INTEGRALS

7 Chapter

Exercise 7.1

1. Find an anti-derivative (or integral) of the following functions by the method of inspection. sin 2x

Ans: We use the method of inspection as follows:

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x \Rightarrow -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Thus, the anti-derivative of $\sin 2x$ is $-\frac{1}{2}\cos 2x$.

2. Find an anti-derivative (or integral) of the following functions by the method of inspection. cos 3x

Ans: We use the method of inspection as follows:

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x \Rightarrow \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Thus, the anti - derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$.

3. Find an anti-derivative (or integral) of the following functions by the method of inspection. e^{2x}

Ans: We use the method of inspection as follows:

$$\frac{d}{dx}(e^{2x}) \Rightarrow 2e^{2x} = \frac{1}{2}\frac{d}{dx}(e^{2x})$$

$$\therefore e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Thus, the anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

4. **Find an anti-derivative (or integral) of the following functions by the method of inspection.** $\mathbf{(ax+b)^2}$

Ans: We use the method of inspection as follows:

$$\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$$

$$\Rightarrow (ax+b)^2 = \frac{1}{3a}\frac{d}{dx}(ax+b)^3$$

$$\therefore (ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Thus, the anti-derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

5. **Find an anti-derivative (or integral) of the following functions by the method of inspection. sin** $\mathbf{2x-4e^{3x}}$

Ans: Weuse the method of inspection as follows:

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \left(\sin 2x - 4e^{3x}\right)$$

Thus, the anti-derivative of $\left(\sin 2x - 4e^{3x}\right)$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$.

6. $\mathbf{\int\left(4e^{3x}+1\right)dx}$

Ans:

$$\int\left(4e^{3x}+1\right)dx$$

$$= 4\int e^{3x}dx + \int 1dx$$

$$= 4\left(\frac{e^{3x}}{3}\right) + x + C$$

$$= \frac{4}{3}e^{3x} + x + C$$

7. $\mathbf{\int x^2\left(1-\frac{1}{x^2}\right)dx}$

Ans:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

$$= \int \left(x^2 - 1\right)dx$$

$$= \frac{x^3}{3} - x + C$$

8. $\int \left(ax^2 + bx + c\right)dx$

Ans:

$$\int \left(ax^2 + bx + c\right)dx$$

$$= a\int x^2 dx + b\int x dx + c\int 1.dx$$

$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

9. $\int \left(2x^2 + e^x\right)dx$

Ans:

$$\int \left(2x^2 + e^x\right)dx$$

$$= 2\int x^2 dx + \int e^x dx$$

$$= 2\left(\frac{x^3}{3}\right) + e^x + C$$

$$= \frac{2}{3}x^3 + e^x + C$$

10. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

Ans:

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int \left(x + \frac{1}{x} - 2\right)dx$$

$$= \int x dx + \int \frac{1}{x} dx - 2\int 1.dx$$

$$= \frac{x^2}{2} + \log|x| - 2x + C$$

11. $\int \frac{\mathbf{x^3+5x^2-4}}{\mathbf{x^2}}\mathbf{dx}$

Ans: $\int \frac{x^3+5x^2-4}{x^2}dx$

$=\int\left(x+5-4x^{-2}\right)dx$

$=\int xdx+5\int 1.dx-4\int x^{-2}dx$

$=\frac{x^2}{2}+5x+\frac{4}{x}+C$

12. $\int \frac{\mathbf{x^3+3x+4}}{\sqrt{\mathbf{x}}}\mathbf{dx}$

Ans: $\int \frac{x^3+3x+4}{\sqrt{x}}dx$

$=\int\left(x^{\frac{5}{2}}+3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\right)dx$

$=\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+\frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}}+C$

$=\frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8\sqrt{x}+C$

13. $\int \frac{\mathbf{x^3-x^2+x-1}}{\mathbf{x-1}}\mathbf{dx}$

Ans: $\int \frac{x^3-x^2+x-1}{x-1}dx$

We obtain, on dividing:

$=\int\left(x^2+1\right)dx$

$=\int x^2dx+\int 1.dx$

$=\frac{x^3}{3}+x+C$

14. $\int \mathbf{(1-x)\sqrt{x}\,dx}$

Ans: $\int(1-x)\sqrt{x}dx$

$$= \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx$$

$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$$

15. $\int \sqrt{x} \left(3x^2 + 2x + 3 \right) dx$

Ans: $\int \sqrt{x} \left(3x^2 + 2x + 3 \right) dx$

$$= 3\int \left(2x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}} \right)$$

$$= 3\int x^{\frac{5}{2}} dx + 2\int x^{\frac{3}{2}} dx + 3\int x^{\frac{1}{2}} dx$$

$$= \frac{6}{7} x^{\frac{7}{2}} + \frac{4}{5} x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

16. $\int \left(2x - 3\cos x + e^x \right) dx$

Ans: $\int \left(2x - 3\cos x + e^x \right) dx$

$$= 2\int x dx - 3\int \cos x dx + \int e^x dx$$

$$= \frac{2x^2}{2} - 3\left(\sin x \right) + e^x + C$$

$$= x^2 - 3\sin x + e^x + C$$

17. $\int \left(2x^2 - 3\sin x + 5\sqrt{x} \right) dx$

Ans: $\int \left(2x^2 - 3\sin x + 5\sqrt{x} \right) dx$

$$= 2\int x^2 dx - 3\int \sin x dx + 5\int x^{\frac{1}{2}} dx$$

$$= \frac{2x^3}{3} - 3\left(-\cos x \right) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$= \frac{2}{3} x^3 + 3\cos x + \frac{10}{3} x^{\frac{3}{2}} + C$$

18. $\int \sec x(\sec x+\tan x)dx$

Ans: $\int \sec x(\sec x+\tan x)dx$

$=\int(\sec^2 x+\sec x\tan x)dx$

$=\int \sec^2 xdx+\int \sec x\tan xdx$

$=\tan x+\sec x+C$

19. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$

Ans: $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$

$=\int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}dx$

$=\int \frac{\sin^2 x}{\cos^2 x}dx$

$=\int \tan^2 xdx$

$=\int \sec^2 xdx-\int 1dx$

$=\tan x-x+C$

20. $\int \frac{2-3\sin x}{\cos^2 x}dx$

Ans: $\int \frac{2-3\sin x}{\cos^2 x}dx$

$=\int\left(\frac{2}{\cos^2 x}-\frac{3\sin x}{\cos^2 x}\right)dx$

$=\int 2\sec^2 xdx-3\int \tan x\sec xdx$

$=2\tan x-3\sec x+C$

21. **The anti – derivative of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ equals**

A. $\frac{1}{3}x^{\frac{1}{3}}+2x^{\frac{1}{2}}+C$

B. $\mathbf{\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^{2} + C}$

C. $\mathbf{\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C}$

D. $\mathbf{\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C}$

Ans:

$$\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

$$= \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

Thus, the correct answer is C.

22. **If $\mathbf{\frac{d}{dx}f(x) = 4x^{3} - \frac{3}{x^{4}}}$ such that $\mathbf{f(2) = 0}$ then $\mathbf{f(x)}$ is**

A. $\mathbf{x^{4} + \frac{1}{x^{3}} - \frac{129}{8}}$

B. $\mathbf{x^{3} + \frac{1}{x^{4}} + \frac{129}{8}}$

C. $\mathbf{x^{4} + \frac{1}{x^{3}} + \frac{129}{8}}$

D. $\mathbf{x^{3} + \frac{1}{x^{4}} - \frac{129}{8}}$

Ans: Given, $\frac{d}{dx}f(x) = 4x^{3} - \frac{3}{x^{4}}$

Anti-derivative of $4x^{3} - \frac{3}{x^{4}} = f(x)$

$$\therefore f(x) = \int 4x^{3} - \frac{3}{x^{4}} = f(x)$$

$$f(x) = 4\int x^{3}dx - 3\int\left(x^{-4}\right)dx$$

$$f(x) = 4\left(\frac{x^{4}}{4}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C$$

$$f(x)=x^4+\frac{1}{x^3}+C$$

Also,

$$f(2)=0$$

$$\therefore f(2)=(2)^4+\frac{1}{(2)^3}+C=0$$

$$\Rightarrow 16+\frac{1}{8}+C=0$$

$$\Rightarrow C=\frac{-129}{8}$$

$$\therefore f(x)=x^4+\frac{1}{x^3}-\frac{129}{8}$$

Thus, the correct answer is A.

Exercise 7.2

1. $\mathbf{\dfrac{2x}{1+x^2}}$

Ans: Substitute $1+x^2=t$

$$\therefore 2xdx=dt$$

$$\Rightarrow \int\frac{2x}{1+x^2}dx=\int\frac{1}{t}dt$$

$$=\log|t|+C$$

$$=\log\left|1+x^2\right|+C$$

$$=\log\left(1+x^2\right)+C$$

2. $\mathbf{\dfrac{(\log x)^2}{x}}$

Ans: Substitute $\log|x|=t$

$$\therefore \frac{1}{x}dx=dt$$

$$\Rightarrow \int\frac{(\log|x|)^2}{x}dx=\int t^2dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(\log|x|)^3}{3} + C$$

3. $\mathbf{\frac{1}{x + x\log x}}$

Ans: $\frac{1}{x + x\log x} = \frac{1}{x(1+\log x)}$

Substitute $1 + \log x = t$

$$\therefore \frac{1}{x}dx = dt$$

$$\Rightarrow \int \frac{1}{x(1+\log x)}dx = \int \frac{1}{t}dt$$

$$= \log|t| + C$$

$$= \log|1 + \log x| + C$$

4. $\mathbf{Sin\,x.\sin(\cos x)}$

Ans: $\text{Sin}\,x.\sin(\cos x)$

Put, $\cos x = t$

$$\therefore -\sin x dx = dt$$

$$\Rightarrow \int \sin x.\sin(\cos x)dx = -\int \sin t dt$$

$$= -[-\cos t] + C$$

$$= \cos t + C$$

$$= \cos(\cos x) + C$$

5. $\mathbf{Sin(ax+b)\cos(ax+b)}$

Ans: $\text{Sin}(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$

Substitute $2(ax+b) = t$

$$\therefore 2a dx = dt$$

$$\Rightarrow \int \frac{\sin 2(ax+b)}{2}dx = \frac{1}{2}\int \frac{\sin t}{2a}dt$$

$$=\frac{1}{4a}\left[-\cos t\right]+C$$

$$=\frac{-1}{4a}\cos 2\left(ax+b\right)+C$$

6. $\sqrt{\mathbf{ax+b}}$

Ans: Substitute $ax+b=t$

$$\Rightarrow adx=dt$$

$$\therefore dx=\frac{1}{a}dt$$

$$\Rightarrow \int\left(ax+b\right)^{\frac{1}{2}}dx=\frac{1}{a}\int t^{\frac{1}{2}}dt$$

$$=\frac{1}{a}\left(\frac{t^{\frac{1}{2}}}{\frac{3}{2}}\right)+C=\frac{2}{3a}\left(ax+b\right)^{\frac{3}{2}}+C$$

7. $\mathbf{x\sqrt{x+2}}$

Ans: Substitute $x+2=t$

$$\therefore dx=dt$$

$$\Rightarrow \int x\sqrt{x+2}=\int\left(t-2\right)\sqrt{t}dt$$

$$=\int\left(t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)dt$$

$$=\int t^{\frac{3}{2}}dt-2\int t^{\frac{1}{2}}dt$$

$$=\frac{2}{5}t^{\frac{5}{2}}-\frac{4}{3}t^{\frac{3}{2}}+C$$

8. $\mathbf{x\sqrt{1+2x^2}}$

Ans: Substitute $1+2x^2=t$

$$\therefore 4xdx=dt$$

$$\Rightarrow \int x\sqrt{1+2x^2}dx=\int\frac{\sqrt{t}dt}{4}$$

$$=\frac{1}{4}\int t^{\frac{1}{2}}dt$$

$$=\frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

$$=\frac{1}{6}\left(1+2x^{2}\right)^{\frac{3}{2}}+C$$

9. $\mathbf{(4x+2)\sqrt{x^2+x+1}}$

Ans: Substitute $x^2+x+1=t$

$$\therefore (2x+1)dx=dt$$

$$\int(4x+2)\sqrt{x^2+x+1}dx$$

$$=\int 2\sqrt{t}dt$$

$$=2\int\sqrt{t}dt$$

$$=2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C=\frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}}+C$$

10. $\mathbf{\frac{1}{x-\sqrt{x}}}$

Ans: $\frac{1}{x-\sqrt{x}}=\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$

Substitute $\left(\sqrt{x-1}\right)=t$

$$\Rightarrow\int\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx=\int\frac{2}{t}dt$$

$$=2\log|t|+C$$

$$=2\log\left|\sqrt{x}-1\right|+C$$

11. $\mathbf{\frac{x}{\sqrt{x+4}},x>0}$

Ans: Substitute $x+4=t$

$\therefore dx = dt$

$$\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(t-4)}{\sqrt{t}}dt = \int\left(\sqrt{t} - \frac{4}{\sqrt{t}}\right)dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C = \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t.t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + C$$

$$= \frac{2}{3}\sqrt{x+4}(x-8) + C$$

12. $\left(x^3 - 1\right)^{\frac{1}{3}} x^5$

Ans: Substitute $x^3 - 1 = t$

$\therefore 3x^2dx = dt$

$$\Rightarrow \int\left(x^3-1\right)^{\frac{1}{3}} x^5dx = \int\left(x^3-1\right)^{\frac{1}{3}} x^3.x^2dx$$

$$= \int t^{\frac{1}{3}}(t+1)\frac{dt}{3} = \frac{1}{3}\int\left(t^{\frac{4}{3}} + t^{\frac{1}{3}}\right)dt$$

$$= \frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}} + \frac{3}{4}t^{\frac{4}{3}}\right] + C$$

$$= \frac{1}{7}\left(x^3-1\right)^{\frac{7}{3}} + \frac{1}{4}\left(x^3-1\right)^{\frac{4}{3}} + C$$

13. $\dfrac{x^2}{\left(2+3x^3\right)^3}$

Ans: Substitute $2 + 3x^3 = t$

$\therefore 9x^2dx = dt$

$$\Rightarrow \int \frac{x^2}{\left(2+3x^3\right)^3}dx = \frac{1}{9}\int\frac{dt}{(t)^3}$$

$$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right] + C$$

$$= \frac{-1}{18\left(2+3x^3\right)^2} + C$$

14. $\dfrac{1}{x(\log x)^m}, x > 0$

Ans: Substitute $\log x = t$

$\frac{1}{x}dx = dt$

$\Rightarrow \int \frac{1}{x(\log x)^m}dx = \int \frac{dt}{(t)^m} = \left(\frac{t^{-m-1}}{1-m}\right) + C$

$= \frac{(\log x)^{1-m}}{(1-m)} + C$

15. $\dfrac{x}{9 - 4x^2}$

Ans: Substitute $9 - 4x^2 = t$

$\therefore -8xdx = dt$

$\Rightarrow \int \frac{x}{9-4x^2}dx = \frac{-1}{8}\int \frac{1}{t}dt$

$= \frac{-1}{8}\log|t| + C$

$= \frac{-1}{8}\log|9-4x^2| + C$

16. e^{2x+3}

Ans: Substitute $2x + 3 = t$

$\therefore 2dx = dt$

$\Rightarrow \int e^{2x+3}dx = \frac{1}{2}\int e^t dt$

$= \frac{1}{2}(e^t) + C$

$= \frac{1}{2}e^{(2x+3)} + C$

17. $\dfrac{x}{e^{x^2}}$

Ans: Substitute $x^2 = t$

$\therefore 2xdx = dt$

$\Rightarrow \int \frac{x}{e^{x^2}}dx = \frac{1}{2}\int \frac{1}{e^t}dt$

$= \frac{1}{2}\int e^{-t}dt$

$$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

$$= \frac{-1}{2e^{x^2}} + C$$

18. $\mathbf{\dfrac{e^{\tan^{-1}x}}{1+x^2}}$

Ans: Substitute $\tan^{-1}x = t$

$$\therefore \frac{1}{1+x^2}dx = dt$$

$$\Rightarrow \int \frac{e^{\tan^{-1}x}}{1+x^2}dx = dt$$

$$= e^t + C$$

$$= e^{\tan^{-1}x} + C$$

19. $\boldsymbol{\dfrac{e^{2x}-1}{e^{2x}+1}}$

Ans: $\dfrac{e^{2x}-1}{e^{2x}+1}$

Dividing numerator and denominator by e^x, we obtain

$$\frac{\frac{(e^{2x}-1)}{e^x}}{\frac{(e^{2x}+1)}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Put $e^x + e^{-x} = t$

$$\therefore e^x - e^{-x}\ dx = dt$$

$$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$= \int \frac{dt}{t}$$

$$= \log[t] + C = \log[e^x + e^{-x}] + C$$

20. Solve the following: $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$.

Ans: Given expression $\frac{e^{2x}-e^{\ 2x}}{e^{2x}+e^{-2x}}$.

Let us substitute $e^{2x}+e^{-2x}=t$, we get

$\left(2e^{2x}+2e^{-2x}\right)dx=dt$

$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt$

Integration of given expression is

$$\Rightarrow \int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}=\int \frac{dt}{2t}$$

$$\Rightarrow \int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}=\frac{1}{2}\int \frac{1}{t}dt$$

$$\Rightarrow \int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}=\frac{1}{2}\log|t|+C$$

Again substitute $t=e^{2x}+e^{-2x}$, we get

$$\therefore \int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}=\frac{1}{2}\log\left|e^{2x}+e^{-2x}\right|+C$$

21. Solve the following: $\tan^{2}(2x-3)$.

Ans: Given expression $\tan^{2}(2x-3)$.

We can apply the identity $\tan^{2}x=\sec^{2}x-1$, we get

$\tan^{2}(2x-3)=\sec^{2}(2x-3)-1$

Substitute $2x-3=t$, we get

$2dx=dt$

Integration of given expression is

$$\Rightarrow \int \tan^{2}(2x-3)dx=\int \sec^{2}(2x-3)-1dx$$

$$\Rightarrow \int \tan^{2}(2x-3)dx=\frac{1}{2}\int \sec^{2}t\,dt-\int 1dx$$

$$\Rightarrow \int \tan^{2}(2x-3)dx=\frac{1}{2}\tan t-x+C$$

Substitute $2x-3=t$

$$\therefore \int \tan^{2}(2x-3)dx=\frac{1}{2}\tan(2x-3)-x+C$$

22. Solve the following: $\sec^{2}(7-4x)$.

Ans: Given expression $\sec^{2}(7-4x)$.

Put $7-4x=t$, we get

$\therefore -4dx=dt$

Integration of given expression is

$$\Rightarrow \int \sec^2(7-4x)dx = -\frac{1}{4}\int \sec^2 t dt$$

$$\Rightarrow \int \sec^2(7-4x)dx = -\frac{1}{4}\tan t + C$$

Substitute $7-4x=t$, we get

$$\therefore \int \sec^2(7-4x)dx = -\frac{1}{4}\tan(7-4x) + C$$

23. Solve the following: $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$.

Ans: Given expression $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$.

Put $\sin^{-1}x = t$, we get

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$$

Integration of given expression is

$$\Rightarrow \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int t dt$$

$$\Rightarrow \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \frac{t^2}{2} + C$$

Substitute $\sin^{-1}x = t$, we get

$$\therefore \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \frac{(\sin^{-1}x)^2}{2} + C$$

24. Solve the following: $\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$.

Ans: Given expression is $\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$.

Given expression can be written as

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$$

Let $3\cos x + 2\sin x = t$, we get

$(-3\sin x + 2\cos x)dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx = \int \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}dx$$

$$\Rightarrow \int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx = \int \frac{dt}{2t}$$

$$\Rightarrow \int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx = \frac{1}{2}\int \frac{1}{t}dt$$

$$\Rightarrow \int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx = \frac{1}{2}\log|t| + C$$

Substitute $3\cos x + 2\sin x = t$

$$\therefore \int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx = \frac{1}{2}\log\left|2\sin x + 3\cos x\right| + C$$

25. **Solve the following: $\frac{1}{\cos^2 x(1-\tan x)^2}$.**

Ans: Given expression $\frac{1}{\cos^2 x(1-\tan x)^2}$.

Given expression can be written as

$$\frac{1}{\cos^2 x(1-\tan x)^2} = \frac{\sec^2 x}{(1-\tan x)^2}$$

Let $(1-\tan x) = t$, we get

$$-\sec^2 x dx = dt$$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\cos^2 x(1-\tan x)^2}dx = \int \frac{\sec^2 x}{(1-\tan x)^2}dx$$

$$\Rightarrow \int \frac{1}{\cos^2 x(1-\tan x)^2}dx = \int \frac{-dt}{t^2}$$

$$\Rightarrow \int \frac{1}{\cos^2 x(1-\tan x)^2}dx = -\int t^{-2}dt$$

$$\Rightarrow \int \frac{1}{\cos^2 x(1-\tan x)^2}dx = \frac{1}{t} + C$$

Substitute $(1-\tan x) = t$,

$$\therefore \int \frac{1}{\cos^2 x(1-\tan x)^2}dx = \frac{1}{(1-\tan x)} + C$$

26. **Solve the following: $\frac{\cos\sqrt{x}}{\sqrt{x}}$.**

Ans: Given expression is $\frac{\cos\sqrt{x}}{\sqrt{x}}$.

Let $\sqrt{x} = t$, we get

$$\frac{1}{2\sqrt{x}}dx = dt$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos t dt$$

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\sin t + C$$

Substitute $\sqrt{x} = t$

$$\therefore \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\sin\sqrt{x} + C$$

27. Solve the following: $\sqrt{\sin 2x}\cos 2x$.

Ans: Given expression is $\sqrt{\sin 2x}\cos 2x$.

Let $\sin 2x = t$, we get

$2\cos 2x dx = dt$

Integration of given expression is

$$\Rightarrow \int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{2}\int \sqrt{t} dt$$

$$\Rightarrow \int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$\Rightarrow \int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{3}t^{\frac{3}{2}} + C$$

Substitute $\sin 2x = t$

$$\therefore \int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

28. Solve the following: $\frac{\cos x}{\sqrt{1+\sin x}}$.

Ans: Given expression $\frac{\cos x}{\sqrt{1+\sin x}}$.

Let $1 + \sin x = t$

$\therefore \cos x dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}}dx = \int \frac{dt}{\sqrt{t}}$$

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = 2\sqrt{t} + C$$

Substitute $1+\sin x = t$,

$$\therefore \int \frac{\cos x}{\sqrt{1+\sin x}} dx = 2\sqrt{1+\sin x} + C$$

29. Solve the following: $\cot x \log \sin x$.

Ans: Given expression $\cot x \log \sin x$.

Let $\log \sin x = t$, we get

$$\frac{1}{\sin x} \cos x dx = dt$$

$$\Rightarrow \cot x dx = dt$$

Integration of given expression is

$$\int \cot x \log \sin x dx = \int t dt$$

$$\Rightarrow \int \cot x \log \sin x dx = \frac{t^2}{2} + C$$

Substitute $\log \sin x = t$,

$$\therefore \int \cot x \log \sin x dx = \frac{1}{2}(\log \sin x)^2 + C$$

30. Solve the following: $\frac{\sin x}{1+\cos x}$.

Ans: Given expression $\frac{\sin x}{1+\cos x}$.

Let $1+\cos x = t$

$$\therefore -\sin x dx = dt$$

Integration of given expression is

$$\Rightarrow \int \frac{\sin x}{1+\cos x} dx = \int -\frac{dt}{t}$$

$$\Rightarrow \int \frac{\sin x}{1+\cos x} dx = -\log|t| + C$$

Substitute $1+\cos x = t$,

$$\therefore \int \frac{\sin x}{1+\cos x} dx = -\log|1+\cos x| + C$$

31. Solve the following: $\frac{\sin x}{(1+\cos x)^2}$.

Ans: Given expression $\frac{\sin x}{(1+\cos x)^2}$.

Let $1+\cos x = t$

$\therefore -\sin x dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2} dx = \int -\frac{dt}{t^2}$$

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2} dx = -\int t^{-2} dt$$

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2} dx = \frac{1}{t} + C$$

Substitute $1+\cos x = t$,

$$\therefore \int \frac{\sin x}{(1+\cos x)^2} dx = \frac{1}{1+\cos x} + C$$

32. Solve the following: $\frac{1}{1+\cot x}$.

Ans: Given expression $\frac{1}{1+\cot x}$.

Let $I = \int \frac{1}{1+\cot x} dx$

Integration of given expression is

$$\Rightarrow I = \int \frac{1}{1+\cot x} dx$$

$$\Rightarrow I = \int \frac{1}{1+\frac{\cos x}{\sin x}} dx$$

$$\Rightarrow I = \int \frac{\sin x}{\sin x + \cos x} dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x} dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{\sin x + \cos x} dx$$

$$\Rightarrow I=\frac{1}{2}\int\frac{(\sin x+\cos x)}{\sin x+\cos x}+\frac{(\sin x-\cos x)}{\sin x+\cos x}dx$$

$$\Rightarrow I=\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{(\sin x-\cos x)}{\sin x+\cos x}dx$$

Let $\sin x+\cos x=t$

$\therefore(\cos x-\sin x)dx=dt$

Substitute in above obtained equation, we get

$$\Rightarrow I=\frac{1}{2}x+\frac{1}{2}\int-\frac{dt}{t}$$

$$\Rightarrow I=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

Substitute $\sin x+\cos x=t$,

$$\therefore I=\frac{x}{2}-\frac{1}{2}\log|\sin x+\cos x|+C$$

33. Solve the following: $\frac{1}{1-\tan x}$.

Ans: Given expression $\frac{1}{1-\tan x}$.

Let $I=\int\frac{1}{1-\tan x}dx$

Integration of given expression is

$$\Rightarrow I=\int\frac{1}{1-\tan x}dx$$

$$\Rightarrow I=\int\frac{1}{1-\frac{\sin x}{\cos x}}dx$$

$$\Rightarrow I=\int\frac{\cos x}{\cos x-\sin x}dx$$

$$\Rightarrow I=\frac{1}{2}\int\frac{2\cos x}{\cos x-\sin x}dx$$

$$\Rightarrow I=\frac{1}{2}\int\frac{(\cos x-\sin x)+(\cos x+\sin x)}{\cos x-\sin x}dx$$

$$\Rightarrow I=\frac{1}{2}\int\frac{(\cos x-\sin x)}{\cos x-\sin x}+\frac{(\cos x+\sin x)}{\cos x-\sin x}dx$$

$$\Rightarrow I=\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{(\cos x+\sin x)}{\cos x-\sin x}dx$$

Let $\cos x - \sin x = t$

$\therefore (-\sin x - \cos x)dx = dt$

Substitute in above obtained equation, we get

$\Rightarrow I = \frac{1}{2}x + \frac{1}{2}\int -\frac{dt}{t}$

$\Rightarrow I = \frac{x}{2} - \frac{1}{2}\log|t| + C$

Substitute $\cos x - \sin x = t$,

$\therefore I = \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$

34. Solve the following: $\frac{\sqrt{\tan x}}{\sin x \cos x}$.

Ans: Given expression $\frac{\sqrt{\tan x}}{\sin x \cos x}$.

Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

Multiply and divide by $\cos x$, we get

$\Rightarrow I = \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$

$\Rightarrow I = \int \frac{\sqrt{\tan x} \times \cos x}{\tan x \times \cos^2 x} dx$

$\Rightarrow I = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

Let $\tan x = t$

$\therefore \sec^2 x dx = dt$

Substitute in above obtained equation, we get

$\Rightarrow I = \int \frac{dt}{\sqrt{t}} dx$

$\Rightarrow I = 2\sqrt{t} + C$

Substitute $\tan x = t$,

$\therefore I = 2\sqrt{\tan x} + C$

35. Solve the following: $\frac{(1 + \log x)^2}{x}$.

Ans: Given expression $\frac{(1+\log x)^2}{x}$.

Let $1+\log x = t$

$\therefore \frac{1}{x}dx = dt$

Integration of given expression is

$\int \frac{(1+\log x)^2}{x}dx = \int t^2 dt$

$\Rightarrow \int \frac{(1+\log x)^2}{x}dx = \frac{t^3}{3} + C$

Substitute $1+\log x = t$

$\therefore \int \frac{(1+\log x)^2}{x}dx = \frac{(1+\log x)^3}{3} + C$

36. Solve the following: $\frac{(x+1)(x+\log x)^2}{x}$.

Ans: Given expression $\frac{(x+1)(x+\log x)^2}{x}$.

Given expression cab be written as

$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2$

$\Rightarrow \frac{(x+1)(x+\log x)^2}{x} = \left(1+\frac{1}{x}\right)(x+\log x)^2$

Let $x+\log x = t$

$\therefore \left(1+\frac{1}{x}\right)dx = dt$

Integration of given expression is

$\Rightarrow \int \frac{(x+1)(x+\log x)^2}{x}dx = \int t^2 dt$

$\Rightarrow \int \frac{(x+1)(x+\log x)^2}{x}dx = \frac{t^3}{3} + C$

Substitute $x+\log x = t$

$\therefore \int \frac{(x+1)(x+\log x)^2}{x}dx = \frac{1}{3}(x+\log x)^3 + C$

37. Solve the following: $\dfrac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$.

Ans: Given expression $\dfrac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$.

Let $x^4 = t$,

$\therefore 4x^3dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx = \frac{1}{4}\int \frac{\sin\left(\tan^{-1}t\right)}{1+t^2}dt \quad \ldots\ldots\ldots(1)$$

Let $\tan^{-1}t = u$

$$\therefore \frac{1}{1+t^2}dt = du$$

Substitute in eq. (1), we get

$$\Rightarrow \int \frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx = \frac{1}{4}\int \sin u\,du$$

$$\Rightarrow \int \frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx = \frac{1}{4}\left(-\cos u\right)+C$$

Substitute $\tan^{-1}t = u$,

$$\Rightarrow \int \frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx = -\frac{1}{4}\cos\left(\tan^{-1}t\right)+C$$

Substitute $x^4 = t$,

$$\therefore \int \frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx = -\frac{1}{4}\cos\left(\tan^{-1}x^4\right)+C$$

38. $\displaystyle\int \frac{10x^9+10^x\log_e 10}{x^{10}+10^x}dx$ equals

(A) $10^x - x^{10} + C$

(B) $10^x + x^{10} + C$

(C) $\left(10^x - x^{10}\right)^{-1} + C$

(D) $\log\left(10^x + x^{10}\right) + C$

Ans: Given expression $\displaystyle\int \frac{10x^9+10^x\log_e 10}{x^{10}+10^x}dx$.

Let $x^{10}+10^x = t$,

$\therefore \left(10x^9+10^x\log_e 10\right)dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t}$$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \log t + C$$

Substitute $x^{10} + 10^x = t$,

$$\therefore \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \log\left(10^x + x^{10}\right) + C$$

Therefore, option D is the correct answer.

39. $\int \frac{dx}{\sin^2 \cos^2 x}$ **equals**

(A) $\tan x + \cot x + C$

(B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$

(D) $\tan x - \cot 2x + C$

Ans: Given expression $\int \frac{dx}{\sin^2 x \cos^2 x}$.

Let $I = \int \frac{dx}{\sin^2 x \cos^2 x}$

$$I = \int \frac{1}{\sin^2 x \cos^2 x} dx$$

We know that $\sin^2 x + \cos^2 x = 1$, we get

$$\Rightarrow I = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$\Rightarrow I = \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$\Rightarrow I = \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx$$

$$\Rightarrow I = \int \sec^2 x dx + \int \mathrm{cosec}^2 x dx$$

$$\Rightarrow I = \tan x - \cot x + C$$

Therefore, option B is the correct answer.

Exercise 7.3

1. Solve the following: $\sin^2\left(2x + 5\right)$

Ans: Given expression $\sin^2\left(2x + 5\right)$.

Given expression can be written as

$$\sin^2(2x+5)=\frac{1-\cos 2(2x+5)}{2}$$

$$\Rightarrow \sin^2(2x+5)=\frac{1-\cos(4x+10)}{2}$$

Integration of given expression is

$$\Rightarrow \int \sin^2(2x+5)dx=\int \frac{1-\cos(4x+10)}{2}dx$$

$$\Rightarrow \int \sin^2(2x+5)dx=\frac{1}{2}\int 1dx-\frac{1}{2}\int \cos(4x+10)dx$$

$$\Rightarrow \int \sin^2(2x+5)dx=\frac{1}{2}x-\frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right)+C$$

$$\therefore \int \sin^2(2x+5)dx=\frac{1}{2}x-\frac{1}{8}\sin(4x+10)+C$$

2. Solve the following: $\sin 3x \cos 4x$

Ans: Given expression $\sin 3x \cos 4x$.

Using the identity $\sin A \cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$ given expression can be written as

$$\sin 3x \cos 4x=\frac{1}{2}\{\sin(3x+4x)+\sin(3x-4x)\}$$

Integration of above expression is

$$\Rightarrow \int \sin 3x \cos 4x dx=\int \frac{1}{2}\{\sin 7x+\sin(-x)\}dx$$

$$\Rightarrow \int \sin 3x \cos 4x dx=\frac{1}{2}\int \{\sin 7x+\sin x\}dx$$

$$\Rightarrow \int \sin 3x \cos 4x dx=\frac{1}{2}\int \sin 7x dx+\frac{1}{2}\int \sin x dx$$

$$\Rightarrow \int \sin 3x \cos 4x dx=\frac{1}{2}\left(\frac{-\cos 7x}{7}\right)-\frac{1}{2}(-\cos x)+C$$

$$\therefore \int \sin 3x \cos 4x dx=\frac{-\cos 7x}{14}+\frac{\cos x}{2}+C$$

3. Solve the following: $\cos 2x \cos 4x \cos 6x$.

Ans: Given expression $\cos 2x \cos 4x \cos 6x$.

Using the identity $\cos A\cos B=\frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$ given expression can be written as

$$\cos 2x(\cos 4x\cos 6x)=\cos 2x\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}$$

Integration of the above expression is

$$\Rightarrow\int\cos 2x(\cos 4x\cos 6x)=\int\cos 2x\left[\frac{1}{2}\{\cos 10x+\cos(-2x)\}\right]dx$$

$$\Rightarrow\int\cos 2x(\cos 4x\cos 6x)=\int\left[\frac{1}{2}\{\cos 2x\cos 10x+\cos 2x\cos(-2x)\}\right]dx$$

$$\Rightarrow\int\cos 2x(\cos 4x\cos 6x)=\frac{1}{2}\int\left[\{\cos 2x\cos 10x+\cos^2 2x\}\right]dx$$

Again applying the identity $\cos A\cos B=\frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$, we get

$$\Rightarrow\int\cos 2x(\cos 4x\cos 6x)=\frac{1}{2}\int\left[\left\{\begin{array}{l}\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\\+\left(\frac{1+\cos 4x}{2}\right)\end{array}\right\}\right]dx$$

$$\Rightarrow\int\cos 2x(\cos 4x\cos 6x)=\frac{1}{4}\int[\cos 12x+\cos 8x+\cos 4x]dx$$

$$\therefore\int\cos 2x\cos 4x\cos 6x=\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+\frac{\sin 4x}{4}\right]+C$$

4. Solve the following: $\sin^3(2x+1)$.

Ans: Given expression $\sin^3(2x+1)$.

Let $I=\int\sin^3(2x+1)dx$

$$\Rightarrow I=\int\sin^2(2x+1)\sin(2x+1)dx$$

$$\Rightarrow I=\int(1-\cos^2(2x+1))\sin(2x+1)dx$$

Let $\cos(2x+1)=t$

$$\therefore -2\sin(2x+1)dx=dt$$

Integration becomes

$$\Rightarrow I=-\frac{1}{2}\int(1-t^2)dt$$

$$\Rightarrow I=-\frac{1}{2}\left(t-\frac{t^3}{3}\right)+C$$

Substitute $\cos(2x+1)=t$,

$\Rightarrow I=-\frac{1}{2}\left(\cos(2x+1)-\frac{\cos^3(2x+1)}{3}\right)+C$

$\therefore \int \sin^3(2x+1)dx=\frac{-\cos(2x+1)}{2}+\frac{\cos^3(2x+1)}{3}+C$

5. Solve the following: $\sin^3 x\cos^3 x$.

Ans: Given expression $\sin^3 x\cos^3 x$.

Let $I=\int \sin^3 x\cos^3 xdx$

$\Rightarrow I=\int \sin^2 x\sin x\cos^3 xdx$

$\Rightarrow I=\int \cos^3 x\left(1-\cos^2 x\right)\sin xdx$

Let $\cos x=t$

$\therefore -\sin dx=dt$

Integration becomes

$\Rightarrow I=-\int t^3\left(1-t^2\right)dt$

$\Rightarrow I=-\int \left(t^3-t^5\right)dt$

$\Rightarrow I=-\left[\frac{t^4}{4}-\frac{t^6}{6}\right]+C$

Substitute $\cos x=t$,

$\Rightarrow I=-\left[\frac{\cos^4 x}{4}-\frac{\cos^6 x}{6}\right]+C$

$\therefore \int \sin^3 x\cos^3 xdx=\frac{\cos^6 x}{6}-\frac{\cos^4 x}{4}+C$

6. Solve the following: $\sin x\sin 2x\sin 3x$.

Ans: Given expression $\sin x\sin 2x\sin 3x$.

Using the identity $\sin A\sin B=\frac{1}{2}\cos(A-B)-\cos(A+B)$, given expression can be written as

$\Rightarrow \sin x\sin 2x\sin 3x=\sin x.\frac{1}{2}\cos(2x-3x)-\cos(2x+3x)$

Integration of given expression is

$\Rightarrow \int \sin x\sin 2x\sin 3xdx=\frac{1}{2}\int \left(\sin x\cos(-x)-\sin x\cos(5x)\right)dx$

$$\Rightarrow \int \sin x \sin 2x \sin 3x dx = \frac{1}{2}\int\left(\sin x \cos x - \sin x \cos 5x\right)dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x dx = \frac{1}{2}\int \frac{\sin 2x}{2}dx - \frac{1}{2}\int\left(\sin x \cos 5x\right)dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x dx = \frac{1}{4}\left(\frac{-\cos 2x}{2}\right) - \frac{1}{2}\int\left\{\frac{1}{2}\left(\sin\left(x+5x\right)+\sin\left(x-5x\right)\right)\right\}dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x dx = \frac{-\cos 2x}{8} - \frac{1}{4}\int\left\{\left(\sin 6x + \sin\left(-4x\right)\right)\right\}dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x dx = \frac{-\cos 2x}{8} - \frac{1}{4}\int\left\{\left(\sin 6x + \sin 4x\right)\right\}dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x dx = \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4}\right] + C$$

$$\therefore \int \sin x \sin 2x \sin 3x dx = \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{4} - \cos 2x\right] + C$$

7. Solve the following: $\sin 4x \sin 8x$.

Ans: Given expression $\sin 4x \sin 8x$.

Using the identity $\sin A \sin B = \frac{1}{2}\cos\left(A-B\right) - \cos\left(A+B\right)$, given expression can be written as

$$\Rightarrow \sin 4x \sin 8x = \frac{1}{2}\cos\left(4x-8x\right) - \cos\left(4x+8x\right)$$

Integration of given expression is

$$\Rightarrow \int \sin 4x \sin 8x dx = \frac{1}{2}\int\left(\cos\left(-4x\right) - \cos\left(12x\right)\right)dx$$

$$\Rightarrow \int \sin 4x \sin 8x dx = \frac{1}{2}\int\left(\cos 4x - \cos 12x\right)dx$$

$$\therefore \int \sin 4x \sin 8x dx = \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right] + C$$

8. Solve the following: $\frac{1-\cos x}{1+\cos x}$.

Ans: Given expression $\frac{1-\cos x}{1+\cos x}$.

Using the identities $2\sin^2\frac{x}{2} = 1-\cos x$ and $\cos x = 2\cos^2\frac{x}{2} - 1$ given expression can be written as

$$\Rightarrow \frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \frac{1-\cos x}{1+\cos x} = \tan^2\frac{x}{2}$$

Integration of given expression is

$$\Rightarrow \int \frac{1-\cos x}{1+\cos x}dx = \int\left[\tan^2\frac{x}{2}\right]dx$$

$$\Rightarrow \int \frac{1-\cos x}{1+\cos x}dx = \int\left[\sec^2\frac{x}{2}-1\right]dx$$

$$\Rightarrow \int \frac{1-\cos x}{1+\cos x}dx = \left[\frac{\tan\frac{x}{2}}{\frac{1}{2}} - x\right] + C$$

$$\therefore \int \frac{1-\cos x}{1+\cos x}dx = 2\tan\frac{x}{2} - x + C$$

9. Solve the following: $\frac{\cos x}{1+\cos x}$.

Ans: Given expression $\frac{\cos x}{1+\cos x}$.

Using the identity $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ and $\cos x = 2\cos^2\frac{x}{2} - 1$ given expression can be written as

$$\Rightarrow \frac{\cos x}{1+\cos x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \frac{\cos x}{1+\cos x} = \frac{1}{2}\left[1 - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right]$$

$$\Rightarrow \frac{\cos x}{1+\cos x} = \frac{1}{2}\left[1 - \tan^2\frac{x}{2}\right]$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos x}{1+\cos x}dx = \frac{1}{2}\int\left[1-\tan^2\frac{x}{2}\right]dx$$

$$\Rightarrow \int \frac{\cos x}{1+\cos x}dx = \frac{1}{2}\int\left[1-\sec^2\frac{x}{2}+1\right]dx$$

$$\Rightarrow \int \frac{\cos x}{1+\cos x}dx = \frac{1}{2}\int\left[2-\sec^2\frac{x}{2}\right]dx$$

$$\Rightarrow \int \frac{\cos x}{1+\cos x}dx = \frac{1}{2}\left[2x-\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]+C$$

$$\therefore \int \frac{\cos x}{1+\cos x}dx = x-\tan\frac{x}{2}+C$$

10. **Solve the following: $\sin^4 x$.**

Ans: Given expression $\sin^4 x$.

Given expression can be written as $\sin^4 x = \sin^2 x \sin^2 x$

$$\Rightarrow \sin^4 x = \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$

$$\Rightarrow \sin^4 x = \frac{1}{4}(1-\cos 2x)^2$$

$$\Rightarrow \sin^4 x = \frac{1}{4}(1+\cos^2 2x-2\cos 2x)$$

$$\Rightarrow \sin^4 x = \frac{1}{4}\left(1+\frac{1+\cos 4x}{2}-2\cos 2x\right)$$

$$\Rightarrow \sin^4 x = \frac{1}{4}\left(1+\frac{1}{2}+\frac{\cos 4x}{2}-2\cos 2x\right)$$

$$\Rightarrow \sin^4 x = \frac{1}{4}\left(\frac{3}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right)$$

Integration of given expression is

$$\Rightarrow \int \sin^4 x dx = \frac{1}{4}\int\left(\frac{3}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right)dx$$

$$\Rightarrow \int \sin^4 x dx = \frac{1}{4}\left[\frac{3}{2}x+\frac{\sin 4x}{8}-\sin 2x\right]+C$$

$$\therefore \int \sin^4 x dx = \frac{3x}{8}+\frac{\sin 4x}{32}-\frac{1}{4}\sin 2x+C$$

11. Solve the following: $\cos^4 2x$.

Ans: Given expression $\cos^4 2x$.

Given expression can be written as

$$\cos^4 2x = \left(\cos^2 2x\right)^2$$

$$\Rightarrow \cos^4 2x = \left(\frac{1+\cos 4x}{2}\right)^2$$

$$\Rightarrow \cos^4 2x = \frac{1}{4}\left(1+\cos^2 4x + 2\cos 4x\right)$$

$$\Rightarrow \cos^4 2x = \frac{1}{4}\left(1+\frac{1+\cos 8x}{2} + 2\cos 4x\right)$$

$$\Rightarrow \cos^4 2x = \frac{1}{4}\left(1+\frac{1}{2}+\frac{\cos 8x}{2} + 2\cos 4x\right)$$

$$\Rightarrow \cos^4 2x = \frac{1}{4}\left(\frac{3}{2}+\frac{\cos 8x}{2} + 2\cos 4x\right)$$

Integration of given expression is

$$\Rightarrow \int \cos^4 2x dx = \int\left(\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right)dx$$

$$\therefore \int \cos^4 2x dx = \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$$

12. Solve the following: $\frac{\sin^2 x}{1+\cos x}$.

Ans: Given expression $\frac{\sin^2 x}{1+\cos x}$.

By applying the identity $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos x = 2\cos^2\frac{x}{2} - 1$, given expression can be written as

$$\Rightarrow \frac{\sin^2 x}{1+\cos x} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \frac{\sin^2 x}{1+\cos x} = \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \frac{\sin^2 x}{1+\cos x} = 2\sin^2 \frac{x}{2}$$

$$\Rightarrow \frac{\sin^2 x}{1+\cos x} = 1-\cos x$$

Integration of given expression is

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos x} dx = \int 1 dx - \int \cos x dx$$

$$\therefore \int \frac{\sin^2 x}{1+\cos x} dx = x - \sin x + C$$

13. Solve the following: $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$.

Ans: Given expression $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$.

We can apply the identity $\cos C - \cos D = -2\sin \frac{C+D}{2} \sin \frac{C-D}{2}$, we get

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{-2\sin \frac{2x+2\alpha}{2} \sin \frac{2x-2\alpha}{2}}{-2\sin \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2}}$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{\sin \frac{2(x+\alpha)}{2} \sin \frac{2(x-\alpha)}{2}}{\sin \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2}}$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2}}$$

We can apply the identity $\sin 2x = 2\sin x \cos x$, we get

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{\left[2\sin \frac{x+\alpha}{2} \cos \frac{x+\alpha}{2}\right]\left[2\sin \frac{x-\alpha}{2} \cos \frac{x-\alpha}{2}\right]}{\sin \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2}}$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = 4\cos \frac{x+\alpha}{2} \cos \frac{x-\alpha}{2}$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = 2\left[\cos \frac{x+\alpha}{2} + \frac{x-\alpha}{2} + \cos \frac{x+\alpha}{2} - \frac{x-\alpha}{2}\right]$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = 2[\cos x + \cos \alpha]$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx = 2\int [\cos x + \cos \alpha] dx$$

$$\therefore \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx = 2[\sin x + x \cos \alpha] + C$$

14. Solve the following: $\frac{\cos x - \sin x}{1 + \sin 2x}$.

Ans: Given expression $\frac{\cos x - \sin x}{1 + \sin 2x}$.

We know that $\sin^2 x + \cos^2 x = 1$.

Given expression can be written as

$$\Rightarrow \frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{\sin^2 x + \cos^2 x + \sin 2x} .$$

We can apply the identity $\sin 2x = 2 \sin x \cos x$, we get

$$\Rightarrow \frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{\sin^2 x + \cos^2 x + 2 \sin x \cos x}$$

$$\Rightarrow \frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{(\sin x + \cos x)^2}$$

Let $\sin x + \cos x = t$

$\therefore (\cos x - \sin x) dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = \int \frac{dt}{t^2}$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = \int t^{-2} dt$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = -t^{-1} + C$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = -\frac{1}{t} + C$$

Substitute $\sin x + \cos x = t$,

$$\therefore \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = -\frac{1}{\sin x + \cos x} + C$$

15. Solve the following: $\tan^3 2x\sec 2x$.

Ans: Given expression $\tan^3 2x\sec 2x$.

Given expression can be written as

$\tan^3 2x\sec 2x = \tan^2 2x\tan 2x\sec 2x$

$\Rightarrow \tan^3 2x\sec 2x = \left(\sec^2 2x - 1\right)\tan 2x\sec 2x$

$\Rightarrow \tan^3 2x\sec 2x = \sec^2 2x\tan 2x\sec 2x - \tan 2x\sec 2x$

Integration of given expression is

$\Rightarrow \int \tan^3 2x\sec 2xdx = \int \sec^2 2x\tan 2x\sec 2xdx - \int \tan 2x\sec 2xdx$

$\Rightarrow \int \tan^3 2x\sec 2xdx = \int \sec^2 2x\tan 2x\sec 2xdx - \frac{\sec 2x}{2} + C$

Let $\sec 2x = t$

$\therefore 2\sec 2x\tan 2xdx = dt$

Above integral becomes

$\Rightarrow \int \tan^3 2x\sec 2xdx = \frac{1}{2}\int t^2dt - \frac{\sec 2x}{2} + C$

$\Rightarrow \int \tan^3 2x\sec 2xdx = \frac{t^3}{6} - \frac{\sec 2x}{2} + C$

Substitute $\sec 2x = t$,

$\therefore \int \tan^3 2x\sec 2xdx = \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$

16. Solve the following: $\tan^4 x$.

Ans: Given expression $\tan^4 x$.

Given expression can be written as

$\Rightarrow \tan^4 x = \tan^2 x\tan^2 x$

$\Rightarrow \tan^4 x = \left(\sec^2 x - 1\right)\tan^2 x$

$\Rightarrow \tan^4 x = \sec^2 x\tan^2 x - \tan^2 x$

$\Rightarrow \tan^4 x = \sec^2 x\tan^2 x - \left(\sec^2 x - 1\right)$

$\Rightarrow \tan^4 x = \sec^2 x\tan^2 x - \sec^2 x + 1$

Integration of given expression is

$\Rightarrow \int \tan^4 xdx = \int \left(\sec^2 x\tan^2 x - \sec^2 x + 1\right)dx$

$\Rightarrow \int \tan^4 xdx = \int \left(\sec^2 x\tan^2 x\right)dx - \int \sec^2 xdx + \int 1dx$

$\Rightarrow \int \tan^4 xdx = \int \sec^2 x\tan^2 xdx - \tan x + x + C$

Let $\tan x = t$

$\therefore \sec^2 xdx = dt$

$$\Rightarrow \int \tan^4 x dx = \int t^2 dt - \tan x + x + C$$

$$\Rightarrow \int \tan^4 x dx = \frac{t^3}{3} - \tan x + x + C$$

Substitute $\tan x = t$,

$$\therefore \int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

17. Solve the following: $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$**.**

Ans: Given expression $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$.

Given expression can be written as

$$\Rightarrow \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$

$$\Rightarrow \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$

$$\Rightarrow \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \tan x \sec x + \cot x \operatorname{cosec} x$$

Integration of given expression is

$$\Rightarrow \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \tan x \sec x dx + \int \cot x \operatorname{cosec} x \, dx$$

$$\therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \sec x - \operatorname{cosec} x + C$$

18. Solve the following: $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$**.**

Ans: Given expression $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$.

By applying the identity $\cos 2x = 1 - 2\sin^2 x$, we get

$$\Rightarrow \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} = \frac{\cos 2x + 1 - \cos 2x}{\cos^2 x}$$

$$\Rightarrow \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\Rightarrow \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} = \sec^2 x$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \sec^2 x dx$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \tan x + C$$

19. Solve the following: $\frac{1}{\sin x \cos^3 x}$.

Ans: Given expression $\frac{1}{\sin x \cos^3 x}$.

We can apply the identity $\sin^2 x + \cos^2 x = 1$, we get

$$\Rightarrow \frac{1}{\sin x \cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}$$

$$\Rightarrow \frac{1}{\sin x \cos^3 x} = \frac{\sin^2 x}{\sin x \cos^3 x} + \frac{\cos^2 x}{\sin x \cos^3 x}$$

$$\Rightarrow \frac{1}{\sin x \cos^3 x} = \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x}$$

$$\Rightarrow \frac{1}{\sin x \cos^3 x} = \tan x \sec^2 x + \frac{\cos^2 x}{\frac{\sin x \cos x}{\cos^2 x}}$$

$$\Rightarrow \frac{1}{\sin x \cos^3 x} = \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x} dx$$

Let $\tan x = t$

$\therefore \sec^2 x dx = dt$

$$\Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \int t dt + \int \frac{1}{t} dt$$

$$\Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \frac{t^2}{2} + \log|t| + C$$

Substitute $\tan x = t$,

$$\therefore \int \frac{1}{\sin x \cos^3 x} dx = \frac{1}{2}\tan^2 x + \log|\tan x| + C$$

20. Solve the following: $\frac{\cos 2x}{(\cos x + \sin x)^2}$.

Ans: Given expression $\frac{\cos 2x}{(\cos x+\sin x)^2}$.

Given expression can be written as

$$\Rightarrow \frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x}$$

We know that $\sin^2 x+\cos^2 x=1$ and $2\sin x\cos x=\sin 2x$, we get

$$\Rightarrow \frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{1+\sin 2x}$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos 2x}{(\cos x+\sin x)^2}dx=\int \frac{\cos 2x}{1+\sin 2x}dx$$

Let $1+\sin 2x=t$

$\therefore 2\cos 2x dx=dt$

Integration becomes

$$\Rightarrow \int \frac{\cos 2x}{(\cos x+\sin x)^2}dx=\frac{1}{2}\int \frac{1}{t}dt$$

$$\Rightarrow \int \frac{\cos 2x}{(\cos x+\sin x)^2}dx=\frac{1}{2}\log|t|+C$$

Substitute $1+\sin 2x=t$

$$\Rightarrow \int \frac{\cos 2x}{(\cos x+\sin x)^2}dx=\frac{1}{2}\log|1+\sin 2x|+C$$

$$\Rightarrow \int \frac{\cos 2x}{(\cos x+\sin x)^2}dx=\frac{1}{2}\log\left|(\cos x+\sin x)^2\right|+C$$

$$\therefore \int \frac{\cos 2x}{(\cos x+\sin x)^2}dx=\log|(\cos x+\sin x)|+C$$

21. Solve the following: $\sin^{-1}(\cos x)$.

Ans: Given expression $\sin^{-1}(\cos x)$.

Let $\cos x=t$

$\therefore \sin x=\sqrt{1-t^2}$

$\Rightarrow -\sin x dx=dt$

$$\Rightarrow dx=-\frac{dt}{\sin x}$$

$$\Rightarrow dx=-\frac{dt}{\sqrt{1-t^2}}$$

Integration of given expression is

$$\Rightarrow \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t\left(\frac{-dt}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \int \sin^{-1}(\cos x)dx = -\int\left(\frac{\sin^{-1} t}{\sqrt{1-t^2}}\right)dt$$

Let $\sin^{-1} t = u$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}}dt = du$$

Integration becomes

$$\Rightarrow \int \sin^{-1}(\cos x)dx = \int 4du$$

$$\Rightarrow \int \sin^{-1}(\cos x)dx = -\frac{u^2}{2} + C$$

Substitute $\sin^{-1} t = u$

$$\Rightarrow \int \sin^{-1}(\cos x)dx = -\frac{(\sin^{-1} t)^2}{2} + C$$

Substitute $\cos x = t$

$$\Rightarrow \int \sin^{-1}(\cos x)dx = -\frac{[\sin^{-1}(\cos x)]^2}{2} + C \quad \text{........(1)}$$

We know that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$$

Substitute in eq. (1), we get

$$\Rightarrow \int \sin^{-1}(\cos x)dx = \frac{-\left(\frac{\pi}{2} - x\right)^2}{2} + C$$

$$\Rightarrow \int \sin^{-1}(\cos x)dx = \frac{1}{2}\left(\frac{\pi^2}{2} + x^2 - \pi x\right) + C$$

$$\Rightarrow \int \sin^{-1}(\cos x)dx = -\frac{\pi^2}{4} - \frac{x^2}{2} + \frac{1}{2}\pi x + C$$

$$\Rightarrow \int \sin^{-1}(\cos x)dx = \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{4}\right)$$

$$\therefore \int \sin^{-1}(\cos x)dx = \frac{\pi x}{2} - \frac{x^2}{2} + C_1$$

22. Solve the following: $\dfrac{1}{\cos(x-a)\cos(x-b)}$.

Ans: Given expression $\dfrac{1}{\cos(x-a)\cos(x-b)}$.

Given expression can be written as

$$\Rightarrow \frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$\Rightarrow \frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$

$$\Rightarrow \frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right]$$

$$\Rightarrow \frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\tan(x-b)-\tan(x-b)\right]$$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \int \frac{1}{\sin(a-b)}\left[\tan(x-b)-\tan(x-b)\right]dx$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \int \frac{1}{\sin(a-b)}\left[-\log\left|\cos(x-b)\right|+\log\cos(x-a)\right]$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \int \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

23. Solve the following: $\displaystyle\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$ is equal to

(A) $\tan x+\cot x+C$
(B) $\tan x+\text{cosec}x+C$
(C) $-\tan x+\cot x+C$
(D) $\tan x+\sec x+C$

Ans: Given expression $\displaystyle\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$.

Given expression can be written as

$$\Rightarrow \int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx = \int \frac{\sin^2 x}{\sin^2 x\cos^2 x}dx - \int \frac{\cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$\Rightarrow \int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx = \int \sec^2 x dx - \int \text{cosec}^2 x dx$$

$$\therefore \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \tan x + \cot x + C$$

Therefore, option A is the correct answer.

24. Solve the following: $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$ **equals**

(A) $-\cot(ex^x) + C$

(B) $\tan(xe^x) + C$

(C) $\tan(e^x) + C$

(D) $\cot(e^x) + C$

Ans: Given expression $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$.

Let $e^x x = t$

$\therefore (e^x.x + e^x.1)dx = dt$

$\Rightarrow e^x(x+1)dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{e^x(1+x)}{\cos^2(e^x x)} dx = \int \frac{dt}{\cos^2 t}$$

$$\Rightarrow \int \frac{e^x(1+x)}{\cos^2(e^x x)} dx = \int \sec^2 t dt$$

$$\Rightarrow \int \frac{e^x(1+x)}{\cos^2(e^x x)} dx = \tan t + C$$

Substitute $e^x x = t$,

$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)} dx = \tan(e^x x) + C$$

Therefore, option B is the correct answer.

Exercise 7.4

1. Solve the following: $\frac{3x^2}{x^6+1}$**.**

Ans: Given expression $\frac{3x^2}{x^6+1}$.

Let $x^3 = t$

$\therefore 3x^2 dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$

We know that $\int \frac{1}{1+x^2} = \tan^{-1} x$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \tan^{-1} t + C$$

Substitute $x^3 = t$,

$$\therefore \int \frac{3x^2}{x^6+1}dx = \tan^{-1}\left(x^3\right) + C$$

2. Solve the following: $\frac{1}{\sqrt{1+4x^2}}$.

Ans: Given expression $\frac{1}{\sqrt{1+4x^2}}$.

Let $2x = t$

$\therefore 2dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

We know that $\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\log\left|t+\sqrt{t^2+1}\right| + C$$

Substitute $2x = t$,

$$\therefore \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C$$

3. Solve the following: $\frac{1}{\sqrt{(2-x)^2+1}}$.

Ans: Given expression $\frac{1}{\sqrt{(2-x)^2+1}}$.

Let $2 - x = t$

$\therefore -dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

We know that $\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\log\left|t+\sqrt{t^2+1}\right|+C$$

Substitute $2-x=t$,

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\log\left|(2-x)+\sqrt{(2-x)^2+1}\right|+C$$

$$\therefore \int \frac{1}{\sqrt{(2-x)^2+1}}dx = \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$$

4. Solve the following: $\frac{1}{\sqrt{9-25x^2}}$.

Ans: Given expression $\frac{1}{\sqrt{9-25x^2}}$.

Let $5x=t$

$\therefore 5dx=dt$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{9-t^2}}dt$$

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}dt$$

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right)+C$$

Substitute $5x=t$,

$$\therefore \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$$

5. Solve the following: $\frac{3x}{1+2x^4}$.

Ans: given expression $\frac{3x}{1+2x^4}$.

Let $\sqrt{2}x^2=t$

$\therefore 2\sqrt{2}dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\tan^{-1}t + C$$

Substitute $\sqrt{2}x^2 = t$,

$$\therefore \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

6. Solve the following: $\mathbf{\frac{x^2}{1-x^6}}$.

Ans: Given expression $\frac{x^2}{1-x^6}$.

Let $x^3 = t$

$\therefore 3x^2dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$

Substitute $x^3 = t$,

$$\therefore \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+x^3}{1-x^3}\right|\right] + C$$

7. Solve the following: $\mathbf{\frac{x-1}{\sqrt{x^2-1}}}$.

Ans: Given expression $\frac{x-1}{\sqrt{x^2-1}}$.

Given expression can be written as

$$\Rightarrow \frac{x-1}{\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}$$

Integration of given expression is

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx$$

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \log\left|x+\sqrt{x^2-1}\right| + C$$

Let $x^2 - 1 = t$

$\therefore 2xdx = dt$

Integration becomes

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}} - \log\left|x+\sqrt{x^2-1}\right| + C$$

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2-1}}dx = \frac{1}{2}\int t^{\frac{1}{2}}dt - \log\left|x+\sqrt{x^2-1}\right| + C$$

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2-1}}dx = \frac{1}{2}\left(2t^{\frac{1}{2}}\right) - \log\left|x+\sqrt{x^2-1}\right| + C$$

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2-1}}dx = \sqrt{t} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Substitute $x^2 - 1 = t$

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

8. Solve the following: $\frac{x^2}{\sqrt{x^6+a^6}}$.

Ans: Given expression $\frac{x^2}{\sqrt{x^6+a^6}}$.

Let $x^3 = t$

$\therefore 3x^2dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$\Rightarrow \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$

Substitute $x^3 = t$,

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

9. Solve the following: $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$.

Ans: Given expression $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$.

Let $\tan x = t$

$\therefore \sec^2 x dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}} dx = \int \frac{dt}{\sqrt{t^2+2^2}}$$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}} dx = \log\left|t+\sqrt{t^2+4}\right|+C$$

Substitute $\tan x = t$,

$$\therefore \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}} dx = \log\left|\tan x+\sqrt{\tan^2 x+4}\right|+C$$

10. Solve the following: $\frac{1}{\sqrt{x^2+2x+2}}$.

Ans: Given expression $\frac{1}{\sqrt{x^2+2x+2}}$.

Given expression can be written as

$$\frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{(x+1)^2+(1)^2}}$$

Let $x+1=t$

$\therefore dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+2}} dx = \int \frac{1}{\sqrt{t^2+1}} dt$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+2}} dx = \log\left|t+\sqrt{t^2+1}\right|+C$$

Substitute $x+1=t$,

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+2}} dx = \log\left|(x+1)+\sqrt{(x+1)^2+1}\right|+C$$

$$\therefore \int \frac{1}{\sqrt{x^2+2x+2}} dx = \log\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

11. Solve the following: $\frac{1}{\sqrt{9x^2+6x+5}}$.

Ans: Given expression $\frac{1}{\sqrt{9x^2+6x+5}}$.

Given expression can be written as

$$\frac{1}{\sqrt{9x^2+6x+5}}=\frac{1}{\sqrt{(3x+1)^2+(2)^2}}$$

Let $3x+1=t$

$\therefore 3dx=dt$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\sqrt{9x^2+6x+5}}dx=\frac{1}{3}\int \frac{1}{\sqrt{t^2+2^2}}dt$$

$$\Rightarrow \int \frac{1}{\sqrt{9x^2+6x+5}}dx=\frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

Substitute $3x+1=t$,

$$\therefore \int \frac{1}{\sqrt{9x^2+6x+5}}dx=\frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{3x+1}{2}\right)\right]+C$$

12. Solve the following: $\frac{1}{\sqrt{7-6x-x^2}}$.

Ans: Given expression $\frac{1}{\sqrt{7-6x-x^2}}$.

Given expression can be written as

$$\Rightarrow \frac{1}{\sqrt{7-6x-x^2}}=\frac{1}{\sqrt{7-(x^2+6x+9-9)}}$$

$$\Rightarrow \frac{1}{\sqrt{7-6x-x^2}}=\frac{1}{\sqrt{16-(x^2+6x+9)}}$$

$$\Rightarrow \frac{1}{\sqrt{7-6x-x^2}}=\frac{1}{\sqrt{16-(x+3)^2}}$$

$$\Rightarrow \frac{1}{\sqrt{7-6x-x^2}}=\frac{1}{\sqrt{4^2-(x+3)^2}}$$

Let $x+3=t$

$\therefore dx=dt$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\sqrt{7-6x-x^2}}dx=\int \frac{1}{\sqrt{4^2-(t)^2}}dt$$

$$\Rightarrow \int \frac{1}{\sqrt{7-6x-x^2}}dx = \sin^{-1}\left(\frac{t}{4}\right) + C$$

Substitute $x+3=t$,

$$\therefore \int \frac{1}{\sqrt{7-6x-x^2}}dx = \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

13. **Solve the following:** $\frac{\mathbf{1}}{\sqrt{\mathbf{(x-1)(x-2)}}}$.

Ans: Given expression $\frac{1}{\sqrt{(x-1)(x-2)}}$.

Given expression can be written as

$$\Rightarrow \frac{1}{\sqrt{(x-1)(x-2)}} = \frac{1}{\sqrt{x^2-3x+2}}$$

$$\Rightarrow \frac{1}{\sqrt{(x-1)(x-2)}} = \frac{1}{\sqrt{x^2-3x+\frac{9}{4}-\frac{9}{4}+2}}$$

$$\Rightarrow \frac{1}{\sqrt{(x-1)(x-2)}} = \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

Let $x-\frac{3}{2}=t$

$\therefore dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \int \frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dx$$

$$\Rightarrow \int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right| + C$$

Substitute $x-\frac{3}{2}=t$,

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right| + C$$

14. **Solve the following:** $\frac{\mathbf{1}}{\sqrt{\mathbf{8+3x-x^2}}}$.

Ans: Given expression $\frac{1}{\sqrt{8+3x-x^2}}$.

Given expression can be written as

$$\Rightarrow \frac{1}{\sqrt{8+3x-x^2}} = \frac{1}{\sqrt{8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}}$$

$$\Rightarrow \frac{1}{\sqrt{8+3x-x^2}} = \frac{1}{\sqrt{8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}}$$

$$\Rightarrow \frac{1}{\sqrt{8+3x-x^2}} = \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}$$

Let $x-\frac{3}{2}=t$

$\therefore dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-(t)^2}}dt$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx = \sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

Substitute $x-\frac{3}{2}=t$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx = \sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$\therefore \int \frac{1}{\sqrt{8+3x-x^2}}dx = \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

15. $\dfrac{\mathbf{1}}{\sqrt{(\mathbf{x-a})(\mathbf{x-b})}}$

Ans: $(x-a)(x-b)=x^2-(a+b)x+ab$

$x^2-(a+b)x+ab=x^2-(a+b)x+\dfrac{(a+b)^2}{4}-\dfrac{(a+b)^2}{4}+ab$

Simplifying,

$=\left[x-\left(\dfrac{a+b}{2}\right)\right]^2-\dfrac{(a-b)^2}{4}$

$\Rightarrow \int \dfrac{1}{\sqrt{(x-a)(x-b)}}dx=\int \dfrac{1}{\sqrt{\left\{x-\left(\dfrac{a+b}{2}\right)\right\}^2-\left(\dfrac{a+b}{2}\right)^2}}dx$

Consider $x-\left(\dfrac{a+b}{2}\right)=t \therefore dx=dt$

$\int \dfrac{1}{\sqrt{\left\{x-\left(\dfrac{a+b}{2}\right)\right\}^2-\left(\dfrac{a+b}{2}\right)^2}}dx=\int \dfrac{1}{\sqrt{t^2-\left(\dfrac{a+b}{2}\right)^2}}dt$

Using the logarithm formula of integration,

$=\log\left|t+\sqrt{t^2-\left(\dfrac{a+b}{2}\right)^2}\right|+C$

Substitute the value of t,

$=\log\left|\left\{x-\left(\dfrac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$

16. $\dfrac{\mathbf{4x+1}}{\sqrt{\mathbf{2x^2+x-3}}}$

Ans: Consider $4x+1=A\dfrac{d}{dx}\left(2x^2+x-3\right)+B$

Simplifying,

$\Rightarrow 4x+1=A(4x+1)+B$

$\Rightarrow 4x+1=4Ax+A+B$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$4A=4\Rightarrow A=1$

$A+B=1\Rightarrow B=0$

Consider $2x^2+x-3=t$

$\therefore (4x+1)dx=dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}}dx=\int \frac{1}{\sqrt{t}}dt$$

Using the power rule of integration,

$=2\sqrt{t}+C$

Substitute the value of t,

$=2\sqrt{2x^2+x-3}+C$

17. $\mathbf{\dfrac{x+2}{\sqrt{x^2-1}}}$

Ans: Consider $x+2=A\frac{d}{dx}\left(x^2-1\right)+B$

$\Rightarrow x+2=A(2x)+B......(1)$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$2A=1\Rightarrow A=\frac{1}{2}$

$B=2$

From (1), we get

$(x+2)=\frac{1}{2}(2x)+2$

$$\int \frac{x+2}{\sqrt{x^2-1}}dx=\int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

$$=\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx+\int \frac{2}{\sqrt{x^2-1}}dx$$

In $\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx$ let $x^2-1=t\Rightarrow 2xdx=dt$

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx=\frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

Integrating using the power rule

$=\frac{1}{2}[2\sqrt{t}]$

Simplifying,

$=\sqrt{t}$

Substitute the value of t,

$= \sqrt{x^2 - 1}$

Then, $\int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x + \sqrt{x^2-1}\right|$

From equation (2), we get

$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$

18. $\mathbf{\frac{5x-2}{1+2x+3x^2}}$

Ans:

Let $5x - 2 = A\frac{d}{dx}\left(1+2x+3x^2\right) + B$

$\Rightarrow\ 5\ x - 2 = A\left(2 + 6\ x\right) + B......\left(1\right)$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$5 = 6A \Rightarrow A = \frac{5}{6}$

$2A + B = -2 \Rightarrow B = -\frac{11}{3}$

Substitute the above values in (1)

$\therefore 5x - 2 = \frac{5}{6}(2+6x) + \left(-\frac{11}{3}\right)$

$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}dx$

$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$

Consider $I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$ and $I_2 = \int \frac{1}{1+2x+3x^2}dx$

$\therefore \int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 ...(1)$

$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$

Put $1 + 2x + 3x^2 = t$

$\Rightarrow (2+6x)dx = dt$

$\therefore I_1 = \int \frac{dt}{t}$

Using the logarithm formula of integration,

$I_1 = \log|t|$

Substitute the value of t,

$I_1 = \log\left|1+2x+3x^2\right| ...(2)$

Then,

$$I_2 = \int \frac{1}{1+2x+3x^2}dx$$

$1+2x+3x^2$ can be rewritten as $1+3\left(x^2+\frac{2}{3}x\right)$

Thus,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

By completing square method,

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

Simplifying,

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

Therefore I_2 can be rewritten as ,

$$I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

Simplifying,

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)...(3)$$

We obtain the below values by substituting equations (2) and (3) in equation (1)

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$$

Simplifying,

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

19. $$\mathbf{\frac{6x+7}{\sqrt{(x-5)(x-4)}}}$$

Ans: Consider $6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$

Differentiating,

$\Rightarrow 6x+7 = A(2x-9)+B$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x+7 = 3(2x-9)+34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

Consider $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \quad ....(1)$$

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Put $x^2-9x+20 = t$

$\Rightarrow (2x-9)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}}$$

Integrating using the power rule

$I_1 = 2\sqrt{t}$

Substitute the value of t,

$I_1 = 2\sqrt{x^2-9x+20}$ .....(2)

$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$$

Consider

$x^2 - 9x + 20$

By completing square methods,

$$= x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2 = \int \frac{1}{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx$$

$$I_2 = \log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| .....(3)$$

We obtain the below values by substituting equations (2) and (3) in (1),

$$\int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx = 3\left[2\sqrt{x^2 - 9x + 20}\right] + 34\log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$$

Simplifying,

$$= 6\sqrt{x^2 - 9x + 20} + 34\log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$$

20. $\mathbf{\frac{x+2}{\sqrt{4x - x^2}}}$

Ans: Consider, $x + 2 = A\frac{d}{dx}\left(4x - x^2\right) + B$

$\Rightarrow x + 2 = A(4 - 2x) + B$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$-2A = 1 \Rightarrow A = -\frac{1}{2}$

$4A + B = 2 \Rightarrow B = 4$

$\Rightarrow (x+2) = -\frac{1}{2}(4 - 2x) + 4$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 \text{ and } +4I_2 \quad ....(1)$$

Then, $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x - x^2 = t$

$\Rightarrow (4-2x)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} ...(2)$$

(Using the logarithm formula of integration,)

$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$

Integrating using the power rule,

$\Rightarrow 4x - x^2 = -\left(-4x + x^2\right)$

By completing square methods,

$= \left(-4x + x^2 + 4 - 4\right)$

$= 4 - (x-2)^2$

$= (2)^2 - (x-2)^2$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2 - (x-2)^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right) ...(3)$$

Using equations (2) and (3) in (1), to obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

21. $\dfrac{\mathbf{x+2}}{\sqrt{\mathbf{x^2+2x+3}}}$

Ans:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$

Simplifying,

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \quad ....(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Put, $x^2+2x+3 = t$

Integrating using the power rule,

$\Rightarrow (2x+2)dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3}..(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx.$$

By completing square methods,

$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2 + (\sqrt{2})^2$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right|...(3)$$

Using equations (2) and (3) in (1), to obtain

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

22. $\mathbf{\frac{x+3}{x^2-2x-5}}$

Ans:

Consider $(x+3) = A\frac{d}{dx}\left(x^2-2x-5\right) + B$

$= \frac{x^2}{2}(\log x)^2 - \int x \log x dx$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$2A = 1 \Rightarrow A = \frac{1}{2}$

$-2A + B = 3 \Rightarrow B = 4$

$\therefore (x+3) = \frac{1}{2}(2x-2) + 4$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

Consider $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$ and $I_2 = \int \frac{1}{x^2-2x-5}dx$

$$\therefore \int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}I_1 + 4I_2 ...(1)$$

Then, $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$

Put $x^2 - 2x - 5 = t$

$\Rightarrow (2x-2)dx = dt$

Using the logarithm formula of integration,

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log\left|x^2 - 2x - 5\right| \;....(2)$$

$$I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$= \int \frac{1}{(x^2-2x+1)-6}dx$$

$$= \int \frac{1}{(x-1)^2 + (\sqrt{6})^2}dx$$

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right)....(3)$$

We obtain the below values by substituting (2) and (3) in (1),

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2 - 2x - 5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}}\right| + C$$

23. $\frac{5x + 3}{\sqrt{x^2 + 4x + 10}}$

Ans:

$$5x + 3 = A\frac{a}{dx}\left(x^2 + 4x + 10\right) + B$$

$$\Rightarrow 5x + 3 = A(2x + 4) + B$$

Equating the coefficients of x and constant term, we get

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

$$4A + B = 3 \Rightarrow B = -7$$

$$\therefore 5x + 3 = \frac{5}{2}(2x + 4) - 7$$

$$\Rightarrow \int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}}dx = \int \frac{\frac{5}{2}(2x + 4) - 7}{\sqrt{x^2 + 4x + 10}}dx$$

$$= \frac{5}{2}\int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}}dx - 7\int \frac{1}{\sqrt{x^2 + 4x + 10}}dx$$

Let $I_1 = \int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2 + 4x + 10}}dx$

$$\therefore \int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}}dx = \frac{5}{2}I_1 - 7I_2 \quad ....(1)$$

Then, $I_1 = \int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}}dx$

Consider $x^2 + 4x + 10 = t \therefore (2x + 4)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2 + 4x + 10} \quad ......(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2 + 4x + 10}}dx$$

$$= \int \frac{1}{\sqrt{\left(x^2 + 4x + 4\right) + 6}}dx = \int \frac{1}{(x + 2)^2 + (\sqrt{6})^2}dx$$

$$= \log\left|(x + 2)\sqrt{x^2 + 4x + 10}\right| ...(3)$$

We obtain the below values by using equations (2) and (3) in (1).

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)\sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)\sqrt{x^2+4x+10}\right| + C$$

24. $\int \frac{dx}{x^2+2x+2}$ **equals**

a) $x\tan^{-1}(x+1)+C$

b) $\tan^{-1}(x+1)+C$

c) $(x+1)\tan^{-1}x+C$

d) $\tan^{-1}x+C$

Ans:

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$

$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the right response is is B.

25. $\int \frac{dx}{\sqrt{9x-4x^2}}$ **equals**

a) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

b) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

c) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

d) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Ans:

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

By completing square methods,

$$=\int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$=\int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$=\frac{1}{2}\int \frac{1}{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}dx$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \quad \left(\int \frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\frac{y}{a}+C\right)$$

Simplifying,

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the right response is B.

Exercise 7.5

1. $\mathbf{\frac{x}{(x+1)(x+2)}}$

Ans:

Let $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$\Rightarrow x=A(x+2)+B(x+1)$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$A+B=1$

$2A+B=0$

On solving, we get

$A=-1$ and $B=2$

$$\therefore \frac{x}{(x+1)(x+2)}=\frac{-1}{(x+1)}+\frac{2}{(x+2)}$$

$$\Rightarrow \int \frac{x}{(x+1)(x+2)}dx=\int \frac{-1}{(x+1)}+\frac{2}{(x+2)}dx$$

Using the logarithm formula of integration,

$= -\log|x+1| + 2\log|x+2| + C$

$= \log(x+2)^2 - \log|x+1| + C$

Simplifying,

$= \log\frac{(x+2)^2}{(x+1)} + C$

2. $\mathbf{\frac{1}{x^2-9}}$

Ans:

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$1 = A(x-3) + B(x+3)$

Equating the coefficients of x and constant term, we get

$A + B = 0$

$1 = -3A + 3B$

$A = -\frac{1}{6}$ and $B = \frac{1}{6}$

$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$

$\Rightarrow \int \frac{1}{(x^2-9)}dx = \int\left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)}\right)dx$

Using the logarithm formula of integration,

$= -\frac{1}{6}\log|x+3| + \frac{1}{6}\log|x-3| + C = \frac{1}{6}\log\frac{|(x-3)|}{|(x+3)|} + C$

3. $\mathbf{\frac{3x-1}{(x-1)(x-2)(x-3)}}$

Ans:

Let $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$3x - 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \ldots(1)$

We obtain the below values by equating the coefficients of x, x^2 and the constant term on both sides.

$A + B + C = 0$

$-5A - 4B - 3C = 3$

$6A + 3B + 2C = -1$

Solving these equations, to obtain

$A = 1, B = -5,$ and $C = 4$

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)}dx = \int \left\{ \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \right\} dx$$

Using the logarithm formula of integration,

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

4. $\mathbf{\dfrac{x}{(x-1)(x-2)(x-3)}}$

Ans:

Let $\dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$

$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

We obtain the below values by equating the coefficients of x , x^2 and the constant term on both sides.

$A + B + C = 0$

$-5A - 4B - 3C = 1$

$6A + 4B + 2C = 0$

Solving these equations, to obtain

$A = \dfrac{1}{2}, B = 2$ and $C = \dfrac{3}{2}$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)}dx = \int \left\{ \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right\} dx$$

Using the logarithm formula of integration,

$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$$

5. $\mathbf{\dfrac{2x}{x^2 + 3x + 2}}$

Ans:

$$\frac{2x}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$2x = A(x+2) + B(x+1)$

We obtain the below values by equating the coefficients of x , x^2 and the constant term on both sides.

$A + B = 2$

$2A + B = 0$

Solving these equations, we get

$A = -2$ and $B = 4$

$$\therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$$

$$\Rightarrow \int \frac{2x}{(x+1)(x+2)}dx = \int \left\{ \frac{4}{(x+2)} - \frac{2}{(x+1)} \right\} dx$$

Using the logarithm formula of integration,

$= 4\log|x+2| - 2\log|x+1| + C$

6. $\mathbf{\dfrac{1-x^2}{x(1-2x)}}$

Ans:

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1-x^2)$ by $x(1-2x)$ to obtain,

$$\frac{1-x^2}{x(1+2x)} = \frac{1}{2} + \frac{1}{2}\left(\frac{2-x}{x(1-2x)} \right)$$

Let $\dfrac{2-x}{x(1-2x)} = \dfrac{A}{x} + \dfrac{B}{(1-2x)} ...(1)$

$\Rightarrow (2-x) = A(1-2x) + Bx$

We obtain the below values by equating the coefficients of x , x^2 and the constant term on both sides.

$-2A + B = -1$ and, $A = 2$

Solving these equations, to obtain $A = 2$ and $B = 3$

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

Substituting in equation (1), we get

$$\frac{1-x^2}{x(1+2)} = \frac{1}{2} + \frac{1}{2}\left\{ \frac{2}{x} + \frac{3}{(1-2x)} \right\}$$

$$\Rightarrow \int \frac{1-x^2}{x(1+2)}dx = \int \frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{(1-2x)} \right)dx$$

Using the power rule and logarithm formula of integration,

$$=\frac{x}{2}+\log|x|+\frac{3}{2(-2)}\log|1-2x|+C=\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$$

7. $\dfrac{\mathbf{x}}{\left(\mathbf{x}^2+\mathbf{1}\right)\left(\mathbf{x}-\mathbf{1}\right)}$

Ans: Let $\dfrac{x}{\left(x^2+1\right)(x-1)}=\dfrac{Ax+B}{\left(x^2+1\right)}+\dfrac{C}{(x-1)}\ldots(1)$

$x=(Ax+B)(x-1)+C\left(x^2+1\right)$

$x=Ax^2-Ax+Bx-B+Cx^2+C$

We obtain the below values by equating the coefficients of x , x^2 and the constant term on both sides.

$A+C=0$

$-A+B=1$

$-B+C=0$

On solving these equations, to obtain $A=-\frac{1}{2}, B=\frac{1}{2}, \text{and } C=\frac{1}{2}$

From equation (1), to obtain

$$\therefore \frac{x}{\left(x^2+1\right)(x-1)}=\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1}+\frac{\frac{1}{2}}{(x-1)}$$

$$\Rightarrow \int\frac{x}{\left(x^2+1\right)(x-1)}=-\frac{1}{2}\int\frac{x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$$

$$=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

Consider $\int\frac{2x}{x^2+1}dx, \text{lel} \left(x^2+1\right)=t \Rightarrow 2xdx=dt$

$$\Rightarrow \int\frac{2x}{x^2+1}dx=\int\frac{dt}{t}=\log|t|=\log\left|x^2+1\right|$$

$$\therefore \int\frac{x}{\left(x^2+1\right)(x-1)}=-\frac{1}{4}\log\left|x^2+1\right|+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

$$=\frac{1}{2}\log|x-1|-\frac{1}{4}\log\left|x^2+1\right|+\frac{1}{2}\tan^{-1}x+C$$

8. $\dfrac{\mathbf{x}}{\left(\mathbf{x}-\mathbf{1}\right)^2\left(\mathbf{x}+\mathbf{2}\right)}$

Ans:

$$\frac{x}{(x-1)^2(x+2)}$$

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

We obtain the below values by equating the coefficients of x , x^2 and the constant term on both sides.

$A + C = 0$

$A + B - 2C = 1$

On solving, to obtain

$A = \frac{2}{9}$ and $C = \frac{-2}{9}$

$B = \frac{1}{3}$

$$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x| 2)}$$

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)}dx = \frac{2}{9}\int \frac{1}{(x-1)}dx + \frac{1}{3}\int \frac{1}{(x-1)^2}dx - \frac{2}{9}\int \frac{1}{(x-2)}dx$$

Using the power rule and logarithm formula of integration,

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + C$$

Simplifying,

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

9. $\mathbf{\frac{3x+5}{x^3-x^2-x+1}}$

Ans:

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$$

let $\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$

$3x+5 = A(x-1)(x+1) + B(x+1) + (x-1)^2$

$3x+5 = A(x-1)^2 + B(x+1) + C\left(x^2+1-2x\right) \quad \ldots(1)$

We obtain the below values by equating the coefficients of x , x^2 and the constant term on both sides.

$A+C=0$

$B-2C=3$

$-A+B+C=5$

On solving, to obtain $B=4$ $A=-\frac{1}{2}$ and $C=\frac{1}{2}$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)}=\frac{-1}{2(x-1)}+\frac{4}{(x-1)^2}+\frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)}dx=-\frac{1}{2}\int \frac{1}{x-1}dx+4\int \frac{1}{(x-1)^2}dx+\frac{1}{2}\int \frac{1}{(x+1)}dx$$

Using the power rule and logarithm formula of integration,

$$=-\frac{1}{2}\log|x-1|+4\left(\frac{-1}{x-1}\right)+\frac{1}{2}\log|x+1|+C$$

$$=\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{(x-1)}+C$$

10. $\mathbf{\frac{2x-3}{\left(x^2-1\right)\left(2x+3\right)}}$

Ans:

$$\frac{2x-3}{\left(x^2-1\right)(2x+3)}=\frac{2x-3}{(x+1)(x-1)(2x+3)}$$

Let $\frac{2x-3}{(x+1)(x-1)(2x+3)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(2x+3)}$

$$\Rightarrow (2x-3)=A(x-1)(2x+3)+B(x+1)(2x+3)+C(x+1)(x-1)$$

$$\Rightarrow (2x-3)-A\left(2x^2+x-3\right)+B\left(2x^2+5x-3\right)+C\left(x^2-1\right)$$

$$\Rightarrow (2x-3)=(2A+2B+C)x^2+(A+5B)x+(-3A+3B-C)$$

We obtain the below values by equating the coefficients of x, x^2 and the constant term on both sides.

$2A+2B+C=0$

$A+5B=2$

$-3A+3B-C=-3$

On solving, to obtain $B=-\frac{1}{10}, A=\frac{5}{2},$ and $C=-\frac{24}{5}$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)}=\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{\left(x^2-1\right)(2x+3)}dx = \frac{5}{2}\int \frac{1}{(x+1)}dx - \frac{1}{10}\int \frac{1}{x-1}dx - \frac{24}{5}\int \frac{1}{(2x+3)}dx$$

Using the logarithm formula of integration,

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$

Simplifying,

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

11. $\mathbf{\dfrac{5x}{(x+1)\left(x^2-4\right)}}$

Ans:

$$\frac{5x}{(x+1)\left(x^2-4\right)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

let $\dfrac{5x}{(x+1)(x+2)(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2)$$

We obtain the below values by equating the coefficients of x, x^2 and the constant term on both sides.

$A + B + C = 0$

$-B + 3C = 5$ and, $-4A - 2B + 2C = 0$

On solving, to obtain

$A = \frac{5}{3}, B = -\frac{5}{2},$ and $C = \frac{5}{6}$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} + -\frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)\left(x^2-4\right)}dx = \frac{5}{3}\int \frac{1}{(x+1)}dx - \frac{5}{2}\int \frac{1}{(x+2)}dx + \frac{5}{6}\int \frac{1}{(x-2)}dx$$

Using the logarithm formula of integration,

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

12. $\mathbf{\dfrac{x^2+x+1}{x^2-1}}$

Ans:

On dividing $\left(x^3+x+1\right)$ by x^2-1, we get

$$\frac{x^3+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}$$

Let $\frac{2x+1}{x^2-1}=\frac{A}{(x+1)}+\frac{B}{(x-1)}$

$$2x+1=A(x-1)+B(x+1)$$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$$A+B=2$$
$$-A+B=1$$

On solving, to obtain

$A=\frac{1}{2}$ and $B=\frac{3}{2}$

$$\therefore \frac{x^3+x+1}{x^2-1}=x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$$

$$\Rightarrow \int \frac{x^3+x+1}{x^2+1}dx=\int xdx+\frac{1}{2}\int\frac{1}{(x+1)}dx+\frac{3}{2}\int\frac{1}{(x-1)}dx$$

$$=\frac{x^2}{2}+\log|x+1|+\frac{3}{2}\log|x-1|+C$$

13. $\mathbf{\frac{2}{(1-x)(1+x^2)}}$

Ans:

Let $\frac{2}{(1-x)(1+x^2)}=\frac{A}{(1-x)}+\frac{Bx+C}{(1+x^2)}$

$$2=A(1+x^2)+(Bx+C)(1-x)$$

$$2=A+Ax^2+Bx-Bx^2+C-Cx$$

We obtain the below values by equating the coefficients of x x^2 and the constant term on both sides.

$$A-B=0$$
$$B-C=0$$
$$A+C=2$$

On solving these equations, to obtain

$A=1, B=1,$ and $C=1$

$$\therefore \frac{2}{(1-x)(1+x^2)}=\frac{1}{1-x}+\frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)}dx=\int\frac{1}{1-x}dx+\int\frac{x}{1+x^2}dx+\int\frac{1}{1+x^2}dx$$

$$= -\int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\log|x-1| + \frac{1}{2}\log\left|1+x^2\right| + \tan^{-1}x + C$$

14. $\mathbf{\frac{3x-1}{(x+2)^2}}$

Ans:

Let $\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$

$\Rightarrow 3x-1 = A(x+2)+B$

We obtain the below values by equating the coefficients of x and the constant term on both sides.

$A = 3$

$2A + B = -1 \Rightarrow B = -7$

$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$

$\Rightarrow \int \frac{3x-1}{(x+2)^2}dx = 3\int \frac{1}{(x+2)}dx - 7\int \frac{x}{(x+2)^2}dx$

Using the power rule and logarithm formula of integration

$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$

$= 3\log|x+2| + \frac{7}{(x+2)} + C$

15. $\mathbf{\frac{1}{x^4-1}}$

Ans:

$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$

Let $\frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$

$1 = A(x-1)(1+x^2) + B(x+1)(1+x^2) + (Cx+D)(x^2-1)$

$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$

$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$

We obtain the below values by equating the coefficients of $x^3, x^2, x,$ and constant term, we get

$A+B+C=0$

$-A+B+D=0$

$A+B-C=0$

$-A+B-D=1$

$A=-\frac{1}{4}, B=\frac{1}{4}, C=0,$ and $D=-\frac{1}{2}$

$$\therefore \frac{1}{(x^4-1)} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x^2+1)}$$

$$\Rightarrow \int \frac{1}{x^4-1}dx = -\frac{1}{4}\log|x-1| + \frac{1}{4}\log|x-1| - \frac{1}{2}\tan^{1} x + C$$

Simplifying,

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{1} x + C$$

16. $\frac{1}{x(x^n+1)}$ **[hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]**

Ans: $\frac{1}{x(x^n+1)}$

Numerator and denominator are multiplied by x^{n-1}, to obtain

$$\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x^{n-1}x(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}$$

Consider $x^n = t \Rightarrow x^{n-1}dx = dt$

$$\therefore \int \frac{1}{x(x^n+1)}dx = \int \frac{x^{n-1}}{x^n(x^n+1)}dx = \frac{1}{n}\int \frac{1}{t(t+1)}dt$$

Let $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$

$1 = A(1+t) + B\,t$

We obtain the below values by equating the coefficients of t and constant,

$A=1$ and $B=-1$

$$\therefore \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x(x^n+1)}dx = \frac{1}{n}\int \left\{\frac{1}{t} - \frac{1}{(1+t)}\right\}dx$$

$=\frac{1}{n}[\log|t|-\log|t+1|]+C$

Substitute the value of t,

$=-\frac{1}{n}\left[\log\left|x^n\right|-\log\left|x^n+1\right|\right]+C$

Simplifying,

$=\frac{1}{n}\log\left|\frac{x^u}{x^{''}+1}\right|+C$

17. $\frac{\cos x}{(1-\sin x)(2-\sin x)}$ **[hint: Put $\sin x=t$]**

Ans:

$\frac{\cos x}{(1-\sin x)(2-\sin x)}$ Put, $\sin x=t \Rightarrow \cos x dx=dt$

$\therefore \int\frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int\frac{dt}{(1-t)(2-t)}$

let $\frac{1}{(1-t)(2-t)}=\frac{A}{(1-t)}+\frac{B}{(2-t)}$

$1=A(2-t)+B(1-t)$

We obtain the below values by equating the coefficients of t and constant,

$-2A-B=0$, and $2A+B=1$

$A=1$ and $B=-1$

$\therefore \frac{1}{(1-t)(2-t)}=\frac{1}{(1-t)}-\frac{1}{(2-t)}$

$\Rightarrow \int\frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int\left\{\frac{1}{1-t}-\frac{1}{(2-t)}\right\}dt=-\log|1-t|+\log|2-t|+C$

Simplifying,

$=\log\left|\frac{2-t}{1-t}\right|+C$

Substitute the value of t,

$=\log\left|\frac{2-\sin x}{1-\sin x}\right|+C$

18. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Ans:

$$\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)}=\frac{\left(4x^2+10\right)}{\left(x^2+3\right)\left(x^2+4\right)}$$

Let $\frac{\left(4x^2+10\right)}{\left(x^2+3\right)\left(x^2+4\right)}=\frac{Ax+B}{\left(x^2+3\right)}+\frac{Cx+D}{\left(x^2+4\right)}$

$$4x^2+10=(Ax+B)\left(x^2+4\right)+(Cx+D)\left(x^2+3\right)$$

$$4x^2+10=Ax^2+4Ax+Bx^2+4B+Cx^3+3Cx+Dx^2+3D$$

$$4x^2+10=(A+C)x^3+(B+D)x^2+(4A+3C)x+(4B+3D)$$

We obtain the below values by equating the coefficients of x^3, x^2, x and constant term,

$A+C=0$

$B+D=4$

$4\,A+3\,C=0$

$4\,B+3\,D=10$

On solving these equations, to obtain $A=0.B=-2.C=0,$ and $D=6$

$$\therefore \frac{\left(4x^2+10\right)}{\left(x^2+3\right)\left(x^2+4\right)}=\frac{-2}{\left(x^2+3\right)}+\frac{6}{\left(x^2+4\right)}$$

$$\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)}=\left(\frac{-2}{\left(x^2+3\right)}+\frac{6}{\left(x^2+4\right)}\right)$$

$$\Rightarrow \int \frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)}dx=\int\left\{1+\frac{2}{\left(x^2+3\right)}-\frac{6}{\left(x^2+4\right)}\right\}dx$$

$$=\int\left\{1+\frac{2}{x^2+(\sqrt{3})^2}-\frac{6}{x^2+2^2}\right\}$$

$$=x+2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)-6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C$$

Simplifying,

$$=x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$$

19. $\mathbf{\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}}$

Ans:

$$\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$$

Put $x^2 - t \to 2xdx - dt$

$$\therefore \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx = \int \frac{dt}{(t+1)(t+3)}$$

Let $\frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$

$I = A\left(t+3\right) + B\left(t+1\right)$

We obtain the below values by equating the coefficients of t and constant,

$1 + B = 0$ and $3A + B = 1$

On solving, we get

$A = \frac{1}{2}$ and $B = -\frac{1}{2}$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} + \frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx = \int \left\{\frac{1}{2(t+1)} - \frac{1}{2(t+3)}\right\}dt$$

$$= \frac{1}{2}\log|(t+1)| - \frac{1}{2}\log|t+3| + C$$

Simplifying,

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

20. $\mathbf{\frac{1}{x\left(x^4-1\right)}}$

Ans:

$$\frac{1}{x\left(x^4-1\right)}$$

Numerator and denominator are multiplied by by x^3, we get

$$\frac{1}{x\left(x^4-1\right)} = \frac{x^3}{x^4\left(x^4-1\right)}$$

$$\therefore \int \frac{1}{x\left(x^4-1\right)}dx = \int \frac{x^3}{x^4\left(x^4-1\right)}dx$$

Consider $x^4 = t \Rightarrow 4x^3dx = dt$

$$\therefore \int \frac{1}{x\left(x^4-1\right)}dx = \frac{1}{4}\int \frac{dt}{t(t-1)}$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$

$1 = A(t-1) + Bt \quad \ldots(1)$

We obtain the below values by equating the coefficients of t and constant,

$A = -1$ and $B = 1$

$\Rightarrow \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$

$\Rightarrow \int \frac{1}{x\left(x^4-1\right)} dx = \frac{1}{4} \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt$

Using the logarithm formula of integration,

$= \frac{1}{4}[-\log|t| + \log|t-1|] + C$

Simplifying,

$= \frac{1}{4} \log\left|\frac{t-1}{t}\right| + C = \frac{1}{4} \log\left|\frac{x^4-1}{x^4}\right| + C$

21. $\mathbf{\frac{1}{e^x-1}}$ **[hint: put $e^x = t$]**

Ans:

$\frac{1}{\left(e^x-1\right)}$

Put $e^x = t \Rightarrow e^x dx = dt$

$\Rightarrow \int \frac{1}{\left(e^x-1\right)} dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$

$1 = A\left(t-1\right) + B\,t$

We obtain the below values by equating the coefficients of t and constant,

$A = -1$ and $B = 1$

$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$

$\Rightarrow \int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$

Substitute the value of t,

$= \log\left|\frac{e^x-1}{e^x}\right| + C$

22. $\int \frac{xdx}{(x-1)(x-2)}$ **equals**

a) $\log\left|\frac{(x-1)^2}{x-2}\right|+C$

b) $\log\left|\frac{(x-2)^2}{x-1}\right|+C$

c) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|+C$

d) $\log|(x-1)(x-2)|+C$

Ans:

Let $\frac{x}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

$x=A(x-2)+B(x-1)$

We obtain the below values by equating the coefficients of x and constant,
$A=-1$ and $B=2$

$\therefore \frac{x}{(x-1)(x-2)}=-\frac{1}{(x1)}+\frac{2}{(x-2)}$

$\Rightarrow \int \frac{x}{(x-1)(x-2)}dx=\int\left\{\frac{-1}{(x-1)}+\frac{2}{(x-2)}\right\}dx$

Using the logarithm formula of integration,

$=-\log|x-1|+2\log|x-2|+C$

Simplifying,

$=\log\left|\frac{(x-2)^2}{x-1}\right|+C$

Thus, the right response is B.

23. $\int \frac{dx}{x(x^2+1)}$ **equals**

a. $\log|x|-\frac{1}{2}\log(x^2+1)+C$

b. $\log|x|+\frac{1}{2}\log(x^2+1)+C$

c. $-\log|x|+\frac{1}{2}\log\left(x^2+1\right)+C$

d. $\frac{1}{2}\log|x|+\log\left(x^2+1\right)+C$

Ans:

Let $\frac{1}{x\left(x^2+1\right)}-\frac{A}{x},\frac{Bx+C}{x^2+1}$

$1=A\left(x^2+1\right)+(Bx+C)x$

We obtain the below values by equating the coefficients of $x^2, x,$ and constant term,

$A+B=0,\ C=0\ A=1$

On solving these equations, to obtain

$A=1, B=-1,$ and $C=0$

$\therefore \frac{1}{x\left(x^2,1\right)}=\frac{1}{x}+\frac{-x}{x^2+1}$

$\Rightarrow \int\frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{1}{x}-\frac{x}{x^2+1}\right\}dx$

$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$

Thus, the right response is A.

Exercise 7.6

1. $x\sin x$

Ans:

Let $I=\int x\sin x dx$

Consider $u=x$ and $v=\sin x$ and integrating by parts, to obtain

$I=\int x\sin x dx-\int\left\{\left(\frac{d}{dx}x\right)\int \sin x dx\right\}dx$

$=x(-\cos x)-\int 1.(-\cos x)dx$

$=-x\cos x+\sin x+C$

2. $x\sin 3x$

Ans:

Let $I=\int x\sin 3x dx$

Consider $u=x$ and $v=\sin 3x$ and integrating by parts, to obtain

$$I = x\int \sin 3x dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin 3x dx\right\}$$

$$= x\left(\frac{-\cos 3x}{3}\right) - \int 1\cdot\left(\frac{-\cos 3x}{3}\right)dx$$

$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x dx = \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

3. $\mathbf{x^2 e^x}$

Ans:

Let $I = \int x^2 e^x dx$

Consider $u = x^2$ and $v = e^x$

$$I = x^2\int e^x dx - \int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx$$

$$= x^2 e^x - \int 2x - e^x dx$$

$$= x^2 e^x - 2\int x \cdot e^x dx$$

Again using integration by parts, to obtain

$$= x^2 e^x - 2\left[x\cdot\int e^x dx - \int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx\right]$$

$$= x^2 e^x - 2\left[xe^x - \int e^x dx\right]$$

Simplifying,

$$= x^2 e^x - 2\left[xe^x - e^x\right]$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

$$= e^x\left(x^2 - 2x + 2\right) + C$$

4. $\mathbf{x\log x}$

Ans:

Let $I = \int x\log x dx$

Consider $u = \log x$ and $v = x$ and integrating by parts, to obtain

$$I = \log x\int x dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x dx\right\}dx$$

$$= \log x\cdot\frac{x^2}{2} - \int\frac{1}{x}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2\log x}{2}\cdot\sqrt{\frac{x}{2}}dx = \frac{x^2\log x}{2} - \frac{x^2}{4} + C$$

5. $\mathbf{x\log 2x}$

Ans:

Let $I=\int x\log 2x dx$

Consider $u=\log 2x$ and $v=x$ and integrating by parts, to obtain

$$I=\log 2x\int xdx-\int\left\{\left(\frac{d}{dx}2\log x\right)\int xdx\right\}dx$$

$$=\log 2x\cdot\frac{x^2}{2}-\int\frac{2}{2x}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\log 2x}{2}-\int\frac{x}{2}dx$$

Integrating using the power rule

$$=\frac{x^2\log 2x}{2}-\frac{x^2}{4}+C$$

6. $\mathbf{x^2\log x}$

Ans:

Let $I=\int x^2\log x dx$

Consider $u=\log x$ and $v=x^2$ and integrating by parts, to obtain

$$I=\log x\int x^2dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x^2dx\right\}dx$$

$$=\log x\left(\frac{x^3}{3}\right)-\int\frac{1}{x}\cdot\frac{x^3}{3}dx$$

Integrating using the power rule

$$=\frac{x^3\log x}{3}-\int\frac{x^2}{3}dx=\frac{x^3\log x}{3}-\frac{x^3}{9}+C$$

7. $\mathbf{x\sin^{-1}x}$

Ans:

Let $I=\int x\sin^{-1}x dx$

Consider $u=\sin^{-1}x$ and $v=x$ and integrating by parts, to obtain

$$I=\sin^{-1}x\int xdx\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int xdx\right\}dx$$

$$=\sin^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

Adding and subtracting by 1

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

Simplifying,

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\int\sqrt{1-x^2}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right\}$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x-\sin^{-1}x\right\}+C$$

Simplifying,

$$=\frac{x^2\sin^{-1}x}{2}+\frac{x}{4}\sqrt{1-x^2}+\frac{1}{4}\sin^{-1}x-\frac{1}{2}\sin^{-1}x+C=\frac{1}{4}\left(2x^2-1\right)in^{-1}x+\frac{x}{4}\sqrt{1-x^2}$$

$+C$

8. $\mathbf{x\tan^{-1}x}$

Ans:

Let $I=\int x\tan^{-1}x dx$

Consider $u=\tan^{-1}x$ and $v=x$ and integrating by parts, to obtain

$$I=\tan^{-1}x\int xdx-\int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int xdx\right\}dx$$

$$=\tan^{-1}x\left(\frac{x^2}{2}\right)\int\frac{1}{1+x^2}\cdot\frac{x^2}{2}dx=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

Adding and subtracting by -1

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\left(\frac{x^2+1}{1+x^2}-\frac{1}{1+x^2}\right)dx=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

Simplifying,

$$=\frac{x^2\,lan^{-1}x}{2}-\frac{1}{2}\left(x-\tan^{-1}x\right)+C=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+C$$

9. $\mathbf{x\cos^{-1}x}$

Ans:

Let $I=\int x\cos^{-1}x\,dx$

Taking $u=\cos^{-1}x$ and $v=x$ and integrating by parts, to obtain

$$I=\cos^{-1}x\int xdx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int xdx\right\}dx$$

$$=\cos^{-1}x\,\frac{x^2}{2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

Adding and subtracting by -1

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

Simplifying,

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\sqrt{1-x^2}+\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\sqrt{1-x^2}dx-\frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}I_1-\frac{1}{2}\cos^{-1}x....(1)$$

Where $I_1=\int\sqrt{1-x^2}dx$

$$\Rightarrow I_1=x\int\sqrt{1-x^2}-\int\frac{d}{dx}\sqrt{1-x^2}\int xdx\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx\Rightarrow I_1=x\sqrt{1-x^2}-\left\{\int\sqrt{1-x^2}dx+\int\frac{-dx}{\sqrt{1-x^2}}\right\}$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{I_1+\cos^{-1}x\right\}\Rightarrow 2I_1=x\sqrt{1-x^2}-\cos^{-1}x$$

$$\therefore I_1=\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x$$

Substituting in (1), we get

$$I=\frac{x\cos^{-1}x}{2}-\frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x\right)-\frac{1}{2}\cos^{-1}x$$

Simplifying,

$$=\frac{\left(2x^2-1\right)}{4}\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C$$

10. $\left(\sin^{-1}x\right)^2$

Ans:

Let $I=\int\left(\sin^{-1}x\right)^{2}\cdot 1dx$

Consider $u=\left(\sin^{-1}x\right)^{2}$ and $v=1$ and integrating by parts, to obtain

$$I=\int\left(\sin^{-1}x\right)\cdot\int 1dx-\int\left\{\frac{d}{dx}\left(\sin^{-1}x\right)^{2}\cdot\int 1.dx\right\}dx$$

$$=\left(\sin^{-1}x\right)^{2}x-\int\frac{2\sin^{-1}x}{\sqrt{1-x^{2}}}\cdot xdx$$

$$=x\left(\sin^{-1}x\right)^{2}+\int\sin^{-1}x\cdot\left(\frac{-2x}{\sqrt{1-x^{2}}}\right)dx$$

$$=x\left(\sin^{-1}x\right)^{2}+\left[\sin^{-1}x\int\frac{-2x}{\sqrt{1-x^{2}}}dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^{2}}}dx\right\}dx\right]$$

$$=x\left(\sin^{-1}x\right)^{2}+\left[\sin^{-1}x\cdot 2\sqrt{1-x^{2}}-\int\frac{1}{\sqrt{1-x^{2}}}\cdot 2\sqrt{1-x^{2}}dx\right]$$

$$=x\left(\sin^{-1}x\right)^{2}+2\sqrt{1-x^{2}}\sin^{-1}x-\int 2dx$$

$$=x\left(\sin^{-1}x\right)^{2}+2\sqrt{1-x^{2}}\sin^{-1}x-2x+C$$

11. $\mathbf{\dfrac{x\cos^{-1}x}{\sqrt{1-x^{2}}}}$

Ans:

Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^{2}}}dx$

Multiplying and dividing by 2

$$I=\frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^{2}}}\cdot\cos^{-1}xdx$$

Consider $u=\cos^{-1}x$ and $v=\left(\frac{-2x}{\sqrt{1-x^{2}}}\right)$ and integrating by parts, to obtain

$$I=\frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^{2}}}dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^{2}}}dx\right\}dx\right]$$

$$=\frac{-1}{2}\left[\cos^{-1}x\cdot 2\sqrt{1-x^{2}}-\int\frac{-1}{\sqrt{1-x^{2}}}\cdot 2\sqrt{1-x^{2}}dx\right]=\frac{-1}{2}\left[2\sqrt{1-x^{2}}\cos^{-1}x+\int 2dx\right]$$

Simplifying,

$$=\frac{-1}{2}\left[2\sqrt{1-x^{2}}\cos^{-1}x+2x\right]+C$$

$$= -\left[\sqrt{1-x^2}\cos^{-1}x + x\right] + C$$

12. $\mathbf{x\sec^2 x}$

Ans:

Let $I = \int x\sec^2 x dx$

Consider $u = x$ and $v = \sec^2 x$ and integrating by parts, to obtain

$$I = x\int \sec^2 x dx - \int\left\{\left\{\frac{d}{dx}x\right\}\int \sec^2 x dx\right\}dx$$

$$= x\tan x - \int 1\cdot \tan x dx$$

$$= x\tan x + \log|\cos x| + C$$

13. $\mathbf{\tan^{-1} x}$

Ans:

Let $I = \int 1\cdot \tan^{-1} x dx$

Consider $u = \tan^{-1} x$ and $v = 1$ and integrating by parts, to obtain

$$I = \tan^{-1}x\int 1 dx - \int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int 1.dx\right\}dx = \tan^{-1}x x - \int \frac{1}{1+x^2} xd$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log\left|1+x^2\right| + C$$

$$= x\tan^{-1}x - \frac{1}{2}\log\left(1+x^2\right) + C$$

14. $\mathbf{x(\log x)^2 dx}$

Ans:

$I = \int x(\log x)^2 dx$

Consider $u = (\log x)^2$ and $v = 1$ and integrating by parts, to obtain

$$I = (\log)^2\int x dx - \int\left[\left\{\left(\frac{d}{dx}\log x\right)^2\right\}\int x dx\right]dx$$

$$= \frac{x^2}{2}(\log x)^2 - \left[\int 2\log x\cdot\frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \int x\log x dx$$

Again using integration by parts, to obtain

$$I=\frac{x^2}{2}(\log x)^2-\left[\log x\int xdx-\int\left\{\left(\frac{d}{dx}\log x\right)\int xdx\right\}dx\right]$$

$$=\frac{x^2}{2}(\log x)^2-\left[\frac{x^2}{2}-\log x-\int\frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$

$$=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{1}{2}\int xdx=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{x^2}{4}+C$$

15. $\left(\mathbf{x^2+1}\right)\mathbf{\log x}$

Ans:

Let $I=\int\left(x^2+1\right)\log xdx=\int x^2\log xdx+\int \log xdx$

Let $I=I_1+I_2\ldots(1)$

Where, $I_1=\int x^2\log xdx$ and $I_2=\int \log xdx$

$I_1=\int x^2\log xdx$

Consider $u=\log x$ and $v=x^2$ and integrating by parts, to obtain

$$I_1=\log x-\int x^2dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x^2dx\right\}dx$$

$$=\log x\cdot\frac{x^3}{3}-\int\frac{1}{x}\cdot\frac{x^3}{3}dx=\frac{x^3}{3}\log x-\frac{1}{3}\int x^2dx$$

$$=\frac{x^3}{3}\log x-\frac{x^3}{9}+C_1\quad\ldots(2)$$

$I_2=\int \log xdx$

Consider $u=\log x$ and $v=1$ and integrating by parts, to obtain

$$I_2=\log x\int 1.dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int 1.dx\right\}$$

$$=\log x\cdot x-\int\frac{1}{x}xdx$$

$$=x\log x-x..(3)$$

Using equations (2) and (3) in (1), we get

$$I=\frac{x^3}{3}\log x-\frac{x^3}{9}+C_1+x\log x-x+C_2$$

$$=\frac{x^3}{3}\log x-\frac{x^3}{9}+x\log x-x+\left(C_1+C_2\right)$$

$$=\left(\frac{x^3}{3}+x\right)\log x-\frac{x^3}{9}-x+C$$

16. $\mathbf{e^x(\sin x+\cos x)}$

Ans:

$$\text{Consider}\, I=\int e^x(\sin x+\cos x)dx$$

$$\text{Consider}\, f(x)=\sin x$$

$$f'(x)=\cos x$$

$$I=\int e^x\{f(x)+f'(x)\}dx$$

$$\text{Since, }\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$$

$$\therefore I=e^x\sin x+C$$

17. $\mathbf{\dfrac{xe^x}{(1+x)^2}}$

Ans:

$$\text{Consider }\ I=\int\frac{xe^x}{(1+x)^2}dx=\int e^x\left\{\frac{x}{(1+x)^2}\right\}dx$$

$$=\int e^x\left\{\frac{1+x-1}{(1+x)^2}\right\}dx=\int e^x\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}dx$$

$$\text{Here, }\ f(x)=\frac{1}{1+x}\qquad f'(x)=\frac{-1}{(1+x)^2}$$

$$\Rightarrow\int\frac{xe^x}{(1+x)^2}dx=\int e^x\{f(x)+f'(x)\}dx$$

$$\text{Since, }\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$$

$$\therefore\int\frac{xe^x}{(1+x)^2}dx=\frac{e^x}{1+x}+C$$

18. **Integrate the function -** $\mathbf{e^x\left(\dfrac{1+\sin x}{1+\cos x}\right)}$

Ans: First simplify $-e^x\left(\dfrac{1+\sin x}{1+\cos x}\right)$

It is known that –

$$1+\sin x=\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$1+\cos x = 2\cos^2\frac{x}{2}$$

$$\therefore e^x\left(\frac{1+\sin x}{1+\cos x}\right) = e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$= e^x\left(\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}\right)$$

$$= \frac{1}{2}e^x\left(\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{\cos^2\frac{x}{2}}\right)$$

$$= \frac{1}{2}e^x\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}e^x\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}e^x\left(\tan\frac{x}{2}+1\right)^2$$

$$= \frac{1}{2}e^x\left(\tan^2\frac{x}{2}+1+2\tan\frac{x}{2}\right)$$

But, $1+\tan^2\frac{x}{2} = \sec^2\frac{x}{2}$

$$= \frac{1}{2}e^x\left(\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right)$$

$$= e^x\left(\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)$$

$$\Rightarrow e^x\left(\frac{1+\sin x}{1+\cos x}\right) = e^x\left(\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)$$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

If we say, $f(x) = \tan\frac{x}{2} \Rightarrow f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$

Thus, we get – $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx = e^x \tan\frac{x}{2} + C$

19. Integrate the function - $e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$

Ans: Say, $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

Suppose, $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

Thus, we get – $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \frac{e^x}{x} + C$

20. Integrate the function - $\frac{(x-3)e^x}{(x-1)^3}$

Ans: $\int e^x \frac{(x-3)}{(x-1)^3} dx = \int e^x \left[\frac{(x-1-2)}{(x-1)^3}\right] dx$

$= \int e^x \left[\frac{(x-1)}{(x-1)^3} - \frac{2}{(x-1)^3}\right] dx$

$= \int e^x \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right] dx$

Suppose, $f(x) = \frac{1}{(x-1)^2} \Rightarrow f'(x) = -\frac{2}{(x-1)^3}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

Thus, $\int e^x \frac{(x-3)}{(x-1)^3} dx = \frac{e^x}{(x-1)^2} + C$

21. Integrate the function - $e^{2x}\sin x$

Ans: Say, $I = \int e^{2x} \sin x dx$

Perform Integration by parts – $\int uv dx = u\int v dx - \int\left(u'\int v dx\right) dx$

With $-u=\sin x \quad v=e^{2x}$

$$I=\int e^{2x}\sin x dx=\sin x\int e^{2x}dx-\int\left[\left(\frac{d}{dx}\sin x\right)\int e^{2x}dx\right]dx$$

$$=\sin x\frac{e^{2x}}{2}-\int\left[(\cos x)\frac{e^{2x}}{2}\right]dx$$

$$=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\int\left(e^{2x}\cos x\right)dx$$

Perform Integration by parts for – $\int\left(e^{2x}\cos x\right)dx$

$$=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\left\{\cos x\int e^{2x}dx-\int\left[\left(\frac{d}{dx}\cos x\right)\int e^{2x}dx\right]dx\right\}$$

$$=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\left\{\cos x\frac{e^{2x}}{2}-\int\left[(-\sin x)\frac{e^{2x}}{2}\right]dx\right\}$$

$$=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\left\{\cos x\frac{e^{2x}}{2}+\frac{1}{2}\int(\sin x)e^{2x}dx\right\}$$

$$=\sin x\frac{e^{2x}}{2}-\frac{e^{2x}\cos x}{4}-\frac{1}{4}\left\{\int(\sin x)e^{2x}dx\right\}$$

But, $I=\int e^{2x}\sin x dx$

$$\Rightarrow I=\sin x\frac{e^{2x}}{2}-\frac{e^{2x}\cos x}{4}-\frac{1}{4}I$$

$$\Rightarrow I+\frac{1}{4}I=\sin x\frac{e^{2x}}{2}-\frac{e^{2x}\cos x}{4}$$

$$\Rightarrow \frac{5}{4}I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}$$

$$\Rightarrow \frac{5}{4}I=\frac{2e^{2x}\sin x}{4}-\frac{e^{2x}\cos x}{4}$$

$$\Rightarrow 5I=e^{2x}(2\sin x-\cos x)$$

Thus, we get – $I=\frac{e^{2x}}{5}(2\sin x-\cos x)+C$

22. Integrate the function - $\sin^{-1}\left(\frac{2x}{1+x^3}\right)$

Ans: Say, $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^3}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^3\theta}\right)$$

But, $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^3\theta}$

$\Rightarrow \sin^{-1}\left(\dfrac{2x}{1+x^3}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^3\theta}\right) = \therefore \sin^{-1}\left(\dfrac{2x}{1+x^3}\right) = \sin^{-1}\left(\sin 2\theta\right) = 2\theta$

Therefore, $\int \sin^{-1}\left(\dfrac{2x}{1+x^3}\right)dx = \int 2\theta\sec^2\theta d\theta$

$= 2\int \theta\sec^2\theta d\theta$

Perform Integration by parts – $\int uvdx = u\int vdx - \int\left(u'\int vdx\right)dx$

With $-u = \theta \quad v = \sec^2\theta$

$2\int \theta\sec^2\theta d\theta = 2\left\{\theta\int \sec^2\theta d\theta - \int\left[\left(\dfrac{d}{d\theta}\theta\right)\int \sec^2\theta d\theta\right]d\theta\right\}$

$= 2\left\{\theta\tan\theta - \int\left[\tan\theta\right]d\theta\right\}$

$= 2\left\{\theta\tan\theta - (-\log|\cos\theta|)\right\} + C$

$= 2\left\{\theta\tan\theta + \log|\cos\theta|\right\} + C$

Replace $\theta = \tan^{-1}x$

$= 2\left\{\tan^{-1}x\tan(\tan^{-1}x) + \log|\cos(\tan^{-1}x)|\right\} + C$

It is known that – $\tan^{-1}x = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$

$= 2\left\{\tan^{-1}x(x) + \log|\cos(\cos^{-1}\dfrac{1}{\sqrt{1+x^2}})|\right\} + C$

$= 2\left\{x\tan^{-1}x + \log|\dfrac{1}{\sqrt{1+x^2}}|\right\} + C$

$= 2\left\{x\tan^{-1}x + \log(1+x^2)^{-\frac{1}{2}}\right\} + C$

Here, $\log m^n = n\log m$

$= 2\left\{x\tan^{-1}x - \dfrac{1}{2}\log(1+x^2)\right\} + C$

$= 2x\tan^{-1}x - \log(1+x^2) + C$

Thus, $\int \sin^{-1}\left(\dfrac{2x}{1+x^3}\right)dx = 2x\tan^{-1}x - \log(1+x^2) + C$

23. Choose the correct answer: $\int x^2 e^{x^3} dx$ equals

(a) $\frac{1}{3}e^{x^3}+C$

(b) $\frac{1}{3}e^{x^2}+C$

(c) $\frac{1}{2}e^{x^3}+C$

(d) $\frac{1}{2}e^{x^2}+C$

Ans: Say, $I=\int x^2e^{x^3}dx$

Suppose, $t=x^3 \Rightarrow dt=3x^2dx$

Rewriting the equation – $I=\int x^2e^{x^3}dx=\frac{1}{3}\int e^t dt$

$\Rightarrow I=\frac{1}{3}\int e^t dt=\frac{1}{3}e^t+C$

Replacing $t=x^3$

$\Rightarrow I=\frac{1}{3}e^{x^3}+C$

The correct option is A.

24. Choose the correct answer: $\int e^x \sec x(1+\tan x)dx$

(a) $e^x\cos x+C$
(b) $e^x\sec x+C$
(c) $e^x\sin x+C$
(d) $e^x\tan x+C$

Ans: Say, $I=\int e^x\sec x(1+\tan x)dx$

$\Rightarrow I=\int e^x(\sec x+\sec x\tan x)dx$

Suppose, $f(x)=\sec x \Rightarrow f'(x)=\sec x\tan x$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^xf(x)+C$

$\Rightarrow I=\int e^x(\sec x+\sec x\tan x)dx=e^x\sec x+C$

Thus, $I=e^x\sec x+C$

The correct option is B.

Exercise 7.7

1. Integrate the function - $\sqrt{4-x^2}$

Ans: Say, $I=\int\sqrt{4-x^2}dx=\int\sqrt{2^2-x^2}dx$

It is known that $-\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\Rightarrow I=\int\sqrt{2^2-x^2}dx=\frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}+C$

$\Rightarrow I=\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$

Thus, $\int\sqrt{4-x^2}dx=\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$

2. Integrate the function - $\sqrt{1-4x^2}$

Ans: Say, $I=\int\sqrt{1-4x^2}dx=\int\sqrt{1^2-(2x)^2}dx$

Let, $2x=t\Rightarrow 2dx=dt$

$x=\frac{t}{2}\Rightarrow dx=\frac{dt}{2}$

So, we get $-\ I=\int\sqrt{1^2-\left[2(\frac{t}{2})\right]^2}\frac{dt}{2}=\frac{1}{2}\int\sqrt{1^2-[t]^2}dt$

$\Rightarrow I=\frac{1}{2}\int\sqrt{1^2-[t]^2}dt$

It is known that $-\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\Rightarrow I=\frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+C$

$\Rightarrow I=\left[\frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t\right]+C$

Replace $-\ t=2x$

$\Rightarrow I=\left[\frac{2x}{4}\sqrt{1-(2x)^2}+\frac{1}{4}\sin^{-1}2x\right]+C$

$\Rightarrow I=\left[\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x\right]+C$

Thus, $\int\sqrt{1-4x^2}dx=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$

3. Integrate the function - $\sqrt{x^2+4x+6}$

Ans: First simplify $-x^2+4x+6$

$x^2+4x+6=x^2+4x+4+2$

$=(x^2+4x+4)+2=(x+2)^2+(\sqrt{2})^2$

$\Rightarrow \sqrt{x^2+4x+6}=\sqrt{(x+2)^2+(\sqrt{2})^2}$

$\therefore \int\sqrt{x^2+4x+6}dx=\int\sqrt{(x+2)^2+(\sqrt{2})^2}dx$

It is known that – $\int\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$\Rightarrow \int\sqrt{(x+2)^2+(\sqrt{2})^2}dx=\frac{x+2}{2}\sqrt{(x+2)^2+(\sqrt{2})^2}$

$+\frac{(\sqrt{2})^2}{2}\log\left|(x+2)+\sqrt{(x+2)^2+(\sqrt{2})^2}\right|+C$

$=\frac{x+2}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

Thus, $\int\sqrt{x^2+4x+6}dx=\frac{x+2}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

4. Integrate the function - $\sqrt{x^2+4x+1}$

Ans: First simplify $-x^2+4x+1$

$x^2+4x+1=x^2+4x+4-3$

$=(x^2+4x+4)-3=(x+2)^2-(\sqrt{3})^2$

$\Rightarrow \sqrt{x^2+4x+1}=\sqrt{(x+2)^2-(\sqrt{3})^2}$

$\therefore \int\sqrt{x^2+4x+1}dx=\int\sqrt{(x+2)^2-(\sqrt{3})^2}dx$

It is known that – $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\Rightarrow \int\sqrt{(x+2)^2-(\sqrt{3})^2}dx=\frac{x+2}{2}\sqrt{(x+2)^2-(\sqrt{3})^2}$

$-\frac{(\sqrt{3})^2}{2}\log\left|(x+2)+\sqrt{(x+2)^2-(\sqrt{3})^2}\right|+C$

$=\frac{x+2}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$

Thus, $\int\sqrt{x^2+4x+1}dx=\frac{x+2}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$

5. Integrate the function - $\sqrt{1-4x-x^2}$

Ans: First simplify – $1-4x-x^2$

$1-4x-x^2=1-4x-x^2-4+4=1+4-(x^2+4x+4)$

$=5-(x^2+4x+4)=(\sqrt{5})^2-(x+2)^2$

$\Rightarrow\sqrt{1-4x-x^2}=\sqrt{(\sqrt{5})^2-(x+2)^2}$

$\therefore\int\sqrt{1-4x-x^2}dx=\int\sqrt{(\sqrt{5})^2-(x+2)^2}dx$

It is known that – $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\Rightarrow\int\sqrt{(\sqrt{5})^2-(x+2)^2}dx=\frac{x+2}{2}\sqrt{(\sqrt{5})^2-(x+2)^2}+\frac{(\sqrt{5})^2}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+C$

$=\frac{x+2}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+C$

Thus, $\int\sqrt{1-4x-x^2}dx=\frac{x+2}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+C$

6. Integrate the function - $\sqrt{x^2+4x-5}$

Ans: First simplify – x^2+4x-5

$x^2+4x-5=x^2+4x-5+4-4=(x^2+4x+4)-5-4$

$=(x^2+4x+4)-9=(x+2)^2-(3)^2$

$\Rightarrow\sqrt{x^2+4x-5}=\sqrt{(x+2)^2-(3)^2}$

$\therefore\int\sqrt{x^2+4x-5}dx=\int\sqrt{(x+2)^2-(3)^2}dx$

It is known that – $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\Rightarrow\int\sqrt{(x+2)^2-(3)^2}dx=\frac{x+2}{2}\sqrt{(x+2)^2-(3)^2}-\frac{(3)^2}{2}\log\left|(x+2)+\sqrt{(x+2)^2-(3)^2}\right|+C$

$=\frac{x+2}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$

Thus, $\int\sqrt{x^2+4x-5}dx=\frac{x+2}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$

7. Integrate the function - $\sqrt{1+3x-x^2}$

Ans: First simplify – $1+3x-x^2$

$1+3x-x^2=1-x^2+3x+\frac{9}{4}-\frac{9}{4}=1+\frac{9}{4}-(x^2-3x+\frac{9}{4})$

$$= \frac{9+4}{4} - (x^2 - 3x + \frac{9}{4}) = \left(\frac{13}{4}\right) - (x^2 - 3x + \frac{9}{4}) = \left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2$$

$$\Rightarrow \sqrt{1 + 3x - x^2} = \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}$$

$$\therefore \int \sqrt{1 + 3x - x^2}\,dx = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}\,dx$$

It is known that – $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$$\Rightarrow \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}\,dx = \frac{\left(x - \frac{3}{2}\right)}{2}\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}$$

$$+ \frac{\left(\frac{\sqrt{13}}{2}\right)^2}{2}\sin^{-1}\frac{\left(x - \frac{3}{2}\right)}{\left(\frac{\sqrt{13}}{2}\right)} + C$$

$$= \frac{2x - 3}{4}\sqrt{1 + 3x - x^2} + \frac{13}{8}\sin^{-1}\frac{2x - 3}{\sqrt{13}} + C$$

Thus, $\int \sqrt{1 + 3x - x^2}\,dx = \frac{2x - 3}{4}\sqrt{1 + 3x - x^2} + \frac{13}{8}\sin^{-1}\frac{2x - 3}{\sqrt{13}} + C$

8. **Integrate the function - $\sqrt{x^2 + 3x}$**

Ans: First simplify $-x^2 + 3x$

$$x^2 + x = x^2 + 3x + \frac{9}{4} - \frac{9}{4} = (x^2 + 3x + \frac{9}{4}) - \frac{9}{4}$$

$$= \left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{4}\right) = \left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

$$\Rightarrow \sqrt{x^2 + 3x} = \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$$

$$\therefore \int \sqrt{x^2 + 3x}\,dx = \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$$

It is known that – $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\Rightarrow\int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}dx=\frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}$$

$$-\frac{\left(\frac{3}{2}\right)^2}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\right|+C$$

$$=\frac{2x+3}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

Thus, $\int\sqrt{x^2+3x}dx=\frac{2x+3}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$

9. Integrate the function - $\sqrt{1+\frac{x^2}{9}}$

Ans: First simplify $-1+\frac{x^2}{9}$

$$1+\frac{x^2}{9}=\frac{1}{9}(9+x^2)=\frac{1}{9}(3^2+x^2)$$

$$\Rightarrow\sqrt{1+\frac{x^2}{9}}=\sqrt{\frac{1}{9}(3^2+x^2)}=\frac{1}{3}\sqrt{(3^2+x^2)}$$

$$\therefore\int\sqrt{1+\frac{x^2}{9}}dx=\int\frac{1}{3}\sqrt{(3^2+x^2)}dx=\frac{1}{3}\int\sqrt{(3^2+x^2)}dx$$

It is known that – $\int\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$\Rightarrow\frac{1}{3}\int\sqrt{(3^2+x^2)}dx=\frac{1}{3}\left\{\frac{x}{2}\sqrt{(x)^2+(3)^2}+\frac{(3)^2}{2}\log\left|(x)+\sqrt{(x)^2+(3)^2}\right|\right\}+C$$

$$=\frac{1}{3}\left\{\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right\}+C$$

$$=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$$

Thus, $\int\sqrt{1+\frac{x^2}{9}}dx=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$

10. Choose the correct answer: $\int \sqrt{1+x^2}dx$ **is equal to –**

(a) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|(x+\sqrt{1+x^2})\right|+C$

(b) $\frac{2}{3}(1+x^2)^{\frac{3}{2}}+C$

(c) $\frac{2}{3}x(1+x^2)^{\frac{3}{2}}+C$

(d) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|(x+\sqrt{1+x^2})\right|+C$

Ans: It is known that – $\int \sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

Thus, $\int \sqrt{x^2+1^2}dx=\frac{x}{2}\sqrt{x^2+1^2}+\frac{1^2}{2}\log\left|x+\sqrt{x^2+1^2}\right|+C$

$\int \sqrt{x^2+1}dx=\frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\log\left|x+\sqrt{x^2+1}\right|+C$

The correct answer is option A.

11. Choose the correct answer: $\int \sqrt{x^2-8x+7}dx$ **is equal to –**

(a) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|(x-4+\sqrt{x^2-8x+7})\right|+C$

(b) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|(x+4+\sqrt{x^2-8x+7})\right|+C$

(c) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|(x-4+\sqrt{x^2-8x+7})\right|+C$

(d) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+\frac{9}{2}\log\left|(x-4+\sqrt{x^2-8x+7})\right|+C$

Ans: First simplify $-x^2-8x+7$

$x^2-8x+7+9-9=x^2-8x+16-9=(x^2-8x+16)-9$

$=(x-4)^2-(3)^2$

$\Rightarrow \sqrt{x^2-8x+7}=\sqrt{(x-4)^2-(3)^2}$

$\therefore \int \sqrt{x^2-8x+7}dx=\int \sqrt{(x-4)^2-(3)^2}dx$

It is known that – $\int \sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\Rightarrow \int \sqrt{(x-4)^2-(3)^2}dx=\frac{(x-4)}{2}\sqrt{(x-4)^2-(3)^2}-\frac{(3)^2}{2}\log\left|(x-4)+\sqrt{(x-4)^2-(3)^2}\right|+C$

$=\frac{x-4}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$

Thus, $\int \sqrt{x^2-8x+7}dx=\frac{x-4}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$

The correct answer is option D

Exercise 7.8

1. **Evaluate the definite integral–** $\int_{-1}^{1}(x+1)dx$

Ans: The second fundamental theorem of integral calculus states that – $\int_a^b f(x)dx = F(b)-F(a)$

Here, $\int (x+1)dx = \frac{x^2}{2}+x$

So, $\int_{-1}^{1}(x+1)dx = \left[\frac{x^2}{2}+x\right]_{-1}^{1}$

$= \left[\frac{1^2}{2}+1\right]-\left[\frac{(-1)^2}{2}+(-1)\right]$

$= \left[\frac{1}{2}+1\right]-\left[\frac{1}{2}-1\right]$

$= \frac{1}{2}+1-\frac{1}{2}+1 = 2$

Thus, $\int_{-1}^{1}(x+1)dx = 2$

2. **Evaluate the definite integral–** $\int_2^3 \frac{1}{x}dx$

Ans: The second fundamental theorem of integral calculus states that – $\int_a^b f(x)dx = F(b)-F(a)$

Here, $\int \frac{1}{x}dx = \log|x|$

So, $\int_2^3 \frac{1}{x}dx = \left[\log|x|\right]_2^3$

$= \left[\log|3|\right]-\left[\log|2|\right]$

$= \log\frac{3}{2}$

Thus, $\int_2^3 \frac{1}{x}dx = \log\frac{3}{2}$

3. **Evaluate the definite integral–** $\int_1^2 (4x^3-5x^2+6x+9)dx$

Ans: The second fundamental theorem of integral calculus states that – $\int_a^b f(x)dx = F(b)-F(a)$

Here, $\int (4x^3 - 5x^2 + 6x + 9)dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9x$

$= x^4 - \frac{5x^3}{3} + 3x^2 + 9x$

So, $\int_1^2 (4x^3 - 5x^2 + 6x + 9)dx = \left[x^4 - \frac{5x^3}{3} + 3x^2 + 9x\right]_1^2$

$= \left[2^4 - \frac{5(2)^3}{3} + 3(2)^2 + 9(2)\right] - \left[1^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right]$

$= \left[16 - \frac{40}{3} + 12 + 18\right] - \left[1 - \frac{5}{3} + 3 + 9\right]$

$= \left[46 - \frac{40}{3}\right] - \left[13 - \frac{5}{3}\right]$

$= 46 - \frac{40}{3} - 13 + \frac{5}{3}$

$= 33 - \frac{35}{3}$

$= \frac{99 - 35}{3}$

$= \frac{64}{3}$

Thus, $\int_1^2 (4x^3 - 5x^2 + 6x + 9)dx = \frac{64}{3}$

4. Evaluate the definite integral– $\int_0^{\frac{\pi}{4}} \sin 2x dx$

Ans: The second fundamental theorem of integral calculus states that – $\int_a^b f(x)dx = F(b) - F(a)$

Here, $\int \sin 2x dx = \frac{-\cos 2x}{2}$

So, $\int_0^{\frac{\pi}{4}} \sin 2x dx = \left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{-\cos 2\left(\frac{\pi}{4}\right)}{2}\right] - \left[\frac{-\cos 0}{2}\right]$

$$= \left[\frac{-\cos\left(\frac{\pi}{2}\right)}{2} \right] + \left[\frac{1}{2} \right]$$

$$= \left[\frac{0}{2} \right] + \left[\frac{1}{2} \right]$$

$$= \frac{1}{2}$$

Thus, $\int_0^{\frac{\pi}{4}} \sin 2x dx = \frac{1}{2}$

5. Evaluate the definite integral– $\int_0^{\frac{\pi}{2}} \cos 2x dx$

Ans: The second fundamental theorem of integral calculus states that – $\int_a^b f(x) dx = F(b) - F(a)$

Here, $\int \cos 2x dx = \frac{\sin 2x}{2}$

So, $\int_0^{\frac{\pi}{2}} \cos 2x dx = \left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$

$$= \left[\frac{\sin 2\left(\frac{\pi}{2}\right)}{2} \right] - \left[\frac{\sin 0}{2} \right]$$

$$= \left[\frac{\sin \pi}{2} \right] + \left[\frac{0}{2} \right]$$

$$= 0 + 0$$

$$= 0$$

Thus, $\int_0^{\frac{\pi}{2}} \cos 2x dx = 0$

6. Evaluate the definite integral– $\int_4^5 e^x dx$

Ans: The second fundamental theorem of integral calculus states that –

$$\int_a^b f(x)dx = F(b) - F(a)$$

Here, $\int e^x dx = e^x$

So, $\int_4^5 e^x dx = \left[e^x\right]_4^5$

$= \left[e^5\right] - \left[e^4\right]$

$= e^4(e-1)$

Thus, $\int_4^5 e^x dx = e^4(e-1)$

7. $\int_0^{\frac{\pi}{4}} \tan x dx$

Ans: We know that,

$\int \tan x dx = -\log|\cos x| + C$

Therefore, by second fundamental theorem of calculus

$$\int_0^{\frac{\pi}{4}} \tan x dx = \left[-\log|\cos x|\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \tan x dx = \left[-\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \tan x dx = \left[-\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|\right]$$

$$\therefore \int_0^{\frac{\pi}{4}} \tan x dx = \frac{1}{2}\log 2$$

8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x dx$

Ans: We know that,

$\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

Therefore, by second fundamental theorem of calculus

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\,dx = \left[\log\left|\operatorname{cosec} x - \cot x\right|\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$\Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\,dx = \left[\log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|\right]$$

$$\Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\,dx = \left[\log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|\right]$$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\,dx = \log\left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right)$$

9. $\displaystyle\int_0^1 \frac{1}{\sqrt{1 - x^2}}\,dx$

Ans: We know that,

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1} x + C$$

Therefore, by second fundamental theorem of calculus

$$\int_0^1 \frac{1}{\sqrt{1 - x^2}}\,dx = \left[\sin^{-1} x\right]_0^1$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1 - x^2}}\,dx = \left[\sin^{-1} 1 - \sin^{-1} 0\right]$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1 - x^2}}\,dx = \left[\frac{\pi}{2} - 0\right]$$

$$\therefore \int_0^1 \frac{1}{\sqrt{1 - x^2}}\,dx = \frac{\pi}{2}$$

10. $\displaystyle\int_0^1 \frac{1}{1 + x^2}\,dx$

Ans: We know that,

$$\int \frac{1}{1 + x^2}\,dx = \tan^{-1} x + C$$

Therefore, by second fundamental theorem of calculus

$$\int_0^1 \frac{1}{1 + x^2}\,dx = \left[\tan^{-1} x\right]_0^1$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx = \left[\frac{\pi}{4} - 0\right]$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}$$

11. $\displaystyle\int_2^3 \frac{1}{x^2-1}dx$

Ans: We know that,

$$\int \frac{1}{x^2-1}dx = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

Therefore, by second fundamental theorem of calculus

$$\int_2^3 \frac{1}{x^2-1}dx = \frac{1}{2}\left[\log\left|\frac{x-1}{x+1}\right|\right]_2^3$$

$$\Rightarrow \int_2^3 \frac{1}{x^2-1}dx = \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$$

$$\Rightarrow \int_2^3 \frac{1}{x^2-1}dx = \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$$

$$\therefore \int_2^3 \frac{1}{x^2-1}dx = \frac{1}{2}\log\frac{3}{2}$$

12. $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x dx$

Ans: We know that,

$$\int \cos^2 x dx = \int\left(\frac{1+\cos 2x}{2}\right)dx$$

$$\Rightarrow \int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

Therefore, by second fundamental theorem of calculus

$$\int_0^{\frac{\pi}{4}} \cos^2 x dx = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \cos^2 x dx = \frac{1}{2}\left[\frac{\pi}{2} - \frac{\sin\pi}{2} - 0 - \frac{\sin 0}{2}\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \cos^2 x dx = \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right]$$

$$\therefore \int_0^{\frac{\pi}{4}} \cos^2 x dx = \frac{\pi}{4}$$

13. $\displaystyle \int_2^3 \frac{xdx}{x^2+1}$

Ans: We know that,

$$\int \frac{xdx}{x^2+1} = \frac{1}{2}\int\left(\frac{2x}{x^2+1}\right)dx$$

$$\Rightarrow \int \frac{xdx}{x^2+1} = \frac{1}{2}\log\left(1+x^2\right)$$

Therefore, by second fundamental theorem of calculus

$$\int_2^3 \frac{xdx}{x^2+1} = \left[\log\left(1+3^2\right) - \log\left(1+2^2\right)\right]_2^3$$

$$\Rightarrow \int_2^3 \frac{xdx}{x^2+1} = \frac{1}{2}\left[\log 10 - \log 5\right]$$

$$\Rightarrow \int_2^3 \frac{xdx}{x^2+1} = \frac{1}{2}\log\frac{10}{5}$$

$$\therefore \int_2^3 \frac{xdx}{x^2+1} = \frac{1}{4}\log 2$$

14. $\displaystyle \int_0^1 \frac{2x+3}{5x^2+1}dx$

Ans: Solving $\displaystyle \int \frac{2x+3}{5x^2+1}dx$,

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx$$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx$$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\log\left(5x^2+1\right)+\frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)x$$

Therefore, by second fundamental theorem of calculus

$$\int_0^1 \frac{2x+3}{5x^2+1}dx = \left\{\frac{1}{5}\log\left(5+1\right)+\frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)x\right\}-\left\{\frac{1}{5}\log\left(1\right)+\frac{3}{\sqrt{5}}\tan^{-1}0\right\}$$

$$\therefore \int_0^1 \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\log 6+\frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

15. $\displaystyle\int_0^1 \mathbf{xe^{x^2}dx}$

Ans: Let $x^2 = t$,

Differentiating it we get,

$2xdx = dt$

Therefore, the integral becomes,

$$\frac{1}{2}\int_0^1 e^t dt$$

$$\frac{1}{2}\int_0^1 e^t dt = \left[\frac{1}{2}e^t\right]_0^1$$

$$\frac{1}{2}\int_0^1 e^t dt = \frac{1}{2}e - \frac{1}{2}e^0$$

$$\frac{1}{2}\left(e-1\right)$$

16. $\displaystyle\int_1^2 \mathbf{\frac{5x^2}{x^2+4x+3}dx}$

Ans: The given integral can be written as

$$\int_1^2 \frac{5x^2}{x^2+4x+3}dx = \int_1^2\left\{5-\frac{20x+15}{x^2+4x+3}\right\}dx$$

$$\int_1^2 \frac{5x^2}{x^2+4x+3}dx = \left[5x\right]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx \qquad \ldots(1)$$

Solving $\displaystyle\int_1^2 \frac{20x+15}{x^2+4x+3}dx$,

Let $20x+15 = A\frac{d}{dx}\left(x^2+4x+3\right)+B$

Equating the coefficients of x and constant term we get,

$A = 10, B = -25$

Let $x^2+4x+3=t$
Differentiating it we get,
$(2x+4)dx=dt$
Therefore, the integral becomes

$$10\int\frac{dt}{t}-25\int\frac{dx}{(x+2)^2-1^2}$$

$$10\int\frac{dt}{t}-25\int\frac{dx}{(x+2)^2-1^2}=10\log t-25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right]$$

$$\Rightarrow\int_1^2\frac{20x+15}{x^2+4x+3}dx=\left[10\log\left(x^2+4x+3\right)-25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]\right]_1^2$$

$$\Rightarrow\int_1^2\frac{20x+15}{x^2+4x+3}dx=10\log 15-10\log 8-25\left[\frac{1}{2}\log\frac{3}{5}-\frac{1}{2}\log\frac{2}{4}\right]$$

$$\Rightarrow\int_1^2\frac{20x+15}{x^2+4x+3}dx=10\log 5+10\log 3-10\log 4-10\log 2$$

$$-\frac{25}{2}\left[\log 3-\log 5-\log 2+\log 4\right]$$

$$\Rightarrow\int_1^2\frac{20x+15}{x^2+4x+3}dx=\frac{45}{2}\log 5-\frac{45}{2}\log 4-\frac{5}{2}\log 3+\frac{5}{2}\log 2$$

$$\therefore\int_1^2\frac{20x+15}{x^2+4x+3}dx=\frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}$$

Substituting it in (1) we get,

$$\int_1^2\frac{5x^2}{x^2+4x+3}dx=5-\left[\frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}\right]$$

$$\therefore\int_1^2\frac{5x^2}{x^2+4x+3}dx=5-\frac{5}{2}\left[9\log\frac{5}{4}-\log\frac{3}{2}\right]$$

17. $\displaystyle\int_0^{\frac{\pi}{4}}\left(2\sec^2 x+x^3+2\right)dx$

Ans: We know that,

$$\int\left(2\sec^2 x+x^3+2\right)dx=2\tan x+\frac{x^4}{4}+2x$$

Therefore, by second fundamental theorem of calculus

$$\int\left(2\sec^2 x+x^3+2\right)dx=\left[2\tan x+\frac{x^4}{4}+2x\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow \int\left(2\sec^2 x + x^3 + 2\right)dx = \left[2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^2 + 2\left(\frac{\pi}{4}\right) - \left(2\tan 0 + 0 + 0\right)\right]$$

$$\Rightarrow \int\left(2\sec^2 x + x^3 + 2\right)dx = 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$\therefore \int\left(2\sec^2 x + x^3 + 2\right)dx = 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

18. $\displaystyle\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$

Ans: We know that,

$$\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx = -\int_0^{\pi}\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)dx$$

$$\Rightarrow -\int_0^{\pi}\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)dx = -\int_0^{\pi}\cos x\,dx$$

$$\int\cos x\,dx = \sin x + C$$

Therefore, by second fundamental theorem of calculus

$$\int_0^{\pi}\cos x\,dx = \sin\pi - \sin 0$$

$$\therefore \int_0^{\pi}\cos x\,dx = 0$$

19. $\displaystyle\int_0^{2}\frac{6x+3}{x^2+4}dx$

Ans: Solving the integral we get,

$$\int\frac{6x+3}{x^2+4}dx = 3\int\frac{2x+1}{x^2+4}dx$$

$$\int\frac{6x+3}{x^2+4}dx = 3\int\frac{2x}{x^2+4}dx + 3\int\frac{1}{x^2+4}dx$$

$$\therefore \int\frac{6x+3}{x^2+4}dx = 3\log\left(x^2+4\right) + \frac{3}{2}\tan^{-1}\frac{x}{2}$$

Therefore, by second fundamental theorem of calculus

$$\int_0^{2}\frac{6x+3}{x^2+4}dx = \left[3\log\left(x^2+4\right) + \frac{3}{2}\tan^{-1}\frac{x}{2}\right]_0^2$$

$$\Rightarrow \int_0^{2}\frac{6x+3}{x^2+4}dx = \left[3\log\left(2^2+4\right) + \frac{3}{2}\tan^{-1}\frac{2}{2} - 3\log\left(0^2+4\right) - \frac{3}{2}\tan^{-1}\frac{0}{2}\right]$$

$$\Rightarrow \int_0^2 \frac{6x+3}{x^2+4}dx = 3\log 8 + \frac{3}{2}\tan^{-1}1 - 3\log 4 - \frac{3}{2}\tan^{-1}0$$

$$\Rightarrow \int_0^2 \frac{6x+3}{x^2+4}dx = 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$$

$$\Rightarrow \int_0^2 \frac{6x+3}{x^2+4}dx = 3\log\frac{8}{4} + \frac{3\pi}{8}$$

$$\therefore \int_0^2 \frac{6x+3}{x^2+4}dx = 3\log 2 + \frac{3\pi}{8}$$

20. $\displaystyle\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$

Ans: Solving the integral we get,

$$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx = x\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx + \left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$

$$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx = xe^x - \int e^x dx - \frac{4}{\pi}\cos\frac{x}{4}$$

$$\therefore \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx = xe^x - e^x - \frac{4}{\pi}\cos\frac{x}{4}$$

Therefore, by second fundamental theorem of calculus

$$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx = \left(1e^1 - e^1 - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0e^0 - e^0 - \frac{4}{\pi}\cos 0\right)$$

$$\therefore \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx = \left(1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}\right)$$

21. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{1+x^2}dx$

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$

Ans: Solving the integral we get,

$$\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = \tan^{-1} x$$

Therefore, by second fundamental theorem of calculus

$$\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$\Rightarrow \int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = \frac{\pi}{3} - \frac{\pi}{4}$$

$$\therefore \int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = \frac{\pi}{12}$$

Thus, the correct option is (D)

22. $\displaystyle\int_0^{\frac{2}{3}} \frac{1}{4+9x^2} dx$

A. $\frac{\pi}{6}$

B. $\frac{\pi}{12}$

C. $\frac{\pi}{24}$

D. $\frac{\pi}{4}$

Ans: Solving the integral we get,

$$\int \frac{1}{4+9x^2} dx = \int \frac{1}{2^2 + (3x)^2} dx$$

Let $3x = t$,

Differentiating it we get,

$3dx = dt$

$$\therefore \int \frac{1}{2^2 + (3x)^2} dx = \frac{1}{6} \tan^{-1}\left(\frac{3x}{2}\right)$$

Therefore, by second fundamental theorem of calculus

$$\int_0^{\frac{2}{3}} \frac{1}{4+9x^2} dx = \frac{1}{6} \tan^{-1}\left(\frac{3}{2} \cdot \frac{2}{3}\right) - \frac{1}{6} \tan^{-1} 0$$

$$\int_0^{\frac{2}{3}} \frac{1}{4+9x^2} dx = \frac{1}{6} \cdot \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{2}{3}} \frac{1}{4+9x^2} dx = \frac{\pi}{24}$$

Thus, the correct answer is (C).

Exercise 7.9

Solve the following integrals.

1. $$\int_0^1 \frac{x}{x^2+1} dx$$

Ans: Let x^2+1

Differentiating $2xdx = dt$,

Therefore, the integral becomes

$$\int_0^1 \frac{x}{x^2+1} dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$\Rightarrow \int_0^1 \frac{x}{x^2+1} dx = \frac{1}{2}\left[\log|t|\right]_1^2$$

$$\Rightarrow \int_0^1 \frac{x}{x^2+1} dx = \frac{1}{2}\left[\log 2 - \log 1\right]$$

$$\Rightarrow \int_0^1 \frac{x}{x^2+1} dx = \frac{1}{2}\log 2$$

2. $$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$$

Ans: The integral can be written as:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi d\phi$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \frac{64}{231}$$

Let $\sin\phi = t$

Differentiating it we get,

$\cos\phi d\phi = dt$

Therefore, the integral becomes

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \int_0^1 \sqrt{t}\left(1+t^4-2t^2\right)dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \int_0^1 \left(t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right)dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{11}{2}}}{\frac{11}{2}}-\frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \frac{2}{3}+\frac{2}{11}-\frac{4}{7}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \frac{154+42-132}{231}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \frac{64}{231}$$

3. $\displaystyle \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Ans: Let $x=\tan\theta$

Differentiating it we get,

$dx=\sec^2\theta d\theta$

Therefore, the integral becomes

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right)\sec^2\theta d\theta$$

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2\int_0^{\frac{\pi}{4}} \theta\sec^2\theta d\theta$$

Let $u=\theta$

And $v=\sec^2\theta$

Using integration by parts we get,

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2\left[\theta\int\sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2\left[\theta\tan\theta - \int\tan\theta d\theta\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2\left[\theta\tan\theta - \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$\Rightarrow \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2\left[\frac{\pi}{4} - \log\left|\frac{1}{\sqrt{2}}\right| - \log 1\right]$$

$$\therefore \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \frac{\pi}{2} - \log 2$$

4. $\int_0^2 x\sqrt{x+2}dx$

(Put $x+2=t^2$)

Ans: Let $x+2=t^2$

Differentiating it, $dx = 2tdt$

Therefore, the integral becomes

$$\int_0^2 x\sqrt{x+2}dx = \int_{\sqrt{2}}^2 2t^2\left(t^2-2\right)dt$$

$$\Rightarrow \int_0^2 x\sqrt{x+2}dx = 2\left[\frac{t^5}{5} - \frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$\Rightarrow \int_0^2 x\sqrt{x+2}dx = 2\left[\frac{35}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3}\right]$$

$$\therefore \int_0^2 x\sqrt{x+2}dx = \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

5. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Ans: Let $\cos x = t$

Differentiating it, $-\sin x dx = dt$

Therefore, the integral becomes

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{dt}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\left[\tan^{-1} t\right]_1^0$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = \frac{\pi}{4}$$

6. $\displaystyle \int_0^2 \frac{1}{x+4-x^2}dx$

Ans: The integral can be written as

$$\int_0^2 \frac{1}{x+4-x^2}dx = \int_0^2 \frac{dx}{-\left(\left(x-\frac{1}{2}\right)^2 - \frac{17}{4}\right)}$$

$$\int_0^2 \frac{1}{x+4-x^2}dx = \int_0^2 \frac{dx}{\left(\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2\right)}$$

Let $x - \frac{1}{2} = t$

Differentiating it we get,

$dx = dt$

Therefore, the integral becomes

$$\int_0^2 \frac{1}{x+4-x^2}dx = \int_{\frac{-1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\left(\frac{\sqrt{17}}{2}\right)^2 - (t)^2\right)}$$

$$\Rightarrow \int_{\frac{-1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\left(\frac{\sqrt{17}}{2}\right)^2 - (t)^2\right)} = \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\left(\frac{\sqrt{17}}{2}\right)+t}{\left(\frac{\sqrt{17}}{2}\right)-t}\right]_{\frac{-1}{2}}^{\frac{3}{2}}$$

$$\Rightarrow \int_{\frac{-1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\left(\frac{\sqrt{17}}{2}\right)^2 - (t)^2\right)} = \frac{1}{\sqrt{17}}\left[\log \frac{\left(\frac{\sqrt{17}}{2}\right)+\frac{3}{2}}{\left(\frac{\sqrt{17}}{2}\right)-\frac{3}{2}} - \log \frac{\left(\frac{\sqrt{17}}{2}\right)-\frac{1}{2}}{\left(\frac{\sqrt{17}}{2}\right)+\frac{1}{2}}\right]$$

$$\Rightarrow \int_{\frac{-1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\left(\frac{\sqrt{17}}{2}\right)^2 - (t)^2\right)} = \frac{1}{\sqrt{17}}\left[\log \frac{\sqrt{17}+3}{\sqrt{17}-3} - \log \frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$\Rightarrow \int_{\frac{-1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\left(\frac{\sqrt{17}}{2}\right)^2 - (t)^2\right)} = \frac{1}{\sqrt{17}}\left[\log \frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}\right]$$

$$\Rightarrow \int_{\frac{-1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\left(\frac{\sqrt{17}}{2}\right)^2 - (t)^2\right)} = \frac{1}{\sqrt{17}}\left[\log \frac{25+17+10\sqrt{17}}{8}\right]$$

$$\therefore \int_0^2 \frac{1}{x+4-x^2} dx = \frac{1}{\sqrt{17}}\left[\log \frac{21+5\sqrt{17}}{4}\right]$$

7. $\displaystyle \int_{-1}^{1} \frac{1}{x^2+2x+5} dx$

Ans: The integral can be written as

$$\int_{-1}^{1} \frac{1}{x^2+2x+5} dx = \int_{-1}^{1} \frac{1}{(x-1)^2+2^2} dx$$

Let $x+1=t$

Differentiating it we get,

$dx = dt$

Therefore, the integral becomes

$$\int_{-1}^{1} \frac{1}{x^2+2x+5} dx = \int_{0}^{2} \frac{dt}{t^2+2^2}$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{x^2+2x+5} dx = \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{x^2+2x+5} dx = \left[\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right]$$

$$\therefore \int_{-1}^{1} \frac{1}{x^2+2x+5}dx = \frac{\pi}{8}$$

8. $\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$

Ans: Let $2x = t$

Differentiating it $2dx = dt$,

Therefore, the integral becomes

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx = \frac{1}{2}\int_{2}^{1}\left(\frac{2}{t}-\frac{2}{t^2}\right)e^{t}dt$$

$$\Rightarrow \int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx = \int_{2}^{1}\left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\int_{2}^{1}\left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt = \int_{2}^{1}\left[f(t)+f'(t)\right]e^{t}dt$$

$$\Rightarrow \int_{2}^{1}\left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt = \left[e^{t}f(t)\right]_{2}^{4}$$

$$\Rightarrow \int_{2}^{1}\left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt = \left[\frac{e^{t}}{t}\right]_{2}^{4}$$

$$\Rightarrow \int_{2}^{1}\left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt = \frac{e^4}{4}-\frac{e^2}{2}$$

$$\therefore \int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx = \frac{e^2\left(e^2-2\right)}{4}$$

9. **The value of the integral** $\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$

A. 6
B. 0
C. 3
D. 4

Ans: Let $x = \sin\theta$

Differentiating it, $dx = \cos\theta d\theta$

Therefore, the integral becomes

$$\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx=\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$\Rightarrow\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta=\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$\Rightarrow\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta=\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\left(\cos\theta\right)^{\frac{5}{3}}}{\sin^{\frac{5}{3}}\theta}\operatorname{cosec}^2\theta d\theta$$

$$\Rightarrow\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta=\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\left(\cot\theta\right)^{\frac{5}{3}}\operatorname{cosec}^2\theta d\theta$$

Let $\cot\theta=t$

Differentiating it, $-\operatorname{cosec}^2\theta d\theta=dt$

Therefore, the integral becomes

$$\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx=\int_{2\sqrt{2}}^{0}t^{\frac{5}{3}}dt$$

$$\Rightarrow\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx=\frac{3}{8}\sqrt{8}^{\frac{8}{3}}$$

$$\Rightarrow\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx=\frac{3}{8}\times16$$

$$\therefore\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx=6$$

Thus, the correct answer is A.

10. **If $f\left(x\right)=\int_0^x t\sin t\,dt$,then $f'\left(x\right)$ is**

A. $\mathbf{\cos x+x\sin x}$

B. $\mathbf{x\sin x}$

C. $\mathbf{x\cos x}$

D. $\mathbf{\sin x+x\cos x}$

Ans: Given $f(x)=\int_0^x t\sin t\,dt$

Using Integration by parts, we get

$$f(x)=t\int_0^x \sin t\,dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int \sin t\,dt\right\}dt$$

$$\Rightarrow f(x)=\left[t(-\cos t)\right]_0^x-\int_0^x -\cot\,tdt$$

$$\Rightarrow f(x)=\left[-t(\cos t)+\sin t\right]_0^x$$

$$\Rightarrow f(x)=-x\cos x+\sin x$$

Therefore,

$$f'(x)=-\left[-x\sin x+\cos x\right]+\cos x$$

$$\therefore f'(x)=x\sin x$$

Thus, the correct answer is B.

Exercise 7.10

Solve the following integrals.

1. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

Ans: Given $I=\int_0^{\frac{\pi}{2}} \cos^2 x dx \qquad \ldots(1)$

We know that,

$$\int_0^a f(x)dx=\int_0^a f(a-x)dx$$

Therefore, the integral becomes

$$I=\int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2}-x\right)dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \sin^2 xdx \qquad \ldots(2)$$

Adding equation (1) and (2),

$$2I=\int_0^{\frac{\pi}{2}}\left(\sin^2 x+\cos^2 x\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{\pi}{4}$$

2. $\displaystyle \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Ans: Given $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \qquad \ldots(1)$

We know that,

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Therefore, the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \qquad \ldots(2)$$

Adding equation (1) and (2),

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \frac{\pi}{4}$$

3. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$$

Ans: Given $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$ …(1)

We know that,

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

Therefore, the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} \left(\frac{\pi}{2} - x\right) dx}{\sin^{\frac{3}{2}} \left(\frac{\pi}{2} - x\right) + \cos^{\frac{3}{2}} \left(\frac{\pi}{2} - x\right)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} \quad \ldots(2)$$

Adding equation (1) and (2),

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} = \frac{\pi}{4}$$

4. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$

Ans: Given $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$ $\quad \ldots(1)$

We know that,

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

Therefore, the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right) dx}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x dx}{\sin^5 x + \cos^5 x} \quad \ldots(2)$$

Adding equation (1) and (2),

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} = \frac{\pi}{4}$$

5. $\displaystyle\int_{-5}^{5} |x+2| dx$

Ans: Let $I=\int_{-5}^{5}|x+2|dx$

Since, $(x+2)\le 0$ for interval $[-5,-2]$.

Therefore, $(x+2)\ge 0$ for interval $[-2,5]$.

As, $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$

Hence, $\int_{-5}^{-2}-(x+2)dx+\int_{-2}^{5}(x+2)dx$.

Thus,

$$I=\int_{-5}^{-2}-(x+2)dx+\int_{-2}^{5}(x+2)dx$$

$$=-\left[\frac{x^2}{2}+2x\right]_{-5}^{2}+\left[\frac{x^2}{2}+2x\right]_{-2}^{5}$$

$$=-\left[\frac{(2)^2}{2}+2(2)-\frac{(-5)^2}{2}-2(-5)\right]+\left[\frac{(5)^2}{2}+2(5)-\frac{(-2)^2}{2}-2(-2)\right]$$

$$=-\left[2-4-\frac{25}{2}+10\right]+\left[\frac{25}{2}+10-2+4\right]$$

$$=-2+4+\frac{25}{2}+10+\frac{25}{2}+10-2+4$$

$$=29$$

6. $\int_2^8 |x-5|dx$

Ans: Let $I=\int_2^8|x-5|dx$

Since, $(x-5)\le 0$ for interval $[2,5]$.

Therefore, $(x-5)\ge 0$ for interval $[5,8]$.

As, $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$

Hence, $\int_2^5-(x-5)dx+\int_5^8(x-5)dx$.

Thus,

$$I=\int_2^5-(x-5)dx+\int_5^8(x-5)dx$$

$$=-\left[\frac{x^2}{2}-5x\right]_2^5+\left[\frac{x^2}{2}-5x\right]_5^8$$

$$=-\left[\frac{(5)^2}{2}-5(5)-\frac{(2)^2}{2}+5(2)\right]+\left[\frac{(8)^2}{2}-5(8)-\frac{(5)^2}{2}+5(5)\right]$$

$$=-\left[\frac{25}{2}-25-2+10\right]+\left[32-40-\frac{25}{2}+25\right]$$

$$=-\frac{25}{2}+25+2-10+32-40-\frac{25}{2}+25$$

$$=9$$

7. $\int_0^1 x(1-x)^n\,dx$

Ans: Let $I=\int_0^1 x(1-x)^n\,dx$

Thus, $I=\int_0^1 (1-x)(1-(1-x))^n\,dx$

Since, $\int_1^a f(x)dx=\int_0^a f(a-x)dx$.

Therefore,

$$\int_0^1 (1-x)(x)^n dx$$

$$=\int_0^1 \left(x^n-x^{n+1}\right)dx$$

$$=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1$$

$$=\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$$

$$=\frac{(n+2)-(n+1)}{(n+1)(n+2)}$$

$$=\frac{1}{(n+1)(n+2)}$$

8. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$

Ans: Let $I=\int_0^{\frac{\pi}{4}}\log(1+\tan x)dx$

Since, $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

Therefore, $I=\int_0^{\frac{\pi}{4}}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$

$$=\int_0^{\frac{\pi}{4}}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]dx$$

$$=\int_0^{\frac{\pi}{4}}\log\left\{1+\frac{1-\tan x}{1+\tan x}\right\}dx$$

$$=\int_0^{\frac{\pi}{4}}\log\frac{2}{1+\tan x}dx$$

$$=\int_0^{\frac{\pi}{4}}\log 2dx-\int_0^{\frac{\pi}{4}}\log(1+\tan x)dx$$

$$=\int_0^{\frac{\pi}{4}}\log 2dx-I$$

$$2I=[x\log 2]_0^{\frac{\pi}{4}}$$

$$2I=\frac{\pi}{4}\log 2$$

$$I=\frac{\pi}{8}\log 2$$

9. $\int_0^2 x\sqrt{2-x}dx$

Ans: Let $I=\int_0^2 x\sqrt{2-x}dx$

Since, $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

Therefore, $I=\int_0^2(2-x)\sqrt{x}dx$

$$= \int_0^2 \left\{ 2x^{1/2} - x^{3/2} \right\} dx$$

$$= \left[2\left(\frac{x^{3/2}}{3/2} \right) - \frac{x^{5/2}}{5/2} \right]_0^2$$

$$= \left[\frac{4}{3} x^{3/2} - \frac{2}{5} x^{5/2} \right]_0^2$$

$$= \frac{4}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2}$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{40\sqrt{2} - 24\sqrt{2}}{15}$$

$$= \frac{16\sqrt{2}}{15}$$

10. $\displaystyle \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\sin 2x \right) dx$

Ans: Let $\displaystyle I = \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\sin 2x \right) dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\left(2\sin x \cos x \right) \right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\sin x - \log\cos x - \log 2 \right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log\sin x - \log\cos x - \log 2 \right) dx \quad \ldots\ldots.(1)$$

Since,

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

Therefore,

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log\cos x - \log\sin x - \log 2 \right) dx \quad \ldots\ldots.(2)$$

On adding equation 1 and 2-

$$2I=\int_{0}^{\frac{\pi}{2}}\left(-\log 2-\log 2\right)dx$$

$$\Rightarrow 2I=-2\log 2\int_{0}^{\frac{\pi}{2}}dx$$

$$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$$

$$\Rightarrow I=-\frac{\pi}{2}\left[\log 2\right]$$

$$\Rightarrow I=\frac{\pi}{2}\left[-\log 2\right]$$

$$\Rightarrow I=\frac{\pi}{2}\left[\log\frac{1}{2}\right]$$

$$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$$

11. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x dx$

Ans: Let $I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x dx$

Since, $\sin^2 x$ is an even function.

Therefore , $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x dx=2\int_{0}^{\frac{\pi}{2}}\sin^2 x dx$

As if $f\left(x\right)$ is an even function, then $\int_{-a}^{a}f\left(x\right)dx=2\int_{0}^{a}f\left(x\right)dx$.

Hence,

$$I=2\int_{0}^{\frac{\pi}{2}}\sin^2 x dx$$

$$=2\int_{0}^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}dx$$

$$=\int_{0}^{\frac{\pi}{2}}\left(1-\cos 2x\right)dx$$

$$=\left[x-\frac{\sin 2x}{2}\right]_{0}^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}$$

12. $\int_{0}^{\pi}\frac{\mathbf{x}}{\mathbf{1+\sin x}}\mathbf{dx}$

Ans: Let $I=\int_{0}^{\pi}\frac{x}{1+\sin x}dx$ …….(1)

Since, $\int_{0}^{a}f\left(x\right)dx=\int_{0}^{a}f\left(a-x\right)dx$

Therefore, $I=\int_{0}^{\pi}\frac{\left(\pi-x\right)}{1+\sin x\left(\pi-x\right)}dx$

$I=\int_{0}^{\pi}\frac{\left(\pi-x\right)}{1+\sin x}dx$ …..(2)

On adding equation 1 and 2-

$$2I=\int_{0}^{\pi}\frac{\pi}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi\int_{0}^{\pi}\frac{\left(1-\sin x\right)}{\left(1+\sin x\right)+\left(1-\sin x\right)}dx$$

$$\Rightarrow 2I=\pi\int_{0}^{\pi}\frac{\left(1-\sin x\right)}{\cos^{2}x}dx$$

$$\Rightarrow 2I=\pi\int_{0}^{\pi}\left\{\sec^{2}x-\tan x\sec x\right\}dx$$

$$\Rightarrow 2I=\pi\left[2\right]$$

$$\Rightarrow I=\pi$$

13. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\mathbf{\sin^{7}xdx}$.

Ans: Let $I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^7 x dx$

Since, $\sin^7 x$ is an even function.

Therefore , $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x dx=0$

As if $f(x)$ is an odd function, then $\int_{-a}^{a}f(x)dx=0$.

Hence, $I=0$

14. $\int_0^{2\pi}\mathbf{\cos^5 x dx}$.

Ans: Let

$I=\int_0^{2\pi}\cos^5 x dx$

$\cos^5(2\pi-x)=\cos^5 x$ .....(1)

If $f(2a-x)=f(x)$ then $\int_0^{2a}f(x)dx=2\int_0^{a}f(x)dx$.

If $f(2a-x)=-f(x)$ then $\int_0^{2a}f(x)dx=0$

Since, $\cos^5(\pi-x)=-\cos^5 x$

Therefore,

$I=2\int_0^{2\pi}\cos^5 x dx$

$\Rightarrow I=2(0)$

$\Rightarrow I=0$

15. $\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$

Ans: Let $I=\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ ......(1)

Since,

$\int_0^{a}f(x)dx=\int_0^{a}f(a-x)dx$

Therefore,

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos x-\sin x}{1+\cos x\sin x}dx \quad \ldots\ldots(2)$$

On adding equation 1 and 2

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\frac{0}{1+\cos x\sin x}dx$$

$$\Rightarrow I=0$$

16. $\int_0^{\pi}\log(1+\cos x)dx$

Ans: Let

$$I=\int_0^{\pi}\log(1+\cos x)dx \quad \ldots\ldots.(1)$$

Since,

$$\int_0^{a}f(x)dx=\int_0^{a}f(a-x)dx$$

Therefore,

$$\Rightarrow I=\int_0^{\pi}\log(1+\cos(\pi-x))dx$$

$$\Rightarrow I=\int_0^{\pi}\log(1-\cos x)dx \quad \ldots\ldots.(2)$$

On adding equation 1 and 2-

$$2I=\int_0^{\pi}\{\log(1-\cos x)+\log(1-\cos x)\}dx$$

$$\Rightarrow 2I=\int_0^{\pi}\log(1-\cos^2 x)dx$$

$$\Rightarrow 2I=\int_0^{\pi}\log\sin^2 xdx$$

$$\Rightarrow 2I=2\int_0^{\pi}\log\sin xdx$$

$$\Rightarrow I=\int_0^{\pi}\log\sin xdx \quad \ldots..(3)$$

Since, $\sin(\pi-x)=\sin x$

Therefore, $I=\int_0^{\frac{\pi}{2}}\log\sin x dx$ ……(4)

$$\Rightarrow I=2\int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2}-x\right)dx$$

$$\Rightarrow I=2\int_0^{\frac{\pi}{2}}\log\cos x dx \quad …….(5)$$

On adding equation 4 and 5-

$$2I=\int_0^{\frac{\pi}{2}}(\log\sin x+\log\cos x)dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log\sin x+\log\cos x+\log 2-\log 2)dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log 2\sin x\cos x-\log 2)dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log 2\sin x\cos x)dx-\int_0^{\frac{\pi}{2}}\log 2dx$$

Let $2x=t$

On differentiating-

$2dx=dt$

If $x=0$ then $t=0$.

Thus,

$$\Rightarrow I=\frac{I}{2}-\frac{\pi}{2}\log 2$$

$$\Rightarrow \frac{I}{2}=-\frac{\pi}{2}\log 2$$

$$\Rightarrow I=-\pi\log 2$$

17. $\displaystyle\int_0^a\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$

Ans: Let $I=\int_0^a\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ …….(1)

Since,

$$\int_0^a f(x)dx=\int_0^a f(a-x)dx$$

Therefore,

$$I=\int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \quad ....(2)$$

On adding equation 1 and 2-

$$2I=\int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx$$

$$\Rightarrow 2I=\int_0^a dx$$

$$\Rightarrow 2I=[x]_0^a$$

$$\Rightarrow 2I=a$$

$$\Rightarrow I=\frac{a}{2}$$

18. $\int_0^4 |x-1|dx$

Ans: Let $I=\int_0^4 |x-1|dx$

Thus, $(x-1)\le 0$ when $0\le x\le 1$ and $(x-1)\ge 0$ when $1\le x\le 4$

Since, $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)$

Therefore, $I=\int_0^1 |x-1|dx+\int_1^4 |x-1|dx$

$$\Rightarrow I=\int_0^1 -(x-1)dx+\int_1^4 (x-1)dx$$

$$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^4$$

$$=1-\frac{1}{2}+\frac{(4)^2}{2}-4-\frac{1}{2}+1$$

$$=1-\frac{1}{2}+8-4-\frac{1}{2}+1$$

$$=5$$

19. **Show that $\int_0^a f(x)g(x)dx=2\int_0^a f(x)dx$, if f and g are defined as $f(x)=f(a-x)$ and $g(x)+g(a-x)=4$.**

Ans: Let $\int_0^a f(x)g(x)dx$ ....(1)

Since,

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Therefore,

$$\Rightarrow \int_0^a f(a-x)g(a-x)dx$$

$$\Rightarrow \int_0^a f(x)g(a-x)dx \quad ......(2)$$

On adding equation 1 and 2-

$$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}dx$$

$$\Rightarrow 2I = \int_0^a f(x)\{g(x) + g(a-x)\}dx$$

As, $g(x) + g(a-x) = 4$.

Thus,

$$\Rightarrow 2I = \int_0^a 4f(x)dx$$

$$\Rightarrow I = 2\int_0^a f(x)dx$$

Hence, $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$.

20. The value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)dx$ is

a. 0
b. 2
c. π
d. 1

Ans: Let $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)dx$

$$\Rightarrow I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(x^3\right)dx+\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(x\cos x\right)dx+\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\tan^5 x\right)dx+\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(1)dx$$

If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)dx$

And $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx=0$

Thus,

$$I=0+0+0+2\int_{0}^{\frac{\pi}{2}} dx$$

$$=2[x]_0^{\frac{\pi}{2}}$$

$$=2\left[\frac{\pi}{2}\right]$$

$$=\pi$$

Hence, the correct option is C.

21. The value of $\int_{0}^{\frac{\pi}{2}}\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

a. 2

b. $\frac{3}{4}$

c. 0

d. –2

Ans: Let $I=\int_{0}^{\frac{\pi}{2}}\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ …..(1)

Since,

$$\int_{0}^{a} f(x)dx=\int_{0}^{a} f(a-x)dx$$

Therefore,

$$\Rightarrow I=\int_{0}^{\frac{\pi}{2}}\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\left(\frac{4+3\cos x}{4+3\sin x}\right)dx \qquad ....(2)$$

On adding equation 1 and 2-

$$2I=\int_0^{\frac{\pi}{2}}\log 1dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}0dx$$

$$\Rightarrow I=0$$

Hence, the correct option is C.

Miscellaneous Exercise

1. $\mathbf{\frac{1}{x-x^3}}$

Ans: Given $\frac{1}{x-x^3}$

So, $\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}$

$$=\frac{1}{x\left(1-x\right)\left(1+x\right)}$$

Let $\frac{1}{x\left(1-x\right)\left(1+x\right)}=\frac{A}{x}+\frac{B}{\left(1-x\right)}+\frac{C}{1+x} \qquad ....(1)$

$$\Rightarrow 1=A\left(1-x^2\right)+Bx\left(1+x\right)+Cx\left(1-x\right)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

On equating the coefficients of x^2, x and constant term –

$$-A+B-C=0$$

$$B+C=0$$

$$A=1$$

Thus, we get $A=1, B=\frac{1}{2}$ and $C=-\frac{1}{2}$

By equation 1-

$$\frac{1}{x\left(1-x\right)\left(1+x\right)}=\frac{1}{x}+\frac{1}{2\left(1-x\right)}-\frac{1}{2\left(1+x\right)}$$

$$\Rightarrow \int\frac{1}{x\left(1-x\right)\left(1+x\right)}dx=\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{\left(1-x\right)}dx-\frac{1}{2}\int\frac{1}{\left(1+x\right)}dx$$

$$= \log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)$$

$$= \log x - \frac{1}{2}\log(1-x)^{\frac{1}{2}} - \frac{1}{2}\log(1+x)^{\frac{1}{2}}$$

$$= \log\left(\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right) + C$$

$$= \log\left(\frac{x^2}{(1-x^2)}\right)^{\frac{1}{2}} + C$$

$$= \frac{1}{2}\log\left(\frac{x^2}{1-x^2}\right) + C$$

2. $\mathbf{\dfrac{1}{\sqrt{x+a}+\sqrt{x+b}}}$

Ans: Given $\dfrac{1}{\sqrt{x+a}+\sqrt{x+b}}$

$$= \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}$$

$$\Rightarrow \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx = \frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$$

3. $\mathbf{\dfrac{1}{x\sqrt{ax-x^2}}}$ $\left[\textbf{Hint: } \mathbf{x=\dfrac{a}{t}}\right]$

Ans: Given

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t}$

On differentiating-

$$dx=-\frac{a}{t^2}dt$$

$$\Rightarrow\int\frac{1}{x\sqrt{ax-x^2}}dx=\int\frac{1}{\frac{a}{t}\sqrt{a.\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$=-\int\frac{1}{at}.\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$=-\frac{1}{a}\int\frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$=-\frac{1}{a}\int\frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

Substituting value of t-

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$=-\frac{2}{a}\left[\frac{\sqrt{a-x}}{\sqrt{x}}\right]+C$$

$$=-\frac{2}{a}\left[\sqrt{\frac{a-x}{x}}\right]+C$$

4. Integrate the expression: $\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$.

Ans:

The given expression is, $\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$.

On multiplying and dividing by x^{-3}, the following can be obtained as shown below,

$$\frac{x^{-3}}{x^2.x^{-3}\left(x^4+1\right)^{\frac{3}{4}}}=\frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2 - x^{-3}}=\frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5\left(x^4\right)^{\frac{-3}{4}}}=\frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{\frac{-3}{4}}=\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{-3}{4}}$$

Now, consider $\frac{1}{x^4}=t$

$$\therefore \frac{-4}{x^5}=\frac{dt}{dx}$$

$$\Rightarrow \frac{dx}{x^5}=\frac{-dt}{4}$$

$$\therefore \int\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx=\int\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{-3}{4}}dx=\frac{-1}{4}\int\left(1+t\right)^{\frac{-3}{4}}dt$$

$$\Rightarrow \int\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx=\frac{-1}{4}\left[\frac{\left(1+t\right)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C,$$ where C is any arbitrary constant.

5. Integrate the expression: $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$. **[Hint:** $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$ **Put,** $x=t^6$

Ans: The given expression is, $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$.

Observe the given hint and obtain as shown below,

$$\therefore \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Consider $x=t^6$

$$\therefore x=t^6 \Rightarrow dx=6t^5$$

$$\therefore \int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=\int\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx=\int\frac{6t^5}{t^2\left(1+t\right)}dt=6\int\frac{t^3}{\left(1+t\right)}dt$$

Now on dividing, we can obtain as shown below,

$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = 6\int \left\{ \left(t^2 - t + 1\right) - \frac{1}{1+t} \right\}dt$$

$$= 6\int \left[\left(\frac{t^3}{3}\right) - \left(\frac{t^2}{2}\right) + t - \log|1+t| \right]dt$$

$$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1 + x^{\frac{1}{6}}\right) + C$$

$$= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1 + x^{\frac{1}{6}}\right) + C$$, where C is any arbitrary constant.

6. Integrate the expression: $\frac{5x}{(x+1)\left(x^2+9\right)}$.

Ans: The given expression is, $\frac{5x}{(x+1)\left(x^2+9\right)}$

Now consider it as shown below,

$$\therefore \frac{5x}{(x+1)\left(x^2+9\right)} = \frac{A}{(x+1)} + \frac{Bx+C}{\left(x^2+9\right)} \quad ...(1)$$

$$\Rightarrow 5x = A\left(x^2+9\right) + Bx + C(x+1)$$

$$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$$

On equating the coefficients of x^2, x and constant term, it can be obtained that,

$A + B = 0 \quad ...(2)$

$B + C = 5 \quad ...(3)$

$9A + C = 0 \quad ...(4)$

And on solving these equations, the values of A, B, C can be obtained as,

$A = \frac{-1}{2}, B = \frac{1}{2}, C = \frac{9}{2}$ respectively.

Now, from equation (1) it can be clearly obtained that,

$$\int \frac{5x}{(x+1)\left(x^2+9\right)}dx = \int \left\{ \frac{-1}{2(x+1)} + \frac{x+9}{2\left(x^2+9\right)} \right\}dx$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{2}\int \frac{x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log\left|x^2+9\right| + \left(\frac{9}{2}\right)\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{x}{3}\right)$$

$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}\tan^{-1}\left(\frac{x}{3}\right) + C$, where C is any arbitrary constant.

7. Integrate the expression, that is, $\frac{\sin x}{\sin(x-\alpha)}$.

Ans: The given expression is, $\frac{\sin x}{\sin(x-\alpha)}$.

Now, substitute $x-\alpha = t$

$\therefore dx = dt$

$$\therefore \int \frac{\sin x}{\sin(x-\alpha)}dx = \int \frac{\sin(t+\alpha)}{\sin t}dt = \int \frac{\sin t\cos\alpha + \cos t\sin\alpha}{\sin t}dt$$

$$= \int \cos\alpha + \cot t\sin\alpha dt$$

$$\Rightarrow \int \frac{\sin x}{\sin(x-\alpha)}dx = t\cos\alpha + \sin\alpha\log|\sin t| + C_1 = (x-\alpha)\cos\alpha$$

$$+\sin\alpha\log|\sin(x-\alpha)| + C_1$$

$$\Rightarrow \int \frac{\sin x}{\sin(x-\alpha)}dx = x\cos\alpha + \sin\alpha\log|\sin(x-\alpha)| - \alpha\cos\alpha + C_1$$

$$= x\cos\alpha + \sin\alpha\log|\sin(x-\alpha)| + C$$

where C_1, C are any arbitrary constants and $C = C_1 - \alpha\cos\alpha$.

8. Integrate the expression, that is, $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$.

Ans: The given expression is, $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$.

$$\therefore \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)} = e^{2\log x} = x^2$$

Now, integrate the given expression as shown below

$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int x^2 dx = \frac{x^3}{3} + C$, where C is any arbitrary constant.

9. Integrate the expression, that is, $\frac{\cos x}{\sqrt{4-\sin^2 x}}$.

Ans: The given expression is, $\frac{\cos x}{\sqrt{4-\sin^2 x}}$.

Now, substitute $\sin x = t$

$\therefore \cos x dx = dt$

$\therefore \int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \frac{dt}{\sqrt{(2)^2-(t)^2}} = \sin^{-1}\left(\frac{t}{2}\right) + C = \sin^{-1}\left(\frac{\sin x}{2}\right) + C$, where C is any arbitrary constant.

10. Integrate the expression, that is, $\frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}$.

Ans: The given expression is, $\frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}$.

$$\therefore \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x} = \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{\sin^2 x + \cos^2 x - \sin^2 x\cos^2 x - \sin^2 x\cos^2 x}$$

$$= \frac{(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x)}{\sin^2 x(1-\cos^2 x) + \cos^2 x(1-\sin^2 x)}$$

$$= \frac{-(-\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)}{\sin^4 x + \cos^4 x} = -\cos 2x$$

Now, integrate the given expression as shown below

$\therefore \int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x} dx = \int -\cos 2x dx = \frac{-\sin 2x}{2} + C$, where C is any arbitrary constant.

11. Integrate the expression, that is, $\frac{1}{\cos(x+a)\cos(x+b)}$.

Ans: The given expression is, $\frac{1}{\cos(x+a)\cos(x+b)}$.

On multiplying and dividing by $\sin(\alpha-\beta)$, the following can be obtained as shown below,

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$\Rightarrow \frac{1}{\sin(a-b)}\left[\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\right]$$

$$\Rightarrow \frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$\Rightarrow \frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]=\frac{1}{\sin(a-b)}\left(\tan(x+a)-\tan(x+b)\right)$$

$$\therefore \int \frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\left(\tan(x+a)-\tan(x+b)\right)dx$$

$$\Rightarrow \int \frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x+b)}{\cos(x-a)}\right|\right]+C$$, where C is any arbitrary constant.

12. Integrate the expression, that is, $\frac{x^3}{\sqrt{1-x^8}}$.

Ans: The given expression is, $\frac{x^3}{\sqrt{1-x^8}}$.

Now, substitute $x^4=t$

$\therefore 4x^3dx=dt$

$$\therefore \int \frac{x^3}{\sqrt{1-x^8}}dx=\frac{1}{4}\int\frac{dt}{\sqrt{1-(t)^2}}=\frac{1}{4}\sin^{-1}t+C=\frac{1}{4}\sin^{-1}\left(x^4\right)+C$$, where C is any arbitrary constant.

13. Integrate the expression, that is, $\frac{e^x}{(1+e^x)(2+e^x)}$.

Ans: The given expression is, $\frac{e^x}{(1+e^x)(2+e^x)}$.

Now, substitute $e^x=t$

$\therefore e^xdx=dt$

$$\therefore \int \frac{e^x}{(1+e^x)(2+e^x)}dx=\int\frac{dt}{(t+1)(t+2)}=\int\left[\frac{1}{(t+1)}-\frac{1}{(t+2)}\right]dt$$

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)}dx=\log|t+1|-\log|t+2|+C=\log\left|\frac{e^x+1}{e^x+2}\right|+C$$, where C is any arbitrary constant.

14. Integrate the expression, $\frac{1}{(x^2+1)\left(x^2+4\right)}$.

Ans: The given expression is, $\frac{1}{(x^2+1)\left(x^2+4\right)}$.

Now consider it as shown below,

$$\therefore \frac{1}{(x^2+1)\left(x^2+4\right)} = \frac{Ax+B}{\left(x^2+1\right)} + \frac{Cx+D}{\left(x^2+4\right)} \quad ...(1)$$

$$\Rightarrow 1 = \left(Ax+B\right)\left(x^2+4\right) + \left(Bx+C\right)\left(x^2+9\right)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

On equating the coefficients of x^3, x^2, x and constant term, it can be obtained that,

$A+C=0 \ ...(2)$

$B+D=0 \ ...(3)$

$4A+C=0 \ ...(4)$

$4B+D=1 \ ...(5)$

And on solving these equations, the values of A, B, C, D can be obtained as,

$A=0, B=\frac{1}{3}, C=0, D=\frac{1}{3}$ respectively.

Now, from equation (1) it can be clearly obtained that,

$$\int \frac{1}{(x^2+1)\left(x^2+4\right)}dx = \frac{1}{3}\int\left\{\frac{1}{\left(x^2+1\right)} - \frac{1}{\left(x^2+4\right)}\right\}dx$$

$$= \frac{1}{3}\tan^{-1}x - \frac{1}{(3)(2)}\tan^{-1}\frac{x}{2} + C$$

$= \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$, where C is any arbitrary constant.

15. Integrate the expression, that is, $\cos^3 xe^{\log \sin x}$.

Ans: The given expression is, $\cos^3 xe^{\log\sin x}$.

Observe as shown below,

$\therefore \cos^3 xe^{\log\sin x} = \cos^3 x \sin x$

Now, consider $\cos x = t$

$\therefore -\sin x dx = dt$

$\therefore \int \cos^3 xe^{\log\sin x}dx = \int \cos^3 x \sin x dx = -\int t^3 dt = \frac{-t^4}{4} + C = \frac{-\cos^4 x}{4} + C$, where C is any arbitrary constant.

16. **Integrate the expression, that is, $e^{3\log x}\left(x^4+1\right)^{-1}$.**

Ans: The given expression is, $e^{3\log x}\left(x^4+1\right)^{-1}$.

Observe as shown below,

$$\therefore e^{3\log x}\left(x^4+1\right)^{-1}=\frac{x^3}{\left(x^4+1\right)}$$

Now, consider $x^4+1=t$

$$\therefore 4x^3dx=dt$$

$$\therefore \int e^{3\log x}\left(x^4+1\right)^{-1}dx=\int\frac{x^3}{x^4+1}dx=\frac{1}{4}\int\frac{dt}{t}=\frac{1}{4}\log|t|+C=\frac{\log\left|x^4+1\right|}{4}+C$$, where C is any arbitrary constant.

17. **Integrate the expression, that is, $f'\left(ax+b\right)\left[f(ax+b)\right]^n$.**

Ans: The given expression is, $f'\left(ax+b\right)\left[f(ax+b)\right]^n$.

Now, consider $\left[f(ax+b)\right]=t$

$$\therefore af'(ax+b)dx=dt$$

$$\therefore \int f'\left(ax+b\right)\left[f(ax+b)\right]^n dx=\frac{1}{a}\int t^n dt=\frac{t^{n+1}}{a\left(n+1\right)}+C=\frac{\left[f(ax+b)\right]^{n+1}}{a\left(n+1\right)}+C$$, where C is any arbitrary constant.

18. **Integrate the expression, that is, $\frac{1}{\sqrt{\sin^3 x\sin\left(x+\alpha\right)}}$.**

Ans: The given expression is, $\frac{1}{\sqrt{\sin^3 x\sin\left(x+\alpha\right)}}$.

$$\therefore \frac{1}{\sqrt{\sin^3 x\sin\left(x+\alpha\right)}}=\frac{1}{\sqrt{\sin^3 x\sin x\cos\alpha+\cos\alpha\sin x}}$$

$$=\frac{1}{\sqrt{\sin^4 x\cos\alpha+\sin^3 x\sin\alpha\cos x}}$$

$$=\frac{1}{\sin^2 x\sqrt{\cos\alpha+\sin\alpha\cot x}}=\frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha+\sin\alpha\cot x}}$$

Now, substitute $\cos\alpha+\sin\alpha\cot x=t$

$$\therefore -\operatorname{cosec}^2 x\sin\alpha dx=dt$$

$$\therefore \int \frac{1}{\sqrt{\sin^3 x \sin\left(x+\alpha\right)}}dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \sin\alpha \cot x}} = \frac{-1}{\sin\alpha}\int \frac{dt}{\sqrt{t}} = \frac{-2\sqrt{t}}{\sin\alpha} + C$$

$$\Rightarrow \int \frac{1}{\sqrt{\sin^3 x \sin\left(x+\alpha\right)}}dx = \frac{-2\sqrt{\cos\alpha + \sin\alpha \cot x}}{\sin\alpha} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\sin\alpha \cos x}{\sin x}} + C$$

where C is any arbitrary constant.

19. Integrate the expression, that is, $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$.

Ans: The given expression is, $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$.

Assume, $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Now, substitute $x = \cos^2\theta$

$\therefore dx = -2\sin\theta\cos\theta dt$

$$\therefore I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(-2\sin\theta\cos\theta\right)d\theta = -\int \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta d\theta$$

$$= -\int \tan\frac{\theta}{2} 2\sin\theta\cos\theta d\theta$$

$$\Rightarrow I = -2\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta d\theta = -4\int \sin^2\frac{\theta}{2}(2\cos^2\frac{\theta}{2} - 1)d\theta$$

$$\Rightarrow I = -8\int \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}d\theta + 4\int \sin^2\frac{\theta}{2}d\theta = 2\int \sin^2\frac{\theta}{2}d\theta + 4\int \sin^2\frac{\theta}{2}d\theta$$

$$\Rightarrow I = -2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta + 4\int\left(\frac{1-\cos\theta}{2}\right)d\theta = -2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{2}\right] + 4\left[\frac{\theta}{2} - \frac{\sin\theta}{2}\right] + C$$

$$\Rightarrow I = -\theta + \frac{\sin 2\theta}{2} + 2\sin\theta + C = \theta + \sqrt{1-\cos^2\theta}\cos\theta - 2\sqrt{1-\cos^2\theta} + C$$

$$\Rightarrow I = \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} - 2\sqrt{1-x} + C = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x - x^2} + C,$$

where C is any arbitrary constant.

20. Integrate the expression, that is, $\frac{2+\sin 2x}{1+\cos 2x}e^x$.

Ans: The given expression is, $\frac{2+\sin 2x}{1+\cos 2x}e^x$.

Assume, $I=\int\frac{2+\sin 2x}{1+\cos 2x}e^x dx$

$\Rightarrow I=\int\frac{2+2\sin x\cos x}{2\cos^2 x}e^x dx=\int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x dx=\int\left(\sec^2 x+\tan x\right)e^x dx$

Now, consider $f\left(x\right)=\tan x$

$\therefore f'\left(x\right)=\sec^2 x dx$

$\therefore I=\int\frac{2+\sin 2x}{1+\cos 2x}e^x dx=\int\left(f\left(x\right)+f'\left(x\right)\right)e^x dx=e^x f\left(x\right)+C=e^x\tan x+C$, where C is any arbitrary constant.

21. Integrate the expression: $\frac{x^2+x+1}{(x+1)^2\left(x+2\right)}$.

Ans: The given expression is, $\frac{x^2+x+1}{(x+1)^2\left(x+2\right)}$.

Now consider it as shown below,

$\therefore\frac{x^2+x+1}{(x+1)^2\left(x+2\right)}=\frac{A}{\left(x+1\right)}+\frac{B}{\left(x+1\right)}+\frac{C}{\left(x+2\right)}\qquad...(1)$

$\Rightarrow x^2+x+1=A\left(x+1\right)\left(x+2\right)+B\left(x+2\right)+C\left(x+1\right)^2$

$\Rightarrow x^2+x+1=A\left(x^2+3x+2\right)+B\left(x+2\right)+C\left(x^2+2x+1\right)$

$\Rightarrow x^2+x+1=\left(A+C\right)x^2+x\left(3A+B+2C\right)+\left(2A+2B+C\right)$

On equating the coefficients of x^2, x and constant term, it can be obtained that,

$A+C=1\ ...(2)$

$3A+B+2C=1\ ...(3)$

$2A+2B+C=1\ ...(4)$

And on solving these equations, the values of A,B,C can be obtained as,

$A=-2, B=1, C=3$ respectively.

Now, from equation (1) it can be clearly obtained that,

$$\int\frac{x^2+x+1}{(x+1)^2\left(x+2\right)}dx=\int\left\{\frac{-2}{\left(x+1\right)}+\frac{1}{\left(x+1\right)}+\frac{3}{\left(x+2\right)}\right\}dx$$

$$=-2\int\frac{1}{x+1}dx+\int\frac{1}{\left(x+1\right)^2}dx+3\int\frac{1}{x+2}dx$$

$= -2\log|x+1| + 3\log|x+2| - \dfrac{1}{(x+1)} + C$, where C is any arbitrary constant.

22. Integrate the expression, that is, $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$.

Ans: The given expression is, $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$.

Assume, $I = \int \tan^{-1}\sqrt{\dfrac{1-x}{1+x}}dx$

Now, consider $x = \cos\theta$

$\therefore dx = -\sin\theta d\theta$

$$\therefore I = \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta)d\theta = -\int \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$$

$$= -\int \tan^{-1}\tan\frac{\theta}{2}\sin\theta d\theta$$

$$\Rightarrow I = -\frac{1}{2}\int \theta\sin\theta d\theta = -\frac{1}{2}\left[\theta(-\cos\theta) - \int 1.(-\cos\theta)d\theta\right] = -\frac{1}{2}\left[\theta(\cos\theta) + \sin\theta\right]$$

$\Rightarrow I = \dfrac{x}{2}\cos^{-1}x - \dfrac{1}{2}\sqrt{1-x^2} + C = \dfrac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C$, where C is any arbitrary constant.

23. Integrate the expression, that is, $\dfrac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4}$.

Ans: The given expression is, $\dfrac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4}$.

Assume, $I = \int \dfrac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4}dx$

$$\Rightarrow I = \frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right) - \log x^2\right] = \frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$= \frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\left[\log\left(1+\frac{1}{x^2}\right)\right]$$

Consider $1 + \dfrac{1}{x^2} = t$

$$\therefore \frac{-2}{x^3}dx = dt$$

Now, integrate the given expression as shown below

$$\therefore I = \int \frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4}dx = \int \frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\left[\log\left(1+\frac{1}{x^2}\right)\right]dx$$

$$= \frac{-1}{2}\int t^{\frac{1}{2}}\log t dt + C$$

Using integration by parts, it can be obtained that,

$$I = \frac{-1}{2}\left[\log t.\int t^{\frac{1}{2}}dt - \left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right] = \frac{-1}{2}\left[\log t.\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$\Rightarrow I = \frac{-1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}dt\right] = \frac{-1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}} = \frac{-1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

$$\Rightarrow I = \frac{-1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}dt\right] = \frac{-1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}} = \frac{-1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

$\Rightarrow I = \frac{-1}{2}\left[1+\frac{1}{x^2}\right]\left(\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right) + C$, where C is any arbitrary constant.

24. Find the value of the expression, that is, $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$.

Ans: The given expression is, $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$.

Assume, $I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$\Rightarrow I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{+2\sin^2\frac{x}{2}}\right)dx = \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{\operatorname{cosec}^2\frac{x}{2}}{2} - \cot\frac{x}{2}\right)dx$$

Now, substitute $f(x) = -\cot\frac{x}{2}$

$$\therefore f'(x) = -\left(-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right)dx = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}dx$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(f(x) + f'(x)\right)dx = \left[e^x f(x)\right]_{\frac{\pi}{2}}^{\pi} = \left[e^x \cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$\Rightarrow I = \left[e^{\pi}\cot\frac{\pi}{2} - e^{\frac{\pi}{2}}\cot\frac{\pi}{4}\right] = \left[0 - e^{\frac{\pi}{2}}\right] = -e^{\frac{\pi}{2}}$$

25. Find the value of the expression, that is, $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$.

Ans: The given expression is, $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$.

Assume, $I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{\sin x \cos x}{\cos^4 x}}{\frac{\sin^4 x + \cos^4 x}{\cos^4 x}} dx = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

Now, substitute $\tan^2 x = t$

$\therefore 2 \tan x \sec^2 x dx = dt$

And also when $x = 0, t = 0$ and when $x = \frac{\pi}{4}, t = 1$.

$$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2} = \frac{1}{2} \left[\tan^{-1} t \right]_0^1 = \frac{1}{2} \left[\tan^{-1}(1) - \tan^{-1}(0) \right] = \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8}$$

26. Find the value of the expression, that is, $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$.

Ans: The given expression is, $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$.

Assume, $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)} dx = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 4 - 3\cos^2 x}{-3\cos^2 x + 4} dx$$

$$\Rightarrow I = -\frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\cos^2 x} dx = -\frac{1}{3} \int_0^{\frac{\pi}{2}} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3} dx$$

$$\Rightarrow I = \frac{-1}{3} [x]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1 + \tan^2 x) - 3} dx = \frac{-\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{(1 + 4\tan^2 x)} dx \quad ...(1)$$

Observe, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{(1 + 4\tan^2 x)} dx$

Now, substitute $2 \tan x = t$

$\therefore 2\sec^2 x dx = dt$

And also when $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$.

$$\therefore \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{(1 + 4\tan^2 x)} dx = \int_0^{\infty} \frac{dt}{(1 + t^2)} dx = \left[\tan^{-1}(t) \right]_0^{\infty} = \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right] = \frac{\pi}{2}$$

Henceforth from equation (1), it can be obtained that,

$$I = -\frac{\pi}{6} + \frac{2}{3} \left(\frac{\pi}{2} \right) = -\frac{\pi}{6} + \frac{2\pi}{6} = \frac{\pi}{6}$$

27. Find the value of the expression, that is, $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$.

Ans: The given expression is, $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$.

Assume, $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sqrt{-(-1+1-2\sin x\cos x)}} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sqrt{1-(\sin^2 x+\cos^2 x-2\sin x\cos x)}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}} dx$$

Now, substitute $(\sin x-\cos x)=t$

$\therefore (\sin x+\cos x)dx = dt$

And also when $x=\frac{\pi}{6}, t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when $x=\frac{\pi}{3}, t=\left(\frac{\sqrt{3}-1}{2}\right)$.

$$\therefore I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = \int_{-\left(\frac{-1+\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}$, it can be thus obtained that $\frac{1}{\sqrt{1-t^2}}$ is an even function,

$\therefore \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$\therefore I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = \left[2\sin^{-1} t\right]_0^{\frac{\sqrt{3}-1}{2}} = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ $x=\frac{\pi}{6}, t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when $x=\frac{\pi}{3}, t=\left(\frac{\sqrt{3}-1}{2}\right)$.

28. Find the value of the expression, that is, $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$.

Ans: The given expression is, $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$.

Assume, $I = \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$\Rightarrow I = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} \times \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} dx = \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$$

$$\Rightarrow I = \int_0^1 \sqrt{1+x}dx + \int_0^1 \sqrt{x}dx = \frac{2}{3}\left[(1+x)^{\frac{2}{3}}\right]_0^1 + \frac{2}{3}\left[(x)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left[(2)^{\frac{2}{3}}-1\right] + \frac{2}{3} = \frac{4\sqrt{2}}{3}$$

29. Find the value of the expression, that is, $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$.

Ans: The given expression is, $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$.

Assume, $I=\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

Now, substitute $\sin x-\cos x=t$

$\therefore (\cos x+\sin x)dx=dt$

And also when $x=0, t=-1$ and when $x=\frac{\pi}{4}, t=0$.

$\therefore (\sin x-\cos x)^2=t^2$

$\Rightarrow 1-2\sin x\cos x=t^2$

$\Rightarrow 1-\sin 2x=t^2$

$\Rightarrow \sin 2x=1-t^2$

$$\therefore I=\int_{-1}^{0}\frac{dt}{9+16(1-t^2)}=\int_{-1}^{0}\frac{dt}{25-16t^2}=\int_{-1}^{0}\frac{dt}{(5)^2-(4t)^2}$$

$$\Rightarrow I=\frac{1}{4}\left[\frac{1}{2(5)}\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}=\frac{1}{40}\left[\log|1|-\log\left|\frac{1}{9}\right|\right]=\frac{1}{40}\log|9|$$

30. Find the value of the expression, that is, $\int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)dx$.

Ans: The given expression is, $\int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)dx$.

Assume, $I=\int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)dx$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)dx=\int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}(\sin x)dx$$

Now, substitute $\sin x=t$

$\therefore \cos x dx=dt$

And also when $x=0, t=0$ and when $x=\frac{\pi}{2}, t=1$.

$\therefore I=2\int_0^1 t\tan^{-1}(t)dt$

Observe, $\int t\tan^{-1}(t)dt$

$$\therefore \int t\tan^{-1}(t)dt=\tan^{-1}t\int t dt-\int\left\{\frac{d(\tan^{-1}t)}{dt}\int t dt\right\}dt=\tan^{-1}t.\frac{t^2}{2}-\int\frac{1}{1+t^2}.\frac{t^2}{2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\int\frac{t^2+1-1}{1+t^2}dt=\frac{t^2\tan^{-1}t}{2}-\int 1dt+\int\frac{1}{1+t^2}dt=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}t+\frac{1}{2}\tan^{-1}t$$

$$\therefore \int_0^1 t.\tan^{-1} t dt = \left[\frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}t + \frac{1}{2}\tan^{-1} t\right]_0^1 = \frac{1}{2}\left[\frac{\pi}{4} - 1 + \frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{1}{2}$$

Henceforth from equation (1), it can be obtained that,

$$I = 2\left[\frac{\pi}{2} - \frac{1}{2}\right] = \frac{\pi}{2} - 1$$

31. **Find the value of the expression, that is, $\int_1^4 \left[|x-1| + |x-2| + |x-3|\right] dx$.**

Ans: The given expression is, $\int_1^4 \left[|x-1| + |x-2| + |x-3|\right] dx$.

Assume, $\int_1^4 \left[|x-1| + |x-2| + |x-3|\right] dx$

$$\Rightarrow I = \int_1^4 |x-1| dx + \int_1^4 |x-2| dx + \int_1^4 |x-3| dx$$

$$\therefore I = I_1 + I_2 + I_3 \ ...(1)$$

where, $I_1 = \int_1^4 |x-1| dx, I_2 = \int_1^4 |x-2| dx, I_3 = +\int_1^4 |x-3| dx$

Now, consider, $I_1 = \int_1^4 |x-1| dx$, where $(x-1) \ge 0 \forall 1 \le x \le 4$

$$\therefore I_1 = \int_1^4 (x-1) dx = \left[\frac{x^2}{2} - x\right]_1^4 = \left[8 - 4 - \frac{1}{2} + 1\right] = \frac{9}{2} \ ...(2)$$

Again, consider, $I_2 = \int_1^4 |x-2| dx$, where $(x-2) \ge 0 \forall 2 \le x \le 4$ and $(x-2) \le 0 \forall 1 \le x \le 2$.

$$\therefore I_2 = \int_1^2 (2-x) dx + \int_2^4 (x-2) dx = \left[2x - \frac{x^2}{2}\right]_1^4 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$\Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2}\right] + \left[8 - 8 - 2 + 4\right] = \frac{1}{2} + 2 = \frac{5}{2} \ ...(3)$$

Also, consider, $I_3 = \int_1^4 |x-3| dx$, where $(x-3) \ge 0 \forall 3 \le x \le 4$ and $(x-3) \le 0 \forall 1 \le x \le 3$.

$$\therefore I_3 = \int_1^3 (3-x) dx + \int_3^4 (x-3) dx = \left[3 - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right] = 2 + \frac{1}{2} = \frac{5}{2} \ ...(4)$$

Now, from equations (1), (2), (3) and (4) it can be obtained that,

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

$$\Rightarrow 2I = \pi\int_0^\pi \frac{\sin x + 1 - 1}{1 + \sin x} dx = \pi\int_0^\pi dx - \pi\int_0^\pi \frac{1}{1 + \sin x} dx = \pi[x]_0^\pi - \pi\int_0^\pi \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi[x]_0^\pi - \pi\int_0^\pi (\sec^2 x - \tan x \sec x) dx = \pi^2 - \pi[\tan x - \sec x]_0^\pi$$

$$\Rightarrow 2I = \pi^2 - \pi\left[0 - (-1) - 0 + 1\right] = \pi^2 - 2\pi$$

$$\Rightarrow I = \frac{\pi(\pi-2)}{2}$$

32. Prove the following equation: $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$.

Ans: The given equation is, $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$.

Assume, $\int_1^3 \frac{dx}{x^2(x+1)}$

Now consider it as shown below,

$$\therefore \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} \quad ...(1)$$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

On equating the coefficients of x^2, x and constant term, it can be obtained that,

$A + C = 0$ ...(2)

$A + B = 0$ ...(3)

$B = 1$ ...(4)

And on solving these equations, the values of A, B, C can be obtained as,

$A = -1, B = 1, C = 1$ respectively.

Now, from equation 1 it can be clearly obtained that,

$$I = \int_1^3 \frac{dx}{x^2(x+1)} = \int_1^3 \left\{ \frac{-1}{(x)} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$\Rightarrow I = \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3 = \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3 = \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log(2) + 1$$

$$\Rightarrow I = \log 4 - \log 3 - \log 2 + \frac{2}{3} = \log 2 - \log 3 + \frac{2}{3} = \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Henceforth, it can be clearly proved.

33. Prove the following equation: $\int_0^4 xe^x dx = 1$.

Ans: The given equation is, $\int_0^4 xe^x dx = 1$.

Assume, $I = \int_0^4 xe^x dx$

Using integration by parts, it can be obtained that,

$$I = x\int_0^4 e^x dx - \int_0^4 \left\{ \left(\frac{d(x)}{dx} \right) \int e^x \right\} = \left[xe^x \right]_0^1 - \left[e^x \right]_0^1 = e - e + 1 = 1$$

Henceforth, it can be clearly proved.

34. Prove the following equation: $\int_{1}^{-1} x^{17}\cos^4 x dx=0$.

Ans: The given equation is, $\int_{1}^{-1} x^{17}\cos^4 x dx = 0$.

Assume, $I = \int_{1}^{-1} x^{17}\cos^4 x dx$

Now, consider $f(x) = x^{17}\cos^4 x$

$\therefore f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

$\Rightarrow f(x)$ is an odd function and henceforth it is clearly known to us that when $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$.

$\therefore I = \int_{1}^{-1} x^{17}\cos^4 x dx = 0$

Henceforth, it can be clearly proved.

35. Prove the following equation: $\int_{0}^{\frac{\pi}{2}} \sin^3 x dx=\frac{2}{3}$.

Ans: The given equation is, $\int_{0}^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$.

Assume, $I = \int_{0}^{\frac{\pi}{2}} \sin^3 x dx$

$\Rightarrow I = \int_{0}^{\frac{\pi}{2}} \sin^2 x \sin x dx = \int_{0}^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx = \int_{0}^{\frac{\pi}{2}} \sin x dx - \int_{0}^{\frac{\pi}{2}} \cos^2 x \sin x dx$

$\Rightarrow I = [-\cos x]_{0}^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_{0}^{\frac{\pi}{2}} = 1 - \frac{1}{3} = \frac{2}{3}$

Henceforth, it can be clearly proved.

36. Prove the following equation: $\int_{0}^{\frac{\pi}{4}} 2\tan^3 x dx=1-\log 2$.

Ans: The given equation is, $\int_{0}^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$.

Assume, $\int_{0}^{\frac{\pi}{4}} 2\tan^3 x dx$

$\Rightarrow I = \int_{0}^{\frac{\pi}{4}} 2\tan^2 x \tan x dx = \int_{0}^{\frac{\pi}{4}} (1-\sec^2 x)\tan x dx = \int_{0}^{\frac{\pi}{4}} \tan x dx - \int_{0}^{\frac{\pi}{4}} \sec^2 x \tan x dx$

$\Rightarrow I = 2\left[\frac{\tan^2 x}{2}\right]_{0}^{\frac{\pi}{4}} + 2[\log\cos x]_{0}^{\frac{\pi}{4}} = 1 + 2\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]_{0}^{\frac{\pi}{2}} = 1 - \log 2 - \log 1$

$\Rightarrow I = 1 - \log 2$

Henceforth, it can be clearly proved.

37. Prove the following equation: $\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$.

Ans: The given equation is, $\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$.

Assume, $I = \int_0^1 \sin^{-1} x dx$

$\Rightarrow I = \int_0^1 \sin^{-1} x.1dx$

Using integration by parts, it can be obtained that,

$$I = \left[\sin^{-1} x.x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} x dx = \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} dx$$

Now, substitute $1 - x^2 = t$

$\therefore (-2x)dx = dt$

And also when $x = 0, t = 1$ and when $x = 1, t = 0$.

$$\therefore I = \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t}} = \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\left[2\sqrt{t}\right]_1^0 = \sin^{-1} 1 - \sqrt{1} = \frac{\pi}{2} - 1$$

Henceforth, it can be clearly proved.

38. The expression, that is, $\int \frac{dx}{e^x + e^{-x}}$ is equal to

A. $\tan^{-1}(e^x) + C$

B. $\tan^{-1}(e^{-x}) + C$

C. $\log(e^x - e^{-x}) + C$

D. $\log(e^x + e^{-x}) + C$

Ans: The given expression is, $\int \frac{dx}{e^x + e^{-x}}$.

Assume, $I = \int \frac{dx}{e^x + e^{-x}}$

Now, consider $e^x = t$

$\therefore e^x dx = dt$

$\therefore I = \int \frac{dx}{e^x + e^{-x}} = \int \frac{1}{1+t^2} dt = \int \tan^{-1} t\, dt + C$, where C is any arbitrary constant.

Hence, the correct answer is option (A).

39. **The expression, that is, $\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$ is**

A. $\frac{-1}{\sin x+\cos x}+C$

B. $\log|\sin x+\cos x|+C$

C. $\log|\sin x-\cos x|+C$

D. $\frac{1}{(\sin x+\cos x)^2}+C$

Ans: The given expression is, $\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$.

Assume, $I=\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$

$\Rightarrow I=\int \frac{(\sin x+\cos x)(\cos x-\sin x)}{(\sin x+\cos x)^2}dx=\int \frac{(\cos x-\sin x)}{(\sin x+\cos x)}dx$

Now, substitute $(\sin x+\cos x)=t$

$\therefore (\cos x-\sin x)dx=dt$

$\therefore I=I=\int \frac{1}{t}dt=\log|t|+C=\log|\cos x+\sin x|+C$, where C is any arbitrary constant.

Hence, the correct answer is option (B).

40. **If $f(a+b-x)=f(x)$, then $\int_a^b xf(x)dx$ is equal to**

A. $\frac{a+b}{2}\int_a^b f(b-x)dx$

B. $\frac{a+b}{2}\int_a^b f(b+x)dx$

C. $\frac{b-a}{2}\int_a^b f(x)dx$

D. $\frac{a+b}{2}\int_a^b f(x)dx$

Ans: Assume, $I=\int_a^b xf(x)dx$ ...(1)

$\Rightarrow I=\int_a^b (a+b-x)f(a+b-x)dx \quad \left[\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

$\Rightarrow I=\int_a^b (a+b-x)f(x)dx=(a+b)\int_a^b f(x)dx-I \quad$ [using (1)]

$$\Rightarrow 2I = (a+b)\int_a^b f(x)dx$$

$$\Rightarrow I = \frac{(a+b)}{2}\int_a^b f(x)dx$$

Hence, the correct answer is option (D).