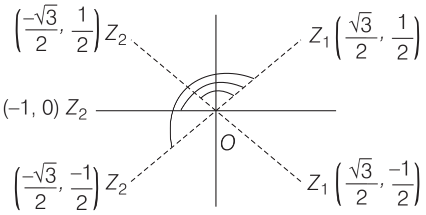
Hence, it belongs to I or IV quadrant.

$\Rightarrow z_1 = P \cap H_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ or $\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

$\therefore \quad z_1 = \frac{\sqrt{3}}{2} + \frac{i}{2}$ or $\frac{\sqrt{3}}{2} - \frac{i}{2} \quad \ldots\text{(i)}$

Similarly, $z_2 = P \cap H_2$ i.e. those points for which

$$\cos\frac{n\pi}{6} < 0$$



$\therefore \quad z_2 = \cos\pi + i\sin\pi, \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}, \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$\Rightarrow \quad z_2 = -1, \frac{-\sqrt{3}}{2} + \frac{i}{2}, \frac{-\sqrt{3}}{2} - \frac{i}{2}$

Thus, $\angle z_1Oz_2 = \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

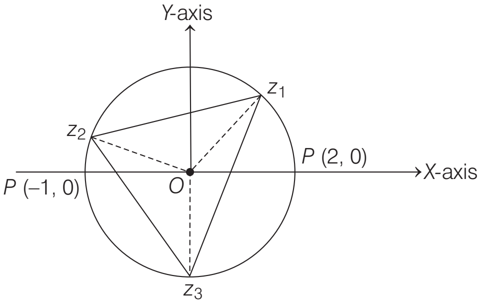
9. $z_1 = 1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$ [let]

$\Rightarrow \quad r\cos\theta = 1, r\sin\theta = \sqrt{3}$

$\Rightarrow \quad r = 2$ and $\theta = \pi/3$

So, $\quad z_1 = 2(\cos\pi/3 + \sin\pi/3)$

Since, $\quad |z_2| = |z_3| = 2$ [given]



Now, the triangle z_1, z_2 and z_3 being an equilateral and the sides z_1z_2 and z_1z_3 make an angle $2\pi/3$ at the centre.

Therefore, $\quad \angle POz_2 = \frac{\pi}{3} + \frac{2\pi}{3} = \pi$

and $\quad \angle POz_3 = \frac{\pi}{3} + \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{5\pi}{3}$

Therefore, $z_2 = 2(\cos\pi + i\sin\pi) = 2(-1 + 0) = -2$

and $z_3 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$

Alternate Solution

Whenever vertices of an equilateral triangle having centroid is given its vertices are of the form $z, z\omega, z\omega^2$.

$\therefore$ If one of the vertex is $z_1 = 1 + i\sqrt{3}$, then other two vertices are $(z_1\omega), (z_1\omega^2)$.

$\Rightarrow \quad (1 + i\sqrt{3})\frac{(-1 + i\sqrt{3})}{2}, (1 + i\sqrt{3})\frac{(-1 - i\sqrt{3})}{2}$

$\Rightarrow \quad \frac{-(1 + 3)}{2}, -\frac{(1 + i^2(\sqrt{3})^2 + 2i\sqrt{3})}{2}$

$\Rightarrow \quad -2, -\frac{(-2 + 2i\sqrt{3})}{2} = 1 - i\sqrt{3}$

$\therefore \quad z_2 = -2$ and $z_3 = 1 - i\sqrt{3}$

10. Given, $D = (1 + i), M = (2 - i)$

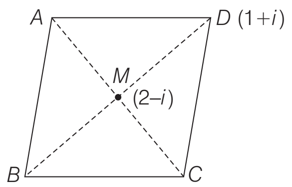
and diagonals of a rhombus bisect each other.

Let $B \equiv (a + ib)$, therefore

$$\frac{a + 1}{2} = 2, \frac{b + 1}{2} = -1$$

$\Rightarrow \quad a + 1 = 4, b + 1 = -2 \Rightarrow a = 3, b = -3$

$\Rightarrow \quad B \equiv (3 - 3i)$



Again, $DM = \sqrt{(2-1)^2 + (-1-1)^2} = \sqrt{1+4} = \sqrt{5}$

But $BD = 2DM \Rightarrow BD = 2\sqrt{5}$

and $2AC = BD \Rightarrow 2AC = 2\sqrt{5}$

$\Rightarrow AC = \sqrt{5}$ and $AC = 2AM$

$\Rightarrow \sqrt{5} = 2AM \Rightarrow AM = \dfrac{\sqrt{5}}{2}$

Now, let coordinate of A be $(x + iy)$.

But in a rhombus $AD = AB$, therefore we have

$$AD^2 = AB^2$$

$$\Rightarrow (x-1)^2 + (y-1)^2 = (x-3)^2 + (y+3)^2$$

$$\Rightarrow x^2 + 1 - 2x + y^2 + 1 - 2y = x^2 + 9 - 6x + y^2 + 9 + 6y$$

$$\Rightarrow 4x - 8y = 16$$

$$\Rightarrow x - 2y = 4$$

$$\Rightarrow x = 2y + 4 \quad \ldots\text{(i)}$$

Again, $AM = \dfrac{\sqrt{5}}{2} \Rightarrow AM^2 = \dfrac{5}{4}$

$$\Rightarrow (x-2)^2 + (y+1)^2 = \frac{5}{4}$$

$$\Rightarrow (2y+2)^2 + (y+1)^2 = \frac{5}{4} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow 5y^2 + 10y + 5 = \frac{5}{4}$$

$$\Rightarrow 20y^2 + 40y + 15 = 0$$

$$\Rightarrow 4y^2 + 8y + 3 = 0$$

$$\Rightarrow (2y+1)(2y+3) = 0$$

$$\Rightarrow 2y + 1 = 0, 2y + 3 = 0$$

$$\Rightarrow y = -\frac{1}{2},\ y = -\frac{3}{2}$$

On putting these values in Eq. (i), we get

$$x = 2\left(-\frac{1}{2}\right) + 4, x = 2\left(-\frac{3}{2}\right) + 4$$

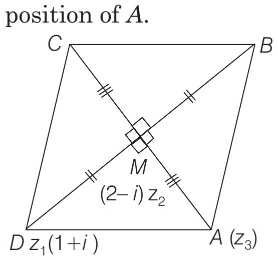
$$\Rightarrow x = 3, x = 1$$

Therefore, A is either $\left(3 - \dfrac{i}{2}\right)$ or $\left(1 - \dfrac{3i}{2}\right)$.

Alternate Solution

Since, M is the centre of rhombus.

$\therefore$ By rotating D about M through an angle of $\pm \pi/2$, we get possible position of A.



$$\Rightarrow \frac{z_3 - (2-i)}{-1+2i} = \frac{1}{2}(\pm i) \quad \Rightarrow \quad \frac{z_3 - (2-i)}{-1+2i} = \frac{1}{2}(\pm i)$$

$$\Rightarrow z_3 = (2-i) \pm \frac{1}{2} i(2i-1) = (2-i) \pm \frac{1}{2}(-2-i)$$

$$= \frac{(4-2i-2-i)}{2}, \frac{4-2i+2+i}{2} = 1 - \frac{3}{2}i, 3 - \frac{i}{2}$$

$\therefore$ A is either $\left(1 - \dfrac{3}{2}i\right)$ or $\left(3 - \dfrac{i}{2}\right)$.

11. Since, z_1, z_2 and z_3 form an equilateral triangle.

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$\Rightarrow (a+i)^2 + (1+ib)^2 + (0)^2 = (a+i)(1+ib) + 0 + 0$$

$$\Rightarrow a^2 - 1 + 2ia + 1 - b^2 + 2ib = a + i(ab+1) - b$$

$$\Rightarrow (a^2 - b^2) + 2i(a+b) = (a-b) + i(ab+1)$$

$$\Rightarrow a^2 - b^2 = a - b$$

and $2(a+b) = ab + 1$

$$\Rightarrow (a = b \text{ or } a + b = 1)$$

and $2(a+b) = ab + 1$

If $a = b$, $2(2a) = a^2 + 1$

$$\Rightarrow a^2 - 4a + 1 = 0$$

$$\Rightarrow a = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

If $a + b = 1, 2 = a(1-a) + 1 \Rightarrow a^2 - a + 1 = 0$

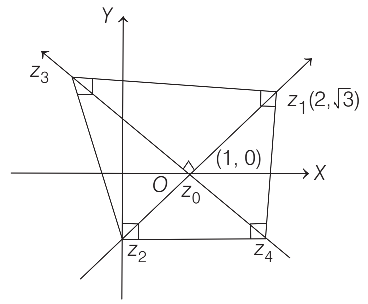
$\Rightarrow a = \dfrac{1 \pm \sqrt{1-4}}{2}$, but a and $b \in R$

$\therefore$ Only solution when $a = b$

$$\Rightarrow a = b = 2 \pm \sqrt{3}$$

$$\Rightarrow a = b = 2 - \sqrt{3} \quad [\because a, b \in (0, 1)]$$

12. Here, centre of circle is (1, 0) is also the mid-point of diagonals of square



$$\Rightarrow \frac{z_1 + z_2}{2} = z_0$$

$$\Rightarrow z_2 = -\sqrt{3}i \quad [\text{where, } z_0 = 1 + 0i]$$

and $\dfrac{z_3 - 1}{z_1 - 1} = e^{\pm i\pi/2}$

$$\Rightarrow z_3 = 1 + (1 + \sqrt{3}i) \cdot \left(\cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2}\right) [\because z_1 = 2 + \sqrt{3}i]$$

$$= 1 \pm i(1 + \sqrt{3}i) = (1 \mp \sqrt{3}) \pm i = (1 - \sqrt{3}) + i$$

and $z_4 = (1 + \sqrt{3}) - i$

13. Let Q be z_2 and its reflection be the point $P(z_1)$ in the given line. If $O(z)$ be any point on the given line then by definition OR is right bisector of QP.

$\therefore$ $OP = OQ$ or $|z - z_1| = |z - z_2|$

$\Rightarrow \quad |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow \quad (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$\Rightarrow \quad z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = z_1\bar{z}_1 - z_2\bar{z}_2$

Comparing with given line $z\bar{b} + \bar{z}b = c$

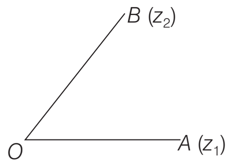
$$\frac{\bar{z}_1 - \bar{z}_2}{\bar{b}} = \frac{z_1 - z_2}{b} = \frac{z_1\bar{z}_1 - z_2\bar{z}_2}{c} = \lambda, \qquad \text{[say]}$$

$$\frac{\bar{z}_1 - \bar{z}_2}{\lambda} = \bar{b}, \frac{z_1 - z_2}{\lambda} = b, \frac{z_1\bar{z}_1 - z_2\bar{z}_2}{\lambda} = c \qquad \text{...(i)}$$

$$\therefore \quad \bar{z}_1 b + z_2\bar{b} = \bar{z}_1\left(\frac{z_1 - z_2}{\lambda}\right) + z_2\left(\frac{\bar{z}_1 - \bar{z}_2}{\lambda}\right)$$

$$= \frac{z\bar{z}_1 - z_2\bar{z}_2}{\lambda} = c \qquad \text{[from Eq. (i)]}$$

14. Since, $z_1 + z_2 = -p$ and $z_1 z_2 = q$



Now, $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}(\cos\alpha + i\sin\alpha)$

$\Rightarrow \quad \dfrac{z_1}{z_2} = \dfrac{\cos\alpha + i\sin\alpha}{1}$

$[\because |z_1| = |z_2|]$

Applying componendo and dividendo, we get

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{\cos\alpha + i\sin\alpha + 1}{\cos\alpha + i\sin\alpha - 1}$$

$$= \frac{2\cos^2(\alpha/2) + 2i\sin(\alpha/2)\cos(\alpha/2)}{-2\sin^2(\alpha/2) + 2i\sin(\alpha/2)\cos(\alpha/2)}$$

$$= \frac{2\cos(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}{2i\sin(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}$$

$$= \frac{\cot(\alpha/2)}{i} = -i\cot\alpha/2 \Rightarrow \frac{-p}{z_1 - z_2} = -i\cot(\alpha/2)$$

On squaring both sides, we get $\dfrac{p^2}{(z_1 - z_2)^2} = -\cot^2(\alpha/2)$

$$\Rightarrow \quad \frac{p^2}{(z_1 + z_2)^2 - 4z_1z_2} = -\cot^2(\alpha/2)$$

$$\Rightarrow \quad \frac{p^2}{p^2 - 4q} = -\cot^2(\alpha/2)$$

$\Rightarrow \quad p^2 = -p^2\cot^2(\alpha/2) + 4q\cot^2(\alpha/2)$

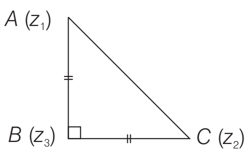
$\Rightarrow \quad p^2(1 + \cot^2\alpha/2) = 4q\cot^2(\alpha/2)$

$\Rightarrow \quad p^2\operatorname{cosec}^2(\alpha/2) = 4q\cot^2(\alpha/2)$

$\Rightarrow \quad p^2 = 4q\cos^2\alpha/2$

15. Since, triangle is a right angled isosceles triangle.

$\therefore$ Rotating z_2 about z_3 in anti-clockwise direction through an angle of $\pi/2$, we get



$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{|z_2 - z_3|}{|z_1 - z_3|}e^{i\pi/2}$$

where, $|z_2 - z_3| = |z_1 - z_3|$

$\Rightarrow \quad (z_2 - z_3) = i(z_1 - z_3)$

On squaring both sides, we get

$$(z_2 - z_3)^2 = -(z_1 - z_3)^2$$

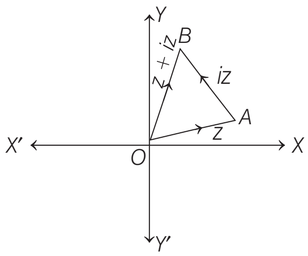
$\Rightarrow \quad z_2^2 + z_3^2 - 2z_2z_3 = -z_1^2 - z_3^2 + 2z_1z_3$

$\Rightarrow \quad z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_3^2 - 2z_1z_2$

$\Rightarrow \quad (z_1 - z_2)^2 = 2\{(z_1z_3 - z_3^2) + (z_2z_3 - z_1z_2)\}$

$\Rightarrow \quad (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

16. We have, $iz = ze^{i\pi/2}$. This implies that iz is the vector obtained by rotating vector z in anti-clockwise direction through 90°. Therefore, $OA \perp AB$. So,



Area of $\Delta OAB = \dfrac{1}{2}OA \times OB = \dfrac{1}{2}|z||iz| = \dfrac{1}{2}|z|^2$

17. Since, z_1, z_2 and origin form an equilateral triangle.

$$\left[\begin{array}{l}\because \text{ if } z_1, z_2, z_3 \text{ from an equilateral triangle, then} \\ z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1\end{array}\right]$$

$\Rightarrow \quad z_1^2 + z_2^2 + 0^2 = z_1z_2 + z_2 \cdot 0 + 0 \cdot z_1$

$\Rightarrow \quad z_1^2 + z_2^2 = z_1z_2$

$\Rightarrow \quad z_1^2 + z_2^2 - z_1z_2 = 0$

18. Since, z_1, z_2, z_3 are the vertices of an equilateral triangle.

$$\therefore \text{ Circumcentre } (z_0) = \text{ Centroid}\left(\frac{z_1 + z_2 + z_3}{3}\right) \qquad \text{...(i)}$$

Also, for equilateral triangle

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 \qquad \text{... (ii)}$$

On squaring Eq. (i), we get

$9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)$

$\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2) \qquad$ [from Eq. (ii)]

$\Rightarrow 3z_0^2 = z_1^2 + z_2^2 + z_3^2$

19. Given, $\alpha_k = \cos\left(\dfrac{k\pi}{7}\right) + i\sin\left(\dfrac{k\pi}{7}\right)$

$$= \cos\left(\frac{2k\pi}{14}\right) + i\sin\left(\frac{2k\pi}{14}\right)$$

$\therefore$ α_k are vertices of regular polygon having 14 sides.

Let the side length of regular polygon be a.

$\therefore |\alpha_{k+1} - \alpha_k| =$ length of a side of the regular polygon

$$= a \qquad \text{...(i)}$$

and $|\alpha_{4k-1} - \alpha_{4k-2}| =$ length of a side of the regular polygon

$$= a \qquad \text{...(ii)}$$

$$\therefore \quad \frac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|} = \frac{12(a)}{3(a)} = 4$$

Topic 5 De-Moivre's Theorem, Cube Roots and *n*th Roots of Unity

1. It is given that, there are two complex numbers z and w, such that $|z\,w| = 1$ and $\arg(z) - \arg(w) = \pi/2$

$\therefore \quad |z||w| = 1 \qquad [\because |z_1 z_2| = |z_1||z_2|]$

and $\arg(z) = \dfrac{\pi}{2} + \arg(w)$

Let $|z| = r$, then $|w| = \dfrac{1}{r}$ ...(i)

and let $\arg(w) = \theta$, then $\arg(z) = \dfrac{\pi}{2} + \theta$ ...(ii)

So, we can assume

$$z = re^{i(\pi/2+\theta)} \quad \text{...(iii)}$$

[$\because$ if $z = x + iy$ is a complex number, then it can be written as $z = re^{i\theta}$ where, $r = |z|$ and $\theta = \arg(z)$]

and $\quad w = \dfrac{1}{r}e^{i\theta}$ ...(iv)

Now, $\quad \bar{z}\cdot w = re^{-i(\pi/2+\theta)}\cdot\dfrac{1}{r}e^{i\theta}$

$= e^{i(-\pi/2-\theta+\theta)} = e^{-i(\pi/2)} = -i \qquad [\because e^{-i\theta} = \cos\theta - i\sin\theta]$

and $\quad z\,\overline{w} = re^{i(\pi/2+\theta)}\cdot\dfrac{1}{r}e^{-i\theta}$

$= e^{i(\pi/2+\theta-\theta)} = e^{i(\pi/2)} = i$

2. **Key Idea** Use, $e^{i\theta} = \cos\theta + i\sin\theta$

Given, $z = \dfrac{\sqrt{3}}{2} + \left(\dfrac{1}{2}\right)i = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6} = e^{i\frac{\pi}{6}}$

so, $(1 + iz + z^5 + iz^8)^9$

$$= \left(1 + ie^{i\frac{\pi}{6}} + e^{i\frac{5\pi}{6}} + ie^{i\frac{8\pi}{6}}\right)^9$$

$$= \left(1 + e^{i\frac{\pi}{2}}\cdot e^{i\frac{\pi}{6}} + e^{i\frac{5\pi}{6}} + e^{i\frac{\pi}{2}}\cdot e^{i\frac{4\pi}{3}}\right)^9 \qquad \left[\because i = e^{i\frac{\pi}{2}}\right]$$

$$= \left(1 + e^{i\frac{2\pi}{3}} + e^{i\frac{5\pi}{6}} + e^{i\frac{11\pi}{6}}\right)^9$$

$$= \left[1 + \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + \left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)\right]^9$$

$$= \left(1 - \frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2}i + \frac{\sqrt{3}}{2} - \frac{i}{2}\right)^9$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^9 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^9$$

$= \cos 3\pi + i\sin 3\pi \qquad$ [$\because$ for any natural number 'n' $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$]

$= -1$

3. Given, $x^2 + x + 1 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{3}i}{2}$

[$\because$ Roots of quadratic equation $ax^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$]

$\Rightarrow z_0 = \omega, \omega^2$ [where $\omega = \dfrac{-1 + \sqrt{3}\,i}{2}$ and $\omega^2 = \dfrac{-1 - \sqrt{3}\,i}{2}$ are the cube roots of unity and $\omega, \omega^2 \neq 1$)

Now consider $z = 3 + 6i\,z_0^{81} - 3i\,z_0^{93}$

$= 3 + 6i - 3i \qquad (\because \omega^{3n} = (\omega^2)^{3n} = 1)$

$= 3 + 3i = 3(1 + i)$

If 'θ' is the argument of z, then

$\tan\theta = \dfrac{\text{Im}(z)}{Re(z)} \qquad [\because z \text{ is in the first quadrant}]$

$= \dfrac{3}{3} = 1 \Rightarrow \theta = \dfrac{\pi}{4}$

4. Given that, $z = \cos\theta + i\sin\theta = e^{i\theta}$

$$\therefore \sum_{\mu=1}^{15} \text{I}\mu(\zeta^{2\mu-1}) = \sum_{\mu=1}^{15} \text{I}\mu(\varepsilon^{\iota\theta})^{2\mu-1} = \sum_{\mu=1}^{15} \text{I}\mu\ \varepsilon^{\iota(2\mu-1)\theta}$$

$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin 29\theta$

$$= \frac{\sin\left(\frac{\theta + 29\,\theta}{2}\right)\sin\left(\frac{15\times 2\,\theta}{2}\right)}{\sin\left(\frac{2\,\theta}{2}\right)}$$

$$= \frac{\sin(15\,\theta)\sin(15\,\theta)}{\sin\theta} = \frac{1}{4\sin 2^\circ}$$

5. Let $z = |a + b\omega + c\omega^2|$

$\Rightarrow z^2 = |a + b\omega + c\omega^2|^2 = (a^2 + b^2 + c^2 - ab - bc - ca)$

or $z^2 = \dfrac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$ ...(i)

Since, a, b, c are all integers but not all simultaneously equal.

$\Rightarrow$ If $a = b$ then $a \neq c$ and $b \neq c$

Because difference of integers = integer

$\Rightarrow (b-c)^2 \geq 1$ {as minimum difference of two consecutive integers is (± 1)} also $(c-a)^2 \geq 1$

and we have taken $a = b \Rightarrow (a-b)^2 = 0$

From Eq. (i), $\quad z^2 \geq \dfrac{1}{2}(0 + 1 + 1)$

$\Rightarrow \quad z^2 \geq 1$

Hence, minimum value of $|z|$ is 1 .

6. Given, $(1 + \omega^2)^n = (1 + \omega^4)^n$

$\Rightarrow \quad (-\omega)^n = (-\omega^2)^n \qquad [\because \omega^3 = 1 \text{ and } 1 + \omega + \omega^2 = 0]$

$\Rightarrow \quad \omega^n = 1$

$\Rightarrow \quad n = 3$ is the least positive value of n.

7. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$

Applying $R_2 \to R_2 - R_1;\ R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2-\omega^2 & \omega^2 - 1 \\ 0 & \omega^2 - 1 & \omega - 1 \end{vmatrix}$$

$$= (-2-\omega^2)(\omega - 1) - (\omega^2 - 1)^2$$
$$= -2\omega + 2 - \omega^3 + \omega^2 - (\omega^4 - 2\omega^2 + 1)$$
$$= 3\omega^2 - 3\omega = 3\omega(\omega - 1) \qquad [\because \omega^4 = \omega]$$

8. Since, $\arg \dfrac{z_1}{z_2} = \dfrac{\pi}{2}$

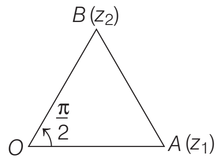
$\Rightarrow \quad \dfrac{z_1}{z_2} = \cos\dfrac{\pi}{2} + i \sin\dfrac{\pi}{2} = i$

$\therefore \quad \dfrac{z_1^n}{z_2^n} = (i)^n \Rightarrow i^n = 1 \qquad [\because |z_2| = |z_1| = 1]$

$\Rightarrow \quad n = 4k$

Alternate Solution

Since, $\quad \arg \dfrac{z_2}{z_1} = \dfrac{\pi}{2}$



$\therefore \quad \dfrac{z_2}{z_1} = \left|\dfrac{z_2}{z_1}\right| e^{i\frac{\pi}{2}}$

$\Rightarrow \quad \dfrac{z_2}{z_1} = i \qquad [\because |z_1| = |z_2| = 1]$

$\Rightarrow \quad \left(\dfrac{z_2}{z_1}\right)^n = i^n$

$\therefore$ z_1 and z_2 are nth roots of unity.

$z_1^n = z_2^n = 1$

$\Rightarrow \quad \left(\dfrac{z_2}{z_1}\right)^n = 1$

$\Rightarrow \quad i^n = 1$

$\Rightarrow \quad n = 4k$, where k is an integer.

9. We know that,

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$\therefore\ 4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$$

$$= 4 + 5\omega^{334} + 3\omega^{365}$$
$$= 4 + 5\cdot(\omega^3)^{111}\cdot\omega + 3\cdot(\omega^3)^{121}\cdot\omega^2$$
$$= 4 + 5\omega + 3\omega^2 \qquad [\because \omega^3 = 1]$$
$$= 1 + 3 + 2\omega + 3\omega + 3\omega^2$$
$$= 1 + 2\omega + 3(1 + \omega + \omega^2) = 1 + 2\omega + 3 \times 0$$
$$[\because 1 + \omega + \omega^2 = 0]$$
$$= 1 + (-1 + \sqrt{3}i) = \sqrt{3}i$$

10. $(1 + \omega - \omega^2)^7 = (-\omega^2 - \omega^2)^7 \qquad [\because 1 + \omega + \omega^2 = 0]$

$$= (-2\omega^2)^7 = (-2)^7\omega^{14} = -128\omega^2$$

11. $(1 + \omega)^7 = (1 + \omega)(1 + \omega)^6$

$$= (1 + \omega)(-\omega^2)^6 = 1 + \omega$$

$\Rightarrow \quad A + B\omega = 1 + \omega$

$\Rightarrow \quad A = 1, B = 1$

12. $\displaystyle\sum_{k=1}^{6}\left(\sin\frac{2k\pi}{7} - i\cos\frac{2k\pi}{7}\right)$

$$= \sum_{k=1}^{6} -i\left(\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}\right)$$

$$= -i\left\{\sum_{k=1}^{6} e^{\frac{i2k\pi}{7}}\right\} = -i\{e^{i2\pi/7} + e^{i4\pi/7} + e^{i6\pi/7} + e^{i8\pi/7} + e^{i10\pi/7} + e^{i12\pi/7}\}$$

$$= -i\left\{e^{i2\pi/7}\frac{(1 - e^{i12\pi/7})}{1 - e^{i2\pi/7}}\right\}$$

$$= -i\left\{\frac{e^{i2\pi/7} - e^{i14\pi/7}}{1 - e^{i2\pi/7}}\right\} \qquad [\because e^{i14\pi/7} = 1]$$

$$= -i\left\{\frac{e^{i2\pi/7} - 1}{1 - e^{i2\pi/7}}\right\} = i$$

13. (P) **PLAN** $e^{i\theta}\cdot e^{i\alpha} = e^{i(\theta + \alpha)}$

Given $\quad z_k = e^{i\frac{2k\pi}{10}} \Rightarrow z_k \cdot z_j = e^{i\left(\frac{2\pi}{10}\right)(k+j)}$

z_k is 10th root of unity.

$\Rightarrow$ $\bar{z}_k$ will also be 10th root of unity.

Taking, z_j as $\bar{z}_k$, we have $z_k \cdot z_j = 1$ (True)

(Q) **PLAN** $\dfrac{e^{i\theta}}{e^{i\alpha}} = e^{i(\theta - \alpha)}$

$$z = z_k / z_1 = e^{i\left(\frac{2k\pi}{10} - \frac{2\pi}{10}\right)} = e^{i\frac{\pi}{5}(k-1)}$$

For $k = 2;\ z = e^{i\frac{\pi}{5}}$ which is in the given set (False)

(R) **PLAN**

(i) $1 - \cos 2\theta = 2\sin^2\theta$

(ii) $\sin 2\theta = 2\sin\theta\cos\theta$ and

(i) $\cos 36° = \dfrac{\sqrt{5} - 1}{4}$

(ii) $\cos 108° = \dfrac{\sqrt{5} + 1}{4}\ \dfrac{|1 - z_1||1 - z_2|\ldots|1 - z_9|}{10}$

NOTE $|1 - z_k| = \left|1 - \cos\dfrac{2\pi k}{10} - i\sin\dfrac{2\pi k}{10}\right|$

$$= \left|2\sin\frac{\pi k}{10}\right|\left|\sin\frac{\pi k}{10} - i\cos\frac{\pi k}{10}\right| = 2\left|\sin\frac{\pi k}{10}\right|$$

Now, required product is

$$\frac{2^9 \sin\frac{\pi}{10}\cdot\sin\frac{2\pi}{10}\cdot\sin\frac{3\pi}{10}\ldots\sin\frac{8\pi}{10}\cdot\sin\frac{9\pi}{10}}{10}$$

$$= \frac{2^9\left(\sin\frac{\pi}{10}\sin\frac{2\pi}{10}\sin\frac{3\pi}{10}\sin\frac{4\pi}{10}\right)^2\sin\frac{5\pi}{10}}{10}$$

$$= \frac{2^9\left(\sin\frac{\pi}{10}\cos\frac{\pi}{10}\cdot\sin\frac{2\pi}{10}\cos\frac{2\pi}{10}\right)^2\cdot 1}{10}$$

$$= \frac{2^9 \left(\frac{1}{2}\sin\frac{\pi}{5}\cdot\frac{1}{2}\sin\frac{2\pi}{5}\right)^2}{10}$$
$$= \frac{2^5\ (\sin 36°\cdot \sin 72°)^2}{10}$$
$$= \frac{2^5}{2^2\times 10}(2\sin 36° \sin 72°)^2$$
$$= \frac{2^2}{5}(\cos 36° - \cos 108°)^2$$
$$= \frac{2^2}{5}\left[\left(\frac{\sqrt{5}-1}{4}\right)+\left(\frac{\sqrt{5}+1}{4}\right)\right]^2 = \frac{2^2}{5}\cdot\frac{5}{4} = 1$$

(S) Sum of nth roots of unity $= 0$

$$1+\alpha+\alpha^2+\alpha^3+\ldots+\alpha^9 = 0$$
$$1+\sum_{k=1}^{9}\alpha^k = 0$$
$$1+\sum_{k=1}^{9}\left(\cos\frac{2k\pi}{10}+i\sin\frac{2k\pi}{10}\right)=0$$
$$1+\sum_{k=1}^{9}\cos\frac{2k\pi}{10}=0$$

So, $$1-\sum_{k=1}^{9}\cos\frac{2k\pi}{10}=2$$

(P) → (i), (Q) → (ii), (R) → (iii), (S) → (iv)

14. Let $$A=\begin{bmatrix}1 & \omega & \omega^2\\ \omega & \omega^2 & 1\\ \omega^2 & 1 & \omega\end{bmatrix}$$

Now, $$A^2=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$ and $\text{Tr}(A)=0$, $|A|=0$

$A^3 = 0$

$$\Rightarrow \begin{vmatrix}z+1 & \omega & \omega^2\\ \omega & z+\omega^2 & 1\\ \omega^2 & 1 & z+\omega\end{vmatrix} = [A+zI]=0$$

$\Rightarrow$ $z^3 = 0$

$\Rightarrow z=0$, the number of z satisfying the given equation is 1.

15. Here, $T_r = (r-1)(r-\omega)(r-\omega)^2] = (r^3-1)$

$\therefore$ $$S_n=\sum_{r=1}^{n}(r^3-1)=\left[\frac{n(n+1)}{2}\right]^2-n$$

16. Since, cube root of unity are $1, \omega, \omega^2$ given by

$$A(1,0), B\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right), C\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$$

$\Rightarrow AB = BC = CA = \sqrt{3}$

Hence, cube roots of unity form an equilateral triangle.

17. Given, $$z^{p+q}-z^p-z^q+1=0 \quad \ldots\text{(i)}$$
$$\Rightarrow (z^p-1)(z^q-1)=0$$

Since, α is root of Eq. (i), either $\alpha^p-1=0$ or $\alpha^q-1=0$

$\Rightarrow$ Either $\frac{\alpha^p-1}{\alpha-1}=0$ or $\frac{\alpha^q-1}{\alpha-1}=0$ $\quad$ [as $\alpha\neq 1$]

$\Rightarrow$ Either $1+\alpha+\alpha^2+\ldots+\alpha^{p-1}=0$

or $1+\alpha+\ldots+\alpha^{q-1}=0$

But $\alpha^p-1=0$ and $\alpha^q-1=0$ cannot occur simultaneously as p and q are distinct primes, so neither p divides q nor q divides p, which is the requirement for $1=\alpha^p=\alpha^q$.

18. Since, $1, a_1, a_2, \ldots, a_{n-1}$ are nth roots of unity.

$\Rightarrow (x^n-1)=(x-1)(x-a_1)(x-a_2)\ldots(x-a_{n-1})$

$\Rightarrow \frac{x^n-1}{x-1}=(x-a_1)(x-a_2)\ldots(x-a_{n-1})$

$\Rightarrow x^{n-1}+x^{n-2}+\ldots+x^2+x+1$
$= (x-a_1)(x-a_2)\ldots(x-a_{n-1})$
$\left[\because \frac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+\ldots+x+1\right]$

On putting $x=1$, we get $1+1+\ldots$ n times
$=(1-a_1)(1-a_2)\ldots(1-a_{n-1})$

$\Rightarrow (1-a_1)(1-a_2)\ldots(1-a_{n-1})=n$

19. Since, n is not a multiple of 3, but odd integers and

$x^3+x^2+x=0 \Rightarrow x=0,\omega,\omega^2$

Now, when $x=0$

$\Rightarrow (x+1)^n-x^n-1=1-0-1=0$

$\therefore$ $x=0$ is root of $(x+1)^n-x^n-1$

Again, when $x=\omega$

$\Rightarrow (x+1)^n-x^n-1=(1+\omega)^n-\omega^n-1=-\omega^{2n}-\omega^n-1=0$

[as n is not a multiple of 3 and odd]

Similarly, $x=\omega^2$ is root of $\{(x+1)^n-x^n-1\}$

Hence, $x=0,\omega,\omega^2$ are the roots of $(x+1)^n-x^n-1$

Thus, x^3+x^2+x divides $(x+1)^n-x^n-1$.

20. Since, α, β are the complex cube roots of unity.

$\therefore$ We take $\alpha=\omega$ and $\beta=\omega^2$.

Now, $xyz=(a+b)(a\alpha+b\beta)(a\beta+b\alpha)$

$=(a+b)[a^2\alpha\beta+ab(\alpha^2+\beta^2)+b^2\alpha\beta]$

$=(a+b)[a^2(\omega\cdot\omega^2)+ab(\omega^2+\omega^4)+b^2(\omega\cdot\omega^2)]$

$=(a+b)(a^2-ab+b^2)$ $\quad$ [$\because 1+\omega+\omega^2=0$ and $\omega^3=1$]

$=a^3+b^3$

21. Priniting error $= e^{i\frac{2\pi}{3}}$

Then, $$\frac{|x|^2+|y|^2+|z|^2}{(a)^2+(b)^2+|c|^2}=3$$

NOTE Here, $w=e^{i\frac{2\pi}{3}}$, then only integer solution exists.