5

Binomial Theorem

Topic 1 Binomial Expansion and General Term

Objective Questions I (Only one correct option)

1. The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1+x+x^2)^9$ is **(2019 Main, 12 April I)**

(a) 84 (b) -126
(c) -84 (d) 126

2. If the coefficients of x^2 and x^3 are both zero, in the expansion of the expression $(1+ax+bx^2)(1-3x)^{15}$ in powers of x, then the ordered pair (a, b) is equal to **(2019 Main, 10 April I)**

(a) (28, 315) (b) $(-21, 714)$
(c) (28, 861) (d) $(-54, 315)$

3. The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\cdot\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to **(2019 Main, 12 April II)**

(a) -72 (b) 36 (c) -36 (d) -108

4. The smallest natural number n, such that the coefficient of x in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$ is ${}^nC_{23}$, is **(2019 Main, 10 April II)**

(a) 35 (b) 23 (c) 58 (d) 38

5. If some three consecutive coefficients in the binomial expansion of $(x+1)^n$ in powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficients is **(2019 Main, 9 April II)**

(a) 964 (b) 227 (c) 232 (d) 625

6. If the fourth term in the binomial expansion of $\left(\frac{2}{x}+x^{\log_8 x}\right)^6$ $(x>0)$ is 20×8^7, then the value of x is **(2019 Main, 9 April I)**

(a) 8^{-2} (b) 8^3
(c) 8 (d) 8^2

7. If the fourth term in the binomial expansion of $\left(\sqrt{x^{\left(\frac{1}{1+\log_{10} x}\right)}}+x^{\frac{1}{12}}\right)^6$ is equal to 200, and $x>1$, then the value of x is **(2019 Main, 8 April II)**

(a) 100 (b) 10^4
(c) 10 (d) 10^3

8. The sum of the coefficients of all even degree terms is x in the expansion of **(2019 Main, 8 April I)**
$(x+\sqrt{x^3-1})^6+(x-\sqrt{x^3-1})^6$, $(x>1)$ is equal to

(a) 29 (b) 32 (c) 26 (d) 24

9. The total number of irrational terms in the binomial expansion of $(7^{1/5}-3^{1/10})^{60}$ is **(2019 Main, 12 Jan II)**

(a) 49 (b) 48 (c) 54 (d) 55

10. The ratio of the 5th term from the beginning to the 5th term from the end in the binomial expansion of $\left(2^{\frac{1}{3}}+\frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is **(2019 Main, 12 Jan I)**

(a) $1: 2(6)^{\frac{1}{3}}$ (b) $1: 4(16)^{\frac{1}{3}}$ (c) $4(36)^{\frac{1}{3}}:1$ (d) $2(36)^{\frac{1}{3}}:1$

11. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$ equals 5670 is **(2019 Main, 11 Jan I)**

(a) 4 (b) 0 (c) 6 (d) 8

12. The positive value of λ for which the coefficient of x^2 in the expression $x^2\left(\sqrt{x}+\frac{\lambda}{x^2}\right)^{10}$ is 720, is **(2019 Main, 10 Jan II)**

(a) 3 (b) $\sqrt{5}$ (c) $2\sqrt{2}$ (d) 4

13. If the third term in the binomial expansion of $(1+x^{\log_2 x})^5$ equals 2560, then a possible value of x is **(2019 Main, 10 Jan I)**

(a) $4\sqrt{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $2\sqrt{2}$

14. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is **(2019 Main, 9 Jan II)**

(a) 12 (b) 10 (c) 15 (d) 14

15. The sum of the coefficients of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x>1)$ is **(2018 Main)**

(a) -1 (b) 0 (c) 1 (d) 2

16. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is **(2017 Main)**

(a) $2^{21} - 2^{11}$ (b) $2^{21} - 2^{10}$
(c) $2^{20} - 2^9$ (d) $2^{20} - 2^{10}$

17. If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is **(2016 Main)**

(a) 64 (b) 2187
(c) 243 (d) 729

18. The sum of coefficients of integral powers of x in the binomial expansion $(1 - 2\sqrt{x})^{50}$ is **(2015 Main)**

(a) $\frac{1}{2}(3^{50} + 1)$ (b) $\frac{1}{2}(3^{50})$ (c) $\frac{1}{2}(3^{50} - 1)$ (d) $\frac{1}{2}(2^{50} + 1)$

19. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is **(2014 Adv.)**

(a) 1051 (b) 1106 (c) 1113 (d) 1120

20. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$ is **(2013 Main)**

(a) 4 (b) 120 (c) 210 (d) 310

21. Coefficient of t^{24} in $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is **(2003, 1M)**

(a) ${}^{12}C_6 + 3$ (b) ${}^{12}C_6 + 1$ (c) ${}^{12}C_6$ (d) ${}^{12}C_6 + 2$

22. In the binomial expansion of $(a - b)^n$, $n \geq 5$ the sum of the 5th and 6th terms is zero. Then, a/b equals **(2001, 1M)**

(a) $\frac{n-5}{6}$ (b) $\frac{n-4}{5}$ (c) $\frac{5}{n-4}$ (d) $\frac{6}{n-5}$

23. If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and -6 respectively, then m is euqal to **(1999, 2M)**

(a) 6 (b) 9 (c) 12 (d) 24

24. The expression $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree **(1992, 2M)**

(a) 5 (b) 6 (c) 7 (d) 8

25. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is **(1983, 1M)**

(a) $\frac{405}{256}$ (b) $\frac{504}{259}$
(c) $\frac{450}{263}$ (d) None of these

26. Given positive integers $r > 1$, $n > 2$ and the coefficient of $(3r)$th and $(r + 2)$th terms in the binomial expansion of $(1 + x)^{2n}$ are equal. Then, **(1980, 2M)**

(a) $n = 2r$ (b) $n = 2r + 1$
(c) $n = 3r$ (d) None of these

27. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to

(a) $\left(16, \frac{251}{3}\right)$ (b) $\left(14, \frac{251}{3}\right)$
(c) $\left(14, \frac{272}{3}\right)$ (d) $\left(16, \frac{272}{3}\right)$

Fill in the Blanks

28. Let n be a positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1 + x)^n$ are in AP, then the value of n is… . **(1994, 2M)**

29. If $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$, then $a = \ldots$ and $n = \ldots$. **(1983, 2M)**

30. For any odd integer $n \geq 1$, $n^3 - (n-1)^3 + \ldots + (-1)^{n-1} 1^3 = \ldots$

31. The larger of $99^{50} + 100^{50}$ and 101^{50} is ... .

Analytical & Descriptive Questions

32. Prove that $\sum_{r=1}^{k} (-3)^{r-1} \, {}^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and n is an even positive integer. **(1993, 5M)**

33. If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ and $a_k = 1$, $\forall\ k \geq n$, then show that $b_n = {}^{2n+1}C_{n+1}$ **(1992, 6M)**

34. Find the sum of the series

$$\sum_{r=0}^{n} (-1)^r \, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} \ldots \text{upto } m \text{ terms}\right].$$

(1985, 5M)

35. Given, $s_n = 1 + q + q^2 + \ldots + q^n$

$$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n, q \neq 1$$

Prove that ${}^{n+1}C_1 + {}^{n+1}C_2 \, s_1 + {}^{n+1}C_3 s_2 + \ldots + {}^{n+1}C_{n+1} \, s_n = 2^n S_n$ **(1984, 4M)**

Integer Answer Type Question

36. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then, the value of n is **(2016 Adv.)**

37. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3) \ldots (1 + x^{100})$ is **(2015 Adv.)**

38. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio $5 : 10 : 14$. Then, n is equal to **(2013 Adv.)**

Topic 2 Properties of Binomial Coefficient

Objective Questions I (Only one correct option)

1. Let $(x+10)^{50}+(x-10)^{50}=a_0+a_1x+a_2x^2+\ldots+a_{50}x^{50}$, for all $x\in R$; then $\frac{a_2}{a_0}$ is equal to **(2019 Main, 11 Jan II)**

(a) 12.25 (b) 12.50 (c) 12.00 (d) 12.75

2. The value of r for which

$${}^{20}C_r\,{}^{20}C_0+{}^{20}C_{r-1}\,{}^{20}C_1+{}^{20}C_{r-2}\,{}^{20}C_2+\ldots+{}^{20}C_0\,{}^{20}C_r$$

is maximum, is **(2019 Main, 11 Jan I)**

(a) 15 (b) 10 (c) 11 (d) 20

3. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to **(2019 Main, 9 Jan I)**

(a) 14 (b) 6 (c) 4 (d) 8

4. For $r=0,1,\ldots,10$, if A_r, B_r and C_r denote respectively the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Then, $\sum_{r=1}^{10}A_r(B_{10}B_r-C_{10}A_r)$ is equal to

(a) $B_{10}-C_{10}$ (b) $A_{10}(B_{10}^2-C_{10}A_{10})$ **(2010)**
(c) 0 (d) $C_{10}-B_{10}$

5. $\binom{30}{0}\binom{30}{10}-\binom{30}{1}\binom{30}{11}+\binom{30}{2}\binom{30}{12}+\ldots+\binom{30}{20}\binom{30}{30}$ is equal to **(2005, 1M)**

(a) ${}^{30}C_{11}$ (b) ${}^{60}C_{10}$
(c) ${}^{30}C_{10}$ (d) ${}^{65}C_{55}$

6. If ${}^{n-1}C_r=(k^2-3)\,{}^nC_{r+1}$, then k belongs to **(2004, 1M)**

(a) $(-\infty,-2]$ (b) $[2,\infty)$
(c) $[-\sqrt{3},\sqrt{3}]$ (d) $(\sqrt{3},2]$

7. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, where $\binom{p}{q}=0$ if $p>q$, is maximum when m is equal to **(2002, 1M)**

(a) 5 (b) 10 (c) 15 (d) 20

8. For $2\le r\le n$, $\binom{n}{r}+2\binom{n}{r-1}+\binom{n}{r-2}$ is equal to **(2000, 2M)**

(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$ (c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

9. If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ equals

(a) $(n-1)\,a_n$ (b) $n\,a_n$ **(1998, 2M)**
(c) $\frac{1}{2}n\,a_n$ (d) None of these

10. If C_r stands for ${}^nC_r$, then the sum of the series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2-2C_1^2+3C_2^2-\ldots+(-1)^n(n+1)C_n^2],$$

where n is an even positive integer, is **(1986, 2M)**

(a) $(-1)^{n/2}(n+2)$ (b) $(-1)^n(n+1)$
(c) $(-1)^{n/2}(n+1)$ (d) None of these

Numerical Value

11. Let $X=({}^{10}C_1)^2+2({}^{10}C_2)^2+3({}^{10}C_3)^2+\ldots+10({}^{10}C_{10})^2$, where ${}^{10}C_r$, $r\in\{1,2,\ldots,10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is .... **(2018 Adv.)**

Fill in the Blank

12. The sum of the coefficients of the polynomial $(1+x-3x^2)^{2163}$ is .... . **(1982, 2M)**

Analytical & Descriptive Questions

13. Prove that

$$2^k\binom{n}{0}\binom{n}{k}-2^{k-1}\binom{n}{1}\binom{n-1}{k-1}+2^{k-2}\binom{n}{2}\binom{n-2}{k-2}-\ldots+(-1)^k\binom{n}{k}\binom{n-k}{0}=\binom{n}{k}$$ **(2003, 4 M)**

14. For any positive integers m,n (with $n\ge m$),

If $\binom{n}{m}={}^nC_m$. Prove that

$$\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\ldots+\binom{m}{m}=\binom{n+1}{m+1}$$

or

Prove that

$$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+\ldots+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}$$ **(IIT JEE 2000, 6M)**

15. Prove that $\frac{3!}{2(n+3)}=\sum_{r=0}^{n}(-1)^r\left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$. **(1997C, 5M)**

16. If n is a positive integer and

$$(1+x+x^2)^n=a_0+a_1x+\ldots+a_{2n}x^{2n}.$$

Then, show that, $a_0^2-a_1^2+\ldots+a_{2n}^2=a_n$. **(1994, 5M)**

17. Prove that $C_0-2^2\cdot C_1+3^2\cdot C_2-\ldots+(-1)^n(n+1)^2\cdot C_n=0$, $n>2$, where $C_r={}^nC_r$. **(1989, 5M)**

18. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, then show that the sum of the products of the C_i's taken two at a time represented by $\Sigma\Sigma\, C_iC_j$ is equal to

$$0\le i<j\le n \quad 2^{2n-1}-\frac{(2n!)}{2(n!)^2}.$$ **(1983, 3M)**

19. Prove that $C_1^2-2\cdot C_2^2+3\cdot C_3^2-\ldots-2n\cdot C_{2n}^2=(-1)^n n\cdot C_n$ **(1979, 4M)**

20. Prove that $({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-\ldots+({}^{2n}C_{2n})^2=(-1)^n\cdot{}^{2n}C_n$. **(1978, 4M)**

Answers

Topic 1

1. (a)	**2.** (a)	3. (c)	4. (d)
5. (c)	**6.** (d)	**7.** (c)	**8.** (d)
9. (c)	**10.** (c)	**11.** (b)	**12.** (d)
13. (b)	**14.** (c)	**15.** (d)	**16.** (d)
17. (d)	**18.** (a)	**19.** (c)	**20.** (c)
21. (d)	**22.** (b)	**23.** (c)	**24.** (c)
25. (a)	**26.** (a)	**27.** (d)	**28.** ($n=7$)
29. ($a=2, n=4$)	**30.** $\frac{1}{4}(n+1)^2(2n-1)$		
31. $(101)^{50}$	**34.** $\left[\frac{2^{mn}-1}{2^{mn}(2^n-1)}\right]$		
36. (5)	**37.** (8)	**38.** ($n=6$)	

Topic 2

1. (a)	**2.** (d)	**3.** (d)	**4.** (d)
5. (c)	**6.** (d)	**7.** (c)	**8.** (d)
9. (c)	**10.** (a)	**11.** (646)	**12.** (−1)

Hints & Solutions

Topic 1 Binomial Expansion and General Term

1. Given expression is

$(1+x)(1-x)^{10}(1+x+x^2)^9$

$=(1+x)(1-x)[(1-x)(1+x+x^2)]^9$

$=(1-x^2)(1-x^3)^9$

Now, coefficient of x^{18} in the product

$$(1+x)(1-x)^{10}(1+x+x^2)^9$$

$=$ coefficient of x^{18} in the product $(1-x^2)(1-x^3)^9$

$=$ coefficient of x^{18} in $(1-x^3)^9$

$-$ coefficient of x^{16} in $(1-x^3)^9$

Since, $(r+1)^{\text{th}}$ term in the expansion of

$(1-x^3)^9$ is ${}^9C_r(-x^3)^r={}^9C_r(-1)^r x^{3r}$

Now, for x^{18}, $3r=18 \Rightarrow r=6$

and for x^{16}, $3r=16$

$\Rightarrow \quad r=\frac{16}{3}\notin N.$

$\therefore$ Required coefficient is ${}^9C_6=\frac{9!}{6!3!}=\frac{9\times8\times7}{3\times2}=84$

2. Given expression is $(1+ax+bx^2)(1-3x)^{15}$. In the expansion of binomial $(1-3x)^{15}$, the $(r+1)$ th term is

$$T_{r+1}={}^{15}C_r(-3x)^r={}^{15}C_r(-3)^r x^r$$

Now, coefficient of x^2, in the expansion of $(1+ax+bx^2)(1-3x)^{15}$ is

${}^{15}C_2(-3)^2+a\,{}^{15}C_1(-3)^1+b\,{}^{15}C_0(-3)^0=0$ (given)

$\Rightarrow (105\times9)-45a+b=0$

$\Rightarrow 45a-b=945$ ...(i)

Similarly, the coefficient of x^3, in the expansion of $(1+ax+bx^2)(1-3x)^{15}$ is

${}^{15}C_3(-3)^3+a\,{}^{15}C_2(-3)^2+b\,{}^{15}C_1(-3)^1=0$ (given)

$\Rightarrow -12285+945a-45b=0$

$\Rightarrow \quad 63a-3b=819$

$\Rightarrow \quad 21a-b=273$ ...(ii)

From Eqs. (i) and (ii), we get

$$24a=672\Rightarrow a=28$$

So, $\quad b=315$

$\Rightarrow \quad (a,b)=(28,315)$

3. **Key Idea** Use the general term (or $(r+1)$th term) in the expansion of binomial $(a+b)^n$

i.e. $\quad T_{r+1}={}^nC_r\,a^{n-r}\,b^r$

Let a binomial $\left(2x^2-\frac{3}{x^2}\right)^6$, it's $(r+1)$th term

$=T_{r+1}={}^6C_r(2x^2)^{6-r}\left(-\frac{3}{x^2}\right)^r$

$={}^6C_r(-3)^r(2)^{6-r}x^{12-2r-2r}$

$={}^6C_r(-3)^r(2)^{6-r}x^{12-4r}$ ...(i)

Now, the term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\left(2x^2-\frac{3}{x^2}\right)^6$

$=$ the term independent of x in the expansion of $\frac{1}{60}\left(2x^2-\frac{3}{x^2}\right)^6$ + the term independent of x in the expansion of $-\frac{x^8}{81}\left(2x^2-\frac{3}{x^2}\right)^6$

$=\frac{{}^6C_3}{60}(-3)^3(2)^{6-3}x^{12-4(3)}$ [put $r=3$]

$+\left(-\frac{1}{81}\right){}^6C_5(-3)^5(2)^{6-5}x^{12-4(5)}x^8$ [put $r=5$]

$=\frac{1}{3}(-3)^3\,2^3+\frac{3^5\times2(6)}{81}$

$=36-72=-36$

4. Given binomial is $\left(x^2+\frac{1}{x^3}\right)^n$, its $(r+1)^{\text{th}}$ term, is

$$T_{r+1} = {}^nC_r(x^2)^{n-r}\left(\frac{1}{x^3}\right)^r = {}^nC_r x^{2n-2r}\frac{1}{x^{3r}}$$
$$= {}^nC_r x^{2n-2r-3r} = {}^nC_r x^{2n-5r}$$

For the coefficient of x,

$2n-5r=1 \Rightarrow 2n=5r+1$ ...(i)

As coefficient of x is given as ${}^nC_{23}$, then either $r=23$ or $n-r=23$.

If $r=23$, then from Eq. (i), we get

$2n=5(23)+1$

$\Rightarrow 2n=115+1 \Rightarrow 2n=116 \Rightarrow n=58.$

If $n-r=23$, then from Eq. (i) on replacing the value of 'r', we get $2n=5(n-23)+1$

$\Rightarrow \quad 2n=5n-115+1 \quad \Rightarrow \quad 3n=114 \Rightarrow n=38$

So, the required smallest natural number $n=38$.

5. **Key Idea** Use general term of Binomial expansion $(x+a)^n$ i.e. $T_{r+1} = {}^nC_{r\,1}x^{n-r}a^r$

Given binomial is $(x+1)^n$, whose general term, is $T_{r+1} = {}^nC_r\,x^r$

According to the question, we have

$${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 2:15:70$$

Now, $\quad \dfrac{{}^nC_{r-1}}{{}^nC_r} = \dfrac{2}{15}$

$$\Rightarrow \quad \frac{\dfrac{n!}{(r-1)!(n-r+1)!}}{\dfrac{n!}{r!(n-r)!}} = \frac{2}{15}$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{2}{15} \Rightarrow 15r = 2n-2r+2$$

$$\Rightarrow \quad 2n-17r+2=0 \qquad \text{...(i)}$$

$$\text{Similarly,} \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{15}{70} \Rightarrow \frac{\dfrac{n!}{r!(n-r)!}}{\dfrac{n!}{(r+1)!(n-r-1)!}} = \frac{3}{14}$$

$$\Rightarrow \quad \frac{r+1}{n-r} = \frac{3}{14} \Rightarrow 14r+14 = 3n-3r$$

$$\Rightarrow \quad 3n-17r-14=0 \qquad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$n-16=0 \Rightarrow n=16$ and $r=2$

$$\text{Now, the average} = \frac{{}^{16}C_1 + {}^{16}C_2 + {}^{16}C_3}{3}$$
$$= \frac{16+120+560}{3} = \frac{696}{3} = 232$$

6. Given binomial is $\left(\frac{2}{x}+x^{\log_8 x}\right)^6$

Since, general term in the expansion of $(x+a)^n$ is $T_{r+1} = {}^nC_r x^{n-r}a^r$

$$\therefore \quad T_4 = T_{3+1} = {}^6C_3\left(\frac{2}{x}\right)^{6-3}(x^{\log_8 x})^3 = 20\times 8^7 \text{ (given)}$$

$$\Rightarrow 20\left(\frac{2}{x}\right)^3 x^{3\log_8 x} = 20\times 8^7 \qquad [\because {}^6C_3 = 20]$$

$$\Rightarrow \quad 2^3\,x^{[3(\log_8 x)-3]} = (2^3)^7 \Rightarrow x^{\left(\frac{3}{3}\log_2 x-3\right)} = (2^3)^6$$

$$\left[\because \log_{a^n}(x) = \frac{1}{n}\log_a x \text{ for } x>0;\, a>0, \neq 1\right]$$

$$\Rightarrow x^{(\log_2 x-3)} = 2^{18}$$

On taking $\log_2$ both sides, we get

$$(\log_2 x-3)\log_2 x = 18$$
$$\Rightarrow \quad (\log_2 x)^2 - 3\log_2 x - 18 = 0$$
$$\Rightarrow (\log_2 x)^2 - 6\log_2 x + 3\log_2 x - 18 = 0$$
$$\Rightarrow \log_2 x(\log_2 x-6) + 3(\log_2 x-6) = 0$$
$$\Rightarrow \quad (\log_2 x-6)(\log_2 x+3) = 0$$
$$\Rightarrow \quad \log_2 x = -3, 6$$
$$\Rightarrow \quad x = 2^{-3}, 2^6 \Rightarrow x = \frac{1}{8}, 8^2$$

7. Given binomial is $\left(\sqrt{x^{\left(\frac{1}{1+\log_{10} x}\right)}} + x^{\frac{1}{12}}\right)^6$

Since, the fourth term in the given expansion is 200.

$$\therefore \quad {}^6C_3\left(x^{\frac{1}{1+\log_{10} x}}\right)^{\frac{3}{2}}\left(x^{\frac{1}{12}}\right)^3 = 200$$

$$\Rightarrow \quad 20\times x^{\left[\frac{3}{2(1+\log_{10} x)}+\frac{1}{4}\right]} = 200$$

$$\Rightarrow \quad x^{\frac{3}{2(1+\log_{10} x)}+\frac{1}{4}} = 10$$

$$\Rightarrow \quad \left[\frac{3}{2(1+\log_{10} x)}+\frac{1}{4}\right]\log_{10} x = 1$$

[applying $\log_{10}$ both sides]

$$\Rightarrow [6+(1+\log_{10} x)]\log_{10} x = 4(1+\log_{10} x)$$
$$\Rightarrow \quad (7+\log_{10} x)\log_{10} x = 4+4\log_{10} x$$
$$\Rightarrow \quad t^2+7t = 4+4t \qquad [\text{let } \log_{10} x = t]$$
$$\Rightarrow \quad t^2+3t-4=0$$
$$\Rightarrow \quad t=1, -4 = \log_{10} x$$
$$\Rightarrow \quad x = 10, 10^{-4}$$

Since, $\quad x>1 \quad x=10$

8. **Key Idea** Use formula :
$(a+b)^n + (a-b)^n = 2[{}^nC_0\,a^n + {}^nC_2 a^{n-2}b^2 + {}^nC_4 a^{n-4}b^4 + \ldots\ldots]$

Given expression is $(x+\sqrt{x^3-1})^6 + (x-\sqrt{x^3-1})^6$

$$= 2\,[{}^6C_0 x^6 + {}^6C_2 x^4(\sqrt{x^3-1})^2 + {}^6C_4 x^2(\sqrt{x^3-1})^4 + {}^6C_6(\sqrt{x^3-1})^6]$$

$$\{\because (a+b)^n + (a-b)^n = 2\,[{}^nC_0 a^n + {}^nC_2 a^{n-2}b^2 + {}^nC_4 a^{n-4}b^4 + \ldots]\}$$

$= 2[{}^6C_0x^6 + {}^6C_2x^4(x^3-1) + {}^6C_4x^2(x^3-1)^2 + {}^6C_6(x^3-1)^3]$

The sum of the terms with even power of x

$= 2[{}^6C_0x^6 + {}^6C_2(-x^4) + {}^6C_4x^8 + {}^6C_4x^2 + {}^6C_6(-1-3x^6)]$

$= 2[{}^6C_0x^6 - {}^6C_2x^4 + {}^6C_4x^8 + {}^6C_4x^2 - 1 - 3x^6]$

Now, the required sum of the coefficients of even powers of x in

$(x+\sqrt{x^3-1})^6 + (x-\sqrt{x^3-1})^6$

$= 2\ [{}^6C_0 - {}^6C_2 + {}^6C_4 + {}^6C_4 - 1 - 3]$

$= 2[1-15+15+15-1-3] = 2(15-3) = 24$

9. The general term in the binomial expansion of $(a+b)^n$ is $T_{r+1} = {}^nC_r\, a^{n-r}b^r$.

So, the general term in the binomial expansion of $(7^{1/5} - 3^{1/10})^{60}$ is

$$T_{r+1} = {}^{60}C_r\,(7^{1/5})^{60-r}(-3^{1/10})^r$$

$$= {}^{60}C_r 7^{\frac{60-r}{5}}(-1)^r 3^{\frac{r}{10}} = (-1)^r\,{}^{60}C_r 7^{12-\frac{r}{5}}\ 3^{\frac{r}{10}}$$

The possible non-negative integral values of 'r' for which $\frac{r}{5}$ and $\frac{r}{10}$ are integer, where $r \le 60$, are $r = 0, 10, 20, 30, 40, 50, 60$.

$\therefore$ There are 7 rational terms in the binomial expansion and remaining $61 - 7 = 54$ terms are irrational terms.

10. Since, rth term from the end in the expansion of a binomial $(x+a)^n$ is same as the $(n-r+2)$th term from the beginning in the expansion of same binomial.

$\therefore$ Required ratio $= \dfrac{T_5}{T_{10-5+2}} = \dfrac{T_5}{T_7} = \dfrac{T_{4+1}}{T_{6+1}}$

$$\Rightarrow \quad \frac{T_5}{T_{10-5+2}} = \frac{{}^{10}C_4(2^{1/3})^{10-4}\left(\dfrac{1}{2(3)^{1/3}}\right)^4}{{}^{10}C_6(2^{1/3})^{10-6}\left(\dfrac{1}{2(3)^{1/3}}\right)^6}$$

$[\because T_{r+1} = {}^nC_r x^{n-r}a^r]$

$$= \frac{2^{6/3}\,(2(3)^{1/3})^6}{2^{4/3}\,(2(3)^{1/3})^4} \qquad [\because {}^{10}C_4 = {}^{10}C_6]$$

$$= 2^{6/3-4/3}\,(2(3)^{1/3})^{6-4}$$

$$= 2^{2/3}\cdot 2^2\cdot 3^{2/3} = 4(6)^{2/3} = 4(36)^{1/3}$$

So, the required ratio is $4(36)^{1/3} : 1$.

11. In the expansion of $\left(\dfrac{x^3}{3} + \dfrac{3}{x}\right)^8$, the middle term is T_{4+1}.

[$\because$ Here, $n = 8$, which is even, therefore middle term $= \left(\dfrac{n+2}{2}\right)$th term]

$$\therefore \quad 5670 = {}^8C_4\left(\frac{x^3}{3}\right)^4\left(\frac{3}{x}\right)^4 = \frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}\,x^8$$

$$\left[\because T_{r+1} = {}^8C_r\left(\frac{x^3}{3}\right)^{8-r}\left(\frac{3}{x}\right)^r\right]$$

$\Rightarrow \quad x^8 = 3^4 \Rightarrow x = \pm\sqrt{3}$

So, sum of all values of x i.e $+\sqrt{3}$ and $-\sqrt{3} = 0$

12. The general term in the expansion of binomial expression $(a+b)^n$ is $T_{r+1} = {}^nC_r\, a^{n-r}b^r$, so the general term in the expansion of binomial expression $x^2\left(\sqrt{x} + \dfrac{\lambda}{x^2}\right)^{10}$ is

$$T_{r+1} = x^2\left({}^{10}C_r(\sqrt{x})^{10-r}\left(\frac{\lambda}{x^2}\right)^r\right) = {}^{10}C_r\ x^2\cdot x^{\frac{10-r}{2}}\ \lambda^r\ x^{-2r}$$

$$= {}^{10}C_r\,\lambda^r\,x^{2+\frac{10-r}{2}-2r}$$

Now, for the coefficient of x^2, put $2 + \dfrac{10-r}{2} - 2r = 2$

$\Rightarrow \quad \dfrac{10-r}{2} - 2r = 0$

$\Rightarrow \quad 10 - r = 4r \Rightarrow r = 2$

So, the coefficient of x^2 is ${}^{10}C_2\,\lambda^2 = 720$ [given]

$\Rightarrow \quad \dfrac{10!}{2!8!}\lambda^2 = 720 \Rightarrow \dfrac{10\cdot9\cdot8!}{2\cdot8!}\lambda^2 = 720$

$\Rightarrow \quad 45\lambda^2 = 720$

$\Rightarrow \quad \lambda^2 = 16 \Rightarrow \lambda = \pm 4$

$\therefore \quad \lambda = 4 \qquad [\lambda > 0]$

13. The $(r+1)$th term in the expansion of $(a+x)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r}x^r$

$\therefore$ 3rd term in the expansion of $(1 + x^{\log_2 x})^5$ is ${}^5C_2(1)^{5-2}(x^{\log_2 x})^2$

$\Rightarrow {}^5C_2(1)^{5-2}(x^{\log_2 x})^2 = 2560$ (given)

$\Rightarrow \quad 10\,(x^{\log_2 x})^2 = 2560$

$\Rightarrow \quad x^{(2\log_2 x)} = 256$

$\Rightarrow \quad \log_2 x^{2\log_2 x} = \log_2 256$

(taking $\log_2$ on both sides)

$\Rightarrow \quad 2(\log_2 x)(\log_2 x) = 8 \qquad (\because \log_2 256 = \log_2 2^8 = 8)$

$(\log_2 x)^2 = 4$

$\Rightarrow \quad \log_2 x = \pm 2$

$\Rightarrow \quad \log_2 x = 2$ or $\log_2 x = -2$

$\Rightarrow \quad x = 4$ or $x = 2^{-2} = \dfrac{1}{4}$

14. Clearly, $\left(\dfrac{1-t^6}{1-t}\right)^3 = (1-t^6)^3\ (1-t)^{-3}$

$\therefore$ Coefficient of t^4 in $(1-t^6)^3\ (1-t)^{-3}$

$=$ Coefficient of t^4 in $(1 - t^{18} - 3t^6 + 3t^{12})(1-t)^{-3}$

$=$ Coefficient of t^4 in $(1-t)^{-3}$

$= {}^{3+4-1}C_4 = {}^6C_4 = 15$

($\because$ coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$)

15.

> **Key Idea** Use formula :
> $= (a+b)^n + (a-b)^n$
> $= 2({}^nC_0\,a^n + {}^nC_2a^{n-2}b^2 + {}^nC_4a^{n-4}b^4 + \ldots)$

We have, $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5$, $x > 1$

$= 2({}^5C_0x^5 + {}^5C_2x^3(\sqrt{x^3-1})^2 + {}^5C_4x(\sqrt{x^3-1})^4)$

$= 2(x^5 + 10x^3(x^3-1) + 5x(x^3-1)^2)$

$= 2(x^5 + 10x^6 - 10x^3 + 5x^7 - 10x^4 + 5x)$

Sum of coefficients of all odd degree terms is

$2\,(1 - 10 + 5 + 5) = 2$

16. $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$

$= ({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_{10})$

$= \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{20}) - (2^{10} - 1)$

$= \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{21} - 1) - (2^{10} - 1)$

$= \frac{1}{2}(2^{21} - 2) - (2^{10} - 1) = 2^{20} - 1 - 2^{10} + 1 = 2^{20} - 2^{10}$

17. Clearly, number of terms in the expansion of

$\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$ is $\frac{(n+2)(n+1)}{2}$ or ${}^{n+2}C_2$.

[assuming $\frac{1}{x}$ and $\frac{1}{x^2}$ distinct]

$\therefore \quad \frac{(n+2)(n+1)}{2} = 28$

$\Rightarrow (n+2)(n+1) = 56 = (6+1)(6+2) \Rightarrow n = 6$

Hence, sum of coefficients $= (1 - 2 + 4)^6 = 3^6 = 729$

Note As $\frac{1}{x}$ and $\frac{1}{x^2}$ are functions of same variables, therefore number of dissimilar terms will be $2n + 1$, i.e. odd, which is not possible. Hence, it contains error.

18. Let T_{t+1} be the general term in the expension of $(1 - 2\sqrt{x})^{50}$

$\therefore \quad T_{r+1} = {}^{50}C_r (1)^{50-r}(-2x^{1/2})^r = {}^{50}C_r 2^r x^{r/2}(-1)^r$

For the integral power of x, r should be even integer.

$\therefore$ Sum of coefficients $= \sum_{r=0}^{25} {}^{50}C_{2r}(2)^{2r}$

$= \frac{1}{2}[(1+2)^{50} + (1-2)^{50}] = \frac{1}{2}(3^{50} + 1)$

Alternate Solution

We have,

$(1 - 2\sqrt{x})^{50} = C_o - C_1 2\sqrt{x} + C_2(\sqrt{2x})^2 + \ldots + C_{50}(2\sqrt{x})^{50}$ ...(i)

$(1 + 2\sqrt{x})^{50} = C_o + C_1 2\sqrt{x} + C_2(2\sqrt{x})^2 + \ldots + C_{50}(2\sqrt{x})^{50}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$(1 - 2\sqrt{x})^{50} + (1 + 2\sqrt{x})^{50}$

$= 2[C_0 + C_2(2\sqrt{x})^2 + \ldots + C_{50}(2\sqrt{x})^{50}]$ ...(iii)

$\Rightarrow \quad \frac{(1 - 2\sqrt{x})^{50} + (1 + 2\sqrt{x})^{50}}{2}$

$= C_0 + C_2(2\sqrt{x})^2 + \ldots + C_{50}(2\sqrt{x})^{50}$

On putting $x = 1$, we get

$\frac{(1 - 2\sqrt{1})^{50} + (1 + 2\sqrt{1})^{50}}{2} = C_0 + C_2 + \ldots + C_{50}(2)^{50}$

$\Rightarrow \quad \frac{(-1)^{50} + (3)^{50}}{2} = C_0 + C_2(2)^2 + \ldots + C_{50}(2)^{50}$

$\Rightarrow \quad \frac{1 + 3^{50}}{2} = C_0 + C_2(2)^2 + \ldots + C_{50}(2)^{50}$

19. Coefficient of x^r in $(1 + x)^n$ is ${}^nC_r$.

In this type of questions, we find different composition of terms where product will give us x^{11}.

Now, consider the following cases for x^{11} in $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$.

Coefficient of $x^0\, x^3\, x^8$; Coefficient of $x^2\, x^9\, x^0$

Coefficient of $x^4\, x^3\, x^4$; Coefficient of $x^8\, x^3\, x^0$

$= {}^4C_0 \times {}^7C_1 \times {}^{12}C_2 + {}^4C_1 \times {}^7C_3 \times {}^{12}C_0 + {}^4C_2 \times {}^7C_1 \times {}^{12}C_1 + {}^4C_4 \times {}^7C_1 \times {}^{12}C_0$

$= 462 + 140 + 504 + 7 = 1113$

20. $\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{(x-1)}{x - x^{1/2}}\right]^{10}$

$= \left[\frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}(\sqrt{x} - 1)}\right]^{10}$

$= \left[\frac{(x^{1/3} + 1)(x^{2/3} + 1 - x^{1/3})}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}(\sqrt{x} - 1)}\right]^{10}$

$= \left[(x^{1/3} + 1) - \frac{(\sqrt{x} + 1)}{\sqrt{x}}\right]^{10} = (x^{1/3} - x^{-1/2})^{10}$

$\therefore$ The general term is

$T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r}(-x^{-1/2})^r = {}^{10}C_r (-1)^r x^{\frac{10-r}{3} - \frac{r}{2}}$

For independent of x, put

$\frac{10 - r}{3} - \frac{r}{2} = 0 \quad \Rightarrow \quad 20 - 2r - 3r = 0$

$\Rightarrow \quad 20 = 5r \quad \Rightarrow \quad r = 4$

$\therefore \quad T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$

21. Here, Coefficient of t^{24} in $\{(1 + t^2)^{12}(1 + t^{12})(1 + t^{24})\}$

= Coefficient of t^{24} in $\{(1 + t^2)^{12} \cdot (1 + t^{12} + t^{24} + t^{36})\}$

= Coefficient of t^{24} in

$\{(1 + t^2)^{12} + t^{12}(1 + t^2)^{12} + t^{24}(1 + t^2)^{12}\}$;

[neglecting $t^{36}(1 + t^2)^{12}$]

= Coefficient of $t^{24} = ({}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0) = 2 + {}^{12}C_6$

22. Given, $T_5 + T_6 = 0$

$\Rightarrow \quad {}^nC_4 a^{n-4} b^4 - {}^nC_5 a^{n-5} b^5 = 0$

$\Rightarrow \quad {}^nC_4 a^{n-4} b^4 = {}^nC_5 a^{n-5} b^5 \quad \Rightarrow \quad \frac{a}{b} = \frac{{}^nC_5}{{}^nC_4} = \frac{n-4}{5}$

23. $(1 + x)^m (1 - x)^n = \left[1 + mx + \frac{m(m-1)}{2}x^2 + \ldots\right]\left[1 - nx + \frac{n(n-1)}{2}x^2 - \ldots\right]$

$= 1 + (m - n)x + \left[\frac{m(m-1)}{2} + \frac{n(n-1)}{2} - mn\right]x^2 + \ldots$

term containing power of $x \geq 3$.

Now, $\quad m - n = 3$ ...(i)

[$\because$ coefficient of $x = 3$, given]

and $\quad \frac{1}{2}m(m-1) + \frac{1}{2}n(n-1) - mn = -6$

$\Rightarrow \quad m(m-1)+n(n-1)-2mn=-12$

$\Rightarrow \quad m^2-m+n^2-n-2mn=-12$

$\Rightarrow \quad (m-n)^2-(m+n)=-12$

$\Rightarrow \quad m+n=9+12=21 \quad \ldots\text{(ii)}$

On solving Eqs. (i) and (ii), we get $m=12$

24. We know that,

$$(a+b)^5+(a-b)^5={}^5C_0a^5+{}^5C_1a^4b+{}^5C_2a^3b^2+{}^5C_3a^2b^3+{}^5C_4ab^4+{}^5C_5b^5+{}^5C_0a^5-{}^5C_1a^4b+{}^5C_2a^3b^2-{}^5C_3a^2b^3+{}^5C_4ab^4-{}^5C_5b^5$$
$$=2\,[a^5+10a^3b^2+5ab^4]$$

$\therefore \quad [x+(x^3-1)^{1/2}]^5+[x-(x^3-1)^{1/2}]^5$
$$=2\,[x^5+10x^3(x^3-1)+5x(x^3-1)^2]$$

Therefore, the given expression is a polynomial of degree 7.

25. The general term in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is

$$t_{r+1}=(-1)^r\,{}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(\frac{3}{x^2}\right)^r=(-1)^r\,{}^{10}C_r\cdot\frac{3^r}{2^{10-r}}\cdot x^{10-3r}$$

For coefficient of x^4, we put $10-3r=4$

$\Rightarrow \quad 3r=6$

$\Rightarrow \quad r=2$

$\therefore$ Coefficient of x^4 in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}=(-1)^2\cdot{}^{10}C_2\cdot\frac{3^2}{2^8}$
$$=\frac{45\times 9}{256}=\frac{405}{256}$$

26. In the expansion $(1+x)^{2n}$, $t_{3r}={}^{2n}C_{3r-1}(x)^{3r-1}$

and $\quad t_{r+2}={}^{2n}C_{r+1}(x)^{r+1}$

Since, binomial coefficients of t_{3r} and t_{r+2} are equal.

$\therefore \quad {}^{2n}C_{3r-1}={}^{2n}C_{r+1}$

$\Rightarrow \quad 3r-1=r+1$ or $2n=(3r-1)+(r+1)$

$\Rightarrow \quad 2r=2$ or $2n=4r$

$\Rightarrow \quad r=1$ or $n=2r$

But $\quad r>1$

$\therefore$ We take, $n=2r$

27. To find the coefficient of x^3 and x^4, use the formula of coefficient of x^r in $(1-x)^n$ is $(-1)^r\,{}^nC_r$ and then simplify.

In expansion of $(1+ax+bx^2)(1-2x)^{18}$.

Coefficient of x^3 = Coefficient of x^3 in $(1-2x)^{18}$
+ Coefficient of x^2 in $a\,(1-2x)^{18}$
+ Coefficient of x in $b(1-2x)^{18}$

$$={}^{18}C_3\cdot 2^3+a\,{}^{18}C_2\cdot 2^2-b\,{}^{18}C_1\cdot 2$$

Given, coefficient of $x^3=0$

$\Rightarrow \quad {}^{18}C_3\cdot 2^3+a\,{}^{18}C_2\cdot 2^2-b\,{}^{18}C_1\cdot 2=0$

$\Rightarrow \quad -\frac{18\times 17\times 16}{3\times 2}\cdot 8+a\cdot\frac{18\times 17}{2}\cdot 2^2-b\cdot 18\cdot 2=0$

$\Rightarrow \quad 17a-b=\frac{34\times 16}{3} \quad \ldots\text{(i)}$

Similarly, coefficient of $x^4=0$

$\Rightarrow \quad {}^{18}C_4\cdot 2^4-a\cdot{}^{18}C_3 2^3+b\cdot{}^{18}C_2\cdot 2^2=0$

$\therefore \quad 32a-3b=240 \quad \ldots\text{(ii)}$

On solving Eqs. (i) and (ii), we get

$$a=16,\; b=\frac{272}{3}$$

28. Let the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^n$ is ${}^nC_1$, ${}^nC_2$, ${}^nC_3$.

According to given condition,

$$2\,({}^nC_2)={}^nC_1+{}^nC_3$$

$\Rightarrow \quad 2\frac{n\,(n-1)}{1\cdot 2}=n+\frac{n\,(n-1)\,(n-2)}{1\cdot 2\cdot 3}$

$\Rightarrow \quad n-1=1+\frac{(n-1)\,(n-2)}{6}$

$\Rightarrow \quad n-1=1+\frac{n^2-3n+2}{6}$

$\Rightarrow \quad 6n-6=6+n^2-3n+2$

$\Rightarrow \quad n^2-9n+14=0$

$\Rightarrow \quad (n-2)\,(n-7)=0$

$\Rightarrow \quad n=2$

or $\quad n=7$

But ${}^nC_3$ is true for $n\ge 3$, therefore $n=7$ is the answer.

29. Given,

$$(1+ax)^n=1+8x+24x^2+\ldots$$

$\Rightarrow \quad 1+anx+\frac{n\,(n-1)}{2!}a^2x^2+\ldots=1+8x+24x^2+\ldots$

$\therefore \quad an=8$ and $a^2\frac{n\,(n-1)}{2}=24$

$\Rightarrow \quad 8\,(8-a)=48$

$\Rightarrow \quad 8-a=6 \Rightarrow a=2$

Hence, $\quad a=2$ and $n=4$

30. Since, n is an odd integer, $(-1)^{n-1}=1$

and $n-1$, $n-3$, $n-5$, etc., are even integers, then

$n^3-(n-1)^3+(n-2)^3-(n-3)^3+\ldots+(-1)^{n-1}\cdot 1^3$

$$=n^3+(n-1)^3+(n-2)^3+\ldots+1^3-2\,[(n-1)^3+(n-3)^3+\ldots+2^3]$$

$$=\Sigma n^3-2\times 2^3\left[\left(\frac{n-1}{2}\right)^3+\left(\frac{n-3}{2}\right)^3+\ldots+1^3\right]$$

[$\because$ $n-1, n-3, \ldots$, are even integers]

$$=\Sigma n^3-16\left[\Sigma\left(\frac{n-1}{2}\right)^3\right]$$

$$=\left[\frac{n\,(n+1)}{2}\right]^2-16\left[\frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\right]^2$$

$$=\frac{1}{4}n^2\,(n+1)^2-\frac{16\,(n-1)^2(n+1)^2}{4\times 4\times 4}$$

$$=\frac{1}{4}\,(n+1)^2\,[n^2-(n-1)^2]=\frac{1}{4}\,(n+1)^2(2n-1)$$

31. Consider, $(101)^{50}-(99)^{50}-(100)^{50}$

$$=(100+1)^{50}-(100-1)^{50}-(100)^{50}$$
$$=\{(100)^{50}\,(1+0.01)^{50}-(1-0.01)^{50}-1)\}$$
$$=(100)^{50}\,\{2\cdot[{}^{50}C_1(0.01)+{}^{50}C_3\,(0.01)^3+\dots]-1\}$$
$$=(100)^{50}\,\{2\,[{}^{50}C_3\,(0.01)^3+{}^{50}C_5\,(0.01)^5+\dots]\}$$

$\therefore$ $(101)^{50}-\{(99)^{50}+(100)^{50}\}>0$

$\Rightarrow$ $(101)^{50}>(99)^{50}+(100)^{50}$

32. Since, n is an even positive integer, we can write

$$n=2m\,,m=1,2,3,\dots$$

Also, $k=\dfrac{3n}{2}=\dfrac{3(2m)}{2}=3m\quad\therefore\quad S=\sum_{r=1}^{3m}(-3)^{r-1}\cdot{}^{6m}C_{2r-1}$

i.e. $S=(-3)^0\;{}^{6m}C_1+(-3)\;{}^{6m}C_3+\dots+(-3)^{3m-1}\cdot{}^{6m}C_{3m-1}$ ...(i)

From the binomial expansion, we write

$$(1+x)^{6m}={}^{6m}C_0+{}^{6m}C_1x+{}^{6m}C_2\,x^2+\dots {}^{6m}C_{6m-1}x^{6m-1}+{}^{6m}C_{6m}x^{6m}\quad\dots(ii)$$

$$(1-x)^{6m}={}^{6m}C_0+{}^{6m}C_1(-x)+{}^{6m}C_2(-x)^2+\dots+{}^{6m}C_{6m-1}\,(-x)^{6m-1}+{}^{6m}C_{6m}(-x)^{6m}\quad\dots(iii)$$

On subtracting Eq. (iii) from Eq. (ii), we get

$$(1+x)^{6m}-(1-x)^{6m}=2\,[{}^{6m}C_1x+{}^{6m}C_3x^3+{}^{6m}C_5x^5+\dots+{}^{6m}C_{6m-1}x^{6m-1}]$$

$$\Rightarrow\ \frac{(1+x)^{6m}-(1-x)^{6m}}{2x}={}^{6m}C_1+{}^{6m}C_3x^2+{}^{6m}C_5x^4+\dots+{}^{6m}C_{6m-1}\,x^{6m-2}$$

Let $x^2=y$

$$\Rightarrow\ \frac{(1+\sqrt{y})^{6m}-(1-\sqrt{y})^{6m}}{2\sqrt{y}}={}^{6m}C_1+{}^{6m}C_3\,y+{}^{6m}C_5\,y^2+\dots+{}^{6m}C_{6m-1}y^{3m-1}$$

For the required sum we have to put $y=-3$ in RHS.

$$\therefore\quad S=\frac{(1+\sqrt{-3})^{6m}-(1-\sqrt{-3})^{6m}}{2\sqrt{-3}}$$
$$=\frac{(1+i\sqrt{3})^{6m}-(1-i\sqrt{3})^{6m}}{2i\sqrt{3}}\quad\dots(iv)$$

Let $z=1+i\,\sqrt{3}\ =r(\cos\theta+i\sin\theta)$

$\Rightarrow$ $r=|z|=\sqrt{1+3}=2$

and $\theta=\pi/3$

Now, $z^{6m}=[r\,(\cos\theta+i\sin\theta)]^{6m}$

$=r^{6m}(\cos 6m\,\theta+i\sin 6m\,\theta)$

Again, $\bar{z}=r\,(\cos\theta-i\sin\theta)$

and $(\bar{z})^{6m}=r^{6m}(\cos 6m\,\theta-i\sin 6m\,\theta)$

$\Rightarrow$ $z^{6m}-\bar{z}^{6m}=r^{6m}(2i\sin 6m\,\theta)$ ...(v)

From Eq. (i),

$$S=\frac{z^{6m}-\bar{z}^{6m}}{2i\,\sqrt{3}}=\frac{r^{6m}(2i\sin 6\,m\,\theta)}{2i\,\sqrt{3}}$$
$$=\frac{2^{6m}\sin 6\,m\,\theta}{\sqrt{3}}$$
$$=0\text{ as }m\in z,\text{ and }\theta=\pi/3$$

33. Let $y=(x-a)^m$, where m is a positive integer, $r\le m$

Now, $\dfrac{dy}{dx}=m(x-a)^{m-1}\ \Rightarrow\ \dfrac{d^2y}{dx^2}=m(m-1)\,(x-a)^{m-2}$

$$\Rightarrow\quad\frac{d^3y}{dx^3}=m(m-1)(m-2)(m-3)(x-a)^{m-4}$$

....

....

On differentiating r times, we get

$$\frac{d^ry}{dx^r}=m(m-1)\dots(m-r+1)(x-a)^{m-r}$$
$$=\frac{m!}{(m-r)!}\,(x-a)^{m-r}=r!({}^mC_r)(x-a)^{m-r}$$

and for $r>m,\ \dfrac{d^ry}{dx^r}=0$

Now, $\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r$ [given]

On differentiating both sides n times w.r.t. x, we get

$$\sum_{r=n}^{2n}a_r(n!)^rC_n(x-2)^{r-n}=\sum_{r=n}^{2n}b_r(n!)^rC_n(x-3)^{r-n}$$

On putting $x=3$, we get $\sum_{r=n}^{2n}a_r(n!)^rC_n=(b_n)n!$

[since, all the terms except first on RHS become zero]

$$\Rightarrow\quad b_n={}^nC_n+{}^{n+1}C_n+{}^{n+2}C_n+\dots+{}^{2n}C_n$$

$[\because a_r=1,\ \forall\ r\ge n]$

$$=({}^{n+2}C_{n+1}+{}^{n+2}C_n)+\dots+{}^{2n}C_n$$
$$={}^{n+3}C_{n+1}+\dots+{}^{2n}C_n=\dots.$$
$$={}^{2n}C_{n+1}+{}^{2n}C_n={}^{2n+1}C_{n+1}$$

34. $\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\dfrac{1}{2^r}+\dfrac{3^r}{2^{2r}}+\dfrac{7^r}{2^{3r}}+\dfrac{15^r}{2^{4r}}+\dots\text{ upto }m\text{ terms}\right]$

$$=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{1}{2}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{3}{4}\right)^r+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{7}{8}\right)^r+\dots\text{ upto }m\text{ terms}$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\dots\text{ upto }m\text{ terms}$$

$$\left[\text{using }\sum_{r=0}^{n}(-1)^r\,{}^nC_rx^r=(1-x)^n\right]$$

$$=\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n+\left(\frac{1}{8}\right)^n+\dots\text{ upto }m\text{ terms}$$

$$=\left(\frac{1}{2}\right)^n\left[\frac{1-\left(\dfrac{1}{2^n}\right)^m}{1-\dfrac{1}{2^n}}\right]=\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

35. ${}^{n+1}C_1+{}^{n+1}C_2s_1+{}^{n+1}C_3\,s_2+\dots+{}^{n+1}C_{n+1}s_n$

$$=\sum_{r=1}^{n+1}{}^{n+1}C_rs_{r-1},$$

where $s_n = 1 + q + q^2 + \ldots + q^n = \dfrac{1 - q^{n+1}}{1 - q}$

$$\therefore \sum_{r=1}^{n+1} {}^{n+1}C_r \left(\frac{1-q^r}{1-q}\right) = \frac{1}{1-q}\left(\sum_{r=1}^{n+1} {}^{n+1}C_r - \sum_{r=1}^{n+1} {}^{n+1}C_r\, q^r\right)$$

$$= \frac{1}{1-q}[(1+1)^{n+1} - (1+q)^{n+1}]$$

$$= \frac{1}{1-q}[2^{n+1} - (1+q)^{n+1}] \quad \ldots \text{(i)}$$

Also, $S_n = 1 + \left(\dfrac{q+1}{2}\right) + \left(\dfrac{q+1}{2}\right)^2 + \ldots + \left(\dfrac{q+1}{2}\right)^n$

$$= \frac{1 - \left(\frac{q+1}{2}\right)^{n+1}}{1 - \left(\frac{q+1}{2}\right)} = \frac{2^{n+1} - (q+1)^{n+1}}{2^n(1-q)} \quad \ldots \text{(ii)}$$

From Eqs. (i) and (ii),

$${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \ldots + {}^{n+1}C_{n+1} s_n = 2^n S_n$$

36. Coefficient of x^2 in the expansion of

$\{(1+x)^2 + (1+x)^3 + \ldots + (1+x)^{49} + (1+mx)^{50}\}$

$\Rightarrow {}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^{49}C_2 + {}^{50}C_2 \cdot m^2 = (3n+1) \cdot {}^{51}C_3$

$\Rightarrow {}^{50}C_3 + {}^{50}C_2 m^2 = (3n+1) \cdot {}^{51}C_3$

$[\because {}^rC_r + {}^{r+1}C_r + \ldots + {}^nC_r = {}^{n+1}C_{r+1}]$

$$\Rightarrow \frac{50 \times 49 \times 48}{3 \times 2 \times 1} + \frac{50 \times 49}{2} \times m^2 = (3n+1)\frac{51 \times 50 \times 49}{3 \times 2 \times 1}$$

$\Rightarrow m^2 = 51n + 1$

$\therefore$ Minimum value of m^2 for which $(51n + 1)$ is integer (perfect square) for $n = 5$.

$\therefore m^2 = 51 \times 5 + 1 \Rightarrow m^2 = 256$

$\therefore m = 16$ and $n = 5$

Hence, the value of n is 5.

37. Coefficient of x^9 in the expansion of

$(1+x)(1+x^2)(1+x^3)\ldots(1+x^{100})$ = Terms having x^9

$= [1^{99} \cdot x^9, 1^{98} \cdot x \cdot x^8, 1^{98} \cdot x^2 \cdot x^7, 1^{98} \cdot x^3 \cdot x^6,$
$1^{98} \cdot x^4 \cdot x^5, 1^{97} \cdot x \cdot x^2 \cdot x^6, 1^{97} \cdot x \cdot x^3 \cdot x^5, 1^{97} \cdot x^2 \cdot x^3 \cdot x^4]$

$\therefore$ Coefficient of $x^9 = 8$

38. Let the three consecutive terms in $(1+x)^{n+5}$ be t_r, t_{r+1}, t_{r+2} having coefficients ${}^{n+5}C_{r-1}, {}^{n+5}C_r, {}^{n+5}C_{r+1}$.

Given, ${}^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5 : 10 : 14$

$$\therefore \frac{{}^{n+5}C_r}{{}^{n+5}C_{r-1}} = \frac{10}{5} \text{ and } \frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_r} = \frac{14}{10}$$

$$\Rightarrow \frac{n+5-(r-1)}{r} = 2 \text{ and } \frac{n-r+5}{r+1} = \frac{7}{5}$$

$\Rightarrow n - r + 6 = 2r$ and $5n - 5r + 25 = 7r + 7$

$\Rightarrow n + 6 = 3r$ and $5n + 18 = 12r$

$$\therefore \frac{n+6}{3} = \frac{5n+18}{12}$$

$\Rightarrow 4n + 24 = 5n + 18 \Rightarrow n = 6$

Topic 2 Properties of Binomial Coefficient

1. We have,

$(x+10)^{50} + (x-10)^{50} = a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$

$\therefore a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$

$= [({}^{50}C_0x^{50} + {}^{50}C_1x^{49}\, 10 + {}^{50}C_2\, x^{48} \cdot 10^2 + \ldots + {}^{50}C_{50}\, 10^{50})$
$+ ({}^{50}C_0x^{50} - {}^{50}C_1x^{49}\, 10 + {}^{50}C_2\, x^{48} 10^2 - \ldots + {}^{50}C_{50}\, 10^{50})]$
$= 2\,[{}^{50}C_0\, x^{50} + {}^{50}C_2x^{48} \cdot 10^2 + {}^{50}C_4x^{46} \cdot 10^4 + \ldots + {}^{50}C_{50} \cdot 10^{50}]$

By comparing coefficients, we get

$a_2 = 2\; {}^{50}C_{48}(10)^{48};\;\; a_0 = 2\; {}^{50}C_{50}(10)^{50} = 2(10)^{50}$

$$\therefore \frac{a_2}{a_0} = \frac{2({}^{50}C_2)(10)^{48}}{2\,(10)^{50}} = 2\,\frac{50 \cdot 49}{1 \cdot 2}\,\frac{(10)^{48}}{2 \cdot (10)^{50}}$$

$[\because {}^{50}C_{48} = {}^{50}C_2]$

$$= \frac{50 \times 49}{2 \cdot (10 \times 10)} = \frac{5 \times 49}{20} = \frac{245}{20} = 12.25$$

2. We know that,

$(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1x + {}^{20}C_2x^2 + \ldots + {}^{20}C_{r-1}x^{r-1} + {}^{20}C_rx^r + \ldots + {}^{20}C_{20}x^{20}$

$\therefore (1+x)^{20} \cdot (1+x)^{20} = ({}^{20}C_0 + {}^{20}C_1x + {}^{20}C_2x^2 + \ldots + {}^{20}C_{r-1}x^{r-1} + {}^{20}C_rx^r + \ldots + {}^{20}c_{20}x^{20})$
$\times ({}^{20}C_0 + {}^{20}C_1x + \ldots + {}^{20}C_{r-1}x^{r-1} + {}^{20}C_rx^r + \ldots + {}^{20}C_{20}x^{20})$

$\Rightarrow (1+x)^{40} = ({}^{20}C_0 \,.\, {}^{20}C_r + {}^{20}C_1\; {}^{20}C_{r-1} \ldots {}^{20}C_r\, {}^{20}C_0)\, x^r + \ldots$

On comparing the coefficient of x^r of both sides, we get

$${}^{20}C_0{}^{20}C_r + {}^{20}C_1{}^{20}C_{r-1} + \ldots + {}^{20}C_r\; {}^{20}C_0 = {}^{40}C_r$$

The maximum value of ${}^{40}C_r$ is possible only when $r = 20$

$[\because {}^nC_{n/2}$ is maximum when n is even$]$

Thus, required value of r is 20.

3. Consider,

$2^{403} = 2^{400+3} = 8 \cdot 2^{400} = 8 \cdot (2^4)^{100} = 8\,(16)^{100} = 8(1+15)^{100}$
$= 8\,(1 + {}^{100}C_1(15) + {}^{100}C_2(15)^2 + \ldots + {}^{100}C_{100}(15)^{100})$

[By binomial theorem,

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\; {}^nC_nx^n,\; n \in N]$

$= 8 + 8\,({}^{100}C_1(15) + {}^{100}C_2(15)^2 + \ldots + {}^{100}C_{100}(15)^{100})$

$= 8 + 8 \times 15\lambda$

where $\lambda = {}^{100}C_1 + \ldots\ldots + {}^{100}C_{100}(15)^{99} \in N$

$$\therefore \frac{2^{403}}{15} = \frac{8 + 8 \times 15\lambda}{15} = 8\lambda + \frac{8}{15}$$

$$\Rightarrow \left\{\frac{2^{403}}{15}\right\} = \frac{8}{15}$$

(where $\{\cdot\}$ is the fractional part function)

$\therefore k = 8$

Alternate Method

$2^{403} = 8 \cdot 2^{400} = 8(16)^{100}$

Note that, when 16 is divided by 15, gives remainder 1.

$\therefore$ When $(16)^{100}$ is divided by 15, gives remainder $1^{100} = 1$ and when $8(16)^{100}$ is divided by 15, gives remainder 8.

$$\therefore \quad \left\{\frac{2^{403}}{15}\right\} = \frac{8}{15}.$$

(where $\{\cdot\}$ is the fractional part function)

$\Rightarrow \quad k = 8$

4. A_r = Coefficient of x^r in $(1+x)^{10} = {}^{10}C_r$

B_r = Coefficient of x^r in $(1+x)^{20} = {}^{20}C_r$

C_r = Coefficient of x^r in $(1+x)^{30} = {}^{30}C_r$

$$\therefore \sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r) = \sum_{r=1}^{10} A_r B_{10} B_r - \sum_{r=1}^{10} A_r C_{10} A_r$$

$$= \sum_{r=1}^{10} {}^{10}C_r\ {}^{20}C_{10}\ {}^{20}C_r - \sum_{r=1}^{10} {}^{10}C_r\ {}^{30}C_{10}\ {}^{10}C_r$$

$$= \sum_{r=1}^{10} {}^{10}C_{10-r}\ {}^{20}C_{10}\ {}^{20}C_r - \sum_{r=1}^{10} {}^{10}C_{10-r}\ {}^{30}C_{10}\ {}^{10}C_r$$

$$= {}^{20}C_{10} \sum_{r=1}^{10} {}^{10}C_{10-r} \cdot {}^{20}C_r - {}^{30}C_{10} \sum_{r=1}^{10} {}^{10}C_{10-r}\ {}^{10}C_r$$

$$= {}^{20}C_{10}\,({}^{30}C_{10} - 1) - {}^{30}C_{10}\,({}^{20}C_{10} - 1)$$

$$= {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$$

5. Let $A = \binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30}$

$$\therefore \quad A = {}^{30}C_0 \cdot {}^{30}C_{10} - {}^{30}C_1 \cdot {}^{30}C_{11} + {}^{30}C_2 \cdot {}^{30}C_{12} - \ldots + {}^{30}C_{20} \cdot {}^{30}C_{30}$$

= Coefficient of x^{20} in $(1+x)^{30}(1-x)^{30}$

= Coefficient of x^{20} in $(1-x^2)^{30}$

= Coefficient of x^{20} in $\sum_{r=0}^{30} (-1)^r\,{}^{30}C_r (x^2)^r$

$= (-1)^{10}\ {}^{30}C_{10}$ [for coefficient of x^{20}, put $r = 10$]

$= {}^{30}C_{10}$

6. Given, ${}^{n-1}C_r = (k^2 - 3)\ {}^nC_{r+1}$

$$\Rightarrow \quad {}^{n-1}C_r = (k^2-3)\frac{n}{r+1}\ {}^{n-1}C_r$$

$$\Rightarrow \quad k^2 - 3 = \frac{r+1}{n}$$

[since, $n \ge r \Rightarrow \frac{r+1}{n} \le 1$ and $n, r > 0$]

$\Rightarrow \quad 0 < k^2 - 3 \le 1 \quad \Rightarrow \quad 3 < k^2 \le 4$

$\Rightarrow \quad k \in [-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$

7. $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$ is the coefficient of x^m in the expansion of $(1+x)^{10}(x+1)^{20}$,

$\Rightarrow \sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$ is the coefficient of x^m in the expansion of $(1+x)^{30}$

i.e. $$\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i} = {}^{30}C_m = \binom{30}{m} \qquad \ldots\text{(i)}$$

and we know that, $\binom{n}{r}$ is maximum, when

$$\binom{n}{r}_{\max} = \begin{cases} r = \dfrac{n}{2}, & \text{if } n \in \text{even.} \\ r = \dfrac{n \pm 1}{2}, & \text{if } n \in \text{odd} \cdot \end{cases}$$

Hence, $\binom{30}{m}$ is maximum when $m = 15$.

8. $$\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} = \left[\binom{n}{r} + \binom{n}{r-1}\right] + \left[\binom{n}{r-1} + \binom{n}{r-2}\right] = \binom{n+1}{r} + \binom{n+1}{r-1} = \binom{n+2}{r}$$

$[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

9. Let $$b = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-(n-r)}{{}^nC_r}$$

$$= n\sum_{r=0}^{n} \frac{1}{{}^nC_r} - \sum_{r=0}^{n} \frac{n-r}{{}^nC_r}$$

$$= na_n - \sum_{r=0}^{n} \frac{n-r}{{}^nC_{n-r}} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= na_n - b \Rightarrow 2b = na_n \Rightarrow b = \frac{n}{2}a_n$$

10. We have,

$$C_0^2 - 2C_1^2 + 3C_2^2 - 4C_3^2 + \ldots + (-1)^n (n+1) C_n^2$$

$$= [C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots + (-1)^n C_n^2] - [C_1^2 - 2C_2^2 + 3C_3^2 - \ldots + (-1)^n nC_n^2]$$

$$= (-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} - (-1)^{\frac{n}{2}-1}\frac{n}{2}\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

$$= (-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}\left(1 + \frac{n}{2}\right)$$

$$\therefore \frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - \ldots + (-1)^r (n+1) C_n^2]$$

$$= \frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}(-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}\frac{(n+2)}{2} = (-1)^{n/2}(n+2)$$

11. We have,

$$X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \ldots + 10\,({}^{10}C_{10})^2$$

$$\Rightarrow \quad X = \sum_{r=1}^{10} r({}^{10}C_r)^2 \Rightarrow X = \sum_{r=1}^{10} r\ {}^{10}C_r\ {}^{10}C_r$$

$$\Rightarrow \quad X = \sum_{r=1}^{10} r \times \frac{10}{r}\ {}^9C_{r-1}\ {}^{10}C_r \qquad \left[\because {}^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow X = 10\sum_{r=1}^{10} {}^9C_{r-1}\ {}^{10}C_r$$

$$\Rightarrow X = 10\sum_{r=1}^{10} {}^9C_{r-1}\ {}^{10}C_{10-r} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$$\Rightarrow X = 10 \times {}^{19}C_9 \qquad [\because {}^{n-1}C_{r-1}\,{}^nC_{n-r} = {}^{2n-1}C_{n-1}]$$

$$\text{Now, } \frac{1}{1430}X = \frac{10\times {}^{19}C_9}{1430} = \frac{{}^{19}C_9}{143} = \frac{{}^{19}C_9}{11\times 13}$$

$$= \frac{19\times 17\times 16}{8} = 19\times 34 = 646$$

12. Sum of coefficients is obtained by putting $x = 1$

i.e. $(1 + 1 - 3)^{2163} = -1$

Thus, sum of the coefficients of the polynomial $(1 + x - 3x^2)^{2163}$ is -1.

13. To show that

$$2^k\cdot{}^nC_0\cdot{}^nC_k - 2^{k-1}\cdot{}^nC_1\cdot{}^{n-1}C_{k-1} + 2^{k-2}\cdot{}^nC_2\cdot{}^{n-2}C_{k-2} - \dots + (-1)^k\ {}^nC_k\,{}^{n-k}C_0 = {}^nC_k$$

Taking LHS

$$2^k\cdot{}^nC_0\cdot{}^nC_k - 2^{k-1}\cdot{}^nC_1\cdot{}^{n-1}C_{k-1} + \dots + (-1)^k\cdot{}^nC_k\cdot{}^{n-k}C_0$$

$$= \sum_{r=0}^{k}(-1)^r\cdot 2^{k-r}\cdot{}^nC_r\cdot{}^{n-r}C_{k-r}$$

$$= \sum_{r=0}^{k}(-1)^r\,2^{k-r}\cdot\frac{n!}{r!(n-r)!}\cdot\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$= \sum_{r=0}^{k}(-1)^r\cdot 2^{k-r}\cdot\frac{n!}{(n-k)!\cdot k!}\cdot\frac{k!}{r!(k-r)!}$$

$$= \sum_{r=0}^{k}(-1)^r\cdot 2^{k-r}\ {}^nC_k\cdot{}^kC_r = 2^k\cdot{}^nC_k\left\{\sum_{r=0}^{k}(-1)^r\cdot\frac{1}{2^r}\cdot{}^kC_r\right\}$$

$$= 2^k\cdot{}^nC_k\left(1-\frac{1}{2}\right)^k = {}^nC_k = \text{RHS}$$

14. Let $S = \binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \dots + \binom{m}{m} = \binom{n+1}{m+1}$ ...(i)

It is obvious that, $n \ge m$. [given]

NOTE This question is based upon additive loop.

$$\text{Now, } S = \binom{m}{m} + \binom{m+1}{m} + \binom{m+2}{m} + \dots + \binom{n}{m}$$

$$= \left\{\binom{m+1}{m+1} + \binom{m+1}{m}\right\} + \binom{m+2}{m} + \dots \binom{n}{m}$$

$$\left[\because \binom{m}{m} = 1 = \binom{m+1}{m+1}\right]$$

$$= \binom{m+2}{m+1} + \binom{m+2}{m} + \dots + \binom{n}{m}$$

$$[\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}]$$

$$= \binom{m+3}{m+1} + \dots + \binom{n}{m}$$

$= \dots\dots\dots\dots$

$$= \binom{n}{m+1} + \binom{n}{m} = \binom{n+1}{m+1}, \text{ which is true.} \qquad \text{...(ii)}$$

Again, we have to prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \dots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$

$$\text{Let } S_1 = \binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \dots + (n-m+1)\binom{m}{m}$$

$$\left.\begin{aligned} = \binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \dots + \binom{m}{m} \\ + \binom{n-1}{m} + \binom{n-2}{m} + \dots + \binom{m}{m} \\ + \binom{n-2}{m} + \dots + \binom{m}{m} \\ \dots\dots \\ + \binom{m}{m} \end{aligned}\right\} n-m+1 \text{ rows}$$

Now, sum of the first row is $\binom{n+1}{m+1}$.

Sum of the second row is $\binom{n}{m+1}$.

Sum of the third row is $\binom{n-1}{m+1}$,

$\dots\dots\dots\dots\dots\dots\dots\dots$

Sum of the last row is $\binom{m}{m} = \binom{m+1}{m+1}$.

$$\text{Thus,} \qquad S = \binom{n+1}{m+1} + \binom{n}{m+1} + \binom{n-1}{m+1} + \dots + \binom{m+1}{m+1} = \binom{n+1+1}{m+2} = \binom{n+2}{m+2}$$

[from Eq. (i) replacing n by $n + 1$ and m by $m + 1$]

15. $\displaystyle\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{{}^{r+3}C_r}$

$$= \sum_{r=0}^{n}(-1)^r\frac{n!\cdot 3!}{(n-r)!\,(r+3)!} = 3!\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!\,(r+3)!}$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}\frac{(-1)^r\cdot(n+3)!}{(n-r)!(r+3)!}$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\cdot\sum_{r=0}^{n}(-1)^r\cdot{}^{n+3}C_{r+3}$$

$$= \frac{3!(-1)^3}{(n+1)(n+2)(n+3)}\sum_{s=3}^{n+3}(-1)^s\cdot{}^{n+3}C_3$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)}\left(\sum_{s=0}^{n+3}(-1)^s\cdot{}^{n+3}C_s - {}^{n+3}C_0 + {}^{n+3}C_1 - {}^{n+3}C_2\right)$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)}\left\{0 - 1 + (n+3) - \frac{(n+3)(n+2)}{2!}\right\}$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)}\cdot\frac{(n+2)(2-n-3)}{2} = \frac{3!}{2(n+3)}$$

16. $(1+x+x^2)^n = a_0 + a_1x + \ldots + a_{2n}x^{2n}$ ...(i)

Replacing x by $-1/x$, we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + \ldots + \frac{a_{2n}}{x^{2n}} \quad \ldots\text{(ii)}$$

Now, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + a_{2n}^2 =$ coefficient of the term independent of x in

$$[a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}] \times \left[a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}}\right]$$

= Coefficient of the term independent of x in

$$(1+x+x^2)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$$

Now, RHS $= (1+x+x^2)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$

$$= \frac{(1+x+x^2)^n\,(x^2-x+1)^n}{x^{2n}} = \frac{[(x^2+1)^2-x^2]^n}{x^{2n}}$$

$$= \frac{(1+2x^2+x^4-x^2)^n}{x^{2n}} = \frac{(1+x^2+x^4)^n}{x^{2n}}$$

Thus, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + a_{2n}^2$

= Coefficient of the term independent of x in

$$\frac{1}{x^{2n}}(1+x^2+x^4)^n$$

= Coefficient of x^{2n} in $(1+x^2+x^4)^n$

= Coefficient of t^n in $(1+t+t^2)^n = a_n$

17. $C_0 - 2^2\cdot C_1 + 3^2\cdot C_2 - \ldots + (-1)^n\,(n+1)^2\cdot C_n$

$$= \sum_{r=0}^{n}(-1)^r(r+1)^2\,{}^nC_r = \sum_{r=0}^{n}(-1)^r(r^2+2r+1)\,{}^nC_r$$

$$= \sum_{r=0}^{n}(-1)^r r^2\cdot{}^nC_r + 2\sum_{r=0}^{n}(-1)^r r\cdot{}^nC_r + \sum_{r=0}^{n}(-1)^r\cdot{}^nC_r$$

$$= \sum_{r=0}^{n}(-1)^r\cdot r\,(r-1)\cdot{}^nC_r + 3\cdot\sum_{r=0}^{n}(-1)^r\cdot r\cdot{}^nC_r + \sum_{r=0}^{n}(-1)^r\,{}^nC_r$$

$$= \sum_{r=2}^{n}(-1)^r n\,(n-1)\,{}^{n-2}C_{r-2} + 3\sum_{r=1}^{n}(-1)^r n\cdot{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r\,{}^nC_r$$

$$= n\,(n-1)\{{}^{n-2}C_0 - {}^{n-2}C_1 + {}^{n-2}C_2 - \ldots + (-1)^n\,{}^{n-2}C_{n-2}\}$$
$$+ 3n\{-{}^{n-1}C_0 + {}^{n-1}C_1 - {}^{n-1}C_2 + \ldots + (-1)^n\,{}^{n-1}C_{n-1}\}$$
$$+ \{{}^nC_0 - {}^nC_1 + {}^nC_2 + \ldots + (-1)^n\,{}^nC_n\}$$

$$= n\,(n-1)\cdot 0 + 3n\cdot 0 + 0,\ \forall n>2 = 0,\ \ \forall n>2$$

18. We know that,

$$2\sum\sum_{0\le i<j\le n} C_iC_j = \sum_{i=0}^{n}\sum_{j=0}^{n}C_iC_j - \sum_{i=0}^{n}\sum_{j=0}^{n}C_iC_j$$

$$= \sum_{i=0}^{n}C_i\sum_{j=0}^{n}C_j - \sum_{i=0}^{n}C_i^2$$

$$= 2^n\,2^n - ({}^{2n}C_n) = 2^{2n} - {}^{2n}C_n$$

$$\therefore \sum\sum_{0\le i<j\le n} C_iC_j = \frac{2^{2n} - {}^{2n}C_n}{2} = 2^{2n-1} - \frac{(2n)!}{2\,(n!)^2}$$

19. We know that, $(1+x)^{2n} = C_0 + C_1x + C_2x^2 + \ldots + C_{2n}x^{2n}$

On differentiating both sides w.r.t. x, we get

$$2n(1+x)^{2n-1} = C_1 + 2\cdot C_2x + 3\cdot C_3x^2 + \ldots + 2nC_{2n}x^{2n-1} \quad \ldots\text{(i)}$$

and $$\left(1-\frac{1}{x}\right)^{2n} = C_0 - C_1\cdot\frac{1}{x} + C_2\cdot\frac{1}{x^2} - C_3\cdot\frac{1}{x^3} + \ldots + C_{2n}\cdot\frac{1}{x^{2n}} \quad \ldots\text{(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$2n\,(1+x)^{2n-1}\left(1-\frac{1}{x}\right)^{2n}$$

$$= [C_1 + 2\cdot C_2x + 3\cdot C_3\,x^2 + \ldots + 2n\cdot C_{2n}x^{2n-1}] \times\left[C_0 - C_1\left(\frac{1}{x}\right) + C_2\left(\frac{1}{x^2}\right) - \ldots.. + C_{2n}\left(\frac{1}{x^{2n}}\right)\right]$$

Coefficient of $\left(\frac{1}{x}\right)$ on the LHS

$$= \text{Coefficient of } \frac{1}{x} \text{ in } 2n\left(\frac{1}{x^{2n}}\right)(1+x)^{2n-1}(x-1)^{2n}$$

$$= \text{Coefficient of } x^{2n-1} \text{ in } 2n(1-x^2)^{2n-1}(1-x)$$

$$= 2n(-1)^{n-1}\cdot(2n-1)\,C_{n-1}(-1)$$

$$= (-1)^n(2n)\frac{(2n-1)!}{(n-1)!\ \ n!} = (-1)^n n\,\frac{(2n)!}{(n!)^2}\cdot n$$

$$= -(-1)^n n\cdot C_n \quad \ldots\text{(iii)}$$

Again, the coefficient of $\left(\frac{1}{x}\right)$ on the RHS

$$= -(C_1^2 - 2\cdot C_2^2 + 3\cdot C_3^2 - \ldots - 2n\,C_{2n}^2) \quad \ldots\text{(iv)}$$

From Eqs. (iii) and (iv),

$$C_1^2 - 2\cdot C_2^2 + 3\cdot C_3^2 - \ldots - 2n\cdot C_{2n}^2 = (-1)^n n\cdot C_n$$

20. $(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n}$

$$= [{}^{2n}C_0 + ({}^{2n}C_1)x + ({}^{2n}C_2)x^2 + \ldots + ({}^{2n}C_{2n})x^{2n}] \times\left[{}^{2n}C_0 - ({}^{2n}C_1)\frac{1}{x} + ({}^{2n}C_2)\frac{1}{x^2} + \ldots + ({}^{2n}C_{2n})\frac{1}{x^{2n}}\right]$$

Independent terms of x on RHS

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2$$

$$\text{LHS} = (1+x)^{2n}\left(\frac{x-1}{x}\right)^{2n} = \frac{1}{x^{2n}}(1-x^2)^{2n}$$

Independent term of x on the LHS $= (-1)^n\cdot{}^{2n}C_n$.