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Area

Topic 1 Area Based on Geometrical Figures Without Using Integration

Objective Questions I (Only one correct option)

1. If the area enclosed between the curves $y = kx^2$ and $x = ky^2$, $(k > 0)$, is 1 square unit. Then, k is **(2019 Main, 10 Jan I)**

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{2}$

2. The area (in sq units) of the region $\{(x, y) : y^2 \ge 2x$ and $x^2 + y^2 \le 4x, x \ge 0, y \ge 0\}$ is **(2016 Main)**

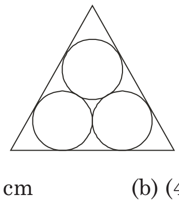
(a) $\pi - \frac{4}{3}$ (b) $\pi - \frac{8}{3}$

(c) $\pi - \frac{4\sqrt{2}}{3}$ (d) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$

3. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then, the area (in sq units) of the quadrilateral $PQRS$ is **(2014 Adv.)**

(a) 3 (b) 6 (c) 9 (d)15

4. The area of the equilateral triangle, in which three coins of radius 1 cm are placed, as shown in the figure, is



(2005, 1M)

(a) $(6 + 4\sqrt{3})$ sq cm (b) $(4\sqrt{3} - 6)$ sq cm

(c) $(7 + 4\sqrt{3})$ sq cm (d) $4\sqrt{3}$ sq cm

5. The area of the quadrilateral formed by the tangents at the end points of latusrectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is **(2003, 1M)**

(a) 27/4 sq units (b) 9 sq units

(c) 27/2 sq units (d) 27 sq units

6. The area (in sq units) bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is **(2002, 2M)**

(a) 1 (b) 2 (c) $2\sqrt{2}$ (d) 4

7. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1,1) and the coordinate axes, lies in the first quadrant. If its area is 2 sq units, then the value of b is **(2001, 2M)**

(a) – 1 (b) 3 (c) – 3 (d) 1

Objective Questions II

(One or more than one correct option)

8. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of ΔOPQ is $3\sqrt{2}$, then which of the following is/are the coordinates of P? **(2015 Adv.)**

(a) $(4, 2\sqrt{2})$ (b) $(9, 3\sqrt{2})$ (c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (d) $(1, \sqrt{2})$

Numerical Value

9. A farmer F_1 has a land in the shape of a triangle with vertices at $P(0, 0)$, $Q(1, 1)$ and $R(2, 0)$. From this land, a neighbouring farmer F_2 takes away the region which lies between the sides PQ and a curve of the form $y = x^n$ $(n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is .... **(2018 Adv.)**

Fill in the Blanks

10. The area of the triangle formed by the positive X-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is … . **(1989, 2M)**

11. The area enclosed within the curve $|x| + |y| = 1$ is .... **(1981, 2M)**

Analytical & Descriptive Question

12. Let $O(0, 0)$, $A(2, 0)$ and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \ge$ min

$\{d\,(P, OB), d(P, AB)\}$, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area. **(1997C, 5M)**

Passage Based Questions

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real-valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real-valued differentiable function $y = g(x)$, satisfying $g(0) = 0$. **(2008, M)**

13. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2})$ is equal to

(a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$ (c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

14. The area of the region bounded by the curve $y = f(x)$, the X-axis and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(a) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\,dx + bf(b) - af(a)$

(b) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\,dx + bf(b) - af(a)$

(c) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\,dx - bf(b) + af(a)$

(d) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]}\,dx - bf(b) + af(a)$

15. $\int_{-1}^{1} g'(x)\,dx$ is equal to

(a) $2g(-1)$ (b) 0 (c) $-2g(1)$ (d) $2g(1)$

Topic 2 Area Using Integration

Objective Questions I (Only one correct option)

1. If the area (in sq units) bounded by the parabola $y^2 = 4\lambda x$ and the line $y = \lambda x, \lambda > 0$, is $\frac{1}{9}$, then λ is equal to **(2019 Main, 12 April II)**

(a) $2\sqrt{6}$ (b) 48 (c) 24 (d) $4\sqrt{3}$

2. If the area (in sq units) of the region $\{(x, y) : y^2 \le 4x, x + y \le 1, \ x \ge 0, y \ge 0\}$ is $a\sqrt{2} + b$, then $a - b$ is equal to **(2019 Main, 12 April I)**

(a) $\frac{10}{3}$ (b) 6 (c) $\frac{8}{3}$ (d) $-\frac{2}{3}$

3. The area (in sq units) of the region bounded by the curves $y = 2^x$ and $y = |x + 1|$, in the first quadrant is **(2019 Main, 10 April II)**

(a) $\frac{3}{2}$ (b) $\log_e 2 + \frac{3}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2} - \frac{1}{\log_e 2}$

4. The area (in sq units) of the region $A = \left\{(x, y) : \frac{y^2}{2} \le x \le y + 4\right\}$ is **(2019 Main, 9 April II)**

(a) 30 (b) $\frac{53}{3}$ (c) 16 (d) 18

5. The area (in sq units) of the region $A = \{(x, y) : x^2 \le y \le x + 2\}$ is **(2019 Main, 9 April I)**

(a) $\frac{13}{6}$ (b) $\frac{9}{2}$ (c) $\frac{31}{6}$ (d) $\frac{10}{3}$

6. Let $S(\alpha) = \{(x, y) : y^2 \le x, 0 \le x \le \alpha\}$ and $A(\alpha)$ is area of the region $S(\alpha)$. If for λ, $0 < \lambda < 4$, $A(\lambda) : A(4) = 2 : 5$, then λ equals **(2019 Main, 8 April II)**

(a) $2\left(\frac{4}{25}\right)^{\frac{1}{3}}$ (b) $4\left(\frac{2}{5}\right)^{\frac{1}{3}}$ (c) $4\left(\frac{4}{25}\right)^{\frac{1}{3}}$ (d) $2\left(\frac{2}{5}\right)^{\frac{1}{3}}$

7. The tangent to the parabola $y^2 = 4x$ at the point where it intersects the circle $x^2 + y^2 = 5$ in the first quadrant, passes through the point

(a) $\left(\frac{1}{4}, \frac{3}{4}\right)$ (b) $\left(\frac{3}{4}, \frac{7}{4}\right)$ (c) $\left(-\frac{1}{3}, \frac{4}{3}\right)$ (d) $\left(-\frac{1}{4}, \frac{1}{2}\right)$

8. The area (in sq units) of the region $A = \{(x, y) \in R \times R \mid 0 \le x \le 3, 0 \le y \le 4, y \le x^2 + 3x\}$ is **(2019 Main, 8 April I)**

(a) $\frac{53}{6}$ (b) 8 (c) $\frac{59}{6}$ (d) $\frac{26}{3}$

9. The area (in sq units) of the region bounded by the parabola, $y = x^2 + 2$ and the lines, $y = x + 1$, $x = 0$ and $x = 3$, is **(2019 Main, 12 Jan I)**

(a) $\frac{15}{2}$ (b) $\frac{17}{4}$ (c) $\frac{21}{2}$ (d) $\frac{15}{4}$

10. The area (in sq units) in the first quadrant bounded by the parabola, $y = x^2 + 1$, the tangent to it at the point (2, 5) and the coordinate axes is **(2019 Main, 11 Jan II)**

(a) $\frac{14}{3}$ (b) $\frac{187}{24}$ (c) $\frac{8}{3}$ (d) $\frac{37}{24}$

11. The area (in sq units) of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is **(2019 Main, 11 Jan I)**

(a) $\frac{7}{8}$ (b) $\frac{9}{8}$ (c) $\frac{5}{4}$ (d) $\frac{3}{4}$

12. The area of the region $\text{A} = \{(x, y); 0 \le y \le x|x| + 1$ and $-1 \le x \le 1\}$ in sq. units, is **(2019 Main, 9 Jan II)**

(a) 2 (b) $\frac{4}{3}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

13. The area (in sq units) bounded by the parabola $y = x^2 - 1$, the tangent at the point (2, 3) to it and the Y-axis is **(2019 Main, 9 Jan I)**

(a) $\frac{8}{3}$ (b) $\frac{56}{3}$ (c) $\frac{32}{3}$ (d) $\frac{14}{3}$

14. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and $\alpha, \beta\ (\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then, the area (in sq units) bounded by the curve $y = (gof)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$, is **(2018 Main)**

(a) $\frac{1}{2}(\sqrt{3} - 1)$ (b) $\frac{1}{2}(\sqrt{3} + 1)$

(c) $\frac{1}{2}(\sqrt{3} - \sqrt{2})$ (d) $\frac{1}{2}(\sqrt{2} - 1)$

15. The area (in sq units) of the region $\{(x, y) : x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is **(2017 Main)**

(a) $\frac{59}{12}$ (b) $\frac{3}{2}$ (c) $\frac{7}{3}$ (d) $\frac{5}{2}$

16. Area of the region $\{(x, y)\} \in R^2 : y \geq \sqrt{|x + 3|}, 5y \leq (x + 9) \leq 15\}$ is equal to **(2016 Adv)**

(a) $\frac{1}{6}$ (b) $\frac{4}{3}$ (c) $\frac{3}{2}$ (d) $\frac{5}{3}$

17. The area (in sq units) of region described by (x, y) $y^2 \leq 2x$ and $y \geq 4x - 1$ is **(2015 JEE Main)**

(a) $\frac{7}{32}$ (b) $\frac{5}{64}$ (c) $\frac{15}{64}$ (d) $\frac{9}{32}$

18. The area (in sq units) of the region described by $A = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } y^2 \leq 1 - x\}$ is **(2014 Main)**

(a) $\frac{\pi}{2} + \frac{4}{3}$ (b) $\frac{\pi}{2} - \frac{4}{3}$ (c) $\frac{\pi}{2} - \frac{2}{3}$ (d) $\frac{\pi}{2} + \frac{2}{3}$

19. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is **(2014 Adv.)**

(a) $4(\sqrt{2} - 1)$ (b) $2\sqrt{2}(\sqrt{2} - 1)$
(c) $2(\sqrt{2} + 1)$ (d) $2\sqrt{2}(\sqrt{2} + 1)$

20. The area (in sq units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, X-axis and lying in the first quadrant, is **(2013 Main, 03)**

(a) 9 (b) 6 (c) 18 (d) $\frac{27}{4}$

21. Let $f : [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1 - x), \forall x \in [-1, 2]$. If $R_1 = \int_{-1}^{2} xf(x)\,dx$ and R_2 are the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$ and the X-axis. Then, **(2011)**

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$
(c) $2R_1 = R_2$ (d) $3R_1 = R_2$

22. If the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts $R_1 (0 \leq x \leq b)$ and $R_2 (b \leq x \leq 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then, b equals

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$ **(2011)**

23. The area of the region between the curves $y = \sqrt{\frac{1 + \sin x}{\cos x}}$ and $y = \sqrt{\frac{1 - \sin x}{\cos x}}$ and bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is **(2008, 3M)**

(a) $\int_0^{\sqrt{2} - 1} \frac{t}{(1 + t^2)\sqrt{1 - t^2}}\,dt$ (b) $\int_0^{\sqrt{2} - 1} \frac{4t}{(1 + t^2)\sqrt{1 - t^2}}\,dt$

(c) $\int_0^{\sqrt{2} + 1} \frac{4t}{(1 + t^2)\sqrt{1 - t^2}}\,dt$ (d) $\int_0^{\sqrt{2} + 1} \frac{t}{(1 + t^2)\sqrt{1 - t^2}}\,dt$

24. The area bounded by the curves $y = (x - 1)^2$, $y = (x + 1)^2$ and $y = \frac{1}{4}$ is **(2005, 1M)**

(a) $\frac{1}{3}$ sq unit (b) $\frac{2}{3}$ sq unit (c) $\frac{1}{4}$ sq unit (d) $\frac{1}{5}$ sq unit

25. The area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 sq unit. Then, the value of a is **(2004, 1M)**

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{3}$

26. The area bounded by the curves $y = f(x)$, the X-axis and the ordinates $x = 1$ and $x = b$ is $(b - 1)\sin(3b + 4)$. Then, $f(x)$ is equal to **(1982, 2M)**

(a) $(x - 1)\cos(3x + 4)$
(b) $8\sin(3x + 4)$
(c) $\sin(3x + 4) + 3(x - 1)\cos(3x + 4)$
(d) None of the above

27. The slope of tanget to a curve $y = f(x)$ at $[x, f(x)]$ is $2x + 1$. If the curve passes through the point (1, 2), then the area bounded by the curve, the X-axis and the line $x = 1$ is

(a) $\frac{3}{2}$ (b) $\frac{4}{3}$ (c) $\frac{5}{6}$ (d) $\frac{1}{12}$

Objective Questions II

(One or more than one correct option)

28. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in R^2 : x^3 \leq y \leq x, 0 \leq x \leq 1\}$ into two equal parts, then **(2017 Adv.)**

(a) $2\alpha^4 - 4\alpha^2 + 1 = 0$ (b) $\alpha^4 + 4\alpha^2 - 1 = 0$
(c) $\frac{1}{2} < \alpha < 1$ (d) $0 < \alpha \leq \frac{1}{2}$

29. If S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then, **(2012)**

(a) $S \geq \frac{1}{e}$ (b) $S \geq 1 - \frac{1}{e}$

(c) $S \leq \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (d) $S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

30. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is **(2009)**

(a) $e - 1$ (b) $\int_1^e \ln(e + 1 - y)\,dy$

(c) $e - \int_0^1 e^x\,dx$ (d) $\int_1^e \ln y\,dy$

31. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals $\frac{9}{2}$? **(1999, 3M)**

(a) -4 (b) -2 (c) 2 (d) 4

Analytical & Descriptive Questions

32. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$,

$f(x)$ is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of $y = f(x)$ with X-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by $f(x)$ and chord AB. **(2005, 5M)**

33. Find the area bounded by the curves $x^2 = y, x^2 = -y$ and $y^2 = 4x - 3$. **(2005, 4M)**

34. A curve passes through (2,0) and the slope of tangent at point $P(x, y)$ equals $\frac{(x+1)^2 + y - 3}{(x+1)}$.

Find the equation of the curve and area enclosed by the curve and the X-axis in the fourth quadrant. **(2004, 5M)**

35. Find the area of the region bounded by the curves

$$y = x^2, y = |2 - x^2| \text{ and } y = 2,$$

which lies to the right of the line $x = 1$. **(2002, 5M)**

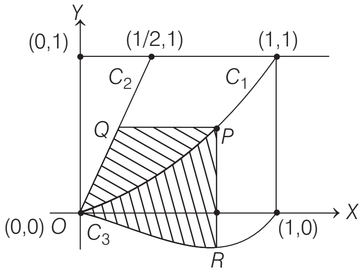
36. Let $b \neq 0$ and for $j = 0, 1, 2 \ldots, n$. If S_j is the area of the region bounded by the Y-axis and the curve $xe^{ay} = \sin by, \frac{j\pi}{b} \le y \le \frac{(j+1)\pi}{(b)}$. Then, show that $S_0, S_1, S_2, \ldots, S_n$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$. **(2001, 5M)**

37. If $f(x)$ is a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

Then, find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left on the line $8x + 1 = 0$. **(1999, 5M)**

38. Let C_1 and C_2 be the graphs of functions $y = x^2$ and $y = 2x, 0 \le x \le 1$, respectively. Let C_3 be the graph of a function $y = f(x), 0 \le x \le 1, f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure). If for every position of P(on C_1) the areas of the shaded regions OPQ and ORP are equal, then determine $f(x)$. **(1998, 8M)**



39. Let $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$, where $0 \le x \le 1$.

Determine the area of the region bounded by the curves $y = f(x)$, X-axis, $x = 0$ and $x = 1$. **(1997, 5M)**

40. Find all the possible values of $b > 0$, so that the area of the bounded region enclosed between the parabolas $y = x - bx^2$ and $y = \frac{x^2}{b}$ is maximum. **(1997C, 5M)**

41. If A_n is the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0, y = 0$ and $x = \frac{\pi}{4}$.

Then, prove that for $n > 2$, $A_n + A_{n+2} = \frac{1}{n+1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$. **(1996, 3M)**

42. Consider a square with vertices at (1,1), (–1, 1), (–1, –1) and (1, –1). If S is the region consisting of all points inside the square which are nearer to the origin than to any edge. Then, sketch the region S and find its area. **(1995, 5M)**

43. In what ratio, does the X-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$? **(1994, 5M)**

44. Sketch the region bounded by the curves $y = x^2$ and $y = 2/(1 + x^2)$. Find its area. **(1992, 4M)**

45. Sketch the curves and identify the region bounded by $x = 1/2$, $x = 2$, $y = \log x$ and $y = 2^x$. Find the area of this region. **(1991, 4M)**

46. Compute the area of the region bounded by the curves $y = ex \log x$ and $y = \frac{\log x}{ex}$, where $\log e = 1$. **(1990, 4M)**

47. Find all maxima and minima of the function $y = x(x-1)^2, 0 \le x \le 2$.

Also, determine the area bounded by the curve $y = x(x-1)^2$, the Y-axis and the line $x = 2$. **(1989, 5M)**

48. Find the area of the region bounded by the curve $C: y = \tan x$, tangent drawn to C at $x = \pi/4$ and the X-axis. **(1988, 5M)**

49. Find the area bounded by the curves $x^2 + y^2 = 25, 4y = |4 - x^2|$ and $x = 0$ above the X-axis. **(1987, 6M)**

50. Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}\, y$ and $x = y$. **(1986, 5M)**

51. Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ and find its area. **(1985, 5M)**

52. Find the area of the region bounded by the X-axis and the curves defined by $y = \tan x, -\frac{\pi}{3} \le x \le \frac{\pi}{3}$ and $y = \cot x, \frac{\pi}{6} \le x \le \frac{\pi}{3}$. **(1984, 4M)**

53. Find the area bounded by the X-axis, part of the curve $y = \left(1 + \frac{8}{x^2}\right)$ and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, then find a. **(1983, 3M)**

54. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

55. For any real $t, x = \frac{e^t + e^{-t}}{2}, y = \frac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. Find the area bounded by this hrperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$. **(1982, 3M)**

Answers

Topic 1

1. (b) **2.** (b) **3.** (d) **4.** (a)
5. (d) **6.** (b) **7.** (c) **8.** (a,d)
9. (4) **10.** $2\sqrt{3}$ sq units
11. 2 sq units **12.** $(2-\sqrt{3})$ sq unit **13.** (b)
14. (a) **15.** (d)

Topic 2

1. (c) **2.** (b) **3.** (d) **4.** (d)
5. (b) **6.** (c) **7.** (b) **8.** (c)
9. (a) **10.** (d) **11.** (b) **12.** (a)
13. (a) **14.** (a)
15. (d) **16.** (c) **17.** (d) **18.** (a)
19. (b) **20.** (a) **21.** (c) **22.** (b)
23. (b) **24.** (a) **25.** (a) **26.** (c)
27. (c) **28.** (a, c) **29.** (b, d) **30.** (b, c, d)
31. (b, d) **32.** $\frac{125}{3}$ sq units **33.** $\frac{1}{3}$ sq unit
34. $y=x^2-2x, \frac{4}{3}$ sq units
35. $\left(\frac{20-12\sqrt{2}}{3}\right)$ sq units **36.** $\left[\frac{\pi(1+e)}{(1+\pi^2)}\cdot\frac{(e^{n+1}-1)}{e-1}\right]$
37. $\left(\frac{761}{192}\right)$ sq units **38.** $f(x)=x^3-x^2, 0\le x\le 1$
39. $\frac{17}{27}$ sq unit **40.** $b=1$ **42.** $\left[\frac{1}{3}(16\sqrt{2}-20)\right]$ sq units
43. 121 : 4 **44.** $\left(\pi-\frac{2}{3}\right)$ sq units
45. $\left(\frac{4-\sqrt{2}}{\log 2}-\frac{5}{2}\log 2+\frac{3}{2}\right)$ sq units **46.** $\left(\frac{e^2-5}{4e}\right)$ sq units
47. $\left(y_{\max}=\frac{4}{27}, y_{\min}=0, \frac{10}{3}\text{ sq units}\right)$
48. $\left[\left(\log\sqrt{2}-\frac{1}{4}\right)\text{sq units}\right]$ **49.** $\left[4+25\sin^{-1}\left(\frac{4}{5}\right)\right]$ sq units
50. $\left(\frac{1}{3}-\pi\right)$ sq units **51.** $\left(\frac{5\pi}{4}-\frac{1}{2}\right)$ sq units
52. $\left(\frac{1}{2}\log_e 3\right)$ sq units **53.** $2\sqrt{2}$ **54.** $\frac{9}{8}$ sq units
55. $\frac{e^{2t_1}-e^{-2t_1}}{4}-\frac{1}{4}(e^{2t_1}-e^{-2t_1}-4t_1)$

Hints & Solutions

Topic 1 Area Based on Geometrical Figures Without Using Integration

1. We know that, area of region bounded by the parabolas $x^2=4ay$ and $y^2=4bx$ is $\frac{16}{3}(ab)$ sq units.

On comparing $y=kx^2$ and $x=ky^2$ with above equations, we get $4a=\frac{1}{k}$ and $4b=\frac{1}{k}$

$$\Rightarrow \quad a=\frac{1}{4k} \text{ and } b=\frac{1}{4k}$$

$\therefore$ Area enclosed between $y=kx^2$ and $x=ky^2$ is

$$\frac{16}{3}\left(\frac{1}{4k}\right)\left(\frac{1}{4k}\right)=\frac{1}{3k^2}$$

$$\Rightarrow \quad \frac{1}{3k^2}=1 \quad [\text{given, area} = 1 \text{ sq.unit}]$$

$$\Rightarrow \quad k^2=\frac{1}{3} \quad \Rightarrow k=\pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad k=\frac{1}{\sqrt{3}} \quad [\because k>0]$$

2. Given equations of curves are $y^2=2x$

which is a parabola with vertex (0, 0) and axis parallel to X-axis. . ..(i)

And $x^2+y^2=4x$

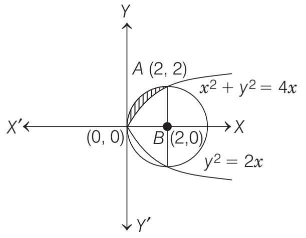
which is a circle with centre (2, 0) and radius = 2 ...(ii)

On substituting $y^2=2x$ in Eq. (ii), we get

$$x^2+2x=4x \Rightarrow x^2=2x \Rightarrow x=0 \text{ or } x=2$$

$$\Rightarrow \quad y=0 \text{ or } y=\pm 2 \quad [\text{using Eq. (i)}]$$

Now, the required area is the area of shaded region, i.e.



$$\text{Required area} = \frac{\text{Area of a circle}}{4}-\int_0^2\sqrt{2x}\,dx$$

$$=\frac{\pi(2)^2}{4}-\sqrt{2}\int_0^2 x^{1/2}dx=\pi-\sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2$$

$$=\pi-\frac{2\sqrt{2}}{3}[2\sqrt{2}-0]=\left(\pi-\frac{8}{3}\right)\text{sq unit}$$

3. **PLAN** (i) $y=mx+a/m$ is an equation of tangent to the parabola $y^2=4ax$.

(ii) A line is a tangent to circle, if distance of line from centre is equal to the radius of circle.

(iii) Equation of chord drawn from exterior point (x_1, y_1) to a circle/parabola is given by $T=0$.

(iv) Area of trapezium $=\frac{1}{2}$ (Sum of parallel sides)

Let equation of tangent to parabola be $y = mx + \frac{2}{m}$

It also touches the circle $x^2 + y^2 = 2$.

$$\therefore \qquad \left| \frac{2}{m\sqrt{1+m^2}} \right| = \sqrt{2}$$

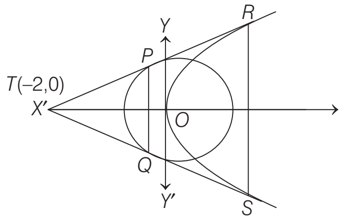
$$\Rightarrow \quad m^4 + m^2 = 2 \;\Rightarrow\; m^4 + m^2 - 2 = 0$$

$$\Rightarrow \qquad (m^2-1)(m^2+2) = 0$$

$$\Rightarrow \qquad m = \pm 1, m^2 = -2 \qquad [\text{rejected } m^2 = -2]$$

So, tangents are $y = x + 2$, $y = -x - 2$.

They, intersect at $(-2, 0)$.



Equation of chord PQ is $-2x = 2 \;\Rightarrow\; x = -1$

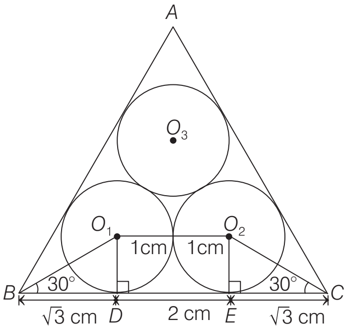
Equation of chord RS is $O = 4(x-2) \;\Rightarrow\; x = 2$

$\therefore$ Coordinates of P, Q, R, S are

$P(-1, 1), Q(-1, -1), R(2, 4), S(2, -4)$

$$\therefore \quad \text{Area of quadrilateral} = \frac{(2+8)\times 3}{2} = 15 \text{ sq units}$$

4. Since, tangents drawn from external points to the circle subtends equal angle at the centre.



$$\therefore \qquad \angle O_1BD = 30°$$

In ΔO_1BD, $\tan 30° = \frac{O_1D}{BD} \;\Rightarrow\; BD = \sqrt{3}$ cm

Also, $\qquad DE = O_1O_2 = 2$ cm and $EC = \sqrt{3}$ cm

Now, $\qquad BC = BD + DE + EC = 2 + 2\sqrt{3}$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}(BC)^2 = \frac{\sqrt{3}}{4}\cdot 4(1+\sqrt{3})^2$$

$$= (6 + 4\sqrt{3}) \text{ sq cm}$$

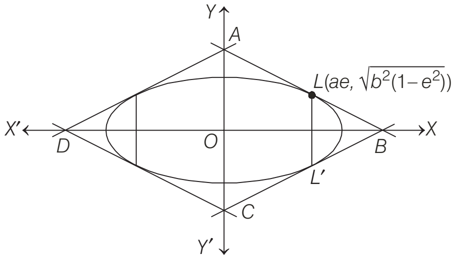
5. Given, $\frac{x^2}{9} + \frac{y^2}{5} = 1$

To find tangents at the end points of latusrectum, we find *ae*.

i.e. $\qquad ae = \sqrt{a^2 - b^2} = \sqrt{4} = 2$

and $\qquad \sqrt{b^2(1-e^2)} = \sqrt{5\left(1 - \frac{4}{9}\right)} = \frac{5}{3}$

By symmetry, the quadrilateral is a rhombus.



So, area is four times the area of the right angled triangle formed by the tangent and axes in the Ist quadrant.

$\therefore$ Equation of tangent at $\left(2, \frac{5}{3}\right)$ is

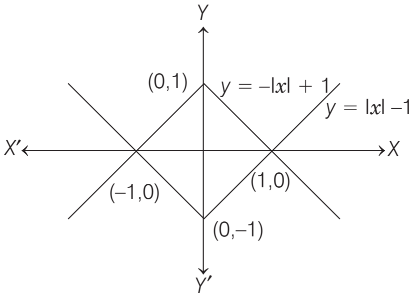
$$\frac{2}{9}x + \frac{5}{3}\cdot\frac{y}{5} = 1 \;\Rightarrow\; \frac{x}{9/2} + \frac{y}{3} = 1$$

$\therefore$ Area of quadrilateral $ABCD$

$$= 4 \qquad [\text{area of } \Delta\, AOB]$$

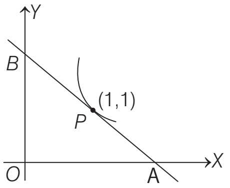
$$= 4\left(\frac{1}{2}\cdot\frac{9}{2}\cdot 3\right) = 27 \text{ sq units}$$

6. The region is clearly square with vertices at the points (1, 0), (0, 1), (– 1, 0) and (0, – 1).



$\therefore$ Area of square $= \sqrt{2} \times \sqrt{2} = 2$ sq units

7. Let $y = f(x) = x^2 + bx - b$



The equation of the tangent at $P(1, 1)$ to the curve $2y = 2x^2 + 2bx - 2b$ is

$$y + 1 = 2x\cdot 1 + b(x+1) - 2b$$

$$\Rightarrow \qquad y = (2+b)x - (1+b)$$

Its meet the coordinate axes at

$$x_A = \frac{1+b}{2+b} \quad \text{and} \quad y_B = -(1+b)$$

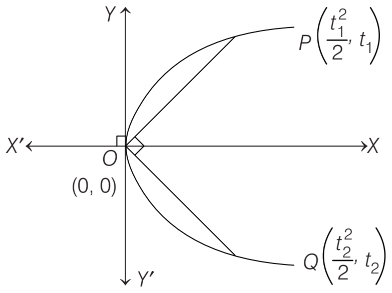
$$\therefore \text{Area of } \Delta\, OAB = \frac{1}{2} OA \times OB$$

$$= -\frac{1}{2} \times \frac{(1+b)^2}{(2+b)} = 2 \qquad [\text{given}]$$

$$\Rightarrow \quad (1+b)^2 + 4(2+b) = 0 \;\Rightarrow\; b^2 + 6b + 9 = 0$$

$$\Rightarrow \qquad (b+3)^2 = 0 \;\Rightarrow\; b = -3$$

8. Since, $\angle POQ = 90°$



$$\Rightarrow \quad \frac{t_1 - 0}{\frac{t_1^2}{2} - 0} \cdot \frac{t_2 - 0}{\frac{t_2^2}{2} - 0} = -1 \quad \Rightarrow \quad t_1 t_2 = -4 \qquad \text{...(i)}$$

$\because \quad$ ar $(\Delta OPQ) = 3\sqrt{2}$

$$\therefore \quad \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ t_1^2/2 & t_1 & 1 \\ t_2^2/2 & t_2 & 1 \end{vmatrix} = \pm 3\sqrt{2} \quad \Rightarrow \quad \frac{1}{2}\left(\frac{t_1^2 t_2}{2} - \frac{t_1 t_2^2}{2}\right) = \pm 3\sqrt{2}$$

$$\Rightarrow \frac{1}{4}(-4t_1 + 4t_2) = \pm 3\sqrt{2} \Rightarrow t_1 + \frac{4}{t_1} = 3\sqrt{2} \quad [\because t_1 > 0 \text{ for } P]$$

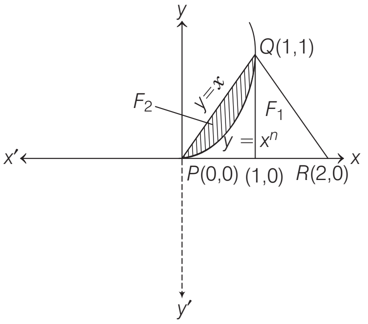
$$\Rightarrow t_1^2 - 3\sqrt{2}t_1 + 4 = 0 \quad \Rightarrow \quad (t_1 - 2\sqrt{2})(t_1 - \sqrt{2}) = 0$$

$$\Rightarrow \qquad t_1 = \sqrt{2} \quad \text{or} \quad 2\sqrt{2}$$

$$\therefore \qquad P(1, \sqrt{2}) \quad \text{or} \quad P(4, 2\sqrt{2})$$

9. We have, $y = x^n, n > 1$

$\because P(0, 0)\, Q(1, 1)$ and $R(2, 0)$ are vertices of ΔPQR.



$\therefore \quad$ Area of shaded region = 30% of area of ΔPQR

$$\Rightarrow \int_0^1 (x - x^n)\, dx = \frac{30}{100} \times \frac{1}{2} \times 2 \times 1$$

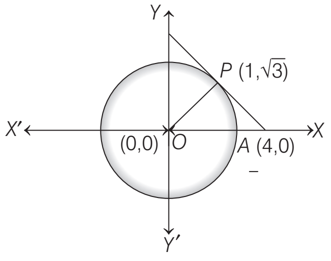
$$\Rightarrow \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{3}{10} \Rightarrow \quad \left(\frac{1}{2} - \frac{1}{n+1}\right) = \frac{3}{10}$$

$$\Rightarrow \quad \frac{1}{n+1} = \frac{1}{2} - \frac{3}{10} = \frac{2}{10} = \frac{1}{5} \Rightarrow n + 1 = 5 \quad \Rightarrow \quad n = 4$$

10. Equation of tangent at the point $(1, \sqrt{3})$ to the curve

$$x^2 + y^2 = 4 \quad \text{is} \quad x + \sqrt{3}y = 4$$

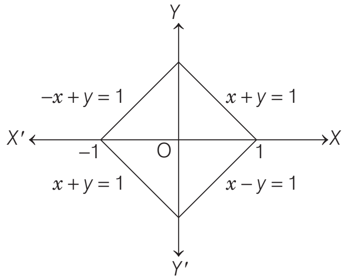
whose X-axis intercept (4, 0).



Thus, area of Δ formed by (0, 0) $(1, \sqrt{3})$ and (4, 0)

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & \sqrt{3} & 1 \\ 4 & 0 & 1 \end{vmatrix} = \frac{1}{2}|(0 - 4\sqrt{3})| = 2\sqrt{3} \text{ sq units}$$

11. The area formed by $|x| + |y| = 1$ is square shown as below :



$\therefore$ Area of square $= (\sqrt{2})^2 = 2$ sq units

12. Let the coordinates of P be (x, y).

Equation of line OA be $y = 0$.

Equation of line OB be $\sqrt{3}\, y = x$.

Equation of line AB be $\sqrt{3}\, y = 2 - x$.

$d\,(P, OA) =$ Distance of P from line $OA = y$

$d\,(P, OB) =$ Distance of P from line $OB = \dfrac{|\sqrt{3}y - x|}{2}$

$d\,(P, AB) =$ Distance of P from line $AB = \dfrac{|\sqrt{3}y + x - 2|}{2}$

Given, $d\,(P, OA) \le \min\{d\,(P, OB), d\,(P, AB)\}$

$$y \le \min\left\{\frac{|\sqrt{3}y - x|}{2}, \frac{|\sqrt{3}\ y + x - 2|}{2}\right\}$$

$$\Rightarrow \qquad y \le \frac{|\sqrt{3}y - x|}{2} \quad \text{and} \quad y \le \frac{|\sqrt{3}y + x - 2|}{2}$$

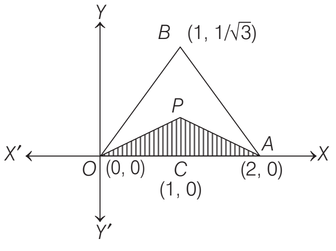
***Case* I** When $y \le \dfrac{|\sqrt{3}y - x|}{2}$ $\qquad$ [since, $\sqrt{3}y - x < 0$]

$$y \le \frac{x - \sqrt{3}y}{2} \quad \Rightarrow \quad (2 + \sqrt{3})y \le x \Rightarrow y \le x \tan 15°$$

***Case* II** When $y \le \dfrac{|\sqrt{3}y + x - 2|}{2}$,

$$2y \le 2 - x - \sqrt{3}y \qquad [\text{since}, \sqrt{3}y + x - 2 < 0]$$

$$\Rightarrow \qquad (2 + \sqrt{3})y \le 2 - x \Rightarrow y \le \tan 15° \cdot (2 - x)$$



From above discussion, P moves inside the triangle as shown below :

$\Rightarrow$ Area of shaded region

$$= \text{Area of } \Delta OQA$$
$$= \frac{1}{2} \text{(Base)} \times \text{(Height)}$$
$$= \frac{1}{2} (2) (\tan 15^\circ) = \tan 15^\circ = (2 - \sqrt{3}) \text{ sq unit}$$

13. Given, $\quad y^3 - 3y + x = 0$

$$\Rightarrow \quad 3y^2 \frac{dy}{dx} - 3 \frac{dy}{dx} + 1 = 0 \quad \ldots(\text{i})$$

$$\Rightarrow 3y^2 \left(\frac{d^2y}{dx^2}\right) + 6y \left(\frac{dy}{dx}\right)^2 - 3 \frac{d^2y}{dx^2} = 0 \quad \ldots(\text{ii})$$

At $x = -10\sqrt{2}$, $y = 2\sqrt{2}$

On substituting in Eq. (i) we get

$$3(2\sqrt{2})^2 \cdot \frac{dy}{dx} - 3 \cdot \frac{dy}{dx} + 1 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{21}$$

Again, substituting in Eq. (ii), we get

$$3(2\sqrt{2})^2 \frac{d^2y}{dx^2} + 6\,(2\sqrt{2}) \cdot \left(-\frac{1}{21}\right)^2 - 3 \cdot \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad 21 \cdot \frac{d^2y}{dx^2} = -\frac{12\sqrt{2}}{(21)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-12\sqrt{2}}{(21)^3} = \frac{-4\sqrt{2}}{7^3 \cdot 3^2}$$

14. Required area $= \int_a^b y\, dx = \int_a^b f(x)\, dx$

$$= [f(x) \cdot x]_a^b - \int_a^b f'(x) x\, dx$$
$$= bf(b) - af(a) - \int_a^b f'(x) x\, dx$$
$$= bf(b) - af(a) + \int_a^b \frac{x dx}{3[\{f(x)\}^2 - 1]}$$
$$\left[\because f'(x) = \frac{dy}{dx} = \frac{-1}{3(y^2 - 1)} = \frac{-1}{3[\{f(x)\}^2 - 1]}\right]$$

15. Let $I = \int_{-1}^{1} g'(x)\, dx = [g(x)]_{-1}^{1} = g(1) - g(-1)$

Since, $\quad y^3 - 3y + x = 0 \quad \ldots(\text{i})$

and $\quad y = g(x)$

$\therefore \quad \{g(x)\}^3 - 3g(x) + x = 0 \quad$ [from Eq. (i)]

At $x = 1$, $\quad \{g(1)\}^3 - 3g(1) + 1 = 0 \quad \ldots(\text{ii})$

At $x = -1$, $\{g(-1)\}^3 - 3g(-1) - 1 = 0 \quad \ldots(\text{iii})$

On adding Eqs. (i) and (ii), we get

$\{g(1)\}^3 + \{g(-1)\}^3 - 3\{g(1) + g(-1)\} = 0$

$\Rightarrow \quad [g(1) + g(-1)][\{g(1)\}^2 + \{g(-1)\}^2 - g(1)g(-1) - 3] = 0$

$$\Rightarrow \quad g(1) + g(-1) = 0$$
$$\Rightarrow \quad g(1) = -g(-1)$$
$$\therefore \quad I = g(1) - g(-1)$$
$$= g(1) - \{-g(1)\} = 2g(1)$$

Topic 2 Area Using Integration

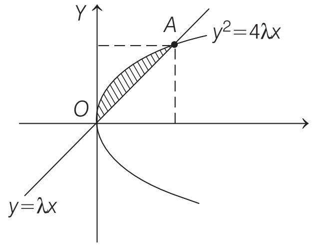
1. Given, equation of curves are

$$y^2 = 4\lambda x \quad \ldots(\text{i})$$

and $$y = \lambda x \quad \ldots(\text{ii})$$

$$\lambda > 0$$

Area bounded by above two curve is, as per figure



the intersection point A we will get on the solving Eqs. (i) and (ii), we get

$$\lambda^2 x^2 = 4\lambda x$$

$$\Rightarrow \quad x = \frac{4}{\lambda}, \text{ so } y = 4.$$

So, $\quad A\left(\frac{4}{\lambda}, 4\right)$

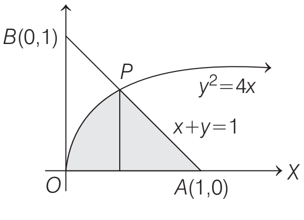
Now, required area is

$$= \int_0^{4/\lambda} (2\sqrt{\lambda x} - \lambda x)\, dx$$
$$= 2\sqrt{\lambda} \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{4/\lambda} - \lambda \left[\frac{x^2}{2}\right]_0^{4/\lambda}$$
$$= \frac{4}{3} \sqrt{\lambda}\, \frac{4\sqrt{4}}{\lambda\sqrt{\lambda}} - \frac{\lambda}{2} \left(\frac{4}{\lambda}\right)^2$$
$$= \frac{32}{3\lambda} - \frac{8}{\lambda} = \frac{32 - 24}{3\lambda} = \frac{8}{3\lambda}$$

It is given that area $= \frac{1}{9}$

$$\Rightarrow \quad \frac{8}{3\lambda} = \frac{1}{9}$$
$$\Rightarrow \quad \lambda = 24$$

2. Given region is $\{(x, y) : y^2 \le 4x, x + y \le 1, x \ge 0, y \ge 0\}$



Now, for point P, put value of $y = 1 - x$ to $y^2 = 4x$, we get

$(1 - x)^2 = 4x \Rightarrow x^2 + 1 - 2x = 4x$

$$\Rightarrow \quad x^2 - 6x + 1 = 0$$
$$\Rightarrow \quad x = \frac{6 \pm \sqrt{36 - 4}}{2}$$
$$= 3 \pm 2\sqrt{2}.$$

Since, x-coordinate of P less than x-coordinate of point $A(1, 0)$.

$\therefore\ x = 3 - 2\sqrt{2}$

Now, required area

$$= \int_0^{3-2\sqrt{2}} 2\sqrt{x}\,dx + \int_{3-2\sqrt{2}}^{1} (1-x)\,dx$$

$$= 2\left|\frac{x^{3/2}}{3/2}\right|_0^{3-2\sqrt{2}} + \left[x - \frac{x^2}{2}\right]_{3-2\sqrt{2}}^{1}$$

$$= \frac{4}{3}(3-2\sqrt{2})^{3/2} + \left(1 - \frac{1}{2}\right) - (3-2\sqrt{2}) + \frac{(3-2\sqrt{2})^2}{2}$$

$$= \frac{4}{3}[(\sqrt{2}-1)^2]^{3/2} + \frac{1}{2} - 3 + 2\sqrt{2} + \frac{1}{2}(9 + 8 - 12\sqrt{2})$$

$$= \frac{4}{3}(\sqrt{2}-1)^3 - \frac{5}{2} + 2\sqrt{2} + \frac{17}{2} - 6\sqrt{2}$$

$$= \frac{4}{3}(2\sqrt{2} - 3(2) + 3(\sqrt{2}) - 1) - 4\sqrt{2} + 6$$

$$= \frac{4}{3}(5\sqrt{2} - 7) - 4\sqrt{2} + 6 = \frac{8\sqrt{2}}{3} - \frac{10}{3}$$

$$= a\sqrt{2} + b \qquad \text{(given)}$$

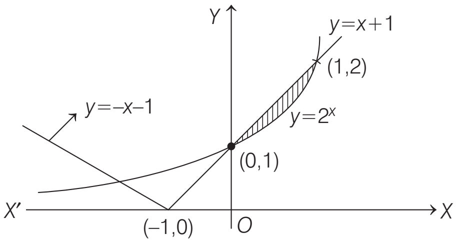
So, on comparing $a = \frac{8}{3}$ and $b = -\frac{10}{3}$

$$\therefore \qquad a - b = \frac{8}{3} + \frac{10}{3} = 6$$

3. Given, equations of curves

$$y = 2^x \text{ and } y = |x+1| = \begin{cases} x+1 & , x \ge -1 \\ -x-1 & , x < -1 \end{cases}$$

$\because$ The figure of above given curves is



In first quadrant, the above given curves intersect each other at (1, 2).

So, the required area $= \int_0^1 ((x+1) - 2^x)\,dx$

$$= \left[\frac{x^2}{2} + x - \frac{2^x}{\log_e 2}\right]_0^1 \qquad \left[\because \int a^x dx = \frac{a^x}{\log_e a} + C\right]$$

$$= \left[\frac{1}{2} + 1 - \frac{2}{\log_e 2} + \frac{1}{\log_e 2}\right]$$

$$= \frac{3}{2} - \frac{1}{\log_e 2}$$

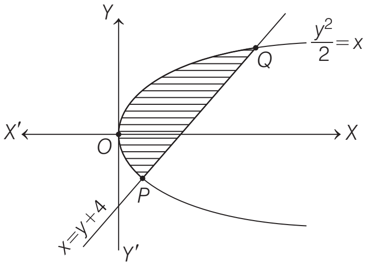
4. Given region $A = \left\{(x, y) : \frac{y^2}{2} \le x \le y + 4\right\}$

$$\therefore \qquad \frac{y^2}{2} = x$$

$$\Rightarrow \qquad y^2 = 2x \qquad \text{...(i)}$$

and $x = y + 4 \Rightarrow y = x - 4$ ...(ii)

Graphical representation of A is



On substituting $y = x - 4$ from Eq. (ii) to Eq. (i), we get

$$(x-4)^2 = 2x$$

$$\Rightarrow \qquad x^2 - 8x + 16 = 2x$$

$$\Rightarrow \qquad x^2 - 10x + 16 = 0$$

$$\Rightarrow \qquad (x-2)(x-8) = 0$$

$$\Rightarrow \qquad x = 2, 8$$

$$\therefore \qquad y = -2, 4 \qquad \text{[from Eq. (ii)]}$$

So, the point of intersection of Eqs. (i) and (ii) are $P(2, -2)$ and $Q(8, 4)$.

Now, the area enclosed by the region A

$$= \int_{-2}^{4}\left[(y+4) - \frac{y^2}{2}\right]dy = \left[\frac{y^2}{2} + 4y - \frac{y^3}{6}\right]_{-2}^{4}$$

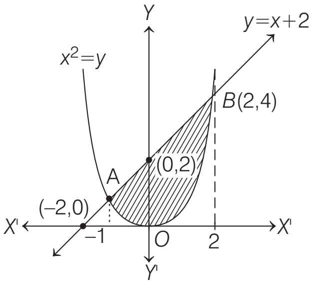
$$= \left(\frac{16}{2} + 16 - \frac{64}{6}\right) - \left(\frac{4}{2} - 8 + \frac{8}{6}\right)$$

$$= 8 + 16 - \frac{32}{3} - 2 + 8 - \frac{4}{3}$$

$= 30 - 12 = 18$ sq unit.

5. Given region is $A = \{(x, y) : x^2 \le y \le x + 2\}$

Now, the region is shown in the following graph



For intersecting points A and B

Taking, $\qquad x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$

$$\Rightarrow \qquad x^2 - 2x + x - 2 = 0$$

$$\Rightarrow \qquad x(x-2) + 1(x-2) = 0$$

$$\Rightarrow \qquad x = -1, 2 \ \Rightarrow\ y = 1, 4$$

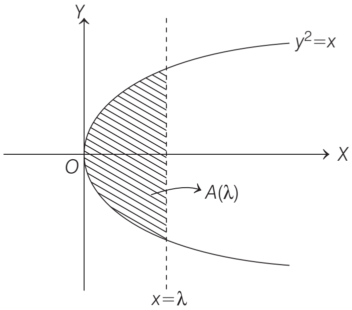
So, $A(-1, 1)$ and B (2, 4).

Now, shaded area $= \int_{-1}^{2} [(x+2) - x^2]\,dx$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$= 8 - \frac{1}{2} - \frac{9}{3} = 8 - \frac{1}{2} - 3 = 5 - \frac{1}{2} = \frac{9}{2}$ sq units

6. Given, $S(\alpha)=\{(x, y): y^2 \le x, 0 \le x \le \alpha\}$ and $A(\alpha)$ is area of the region $S(\alpha)$



Clearly, $A(\lambda)=2\int_0^{\lambda}\sqrt{x}\,dx=2\left[\frac{x^{3/2}}{3/2}\right]_0^{\lambda}=\frac{4}{3}\lambda^{3/2}$

Since, $\frac{A(\lambda)}{A(4)}=\frac{2}{5}, (0<\lambda<4)$

$\Rightarrow \quad \frac{\lambda^{3/2}}{4^{3/2}}=\frac{2}{5} \quad \Rightarrow \quad \left(\frac{\lambda}{4}\right)^3=\left(\frac{2}{5}\right)^2$

$\Rightarrow \quad \frac{\lambda}{4}=\left(\frac{4}{25}\right)^{1/3} \quad \Rightarrow \quad \lambda=4\left(\frac{4}{25}\right)^{1/3}$

7. Given equations of the parabola $y^2=4x$ ...(i)

and circle $x^2+y^2=5$ ...(ii)

So, for point of intersection of curves (i) and (ii), put $y^2=4x$ in Eq. (ii), we get

$$x^2+4x-5=0$$
$$\Rightarrow \quad x^2+5x-x-5=0$$
$$\Rightarrow \quad (x-1)(x+5)=0$$
$$\Rightarrow \quad x=1,-5$$

For first quadrant $x=1$, so $y=2$.

Now, equation of tangent of parabola (i) at point (1, 2) is $T=0$

$\Rightarrow \quad 2y=2(x+1)$

$\Rightarrow \quad x-y+1=0$

The point $\left(\frac{3}{4},\frac{7}{4}\right)$ satisfies, the equation of line

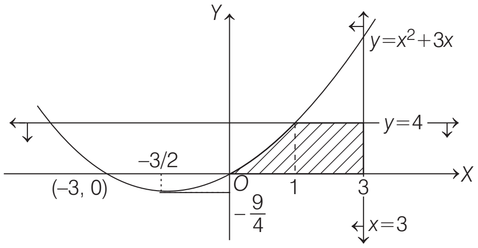
$$x-y+1=0$$

8. Given, $y \le x^2+3x$

$\Rightarrow \quad y \le \left(x+\frac{3}{2}\right)^2-\frac{9}{4} \Rightarrow \left(x+\frac{3}{2}\right)^2 \ge \left(y+\frac{9}{4}\right)$

Since, $0 \le y \le 4$ and $0 \le x \le 3$

$\therefore$ The diagram for the given inequalities is



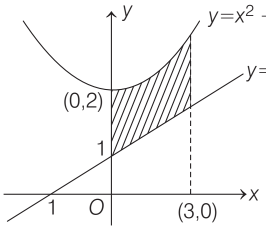
and points of intersection of curves $y=x^2+3x$ and $y=4$ are (1, 4) and (–4, 4)

Now required area

$=\int_0^1 (x^2+3x)dx+\int_1^3 4\,dx=\left[\frac{x^3}{3}+\frac{3x^2}{2}\right]_0^1+[4x]_1^3$

$=\frac{1}{3}+\frac{3}{2}+4(3-1)=\frac{2+9}{6}+8=\frac{11}{6}+8=\frac{59}{6}$ sq units

9. Given equation of parabola is $y=x^2+2$, and the line is $y=x+1$



The required area = area of shaded region

$=\int_0^3 ((x^2+2)-(x+1))\,dx=\int_0^3 (x^2-x+1)\,dx$

$=\left[\frac{x^3}{3}-\frac{x^2}{2}+x\right]_0^3=\left(\frac{27}{3}-\frac{9}{2}+3\right)-0$

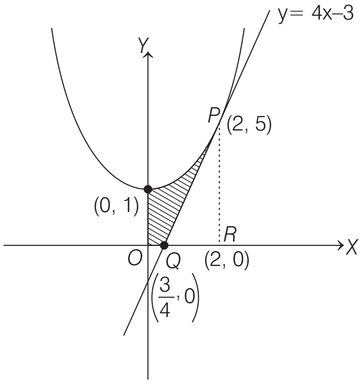
$=9-\frac{9}{2}+3=12-\frac{9}{2}=\frac{15}{2}$ sq units

10. Given, equation of parabola is $y=x^2+1$, which can be written as $x^2=(y-1)$. Clearly, vertex of parabola is (0, 1) and it will open upward.

Now, equation of tangent at (2, 5) is $\frac{y+5}{2}=2x+1$

[$\because$ Equation of the tangent at (x_1, y_1) is given by $T=0$. Here, $\frac{1}{2}(y+y_1)=xx_1+1$]

$$y=4x-3$$



Required area = Area of shaded region

$=\int_0^2 y(\text{parabola})\,dx-(\text{Area of }\Delta PQR)$

$=\int_0^2 (x^2+1)\,dx-(\text{Area of }\Delta PQR)$

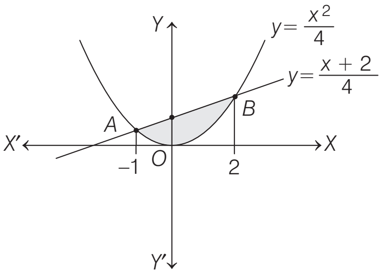
$=\left(\frac{x^3}{3}+x\right)_0^2-\frac{1}{2}\left(2-\frac{3}{4}\right)\cdot 5$

[∵ Area of a triangle $= \frac{1}{2} \times$ base $\times$ height]

$$= \left(\frac{8}{3} + 2\right) - 0 - \frac{1}{2}\left(\frac{5}{4}\right)5$$

$$= \frac{14}{3} - \frac{25}{8} = \frac{112 - 75}{24} = \frac{37}{24}$$

11. Given equation of curve is $x^2 = 4y$, which represent a parabola with vertex (0, 0) and it open upward.



Now, let us find the points of intersection of $x^2 = 4y$ and $4y = x + 2$

For this consider, $x^2 = x + 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x - 2)(x + 1) = 0$

$\Rightarrow x = -1, x = 2$

When $x = -1$, then $y = \frac{1}{4}$

and when $x = 2$, then $y = 1$

Thus, the points of intersection are $A\left(-1, \frac{1}{4}\right)$ and B (2, 1).

Now, required area = area of shaded region

$$= \int_{-1}^{2} \{y\,(\text{line}) - y\ (\text{parabola})\}\, dx$$

$$= \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[8 - \frac{1}{2} - 3\right] = \frac{1}{4}\left[5 - \frac{1}{2}\right] = \frac{9}{8} \text{ sq units.}$$

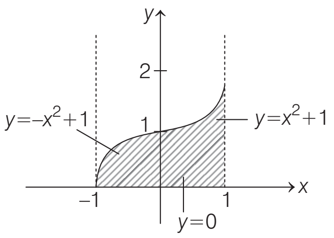
12. We have,

$A = \{(x, y) : 0 \le y \le x\,|\,x\,| + 1 \text{ and } -1 \le x \le 1\}$

When $x \ge 0$, then $0 \le y \le x^2 + 1$

and when $x < 0$, then $0 \le y \le -x^2 + 1$

Now, the required region is the shaded region.



[∵ $y = x^2 + 1 \Rightarrow x^2 = (y - 1)$, parabola with vertex (0, 1) and $y = -x^2 + 1 \Rightarrow x^2 = -(y - 1)$, parabola with vertex (0,1) but open downward]

We need to calculate the shaded area, which is equal to

$$\int_{-1}^{0}(-x^2 + 1)dx + \int_{0}^{1}(x^2 + 1)\,dx$$

$$= \left[-\frac{x^3}{3} + x\right]_{-1}^{0} + \left[\frac{x^3}{3} + x\right]_{0}^{1}$$

$$= \left(0 - \left[-\frac{(-1)^3}{3} + (-1)\right]\right) + \left(\left[\frac{1}{3} + 1\right] - 0\right)$$

$$= -\left(\frac{1}{3} - 1\right) + \frac{4}{3}$$

$$= \frac{2}{3} + \frac{4}{3} = 2$$

13. Given, equation of parabola is $y = x^2 - 1$, which can be rewritten as $x^2 = y + 1$ or $x^2 = (y - (-1))$.

$\Rightarrow$ Vertex of parabola is $(0, -1)$ and it is open upward.

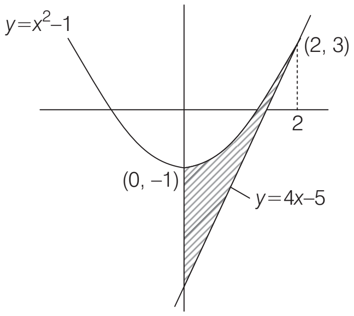
Equation of tangent at (2, 3) is given by $T = 0$

$\Rightarrow \frac{y + y_1}{2} = x\,x_1 - 1$, where, $x_1 = 2$

and $\quad y_1 = 3.$

$\Rightarrow \quad \frac{y + 3}{2} = 2x - 1$

$\Rightarrow \quad y = 4x - 5$



Now, required area = area of shaded region

$$= \int_0^2 (y(\text{parabola}) - y(\text{tangent}))\, dx$$

$$= \int_0^2 [(x^2 - 1) - (4x - 5)]\, dx$$

$$= \int_0^2 (x^2 - 4x + 4)\, dx = \int_0^2 (x - 2)^2\, dx$$

$$= \left|\frac{(x-2)^3}{3}\right|_0^2 = \frac{(2-2)^3}{3} - \frac{(0-2)^3}{3} = \frac{8}{3} \text{ sq units.}$$

14. We have,

$\Rightarrow \quad 18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow \quad 18x^2 - 6\pi x - 3\pi x + \pi^2 = 0$

$\quad (6x - \pi)(3x - \pi) = 0$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{3}$

Now, $\alpha < \beta \quad \alpha = \frac{\pi}{6},$

$\beta = \frac{\pi}{3}$

Given, g(x) = cos x^2 and f(x) = $\sqrt{x}$

$$y = gof(x)$$

$$\therefore \quad y = g(f(x)) = \cos x$$

Area of region bounded by $x = \alpha, x = \beta$, $y = 0$ and curve $y = g(f(x))$ is

$$A = \int_{\pi/6}^{\pi/3} \cos x\, dx$$

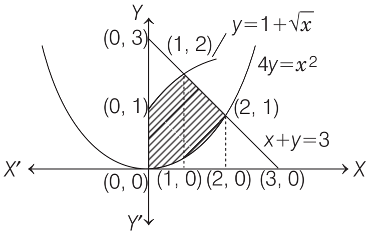
$$A = [\sin x]_{\pi/6}^{\pi/3}$$

$$A = \sin\frac{\pi}{3} - \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$A = \left(\frac{\sqrt{3}-1}{2}\right)$$

15. Required area

$$= \int_0^1 (1+\sqrt{x})dx + \int_1^2 (3-x)dx - \int_0^2 \frac{x^2}{4}\,dx$$



$$= \left[x + \frac{x^{3/2}}{3/2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^2 - \left[\frac{x^3}{12}\right]_0^2$$

$$= \left(1 + \frac{2}{3}\right) + \left(6 - 2 - 3 + \frac{1}{2}\right) - \left(\frac{8}{12}\right)$$

$$= \frac{5}{3} + \frac{3}{2} - \frac{2}{3} = 1 + \frac{3}{2} = \frac{5}{2} \text{ sq units}$$

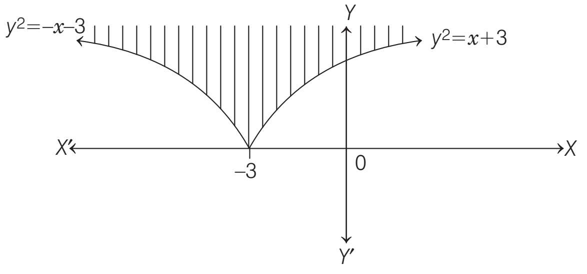
16. Here, $\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$

$$\therefore \quad y \geq \sqrt{x+3}$$

$$\Rightarrow \quad y \geq \begin{cases} \sqrt{x+3}\,, \text{ when } x \geq -3 \\ \sqrt{-x-3}\,, \text{ when } x \leq -3 \end{cases}$$

$$\text{or} \quad y^2 \geq \begin{cases} x+3\,, \text{ when } x \geq -3 \\ -3-x, \text{ when } x \leq -3 \end{cases}$$

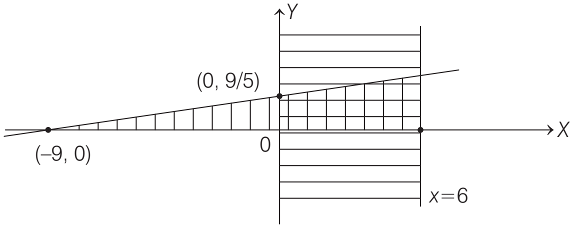
Shown as



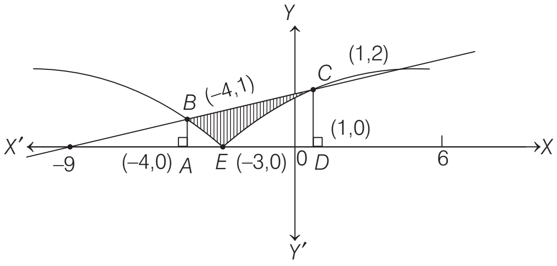
Also, $\quad 5y \leq (x+9) \leq 15$

$\Rightarrow \quad (x+9) \geq 5y$ and $x \leq 6$

Shown as



$\therefore \quad \{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$



$\therefore$ Required area = Area of trapezium *ABCD*
– Area of *ABE* under parabola
– Area of *CDE* under parabola

$$= \frac{1}{2}(1+2)(5) - \int_{-4}^{-3} \sqrt{-(x+3)}\,dx - \int_{-3}^{1} \sqrt{(x+3)}\,dx$$

$$= \frac{15}{2} - \left[\frac{(-3-x)^{3/2}}{-\frac{3}{2}}\right]_{-4}^{-3} - \left[\frac{(x+3)^{3/2}}{\frac{3}{2}}\right]_{-3}^{1}$$

$$= \frac{15}{2} + \frac{2}{3}[0-1] - \frac{2}{3}[8-0] = \frac{15}{2} - \frac{2}{3} - \frac{16}{3} = \frac{15}{2} - \frac{18}{3} = \frac{3}{2}$$

17. Given region is $\{(x, y) : y^2 \leq 2x$ and $y \geq 4x - 1\}$

$y^2 \leq 2x$ repressents a region inside the parabola

$$y^2 = 2x \qquad \text{...(i)}$$

and $y \geq 4x - 1$ represents a region to the left of the line

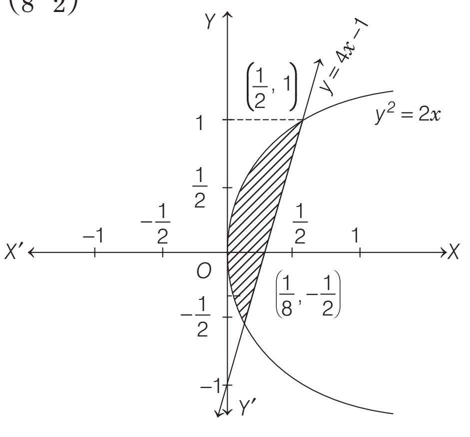
$$y = 4x - 1 \qquad \text{...(ii)}$$

The point of intersection of the curves (i) and (ii) is

$$(4x-1)^2 = 2x \Rightarrow 16x^2 + 1 - 8x = 2x$$

$$\Rightarrow \quad 16x^2 - 10x + 11 = 0 \Rightarrow x = \frac{1}{2}, \frac{1}{8}$$

So, the points where these curves intersect are $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{8}, \frac{1}{2}\right)$.



$$\therefore \text{ Required area} = \int_{-1/2}^{1}\left(\frac{y+1}{4} - \frac{y^2}{2}\right)dy$$

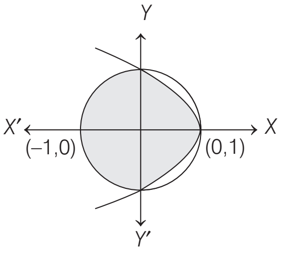
$$= \frac{1}{4}\left(\frac{y^2}{2} + y\right)_{-1/2}^{-1} - \frac{1}{6}(y^3)_{-1/2}^{1}$$

$$= \frac{1}{4}\left\{\left(\frac{1}{2}+1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right\} - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\}$$

$$= \frac{1}{4} \times \frac{15}{8} - \frac{3}{16} = \frac{9}{32} \text{ sq units}$$

18. Given, $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$

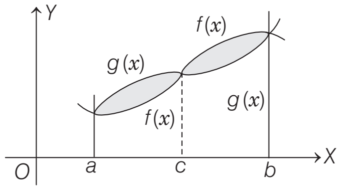


$$\text{Required area} = \frac{1}{2}\pi r^2 + 2\int_0^1 (1 - y^2)dy$$

$$= \frac{1}{2}\pi(1)^2 + 2\left(y - \frac{y^3}{3}\right)_0^1$$

$$= \left(\frac{\pi}{2} + \frac{4}{3}\right) \text{ sq units}$$

19. **PLAN** To find the bounded area between $y = f(x)$ and $y = g(x)$ between $x = a$ to $x = b$.



$$\therefore \text{ Area bounded} = \int_a^c [g(x) - f(x)]dx + \int_c^b [f(x) - g(x)]dx$$

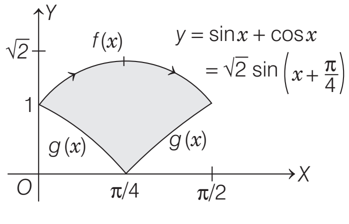
$$= \int_a^b |f(x) - g(x)|dx$$

Here, $f(x) = y = \sin x + \cos x$, when $0 \le x \le \frac{\pi}{2}$

and $g(x) = y = |\cos x - \sin x|$

$$= \begin{cases} \cos x - \sin x, & 0 \le x \le \frac{\pi}{4} \\ \sin x - \cos x, & \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$$

could be shown as



$$\therefore \text{ Area bounded} = \int_0^{\pi/4} \{(\sin x + \cos x) - (\cos x - \sin x)\}\, dx$$

$$+ \int_{\pi/4}^{\pi/2} \{(\sin x + \cos x) - (\sin x - \cos x)\}\, dx$$

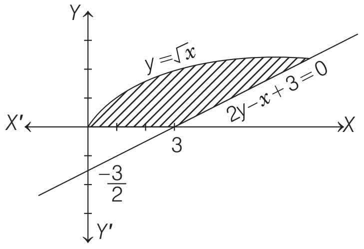
$$= \int_0^{\pi/4} 2\sin x\, dx + \int_{\pi/4}^{\pi/2} 2\cos x\, dx$$

$$= -2\,[\cos x]_0^{\pi/4} + 2\,[\sin x \cdot n]_{\pi/4}^{\pi/2}$$

$$= 4 - 2\sqrt{2} = 2\sqrt{2}(\sqrt{2} - 1) \text{ sq units}$$

20. Given curves are $y = \sqrt{x}$ ...(i)

and $2y - x + 3 = 0$ ...(ii)



On solving Eqs. (i) and (ii), we get

$$2\sqrt{x} - (\sqrt{x})^2 + 3 = 0$$

$$\Rightarrow (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$$

$$\Rightarrow (\sqrt{x} - 3)(\sqrt{x} + 1) = 0 \Rightarrow \sqrt{x} = 3$$

[since, $\sqrt{x} = -1$ is not possible]

$\therefore$ $y = 3$

Hence, required area

$$= \int_0^3 (x_2 - x_1)\, dy = \int_0^3 \{(2y + 3) - y^2\}\, dy$$

$$= \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3 = 9 + 9 - 9 = 9 \text{ sq units}$$

21. $R_1 = \int_{-1}^{2} x\, f(x)\, dx$ ...(i)

Using $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

$$R_1 = \int_{-1}^{2} (1 - x)\, f(1 - x)\, dx$$

$\therefore$ $R_1 = \int_{-1}^{2} (1 - x)\, f(x)\, dx$ ...(ii)

[$f(x) = f(1 - x)$, given]

Given, R_2 is area bounded by $f(x)$, $x = -1$ and $x = 2$.

$\therefore$ $R_2 = \int_{-1}^{2} f(x)\, dx$ ...(iii)

On adding Eqs. (i) and (ii), we get

$2R_1 = \int_{-1}^{2} f(x)\, dx$ ...(iv)

From Eqs. (iii) and (iv), we get

$$2R_1 = R_2$$

22. Here, area between 0 to b is R_1 and b to 1 is R_2.

$$\therefore \int_0^b (1 - x)^2\, dx - \int_b^1 (1 - x)^2\, dx = \frac{1}{4}$$

$$\Rightarrow \left[\frac{(1 - x)^3}{-3}\right]_0^b - \left[\frac{(1 - x)^3}{-3}\right]_b^1 = \frac{1}{4}$$

$$\Rightarrow -\frac{1}{3}\,[(1-b)^3-1]+\frac{1}{3}\,[0-(1-b)^3]=\frac{1}{4}$$

$$\Rightarrow \quad -\frac{2}{3}\,(1-b)^3=-\frac{1}{3}+\frac{1}{4}=-\frac{1}{12} \quad \Rightarrow \quad (1-b)^3=\frac{1}{8}$$

$$\Rightarrow \quad (1-b)=\frac{1}{2} \quad \Rightarrow \quad b=\frac{1}{2}$$

23. Required area $= \int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}}-\sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$$\left[\because \frac{1+\sin x}{\cos x}>\frac{1-\sin x}{\cos x}>0\right]$$

$$=\int_0^{\pi/4}\left(\sqrt{\frac{1+\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}-\sqrt{\frac{1-\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}\right)dx$$

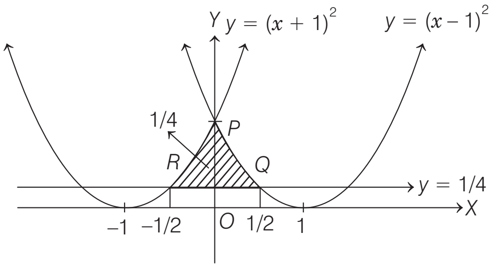
$$=\int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}}-\sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$=\int_0^{\pi/4}\frac{1+\tan\frac{x}{2}-1+\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\,dx=\int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\,dx$$

Put $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx=dt=\int_0^{\tan\frac{\pi}{8}}\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}}$

As $\quad \int_0^{\sqrt{2}-1}\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}} \qquad \left[\because \tan\frac{\pi}{8}=\sqrt{2}-1\right]$

24. The curves $y=(x-1)^2$, $y=(x+1)^2$ and $y=1/4$ are shown as



where, points of intersection are

$$(x-1)^2=\frac{1}{4} \quad\Rightarrow\quad x=\frac{1}{2} \text{ and } (x+1)^2=\frac{1}{4}\Rightarrow x=-\frac{1}{2}$$

i.e. $\quad Q\left(\frac{1}{2},\frac{1}{4}\right)$ and $R\left(-\frac{1}{2},\frac{1}{4}\right)$

$\therefore\quad$ Required area $=2\int_0^{1/2}\left[(x-1)^2-\frac{1}{4}\right]dx$

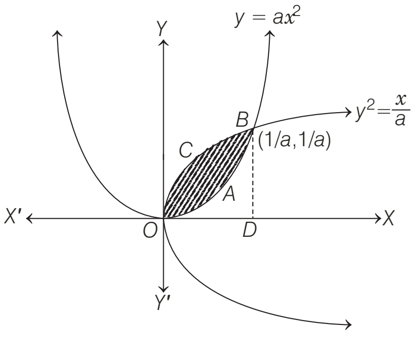
$$=2\left[\frac{(x-1)^3}{3}-\frac{1}{4}x\right]_0^{1/2}$$

$$=2\left[-\frac{1}{8\cdot 3}-\frac{1}{8}-\left(-\frac{1}{3}-0\right)\right]=\frac{8}{24}=\frac{1}{3}\text{ sq unit}$$

25. As from the figure, area enclosed between the curves is *OABCO*.

Thus, the point of intersection of

$$y=ax^2 \text{ and } x=ay^2$$



$$\Rightarrow \qquad x=a\,(ax^2)^2$$

$$\Rightarrow \qquad x=0,\frac{1}{a} \quad\Rightarrow\quad y=0,\ \frac{1}{a}$$

So, the points of intersection are (0, 0) and $\left(\frac{1}{a},\frac{1}{a}\right)$.

$\therefore$ Required area *OABCO* = Area of curve *OCBDO* – Area of curve *OABDO*

$$\Rightarrow \quad \int_0^{1/a}\left(\sqrt{\frac{x}{a}}-ax^2\right)dx=1 \qquad \text{[given]}$$

$$\Rightarrow \quad \left[\frac{1}{\sqrt{a}}\cdot\frac{x^{3/2}}{3/2}-\frac{ax^3}{3}\right]_0^{1/a}=1$$

$$\Rightarrow \quad \frac{2}{3a^2}-\frac{1}{3a^2}=1$$

$$\Rightarrow \quad a^2=\frac{1}{3} \quad\Rightarrow\quad a=\frac{1}{\sqrt{3}} \qquad [\because a>0]$$

26. Since, $\int_1^b f(x)\,dx=(b-1)\sin(3b+4)$

On differentiating both sides w.r.t. b, we get

$$f(b)=3(b-1)\cdot\cos(3b+4)+\sin(3b+4)$$

$$\therefore \quad f(x)=\sin(3x+4)+3(x-1)\cos(3x+4)$$

27. Given, $\frac{dy}{dx}=2x+1$

On integrating both sides

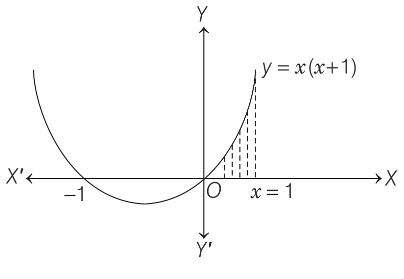
$$\int dy=\int(2x+1)\,dx$$

$\Rightarrow \qquad y=x^2+x+C$ which passes through (1,2)

$\therefore \qquad 2=1+1+C$

$\Rightarrow \qquad C=0$

$\therefore \qquad y=x^2+x$



Thus, the required area bounded by X-axis, the curve and $x=1$

$$=\int_0^1 (x^2+x)dx=\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1$$

$$=\frac{1}{3}+\frac{1}{2}=\frac{5}{6}\text{ sq unit}$$

28. $\int_0^1 (x-x^3)dx=2\int_0^\alpha (x-x^3)dx$

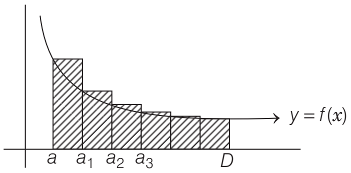
$$\frac{1}{4}=2\left(\frac{\alpha^2}{2}-\frac{\alpha^4}{4}\right)$$

$$2\alpha^4-4\alpha^2+1=0$$

$$\Rightarrow \quad \alpha^2=\frac{4-\sqrt{16-8}}{4} \qquad (\because \alpha\in(0,1))$$

$$\alpha^2=1-\frac{1}{\sqrt{2}}$$

29. **PLAN** (i) Area of region $f(x)$ bounded between $x=a$ to $x=b$ is



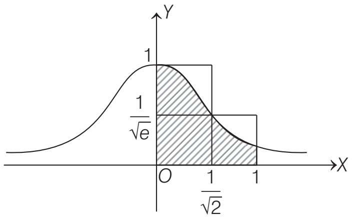
$\int_a^b f(x)dx=$ Sum of areas of rectangle shown in shaded part.

(ii) If $f(x)\geq g(x)$ when defined in $[a,b]$, then

$$\int_a^b f(x)dx\geq\int_a^b g(x)dx$$

Description of Situation As the given curve $y=e^{-x^2}$ cannot be integrated, thus we have to bound this function by using above mentioned concept.

Graph for $y=e^{-x^2}$



Since, $x^2\leq x$ when $x\in[0,1]$

$$\Rightarrow \quad -x^2\geq -x \text{ or } e^{-x^2}\geq e^{-x}$$

$$\therefore \quad \int_0^1 e^{-x^2}dx\geq\int_0^1 e^{-x}\,dx$$

$$\Rightarrow \quad S\geq-(e^{-x})_0^1=1-\frac{1}{e} \qquad \ldots\text{(i)}$$

Also, $\int_0^1 e^{-x^2}dx\leq$ Area of two rectangles

$$\leq\left(1\times\frac{1}{\sqrt{2}}\right)+\left(1-\frac{1}{\sqrt{2}}\right)\times\frac{1}{\sqrt{e}}$$

$$\leq\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right) \qquad \ldots\text{(ii)}$$

$$\therefore \quad \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)\geq S\geq 1-\frac{1}{e} \qquad \text{[from Eqs. (i) and (ii)]}$$

30. Shaded area $=e-\left(\int_0^1 e^x\,dx\right)=1$

Also, $\int_1^e \ln(e+1-y)\,dy$ [put $e+1-y=t\Rightarrow -dy=dt$]

$$=\int_e^1 \ln\ t\,(-dt)=\int_1^e \ln t\,dt=\int_1^e \ln y\,dy=1$$

31. ***Case* I** When $m=0$

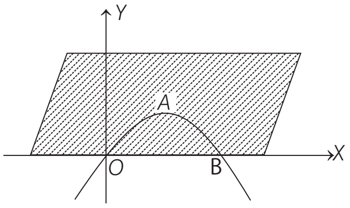
In this case, $y=x-x^2$ ...(i)

and $y=0$ ...(ii)

are two given curves, $y>0$ is total region above X-axis.

Therefore, area between $y=x-x^2$ and $y=0$

is area between $y=x-x^2$ and above the X-axis



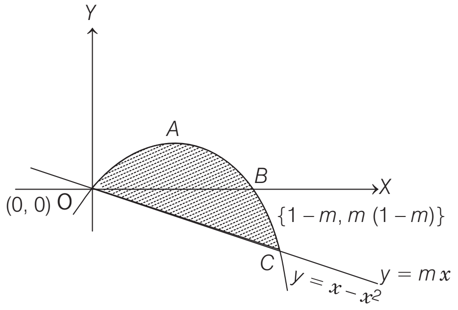
$$\therefore \quad A=\int_0^1 (x-x^2)\,dx=\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}\neq\frac{9}{2}$$

Hence, no solution exists.

***Case* II** When $m<0$

In this case, area between $y=x-x^2$ and $y=mx$ is $OABCO$ and points of intersection are (0,0) and $\{1-m, m(1-m)\}$.

$\therefore$ Area of curve $OABCO=\int_0^{1-m}[x-x^2-mx]\,dx$



$$=\left[(1-m)\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{1-m}$$

$$=\frac{1}{2}(1-m)^3-\frac{1}{3}(1-m)^3=\frac{1}{6}(1-m)^3$$

$$\therefore \quad \frac{1}{6}(1-m)^3=\frac{9}{2} \qquad \text{[given]}$$

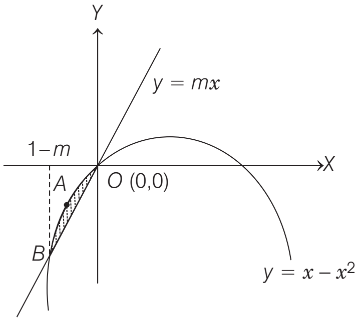
$$\Rightarrow \quad (1-m)^3=27$$

$$\Rightarrow \quad 1-m=3$$

$$\Rightarrow \quad m=-2$$

***Case* III** When $m > 0$

In this case, $y = mx$ and $y = x - x^2$ intersect in (0,0) and $\{(1-m),\ m(1-m)\}$ as shown in figure



$\therefore$ Area of shaded region $= \int_{1-m}^{0} (x - x^2 - mx)\, dx$

$$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_{1-m}^{0}$$

$$= -\frac{1}{2}(1-m)(1-m)^2 + \frac{1}{3}(1-m)^3$$

$$= -\frac{1}{6}(1-m)^3$$

$\Rightarrow \quad \frac{9}{2} = -\frac{1}{6}(1-m)^3$ [given]

$\Rightarrow \quad (1-m)^3 = -27$

$\Rightarrow \quad (1-m) = -3$

$\Rightarrow \quad m = 3 + 1 = 4$

Therefore, (b) and (d) are the answers.

32. Given, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$

$\Rightarrow \quad 4a^2 f(-1) + 4a\, f(1) + f(2) = 3a^2 + 3a,$ ...(i)

$4b^2 f(-1) + 4b\, f(1) + f(2) = 3b^2 + 3b$ ...(ii)

and $\quad 4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c$ ...(iii)

where, $f(x)$ is quadratic expression given by,

$f(x) = ax^2 + bx + c$ and Eqs. (i), (ii) and (iii).

$\Rightarrow \quad 4x^2 f(-1) + 4x\, f(1) + f(2) = 3x^2 + 3x$

or $\{4\, f(-1) - 3\}\, x^2 + \{4f(1) - 3\}\, x + f(2) = 0$ ...(iv)

As above equation has 3 roots a, b and c.

So, above equation is identity in x.

i.e. coefficients must be zero.

$\Rightarrow \quad f(-1) = 3/4, f(1) = 3/4, f(2) = 0$ ...(v)

$\because \quad f(x) = ax^2 + bx + c$

$\therefore \quad a = -1/4, b = 0$ and $c = 1$, using Eq. (v)

Thus, $\quad f(x) = \frac{4 - x^2}{4}$ shown as,

Let $\quad A(-2, 0), B = (2t, -t^2 + 1)$

Since, AB subtends right angle at vertex V (0, 1).

$\Rightarrow \quad \frac{1}{2} \cdot \frac{-t^2}{2t} = -1$

$\Rightarrow \quad t = 4$

$\therefore \quad B(8, -15)$

So, equation of chord AB is $y = \frac{-(3x+6)}{2}$.

$\therefore$ Required area $= \left|\int_{-2}^{8} \left(\frac{4 - x^2}{4} + \frac{3x + 6}{2}\right) dx\right|$

$$= \left|\left(x - \frac{x^3}{12} + \frac{3x^2}{4} + 3x\right)_{-2}^{8}\right|$$

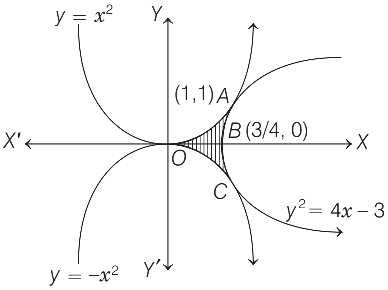
$$= \left|\left[8 - \frac{128}{3} + 48 + 24 - \left(-2 + \frac{2}{3} + 3 - 6\right)\right]\right|$$

$$= \frac{125}{3} \text{ sq units}$$

33. The region bounded by the curves $y = x^2$, $y = -x^2$ and $y^2 = 4x - 3$ is symmetrical about X-axis, where $y = 4x - 3$ meets at (1, 1).

$\therefore$ Area of curve ($OABCO$)

$$= 2\left[\int_0^1 x^2\, dx - \int_{3/4}^{1} (\sqrt{4x - 3})\, dx\right]$$



$$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{(4x-3)^{3/2}}{3 \cdot 4/2}\right)_{3/4}^{1}\right]$$

$$= 2\left(\frac{1}{3} - \frac{1}{6}\right)$$

$$= 1 \cdot \frac{1}{6} = \frac{1}{3} \text{ sq unit}$$

34. Here, slope of tangent,

$$\frac{dy}{dx} = \frac{(x+1)^2 + y - 3}{(x+1)}$$

$\Rightarrow \quad \frac{dy}{dx} = (x+1) + \frac{(y-3)}{(x+1)},$

Put $x + 1 = X$ and $y - 3 = Y$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dY}{dX}$

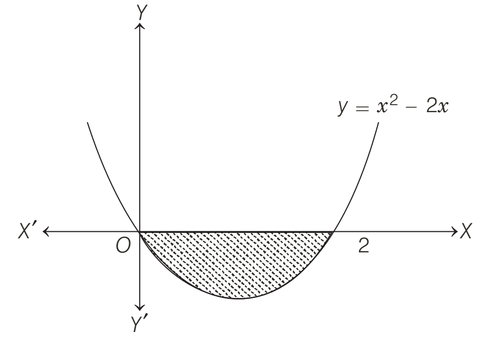
$\therefore \quad \frac{dY}{dX} = X + \frac{Y}{X}$

$\Rightarrow \quad \frac{dY}{dX} - \frac{1}{X} Y = X$

$\text{IF} = e^{\int -\frac{1}{X} dX} = e^{-\log X} = \frac{1}{X}$

$\therefore$ Solution is, $Y \cdot \dfrac{1}{X} = \int X \cdot \dfrac{1}{X}\, dX + c$

$\Rightarrow \quad \dfrac{Y}{X} = X + c$



$y - 3 = (x+1)^2 + c(x+1)$, which passes through (2, 0).

$\Rightarrow \quad -3 = (3)^2 + 3c$

$\Rightarrow \quad c = -4$

$\therefore$ Required curve

$$y = (x+1)^2 - 4(x+1) + 3$$

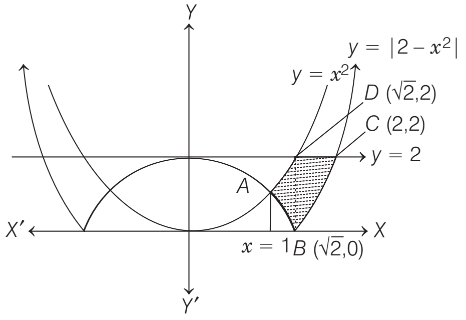
$\Rightarrow \quad y = x^2 - 2x$

$\therefore$ Required area $= \left| \int_0^2 (x^2 - 2x)dx \right| = \left| \left(\dfrac{x^3}{3} - x^2 \right)_0^2 \right|$

$$= \frac{8}{3} - 4 = \frac{4}{3} \text{ sq units}$$

35. The points in the graph are

$A\ (1, 1), B\ (\sqrt{2}, 0), C\ (2, 2), D\ (\sqrt{2}, 2)$



$\therefore$ Required area

$$= \int_1^{\sqrt{2}} \{x^2 - (2 - x^2)\}\, dx + \int_{\sqrt{2}}^2 \{2 - (x^2 - 2)\}\, dx$$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)\, dx + \int_{\sqrt{2}}^2 (4 - x^2)\, dx$$

$$= \left[\frac{2x^3}{3} - 2x \right]_1^{\sqrt{2}} + \left[4x - \frac{x^3}{3} \right]_{\sqrt{2}}^2$$

$$= \left[\frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{2}{3} + 2 \right] + \left[8 - \frac{8}{3} - 4\sqrt{2} + \frac{2\sqrt{2}}{3} \right]$$

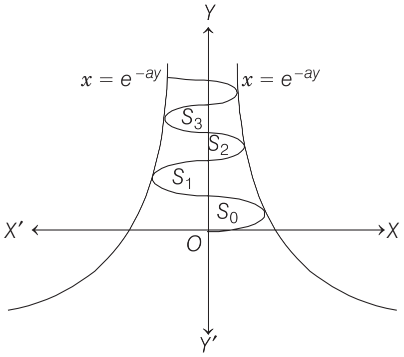
$$= \left(\frac{20 - 12\sqrt{2}}{3} \right) \text{ sq units}$$

36. Given, $x = (\sin by)\, e^{-ay}$

Now, $\quad -1 \le \sin \text{by} \le 1$

$\Rightarrow \quad -e^{-ay} \le e^{-ay} \sin by \le e^{-ay}$

$\Rightarrow \quad -e^{-ay} \le x \le e^{-ay}$



In this case, if we take a and b positive, the values $-e^{-ay}$ and e^{-ay} become left bond and right bond of the curve and due to oscillating nature of sin by, it will oscillate between $x = e^{-ay}$ and $x = -e^{-ay}$

Now, $\quad S_j = \int_{j\pi/b}^{(j+1)\pi/b} \sin by \cdot e^{-ay}\, dy$

$$\left[\begin{array}{l} \text{since, } I = \int \sin by \cdot e^{-ay} dy \\ I = \dfrac{-e^{-ay}}{a^2 + b^2} \quad (a \sin by + b \cos by) \end{array} \right]$$

$$\therefore \quad S_j = \left| \frac{-1}{a^2 + b^2} \left[e^{\frac{-a(j+1)\pi}{b}} \{a \sin(j+1)\pi + b \cos(j+1)\pi\} - e^{\frac{-aj\pi}{b}} (a \sin j\pi + b \cos j\pi) \right] \right|$$

$$S_j = \left| -\frac{1}{a^2 + b^2} \left[e^{-\frac{a}{b}(j+1)\pi} \{0 + b(-1)^{j+1}\} - e^{-aj\pi/b} \{0 + b(-1)^j\} \right] \right|$$

$$= \left| \frac{b\,(-1)^j\, e^{-\frac{a}{b} j\pi}}{a^2 + b^2} \left(e^{-\frac{a}{b}\pi} + 1 \right) \right|$$

$$[\because (-1)^{j+2} = (-1)^2\,(-1)^j = (-1)^j]$$

$$= \frac{b\, e^{-\frac{a}{b} j\pi}}{a^2 + b^2} \left(e^{-\frac{a}{b}\pi} + 1 \right)$$

$$\therefore \quad \frac{S_j}{S_{j-1}} = \frac{\dfrac{b e^{-\frac{a}{b} j\pi} \left(e^{\frac{-a\pi}{b}} + 1 \right)}{a^2 + b^2}}{\dfrac{b\, e^{-\frac{a}{b}(j-1)\pi} \left(e^{\frac{-a\pi}{b}} + 1 \right)}{a^2 + b^2}} = \frac{e^{-\frac{a}{b} j\pi}}{e^{-\frac{a}{b}(j-1)\pi}}$$

$$= e^{-\frac{a}{b}\pi} = \text{constant}$$

$\Rightarrow \quad S_0, S_1, S_2, \ldots, S_j$ form a GP.

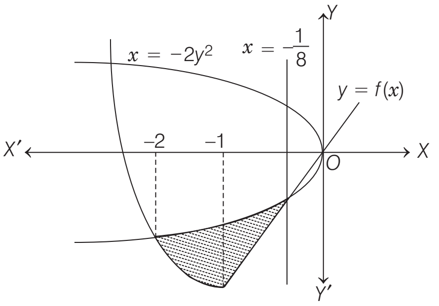
For $a = -1$ and $b = \pi$

$$S_j = \frac{\pi \cdot e^{\frac{1}{\pi}.\pi j}}{(1+\pi^{2)}}\left(e^{\frac{1}{\pi}\cdot\pi}+1\right) = \frac{\pi \cdot e^j}{(1+\pi^{2)}}(1+e)$$

$$\Rightarrow \quad \sum_{j=0}^{n} S_j = \frac{\pi\cdot(1+e)}{(1+\pi)^2}\sum_{j=0}^{n} e^j = \frac{\pi(1+e)}{(1+\pi^2)}(e^0+e^1+\ldots+e^n)$$

$$= \frac{\pi(1+e)}{(1+\pi^2)}\cdot\frac{(e^{n+1}-1)}{e-1}$$

37. Given, $f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2+ax+b, & |x| > 1 \end{cases}$



$$\Rightarrow \quad f(x) = \begin{cases} x^2+ax+b, & \text{if } x<-1 \\ 2x, & \text{if } -1 \le x < 1 \\ x^2+ax+b, & \text{if } x \ge 1 \end{cases}$$

f is continuous on R, so f is continuous at –1 and 1.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$$

and $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$\Rightarrow \quad 1-a+b=-2$ and $2=1+a+b$

$\Rightarrow \quad a-b=3$ and $a+b=1$

$\therefore \quad a=2, \quad b=-1$

Hence, $f(x) = \begin{cases} x^2+2x-1, & \text{if } x<-1 \\ 2x, & \text{if } -1\le x<1 \\ x^2+2x-1, & \text{if } x \ge 1 \end{cases}$

Next, we have to find the points $x=-2y^2$ and $y=f(x)$. The point of intersection is (–2, –1).

$$\therefore \quad \text{Required area} = \int_{-2}^{-1/8}\left[\sqrt{\frac{-x}{2}} - f(x)\right]dx$$

$$= \int_{-2}^{-1/8}\sqrt{\frac{-x}{2}}\,dx - \int_{-2}^{-1}(x^2+2x-1)dx - \int_{-1}^{-1/8} 2x\,dx$$

$$= -\frac{2}{3\sqrt{2}}[(-x)^{3/2}]_{-2}^{-1/8} - \left[\left(\frac{x^3}{3}+x^2-x\right)\right]_{-2}^{-1} - [x^2]_{-1}^{-1/8}$$

$$= -\frac{2}{3\sqrt{2}}\left[\left(\frac{1}{8}\right)^{3/2} - 2^{3/2}\right] - \left(-\frac{1}{3}+1+1\right) + \left(-\frac{8}{3}+4+2\right) - \left[\frac{1}{64}-1\right]$$

$$= \frac{\sqrt{2}}{3}[2\sqrt{2}-2^{-9/2}] + \frac{5}{3} + \frac{63}{64}$$

$$= \frac{63}{16\times 3} + \frac{509}{64\times 3} = \frac{761}{192} \text{ sq units}$$

38. Refer to the figure given in the question. Let the coordinates of P be (x, x^2), where $0 \le x \le 1$.

For the area ($OPRO$),

Upper boundary: $y = x^2$ and

lower boundary : $y = f(x)$

Lower limit of $x : 0$

Upper limit of $x : x$

$$\therefore \quad \text{Area } (OPRO) = \int_0^x t^2\,dt - \int_0^x f(t)\,dt$$

$$= \left[\frac{t^3}{3}\right]_0^x - \int_0^x f(t)\,dt$$

$$= \frac{x^3}{3} - \int_0^x f(t)\,dt$$

For the area ($OPQO$),

The upper curve : $x = \sqrt{y}$

and the lower curve : $x = y/2$

Lower limit of $y : 0$

and upper limit of $y : x^2$

$$\therefore \quad \text{Area } (OPQO) = \int_0^{x^2}\sqrt{t}\,dt - \int_0^{x^2}\frac{t}{2}\,dt$$

$$= \frac{2}{3}[t^{3/2}]_0^{x^2} - \frac{1}{4}[t^2]_0^{x^2}$$

$$= \frac{2}{3}x^3 - \frac{x^4}{4}$$

According to the given condition,

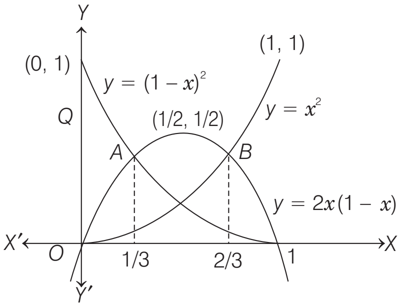
$$\frac{x^3}{3} - \int_0^x f(t)\,dt = \frac{2}{3}x^3 - \frac{x^4}{4}$$

On differentiating both sides w.r.t. x, we get

$$x^2 - f(x)\cdot 1 = 2x^2 - x^3$$

$$\Rightarrow \quad f(x) = x^3 - x^2, 0 \le x \le 1$$

39. We can draw the graph of $y = x^2$, $y = (1-x^2)$ and $y = 2x(1-x)$ in following figure



Now, to get the point of intersection of $y = x^2$ and $y = 2x(1-x)$, we get

$$x^2 = 2x(1-x)$$

$$\Rightarrow \quad 3x^2 = 2x$$

$$\Rightarrow \quad x(3x-2) = 0$$

$$\Rightarrow \quad x = 0, 2/3$$

Similarly, we can find the coordinate of the points of intersection of

$y = (1-x^2)$ and $y = 2x(1-x)$ are $x = 1/3$ and $x = 1$

From the figure, it is clear that,

$$f(x) = \begin{cases} (1-x)^2, & \text{if } 0 \le x \le 1/3 \\ 2x(1-x), & \text{if } 1/3 \le x \le 2/3 \\ x^2, & \text{if } 2/3 \le x \le 1 \end{cases}$$

$\therefore$ The required area

$$A = \int_0^1 f(x)\,dx$$

$$= \int_0^{1/3} (1-x)^2\,dx + \int_{1/3}^{2/3} 2x(1-x)\,dx + \int_{2/3}^{1} x^2\,dx$$

$$= \left[-\frac{1}{3}(1-x)^3\right]_0^{1/3} + \left[x^2 - \frac{2x^3}{3}\right]_{1/3}^{2/3} + \left[\frac{1}{3}x^3\right]_{2/3}^{1}$$

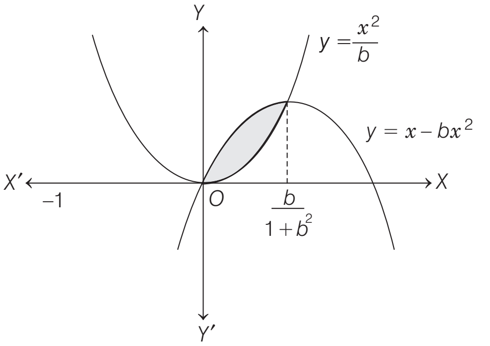
$$= \left[-\frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3}\right] + \left[\left(\frac{2}{3}\right)^2 - \frac{2}{3}\left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^2 + \frac{2}{3}\left(\frac{1}{3}\right)^3\right] + \left[\frac{1}{3}(1) - \frac{1}{3}\left(\frac{2}{3}\right)^3\right]$$

$$= \frac{19}{81} + \frac{13}{81} + \frac{19}{81} = \frac{17}{27} \text{ sq unit}$$

40. Eliminating y from $y = \frac{x^2}{b}$ and $y = x - bx^2$, we get

$$x^2 = bx - b^2x^2$$

$\Rightarrow$ $$x = 0, \frac{b}{1+b^2}$$



Thus, the area enclosed between the parabolas

$$A = \int_0^{b/(1+b)^2} \left(x - bx^2 - \frac{x^2}{b}\right) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \cdot \frac{1+b^2}{b}\right]_0^{b/(1+b)^2} = \frac{1}{6} \cdot \frac{b^2}{(1+b^2)^2}$$

On differentiating w.r.t. b, we get

$$\frac{dA}{db} = \frac{1}{6} \cdot \frac{(1+b^2)^2 \cdot 2b - 2b^2 \cdot (1+b^2) \cdot 2b}{(1+b^2)^4}$$

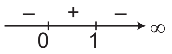
$$= \frac{1}{3} \cdot \frac{b(1-b^2)}{(1+b^2)^3}$$

For maximum value of A, put $\frac{dA}{db} = 0$

$\Rightarrow$ $$b = -1, 0, 1, \text{ since } b > 0$$

$\therefore$ We consider only $b = 1$.

Sign scheme for $\frac{dA}{db}$ around $b = 1$ is as shown below :



From sign scheme, it is clear that A is maximum at $b = 1$.

41. We have, $A_n = \int_0^{\pi/4} (\tan x)^n\,dx$

Since, $0 < \tan x < 1$, when $0 < x < \pi/4$

We have, $0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$

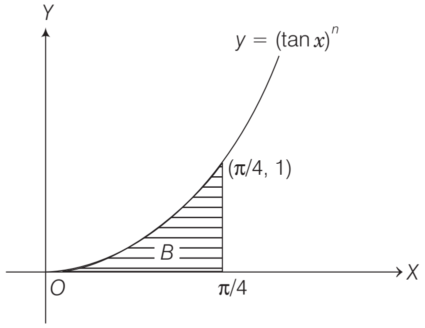
$\Rightarrow$ $\int_0^{\pi/4} (\tan x)^{n+1}\,dx < \int_0^{\pi/4} (\tan x)^n\,dx$

$\Rightarrow$ $A_{n+1} < A_n$

Now, for $n > 2$

$$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}]\,dx$$

$$= \int_0^{\pi/4} (\tan x)^n (1 + \tan^2 x)\,dx$$



$$= \int_0^{\pi/4} (\tan x)^n \sec^2 x\,dx$$

$$= \left[\frac{1}{(n+1)} (\tan x)^{n+1}\right]_0^{\pi/4}$$

$$= \frac{1}{(n+1)}(1 - 0) = \frac{1}{n+1}$$

Since, $A_{n+2} < A_{n+1} < A_n$,

then $A_n + A_{n+2} < 2A_n$

$\Rightarrow$ $\frac{1}{n+1} < 2A_n$

$\Rightarrow$ $\frac{1}{2n+2} < A_n$ ...(i)

Also, for $n > 2$ $A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$

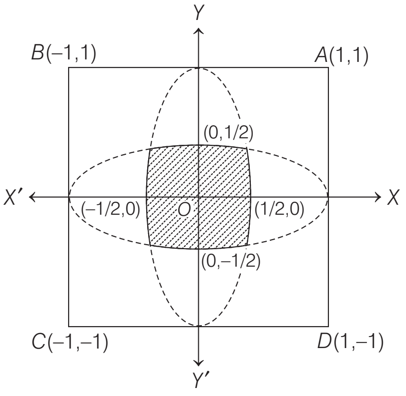
$\Rightarrow$ $2A_n < \frac{1}{n-1}$

$\Rightarrow$ $A_n < \frac{1}{2n-2}$ ...(ii)

From Eqs. (i) and (ii), $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$

42. The equations of the sides of the square are as follow :

$AB: y = 1, BC : x = -1, CD : y = -1, DA : x = 1$



Let the region be S and (x, y) is any point inside it.

Then, according to given conditions,

$$\sqrt{x^2 + y^2} < |1 - x|, |1 + x|, |1 - y|, |1 + y|$$

$$\Rightarrow \quad x^2 + y^2 < (1 - x)^2, (1 + x)^2, (1 - y)^2, (1 + y)^2$$

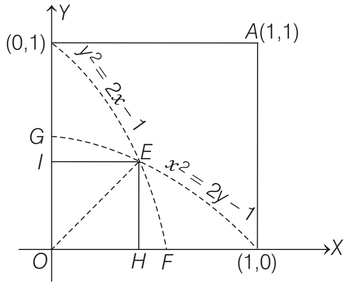
$$\Rightarrow \quad x^2 + y^2 < x^2 - 2x + 1, x^2 + 2x + 1, y^2 - 2y + 1, y^2 + 2y + 1$$

$$\Rightarrow \quad y^2 < 1 - 2x, y^2 < 1 + 2x, x^2 < 1 - 2y \text{ and } x^2 < 2y + 1$$

Now, in $y^2 = 1 - 2x$ and $y^2 = 1 + 2x$, the first equation represents a parabola with vertex at (1/2,0) and second equation represents a parabola with vertex (−1/2, 0) and in $x^2 = 1 - 2y$ and $x^2 = 1 + 2y$, the first equation represents a parabola with vertex at (0, 1/2) and second equation represents a parabola with vertex at (0, −1 /2) .

Therefore, the region S is lying inside the four parabolas

$$y^2 = 1 - 2x, y^2 = 1 + 2x, x^2 = 1 + 2y, x^2 = 1 - 2y$$



where, S is the shaded region.

Now, S is symmetrical in all four quadrants, therefore $S = 4 \times$ Area lying in the first quadrant.

Now, $y^2 = 1 - 2x$ and $x^2 = 1 - 2y$ intersect on the line $y = x$. The point of intersection is $E\ (\sqrt{2} - 1,\ \sqrt{2} - 1)$.

Area of the region $OEFO$

$$= \text{Area of } \Delta\, OEH + \text{Area of } HEFH$$

$$= \frac{1}{2} (\sqrt{2} - 1)^2 + \int_{\sqrt{2}-1}^{1/2} \sqrt{1 - 2x}\, dx$$

$$= \frac{1}{2} (\sqrt{2} - 1)^2 + \left[(1 - 2x)^{3/2}\, \frac{2}{3} \cdot \frac{1}{2} (-1) \right]_{\sqrt{2}-1}^{1/2}$$

$$= \frac{1}{2} (2 + 1 - 2\sqrt{2}) + \frac{1}{3} (1 + 2 - 2\sqrt{2})^{3/2}$$

$$= \frac{1}{2} (3 - 2\sqrt{2}) + \frac{1}{3} (3 - 2\sqrt{2})^{3/2}$$

$$= \frac{1}{2} (3 - 2\sqrt{2}) + \frac{1}{3} [(\sqrt{2} - 1)^2]^{3/2}$$

$$= \frac{1}{2} (3 - 2\sqrt{2}) + \frac{1}{3} (\sqrt{2} - 1)^3$$

$$= \frac{1}{2} (3 - \sqrt{2}) + \frac{1}{3} [2\sqrt{2} - 1 - 3\sqrt{2}\, (\sqrt{2} - 1)]$$

$$= \frac{1}{2} (3 - 2\sqrt{2}) + \frac{1}{3} [5\sqrt{2} - 7]$$

$$= \frac{1}{6} [9 - 6\sqrt{2} + 10\sqrt{2} - 14] = \frac{1}{6} [4\sqrt{2} - 5] \text{ sq units}$$

Similarly, area $OEGO = \frac{1}{6} (4\sqrt{2} - 5)$ sq units

Therefore, area of S lying in first quadrant

$$= \frac{2}{6} (4\sqrt{2} - 5) = \frac{1}{3} (4\sqrt{2} - 5) \text{ sq units}$$

Hence, $S = \frac{4}{3} (4\sqrt{2} - 5) = \frac{1}{3} (16\sqrt{2} - 20)$ sq units

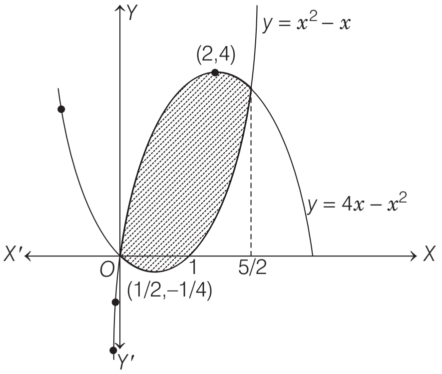
43. Given parabolas are $y = 4x - x^2$

and $\quad y = -(x - 2)^2 + 4$

or $\quad (x - 2)^2 = -(y - 4)$

Therefore, it is a vertically downward parabola with vertex at (2,4) and its axis is $x = 2$

and $\quad y = x^2 - x \Rightarrow y = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$



$$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 = y + \frac{1}{4}$$

This is a parabola having its vertex at $\left(\frac{1}{2}, -\frac{1}{4}\right)$.

Its axis is at $x = \frac{1}{2}$ and opening upwards.

The points of intersection of given curves are

$$4x - x^2 = x^2 - x \Rightarrow 2x^2 = 5x$$

$$\Rightarrow \quad x (2 - 5x) = 0 \quad \Rightarrow x = 0, \frac{5}{2}$$

Also, $y = x^2 - x$ meets X-axis at (0,0) and (1, 0).

$$\therefore \text{ Area, } A_1 = \int_0^{5/2} [(4x - x^2) - (x^2 - x)]\, dx$$

$$= \int_0^{5/2} (5x - 2x^2)\, dx$$

$$= \left[\frac{5}{2}x^2 - \frac{2}{3}x^3\right]_0^{5/2} = \frac{5}{2}\left(\frac{5}{2}\right)^2 - \frac{2}{3}\cdot\left(\frac{5}{2}\right)^3$$

$$= \frac{5}{2}\cdot\frac{25}{4} - \frac{2}{3}\cdot\frac{125}{8}$$

$$= \frac{125}{8}\left(1 - \frac{2}{3}\right) = \frac{125}{24} \text{ sq units}$$

This area is considering above and below X-axis both. Now, for area below X-axis separately, we consider

$$A_2 = -\int_0^1 (x^2 - x)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq units}$$

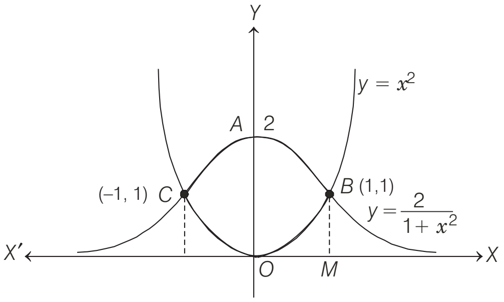
Therefore, net area above the X-axis is

$$A_1 - A_2 = \frac{125 - 4}{24} = \frac{121}{24} \text{ sq units}$$

Hence, ratio of area above the X-axis and area below X-axis

$$= \frac{121}{24} : \frac{1}{6} = 121 : 4$$

44. The curve $y = x^2$ is a parabola. It is symmetric about Y-axis and has its vertex at (0, 0) and the curve $y = \dfrac{2}{1 + x^2}$ is a bell shaped curve. X-axis is its asymptote and it is symmetric about Y-axis and its vertex is (0, 2).



Since, $\quad y = x^2 \qquad$ ...(i)

and $\quad y = \dfrac{2}{1 + x^2} \qquad$ ...(ii)

$\Rightarrow \quad y = \dfrac{2}{1 + y}$

$\Rightarrow \quad y^2 + y - 2 = 0$

$\Rightarrow \quad (y - 1)(y + 2) = 0 \Rightarrow y = -2, 1$

But $\quad y \geq 0$, so $y = 1 \Rightarrow x = \pm 1$

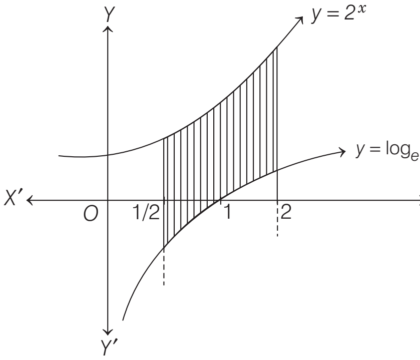
Therefore, coordinates of C are (–1, 1) and coordinates of B are (1,1).

$\therefore$ Required area $OBACO = 2 \times$ Area of curve $OBAO$

$$= 2\left[\int_0^1 \frac{2}{1 + x^2}\, dx - \int_0^1 x^2\, dx\right]$$

$$= 2\left[[2\tan^{-1} x]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right] = 2\left[\frac{2\pi}{4} - \frac{1}{3}\right] = \left(\pi - \frac{2}{3}\right) \text{ sq unit}$$

45. The required area is the shaded portion in following figure



$\therefore$ The required area

$$= \int_{1/2}^2 (2^x - \log x)\, dx = \left(\frac{2^x}{\log 2} - (x\log x - x)\right)_{1/2}^2$$

$$= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right) \text{ sq units}$$

46. Both the curves are defined for $x > 0$.

Both are positive when $x > 1$ and negative when $0 < x < 1$.

We know that, $\lim\limits_{x \to 0^+} (\log x) \to -\infty$

Hence, $\lim\limits_{x \to 0^+} \dfrac{\log x}{ex} \to -\infty$. Thus, Y-axis is asymptote of second curve.

And $\lim\limits_{x \to 0^+} ex \log x \qquad [(0) \times \infty \text{ form}]$

$$= \lim_{x \to 0^+} \frac{e \log x}{1/x} \qquad \left[-\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to 0^+} \frac{e\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = 0 \qquad \text{[using L'Hospital's rule]}$$

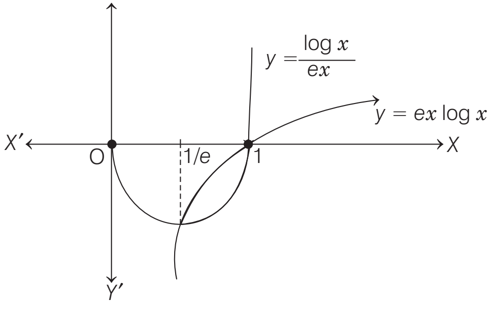
Thus, the first curve starts from (0, 0) but does not include (0, 0).

Now, the given curves intersect, therefore

$$ex \log x = \frac{\log x}{ex}$$

i.e. $\quad (e^2x^2 - 1)\log x = 0$

$\Rightarrow \quad x = 1, \dfrac{1}{e} \qquad [\because x > 0]$



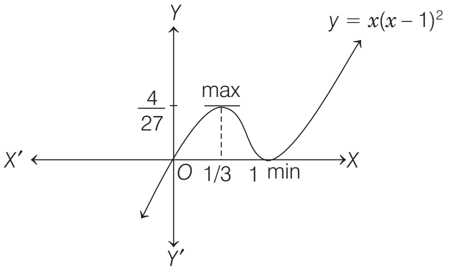
∴ The required area

$$= \int_{1/e}^{1} \left(\frac{(\log x)}{ex} - ex \log x \right) dx$$

$$= \frac{1}{e} \left[\frac{(\log x)^2}{2} \right]_{1/e}^{1} - e \left[\frac{x^2}{4} (2 \log x - 1) \right]_{1/e}^{1} = \left(\frac{e^2 - 5}{4e} \right) \text{ sq units}$$

47. Given, $y = x (x - 1)^2$

$$\Rightarrow \quad \frac{dy}{dx} = x \cdot 2 (x - 1) + (x - 1)^2$$



$$= (x - 1) \cdot (2x + x - 1)$$
$$= (x - 1) (3x - 1)$$

$$\begin{array}{ccccc} + & \bullet & - & \bullet & + \\ \hline & 1/3 & & 1 & \end{array}$$

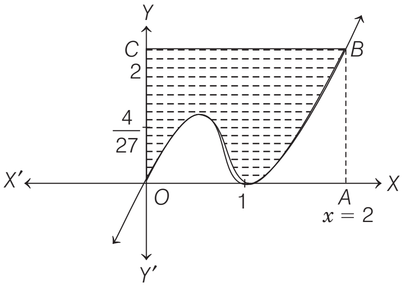
∴ Maximum at $x = 1/3$

$$y_{\max} = \frac{1}{3} \left(-\frac{2}{3} \right)^2 = \frac{4}{27}$$

Minimum at $x = 1$

$$y_{\min} = 0$$

Now, to find the area bounded by the curve $y = x (x - 1)^2$, the Y-axis and line $x = 2$.



∴ Required area = Area of square $OABC - \int_0^2 y \, dx$

$$= 2 \times 2 - \int_0^2 x (x - 1)^2 \, dx$$

$$= 4 - \left[\left[\frac{x (x - 1)^3}{3} \right]_0^2 - \frac{1}{3} \int_0^2 (x - 1)^3 \cdot 1 \, dx \right]$$

$$= 4 - \left[\frac{x}{3} (x - 1)^3 - \frac{(x - 1)^4}{12} \right]_0^2$$

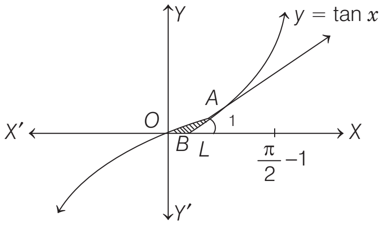
$$= 4 - \left[\frac{2}{3} - \frac{1}{12} + \frac{1}{12} \right] = \frac{10}{3} \text{ sq units}$$

48. Given, $y = \tan x \Rightarrow \dfrac{dy}{dx} = \sec^2 x$

$$\therefore \quad \left(\frac{dy}{dx} \right)_{x = \frac{\pi}{4}} = 2$$

Hence, equation of tangent at $A \left(\dfrac{\pi}{4}, 1 \right)$ is

$$\frac{y - 1}{x - \pi/4} = 2 \quad \Rightarrow \quad y - 1 = 2x - \frac{\pi}{2}$$



$$\Rightarrow \quad (2x - y) = \left(\frac{\pi}{2} - 1 \right)$$

∴ Required area is $OABO$

$$= \int_0^{\pi/4} (\tan x) \, dx - \text{area of } \Delta \, ALB$$

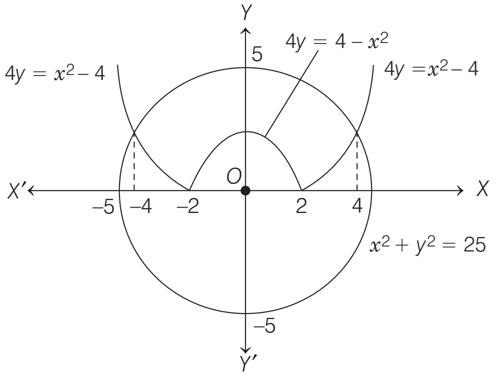
$$= [\log |\sec x|]_0^{\pi/4} - \frac{1}{2} \cdot BL \cdot AL$$

$$= \log \sqrt{2} - \frac{1}{2} \left(\frac{\pi}{4} - \frac{\pi - 2}{4} \right) \cdot 1$$

$$= \left(\log \sqrt{2} - \frac{1}{4} \right) \text{ sq unit}$$

49. Given curves, $x^2 + y^2 = 25, 4y = |4 - x^2|$ could be sketched as below, whose points of intersection are

$$x^2 + \frac{(4 - x^2)^2}{16} = 25$$



$$\Rightarrow \quad (x^2 + 24) (x^2 - 16) = 0$$

$$\Rightarrow \quad x = \pm 4$$

∴ Required area $= 2 \left[\int_0^4 \sqrt{25 - x^2} \, dx - \int_0^2 \left(\dfrac{4 - x^2}{4} \right) dx - \int_2^4 \left(\dfrac{x^2 - 4}{4} \right) dx \right]$

$$= 2 \left[\left[\frac{x}{2} \sqrt{25 - x^2} + \frac{25}{2} \sin^{-1} \left(\frac{x}{5} \right) \right]_0^4 - \frac{1}{4} \left[4x - \frac{x^3}{3} \right]_0^2 - \frac{1}{4} \left[\frac{x^3}{3} - 4x \right]_2^4 \right]$$

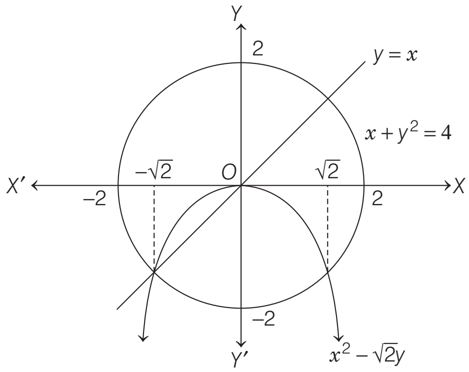
$$=2\left[\left[6+\frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right)\right]-\frac{1}{4}\left[8-\frac{8}{3}\right]\right.$$

$$\left.-\frac{1}{4}\left[\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right]\right]$$

$$=2\left[6+\frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right)-\frac{4}{3}-\frac{4}{3}-\frac{4}{3}\right]$$

$$=\left[4+25\ \sin^{-1}\left(\frac{4}{5}\right)\right]\text{sq units}$$

50. Given curves are $x^2+y^2=4, x^2=-\sqrt{2}y$ and $x=y$.



Thus, the required area

$$=\left|\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{4-x^2}\ dx\right|-\left|\int_{-\sqrt{2}}^{0}x\,dx\right|-\left|\int_{0}^{\sqrt{2}}\frac{-x^2}{\sqrt{2}}\ dx\right|$$

$$=2\int_0^{\sqrt{2}}\sqrt{4-x^2}\ dx-\left|\left(\frac{x^2}{2}\right)_{-\sqrt{2}}^{0}\right|-\left|\frac{x^3}{3\sqrt{2}}\right|_0^{\sqrt{2}}$$

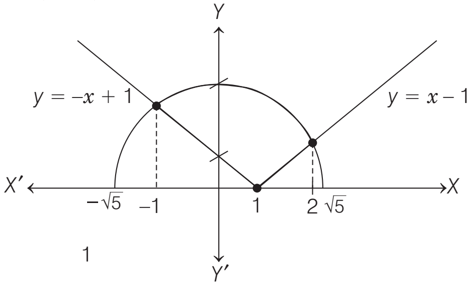
$$=2\left\{\frac{x}{2}\sqrt{4-x^2}-\frac{4}{2}\sin^{-1}\frac{x}{2}\right\}_0^{\sqrt{2}}-1-\frac{2}{3}$$

$$=(2-\pi)-\frac{5}{3}$$

$$=\left(\frac{1}{3}-\pi\right)\text{sq units}$$

51. Given curves $y=\sqrt{5-x^2}$ and $y=|x-1|$ could be sketched as shown, whose point of intersection are

$5-x^2=(x-1)^2$



$$\Rightarrow\qquad 5-x^2=x^2-2x+1$$

$$\Rightarrow\qquad 2x^2-2x-4=0$$

$$\Rightarrow\qquad x=2,-1$$

∴ Required area

$$=\int_{-1}^{2}\sqrt{5-x^2}\ dx-\int_{-1}^{1}(-x+1)\ dx-\int_{1}^{2}(x-1)\ dx$$

$$=\left[\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2}-\left[\frac{-x^2}{2}+x\right]_{-1}^{1}-\left[\frac{x^2}{2}-x\right]_{1}^{2}$$

$$=\left(1+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right)-\left[-1+\frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right]$$

$$-\left(-\frac{1}{2}+1+\frac{1}{2}+1\right)-\left(2-2-\frac{1}{2}+1\right)$$

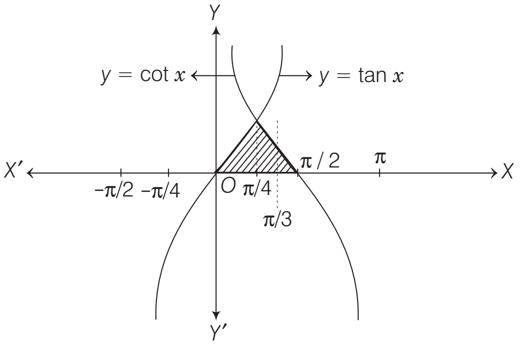
$$=\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{5}}\right)-\frac{1}{2}$$

$$=\frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\sqrt{1-\frac{1}{5}}+\frac{1}{\sqrt{5}}\sqrt{1-\frac{4}{5}}\right)-\frac{1}{2}$$

$$=\frac{5}{2}\sin^{-1}(1)-\frac{1}{2}=\left(\frac{5\pi}{4}-\frac{1}{2}\right)\text{sq units}$$

52. Given, $y=\begin{cases}\tan x, & -\frac{\pi}{3}\le x\le\frac{\pi}{3}\\ \cot x, & \frac{\pi}{6}\le x\le\frac{\pi}{2}\end{cases}$

which could be plotted as Y-axis.



$$\therefore\ \text{Required area}=\int_0^{\pi/4}(\tan x)\ dx+\int_{\pi/4}^{\pi/3}(\cot x)\ dx$$

$$=[-\log|\cos x|]_0^{\pi/4}+[\log\sin x]_{\pi/4}^{\pi/3}$$

$$=-\left(\log\frac{1}{\sqrt{2}}-0\right)+\left(\log\frac{\sqrt{3}}{2}-\log\frac{1}{\sqrt{2}}\right)$$

$$=\log\frac{\sqrt{3}}{2}-2\log\frac{1}{\sqrt{2}}$$

$$=\log\frac{\sqrt{3}}{2}-\log\frac{1}{2}=\left(\frac{1}{2}\log_e 3\right)\text{sq units}$$

53. Here, $\int_2^a\left(1+\frac{8}{x^2}\right)dx=\int_a^4\left(1+\frac{8}{x^2}\right)dx$

$$\Rightarrow\qquad \left[x-\frac{8}{x}\right]_2^a=\left[x-\frac{8}{x}\right]_a^4$$

$$\Rightarrow\qquad \left(a-\frac{8}{a}\right)-(2-4)=(4-2)-\left(a-\frac{8}{a}\right)$$

$$\Rightarrow \quad a - \frac{8}{a} + 2 = 2 - a + \frac{8}{a} \Rightarrow 2a - \frac{16}{a} = 0$$

$$\Rightarrow \quad 2\,(a^2 - 8) = 0$$

$$\Rightarrow \quad a = \pm 2\sqrt{2} \qquad \text{[neglecting –ve sign]}$$

$$\therefore \quad a = 2\sqrt{2}$$

54. The point of intersection of the curves $x^2 = 4y$ and $x = 4y - 2$ could be sketched are $x = -1$ and $x = 2$.

$\therefore$ Required area

$$= \int_{-1}^{2} \left\{ \left(\frac{x+2}{4} \right) - \left(\frac{x^2}{4} \right) \right\} dx$$

$$= \frac{1}{4} \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$

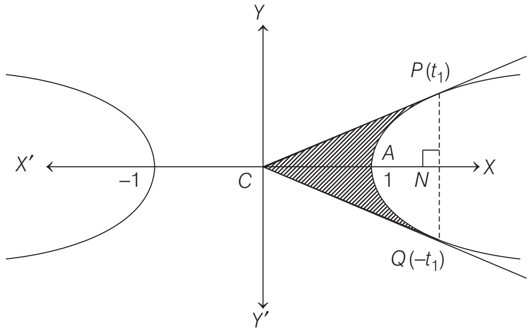
$$= \frac{1}{4} \left[\left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right]$$

$$= \frac{1}{4} \left[\frac{10}{3} - \left(\frac{-7}{6} \right) \right] = \frac{1}{4} \cdot \frac{9}{2} = \frac{9}{8} \text{sq units}$$

55. Let $P = \left(\frac{e^{t_1} + e^{-t_1}}{2}, \frac{e^{t_1} - e^{-t_1}}{2} \right)$

and $Q = \left(\frac{e^{-t} + e^{t_1}}{2}, \frac{e^{-t_1} - e^{t}}{2} \right)$

We have to find the area of the region bounded by the curve $x^2 - y^2 = 1$ and the lines joining the centre $x = 0$, $y = 0$ to the points (t_1) and $(-t_1)$.



Required area

$$= 2 \left[\text{area of } \Delta PCN - \int_{1}^{\frac{e^{t_1} + e^{-t_1}}{2}} y dx \right]$$

$$= 2 \left[\frac{1}{2} \left(\frac{e^{t_1} + e^{-t_1}}{2} \right) \left(\frac{e^{t_1} - e^{-t_1}}{2} \right) - \int_{1}^{t_1} y \frac{dy}{dt} \cdot dt \right]$$

$$= 2 \left[\frac{e^{2t_1} - e^{-2t_1}}{8} - \int_{0}^{t_1} \left(\frac{e^{t} - e^{-t}}{2} \right) dt \right]$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{2} \int_{0}^{t_1} (e^{2t} + e^{-2t} - 2) dt$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{2} \left[\frac{e^{2t}}{2} - \frac{e^{-2t}}{2} - 2t \right]$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{4} (e^{2t_1} - e^{-2t_1} - 4t_1)$$