determinants

4 Chapter

Exercise 4.1

1. Evaluate the determinant: $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$.

Ans: Solving the determinant $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$, we have:

$$\Rightarrow \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1)-4(-5)$$

$$\Rightarrow \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = -2+20$$

$$\therefore \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 18$$

2. Evaluate the determinants.

i. $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Ans: Solving the determinant $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$, we have:

$$\Rightarrow \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos\theta)(\cos\theta)-(-\sin\theta)(\sin\theta)$$

$$\Rightarrow \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta$$

We know,

$$\cos^2\theta + \sin^2\theta = 1$$

$$\therefore \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1$$

ii. $\begin{vmatrix} \mathbf{x^2 - x + 1} & \mathbf{x - 1} \\ \mathbf{x + 1} & \mathbf{x + 1} \end{vmatrix}$

Ans: Solving the determinant $\begin{vmatrix} x^2\text{-}x+1 & x\text{-}1 \\ x+1 & x+1 \end{vmatrix}$, we have:

$$\Rightarrow \begin{vmatrix} x^2\text{-}x+1 & x\text{-}1 \\ x+1 & x+1 \end{vmatrix} = \left(x^2\text{-}x+1\right)\left(x+1\right) - \left(x\text{-}1\right)\left(x+1\right)$$

$$\Rightarrow \begin{vmatrix} x^2\text{-}x+1 & x\text{-}1 \\ x+1 & x+1 \end{vmatrix} = x^3\text{-}x^2+x+x^2\text{-}x+1\text{-}\left(x^2\text{-}1\right)$$

So,

$$\begin{vmatrix} x^2\text{-}x+1 & x\text{-}1 \\ x+1 & x+1 \end{vmatrix} = x^3+1\text{-}x^2+1$$

$$\therefore \begin{vmatrix} x^2\text{-}x+1 & x\text{-}1 \\ x+1 & x+1 \end{vmatrix} = x^3\text{-}x^2+2$$

3. **If** $\mathbf{A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}}$**, then show that** $\mathbf{|2A| = 4|A|}$.

Ans: Given that, $A=\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

Multiplying A by 2, we have:

$\Rightarrow 2A= 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$\Rightarrow 2A=\begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$\therefore$ L.H.S $=|2A|=\begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix}$

$\Rightarrow |2A|=2\times 4 - 4\times 8$

$\Rightarrow |2A|=8-32$

$\therefore |2A|=-24$

The value of determinant A is

$\Rightarrow |A|=\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$

$\Rightarrow |A|=2-8$

$\therefore |A|=-6$

R.H.S is given as $4|A|$.

$\therefore 4|A|=4\times(-6)=-24$

Hence, we have L.H.S $=$ R.H.S

$\therefore |2A|=4|A|$.

4. **If** $\mathbf{A}=\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{1}\\ \mathbf{0} & \mathbf{1} & \mathbf{2}\\ \mathbf{0} & \mathbf{0} & \mathbf{4}\end{bmatrix}$ **, then show that** $\mathbf{3A|=27|A|}$**.**

Ans: Given, $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}$

Determining the value of determinant A, by expanding along the first column, i.e., C_1, we get:

$$\Rightarrow |A|=1\begin{vmatrix}1 & 2\\ 0 & 4\end{vmatrix}-0\begin{vmatrix}0 & 1\\ 0 & 4\end{vmatrix}+0\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix}$$

$$\Rightarrow |A|=1(4-0)-0+0$$

$$\therefore |A|=4$$

Hence, $27|A|=27\times 4$

$$\Rightarrow 27|A|=108 \quad \ldots\ldots(1)$$

The value of $|3A|$ is obtained as:

$$\Rightarrow 3A=3\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}$$

$$\Rightarrow 3A=\begin{bmatrix}3 & 0 & 3\\ 0 & 3 & 6\\ 0 & 0 & 12\end{bmatrix}$$

$$\therefore |3A|=3\begin{vmatrix}3 & 6\\ 0 & 12\end{vmatrix}-0\begin{vmatrix}0 & 3\\ 0 & 12\end{vmatrix}+0\begin{vmatrix}0 & 3\\ 3 & 6\end{vmatrix}$$

$\Rightarrow 3(36\text{-}0)+0+0$

$\Rightarrow |3A|=3\times 36$

Thus, $|3A|=108$ ……(2)

From equations (1) and (2), we have:

$|3A|=27|A|$

Hence proved.

5. Evaluate the determinants

i. $\begin{vmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-5} & \mathbf{0} \end{vmatrix}$

Ans: Let $A=\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Determining the value of A by expanding along the third row, we have:

$\Rightarrow |A|=3\begin{vmatrix} -1 & -2 \\ 0 & -1 \end{vmatrix}-(-5)\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix}+0\begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix}$

$\Rightarrow |A|=(3\text{-}15)$

$\therefore |A|= -12$

ii. $\begin{vmatrix} \mathbf{3} & \mathbf{-4} & \mathbf{5} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} \end{vmatrix}$

Ans: Let $A=\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Determining the value of A by expanding along the first row, we have:

$$\Rightarrow |A|=3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}+4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}+5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$\Rightarrow |A|=3(1+6)+4(1+4)+5(3-2)$$

$$\Rightarrow |A|=21+20+5$$

$$\therefore |A|=46$$

iii. $\begin{vmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{0} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{3} & \mathbf{0} \end{vmatrix}$.

Ans: Let $A=\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

Determining the value of A by expanding along the first row, we have:

$$\Rightarrow |A|=0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix}-1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix}+2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$$

$$\Rightarrow |A|=0(9)-(-6)+2(-3)$$

$$\therefore |A|=0$$

iv. $\begin{vmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{0} & \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-5} & \mathbf{0} \end{vmatrix}$

Ans: Let $A=\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Determining the value of A by expanding along the first column, we have:

$$\Rightarrow |A|=2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix}-0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix}+3\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix}$$

$$\Rightarrow |A|=2(-5)-0+3(5)$$

$$\Rightarrow |A|=-10+15$$

$$\therefore |A|=5$$

6. **If** $A=\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{1} & \mathbf{-3} \\ \mathbf{5} & \mathbf{4} & \mathbf{-9} \end{bmatrix}$**, find** $|A|$**.**

Ans: Given, $A=\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$

Determining the value of A by expanding along the first row, we have:

$$\Rightarrow |A|=1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix}-1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix}-2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

$$\Rightarrow |A| = 1(-9+12) - 1(-18+15) - 2(8-5)$$

$$\Rightarrow |A| = 3+3-6$$

$$\therefore |A| = 0$$

7. Find values of x , if

i. $\begin{vmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{5} & \mathbf{1} \end{vmatrix} = \begin{vmatrix} \mathbf{2x} & \mathbf{4} \\ \mathbf{6} & \mathbf{x} \end{vmatrix}$

Ans: Given, $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Solving it, we have:

$$\Rightarrow (2\times1)-(5\times4) = (2x\times x)-(6\times4)$$

$$\Rightarrow 2-20 = 2x^2-24$$

$$\Rightarrow -18+24 = 2x^2$$

$$\Rightarrow 3 = x^2$$

Applying square root on both the sides, we obtain:

$$\Rightarrow x = \pm\sqrt{3}$$

ii. $\begin{vmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} \end{vmatrix} = \begin{vmatrix} \mathbf{x} & \mathbf{3} \\ \mathbf{2x} & \mathbf{5} \end{vmatrix}$

Ans: Given, $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solving it, we have:

$\Rightarrow (2\times5)-(3\times4)=(x\times5)-(3\times2x)$

$\Rightarrow 10-12=5x-6x\,9$

$\Rightarrow -2=-x$

Multiplying by (-1) on both the sides, we obtain:

$\Rightarrow x = 2$

8. **If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to**

A. **6**
B. **±6**
C. **-6**
D. **0**

Ans: Given, $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

Solving it, we have:

$\Rightarrow x^2-36=36-36$

$\Rightarrow x^2-36=0$

$\Rightarrow x^2=36$

Applying square root on both the sides, we obtain:

$\Rightarrow x=\pm6$

Hence, **B.** ±6 is the correct answer.

Exercise 4.2

1. **Using the property of determinants and without expanding, prove that:**

$$\begin{vmatrix} \mathbf{x} & \mathbf{a} & \mathbf{x+a} \\ \mathbf{y} & \mathbf{b} & \mathbf{y+b} \\ \mathbf{z} & \mathbf{c} & \mathbf{z+c} \end{vmatrix} \mathbf{=0}.$$

Ans: Given matrix $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$.

Applying the Sum Property of determinants, we have

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix}$$

We know, if two rows or columns of a determinant are identical, then the value of the determinant is zero.

Since, the two columns in both the determinants are identical, thus its determinant would be zero.

$$\Rightarrow \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0 + 0$$

$$\therefore \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

2. Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} \mathbf{a-b} & \mathbf{b-c} & \mathbf{c-a} \\ \mathbf{b-c} & \mathbf{c-a} & \mathbf{a-b} \\ \mathbf{c-a} & \mathbf{a-b} & \mathbf{b-c} \end{vmatrix} \mathbf{=0}$$

Ans: Let $\Delta=\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Applying row operation, $R_1 \to R_1+R_2$

$$\Rightarrow \Delta=\begin{vmatrix} a-b+b-c & b-c+c-a & c-a+a-b \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$\Rightarrow \Delta=\begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$\Rightarrow \Delta=\begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ -(a-c) & -(b-a) & -(c-b) \end{vmatrix}$$

Multiplying the third row by (-1), we get:

$$\Rightarrow \Delta=\begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ a-c & b-a & c-b \end{vmatrix}$$

We know, if two rows or columns of a determinant are identical, then the value of the determinant is zero.

Since, the two rows R_1 and R_3 are identical.

$\therefore \Delta=0$

Hence, $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}=0$.

3. Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} \mathbf{2} & \mathbf{7} & \mathbf{65} \\ \mathbf{3} & \mathbf{8} & \mathbf{75} \\ \mathbf{5} & \mathbf{9} & \mathbf{86} \end{vmatrix} = \mathbf{0}$$

Ans: Let $\Delta = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

$$\therefore \Delta = \begin{vmatrix} 2 & 7 & 63+2 \\ 3 & 8 & 72+3 \\ 5 & 9 & 81+5 \end{vmatrix}$$

Applying the Sum Property of determinants, we get

$$\Rightarrow \Delta = \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} + \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix}$$

The two columns of the second determinant are identical, thus it’s value becomes zero.

Hence,

$$\Rightarrow \Delta = \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} + 0$$

$$\Rightarrow \Delta = \begin{vmatrix} 2 & 7 & 9(7) \\ 3 & 8 & 9(8) \\ 5 & 9 & 9(9) \end{vmatrix}$$

Taking 9 common from the third column, we have

$$\Rightarrow \Delta = 9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

Since, the two columns C_2 and C_3

.are identical.

$\therefore \Delta = 0$

Hence, $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

4. Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} \mathbf{1} & \mathbf{bc} & \mathbf{a(b+c)} \\ \mathbf{1} & \mathbf{ca} & \mathbf{b(c+a)} \\ \mathbf{1} & \mathbf{ab} & \mathbf{c(a+b)} \end{vmatrix} = \mathbf{0}.$$

Ans: Let $\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

Applying the column operation, $C_3 \rightarrow C_3 + C_2$.

$$\Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

Taking $(ab+bc+ca)$ common from the third column, we get:

$$\Delta=\left(ab+bc+ca\right)\begin{vmatrix}1 & bc & 1\\ 1 & ca & 1\\ 1 & ab & 1\end{vmatrix}$$

Since, the two columns C_1 and C_3 are identical.

$\therefore \Delta=0$

Hence, $\begin{vmatrix}1 & bc & a\left(b+c\right)\\ 1 & ca & b\left(c+a\right)\\ 1 & ab & c\left(a+b\right)\end{vmatrix}=0$.

5. Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix}\mathbf{b+c} & \mathbf{q+r} & \mathbf{y+z}\\ \mathbf{c+a} & \mathbf{r+p} & \mathbf{z+x}\\ \mathbf{a+b} & \mathbf{p+q} & \mathbf{x+y}\end{vmatrix}=\mathbf{2}\begin{vmatrix}\mathbf{a} & \mathbf{p} & \mathbf{x}\\ \mathbf{b} & \mathbf{q} & \mathbf{y}\\ \mathbf{c} & \mathbf{r} & \mathbf{z}\end{vmatrix}.$$

Ans: Let $\Delta=\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}$

Applying the Sum Property, we get

$$\Rightarrow \Delta=\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a & p & x\end{vmatrix}+\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ b & q & y\end{vmatrix}$$

Suppose $\Delta=\Delta_1+\Delta_2$ ……(1)

Now, $\Delta_1=\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a & p & x\end{vmatrix}$

Applying the row operation, $R_2 \to R_2 - R_3$

$$\Rightarrow \Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c & r & z \\ a & p & x \end{vmatrix}$$

Again, applying the row operation, $R_1 \to R_1 - R_2$

$$\Rightarrow \Delta_1 = \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

We know, that if any two rows or columns of a determinant are interchanged, the value of the determinant is multiplied by (-1).

Hence, interchanging the rows, $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$, we have

$$\Rightarrow \Delta_1 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\therefore \Delta_1 = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad \ldots\ldots(2)$$

We have, $\Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$

Applying the row operation, $R_1 \to R_1 - R_3$

$$\Rightarrow \Delta_2 = \begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying the row operation, $R_2 \to R_2 - R_1$

$$\Rightarrow \Delta_2 = \begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix}$$

Interchanging the rows, $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$

$$\Rightarrow \Delta_2 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\therefore \Delta_2 = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad \ldots\ldots (3)$$

From (2) and (3), we get:

$$\Rightarrow \Delta_1 = \Delta_2 = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

From (1), we have:

$$\Rightarrow \Delta = 2\Delta_1$$

$$\therefore \Delta = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}.$$

Hence, $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}.$

6. By using properties of determinants, show that:

$$\begin{vmatrix} \mathbf{0} & \mathbf{a} & \mathbf{-b} \\ \mathbf{-a} & \mathbf{0} & \mathbf{-c} \\ \mathbf{b} & \mathbf{c} & \mathbf{0} \end{vmatrix} = \mathbf{0}.$$

Ans: Given, $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

We know, that if we multiply the elements of a matrix by a scalar c, then we will multiply the matrix by the scalar, $\frac{1}{c}$.

Applying $R_1 \to cR_1$:

$$\Rightarrow \Delta = \frac{1}{c}\begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 - bR_2$

$$\Delta = \frac{1}{c}\begin{vmatrix} ab & ac & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Taking a common from the first row, we have:

$$\Rightarrow \Delta = \frac{a}{c}\begin{vmatrix} b & c & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Since, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0$

Hence, $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$.

7. By using properties of determinants, show that:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Ans: Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

Taking out a, b, c from R_1, R_2 and R_3 respectively, we have:

$$\Rightarrow \Delta = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Similarly, taking out a, b, c from C_1, C_2 and C_3 respectively, we have:

$$\Rightarrow \Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying the row operations $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$

$$\Rightarrow \Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

Solving it along the first column, C_1 we get:

$$\Rightarrow \Delta = a^2b^2c^2(-1)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$\Rightarrow \Delta = a^2b^2c^2(0-4)$$

$$\therefore \Delta = 4a^2b^2c^2$$

Hence, $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$.

8. By using properties of determinants, show that:

i. $\begin{vmatrix} \mathbf{1} & \mathbf{a} & \mathbf{a^2} \\ \mathbf{1} & \mathbf{b} & \mathbf{b^2} \\ \mathbf{1} & \mathbf{c} & \mathbf{c^2} \end{vmatrix} = \mathbf{(a-b)(b-c)(c-a)}$

Ans: Let $\Delta = \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}$

Applying the row operations $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

We know, $a^2 - b^2 = (a+b)(a-b)$

Thus,

$$\Rightarrow \Delta = (c-a)(b-c)\begin{vmatrix} 0 & -1 & -(a+c) \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$

$$\Rightarrow \Delta = (b-c)(c-a)\begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow \Delta = (b-c)(c-a)\begin{vmatrix} 0 & 0 & -(a-b) \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Taking out $(a-b)$ common from R_1

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Expanding along C_1,

$$\Rightarrow \Delta = (a\text{-}b)(b\text{-}c)(c\text{-}a)\begin{vmatrix} 0 & 1 \\ 1 & b+c \end{vmatrix}$$

$$\therefore \Delta = (a\text{-}b)(b\text{-}c)(c\text{-}a)$$

Hence, $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

ii. $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^3} & \mathbf{b^3} & \mathbf{c^3} \end{vmatrix} = \mathbf{(a-b)(b-c)(c-a)(a+b+c)}$

Ans: Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Applying the column operations, $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix}$$

We know, $x^3 - y^3 = (x-y)(x^2+y^2+xy)$.

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

Taking out $(c-a)$ and $(b-c)$ common from C_1 and C_2 respectively,

$$\Rightarrow \Delta = (c-a)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & c \\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$

$$\Rightarrow \Delta = (c-a)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (b^2-a^2)+c(b-a) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Taking out $(a-c)$ common from C_1, we get:

$$\Rightarrow \Delta=(a-b)(c-a)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & c \\ -(a+b+c) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Again taking out $(a+b+c)$ common from C_1, we get:

$$\Rightarrow \Delta=(a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & c \\ -1 & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Expanding along C_1, we get:

$$\Rightarrow \Delta=(a-b)(b-c)(c-a)(a+b+c)(-1)\begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix}$$

$$\therefore \Delta=(a-b)(b-c)(c-a)(a+b+c)$$

Hence, $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$

9. By using properties of determinants, show that:

$$\begin{vmatrix} \mathbf{x} & \mathbf{x^2} & \mathbf{yz} \\ \mathbf{y} & \mathbf{y^2} & \mathbf{zx} \\ \mathbf{z} & \mathbf{z^2} & \mathbf{xy} \end{vmatrix}=\mathbf{(x-y)(y-z)(z-x)(xy+yz+zx)}.$$

Ans: Let $\Delta=\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$

Applying the row operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta=\begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & zx-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix}$$

$$\Rightarrow \Delta=\begin{vmatrix} x & x^2 & yz \\ -(x-y) & -(x-y)(x+y) & z(x-y) \\ (z-x) & (z-x)(z+x) & -y(z-x) \end{vmatrix}$$

Taking out $(x-y)$ and $(z-x)$ common from R_2 and R_3 respectively

$$\Rightarrow \Delta=(x-y)(z-x)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & z+x & -y \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\Rightarrow \Delta=(x-y)(z-x)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & z-y & z-y \end{vmatrix}$$

Taking out $(z-y)$ common from R_3, we get:

$$\Rightarrow \Delta=(x-y)(z-x)(z-y)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & 1 & 1 \end{vmatrix}$$

Expanding along R_3

$$\Rightarrow \Delta=\left[(x-y)(z-x)(z-y)\right]\left[(-1)\begin{vmatrix} x & yz \\ -1 & z \end{vmatrix}+1\begin{vmatrix} x & x^2 \\ -1 & -x-y \end{vmatrix}\right]$$

$$\Rightarrow \Delta=(x\text{-}y)(z\text{-}x)(z\text{-}y)\left[(\text{-}xz\text{-}yz)+(\text{-}x^2\text{-}xy+x^2)\right]$$

$$\Rightarrow \Delta=-(x-y)(z-x)(z-y)(xy+yz+zx)$$

$$\therefore \Delta=(x-y)(y-z)(z-x)(xy+yz+zx)$$

Hence, $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}=(x-y)(y-z)(z-x)(xy+yz+zx)$

10. By using properties of determinants, show that:

i. $\begin{vmatrix} \mathbf{x+4} & \mathbf{2x} & \mathbf{2x} \\ \mathbf{2x} & \mathbf{x+4} & \mathbf{2x} \\ \mathbf{2x} & \mathbf{2x} & \mathbf{x+4} \end{vmatrix}=\mathbf{(5x+4)(4\text{-}x)^2}$

Ans: Let $\Delta=\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

Applying the row operation, $R_1 \to R_1+R_2+R_3$

$$\Rightarrow \Delta=\begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Taking out $(5x+4)$ common from R_1

$$\Rightarrow \Delta=(5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 0 & x+4 \end{vmatrix}$$

Applying the column operations, $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$\Rightarrow \Delta=(5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$$

Taking out $(4-x)$ common from C_2 and C_3 respectively,

$$\Rightarrow \Delta=(5x+4)(4\text{-}x)(4\text{-}x)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2x & 0 & 1 \end{vmatrix}$$

Expanding along C_3

$$\Rightarrow \Delta=(5x+4)(4-x)^2\begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

$$\therefore \Delta=(5x+4)(4-x)^2$$

Hence, $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}=(5x+4)(4\text{-}x)^2$

ii. $\begin{vmatrix} \mathbf{y+k} & \mathbf{y} & \mathbf{y} \\ \mathbf{y} & \mathbf{y+k} & \mathbf{y} \\ \mathbf{y} & \mathbf{y} & \mathbf{y+k} \end{vmatrix}=\mathbf{k^2(3x+k)}$

Ans: Let $\Delta=\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

Applying the row operation, $R_1 \rightarrow R_1+R_2+R_3$

$$\Rightarrow \Delta=\begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Taking out $(3y+k)$ common from R_1

$$\Rightarrow \Delta=(3y+k)\begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying the column operations, $C_2 \rightarrow C_2-C_1$ and $C_3 \rightarrow C_3-C_1$

$$\Rightarrow \Delta=(3y+k)\begin{vmatrix} 1 & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix}$$

Taking out (k) common from C_2 and C_3 respectively,

$$\Rightarrow \Delta=k^2(3x+k)\begin{vmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y & 0 & 1 \end{vmatrix}$$

Expanding along C_3

$$\Rightarrow \Delta=k^2(3x+k)\begin{vmatrix} 1 & 0 \\ y & 1 \end{vmatrix}$$

$$\therefore \Delta=k^2(3x+k)$$

Hence, $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3x+k)$.

11. By using properties of determinants, show that:

i. $\begin{vmatrix} \mathbf{a-b-c} & \mathbf{2a} & \mathbf{2a} \\ \mathbf{2b} & \mathbf{b-c-a} & \mathbf{2b} \\ \mathbf{2c} & \mathbf{2c} & \mathbf{c-a-b} \end{vmatrix} = \mathbf{(a+b+c)^3}$

Ans: Let $\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying the row operation, $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking out $(a+b+c)$ common from R_1

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying the column operations, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

Taking out $(a+b+c)$ common from C_2 and C_3 respectively,

$$\Rightarrow \Delta=\left(a+b+c\right)^3\begin{vmatrix}1 & 0 & 0\\ 2b & -1 & 0\\ 2c & 0 & -1\end{vmatrix}$$

Expanding along C_3

$$\Rightarrow \Delta=\left(a+b+c\right)^3\left(-1\right)\left(-1\right)$$

$$\therefore \Delta=\left(a+b+c\right)^3$$

Hence, $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}=\left(a+b+c\right)^3$.

ii. $\begin{vmatrix}\mathbf{x+y+2z} & \mathbf{x} & \mathbf{y}\\ \mathbf{z} & \mathbf{y+z+2z} & \mathbf{y}\\ \mathbf{z} & \mathbf{x} & \mathbf{z+x+2y}\end{vmatrix}\mathbf{=2\left(x+y+z\right)^3}$

Ans: Let $\Delta=\begin{vmatrix}x+y+2z & x & y\\ z & y+z+2z & y\\ z & x & z+x+2y\end{vmatrix}$

Applying the column operation, $C_1 \rightarrow C_1+C_2+C_3$

$$\Rightarrow \Delta=\begin{vmatrix}2\left(x+y+z\right) & x & y\\ 2\left(x+y+z\right) & y+z+2z & y\\ 2\left(x+y+z\right) & x & z+x+2y\end{vmatrix}$$

Taking out $2\left(x+y+z\right)$ common from C_1

$$\Rightarrow \Delta=2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2z & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying the row operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta=2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

Taking out $(x+y+z)$ common from R_2 and R_3 respectively,

$$\Rightarrow \Delta=2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3

$$\Rightarrow \Delta=2(x+y+z)^3(1)(1\text{-}0)$$

$$\therefore \Delta=2(x+y+z)^3$$

Hence, $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2z & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3.$

12. By using properties of determinants, show that:

$$\begin{vmatrix} \mathbf{1} & \mathbf{x} & \mathbf{x^2} \\ \mathbf{x^2} & \mathbf{1} & \mathbf{x} \\ \mathbf{x} & \mathbf{x^2} & \mathbf{1} \end{vmatrix} = \mathbf{(1\text{-}x^3)^2}.$$

Ans: Let $\Delta=\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

Applying $R_1 \to R_1+R_2+R_3$

$$\Rightarrow \Delta=\begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Taking out $\left(1+x+x^2\right)$ common from R_1

$$\Rightarrow \Delta=\left(1+x+x^2\right)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Rightarrow \Delta=\left(1+x+x^2\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1\text{-}x^2 & x\text{-}x^2 \\ x & x^2\text{-}x & 1\text{-}x \end{vmatrix}$$

Taking out $(1-x)$ common from C_2 and C_3 respectively,

$$\Rightarrow \Delta=\left(1+x+x^2\right)(1\text{-}x)(1\text{-}x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & \text{-}x & 1 \end{vmatrix}$$

$$\Rightarrow \Delta=\left(1\text{-}x^3\right)(1\text{-}x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & \text{-}x & 1 \end{vmatrix}$$

Expanding along R_1.

$$\Rightarrow \Delta=\left(1-x^3\right)\left(1-x\right)\left(1\right)\begin{vmatrix}1+x & x \\ -x & 1\end{vmatrix}$$

$$\Rightarrow \Delta=\left(1-x^3\right)\left(1-x\right)\left(1+x+x^2\right)$$

$$\Rightarrow \Delta=\left(1-x^3\right)\left(1-x^3\right)$$

$$\therefore \Delta=\left(1-x^3\right)^2$$

Hence, $\begin{vmatrix}1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1\end{vmatrix}=\left(1-x^3\right)^2$.

13. By using properties of determinants, show that:

$$\begin{vmatrix}\mathbf{1+a^2-b^2} & \mathbf{2ab} & \mathbf{-2b} \\ \mathbf{2ab} & \mathbf{1-a^2+b^2} & \mathbf{2a} \\ \mathbf{2b} & \mathbf{-2a} & \mathbf{1-a^2-b^2}\end{vmatrix}=\left(\mathbf{1+a^2+b^2}\right)^{\mathbf{3}}.$$

Ans: Let $\Delta=\begin{vmatrix}1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$

Applying the row operations, $R_1 \to R_1+bR_3$ and $R_2 \to R_2-R_3$

$$\Rightarrow \Delta=\begin{vmatrix}1+a^2+b^2 & 0 & -b\left(1+a^2+b^2\right) \\ 0 & 1+a^2+b^2 & a\left(1+a^2+b^2\right) \\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

Taking out $\left(1+a^2+b^2\right)$ common from R_1 and R_2 respectively,

$$\Rightarrow \Delta=\left(1+a^2+b^2\right)^2\begin{vmatrix}1 & 0 & -b\\ 0 & 1 & a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

Expanding along R_1

$$\Rightarrow \Delta=\left(1+a^2+b^2\right)\left[(1)\begin{vmatrix}1 & a\\ -2a & 1-a^2-b^2\end{vmatrix}-b\begin{vmatrix}0 & 1\\ 2b & -2a\end{vmatrix}\right]$$

$$\Rightarrow \Delta=\left(1+a^2+b^2\right)^2\left[1\text{-}1\text{-}a^2\text{-}b^2+2a^2\text{-}b\left(\text{-}2b\right)\right]$$

$$\Rightarrow \Delta=\left(1+a^2+b^2\right)^2\left(1+a^2+b^2\right)$$

$$\therefore \Delta=\left(1+a^2+b^2\right)^3$$

Hence, $\begin{vmatrix}1+a^2-b^2 & 2ab & -2b\\ 2ab & 1-a^2+b^2 & 2a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}=\left(1+a^2+b^2\right)^3$.

14. By using properties of determinants, show that:

$$\begin{vmatrix}\mathbf{a^2+1} & \mathbf{ab} & \mathbf{ac}\\ \mathbf{ab} & \mathbf{b^2+1} & \mathbf{bc}\\ \mathbf{ca} & \mathbf{cb} & \mathbf{c^2+1}\end{vmatrix}=\mathbf{1+a^2+b^2+c^2}.$$

Ans: Let $\Delta=\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}$

Taking out a,b and c from R_1 , R_2 and R_3 respectively

$$\Rightarrow \Delta=abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Applying the row operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta=abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

Applying $C_1 \to aC_1$, $C_2 \to bC_2$ and $C_3 \to cC_3$

$$\Rightarrow \Delta=abc\times\frac{1}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expanding along C_3

$$\Rightarrow \Delta=-1\begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix}+1\begin{vmatrix} a^2+1 & b^2 \\ -1 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta=1\left(-c^2\right)+\left(a^2+1+b^2\right)$$

$$\therefore \Delta=1+a^2+b^2+c^2$$

Hence, $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}=1+a^2+b^2+c^2$.

15. **Choose the correct answer. Let A be a square matrix of order 3×3, then**

A. $|kA|$ **is equal to** $k|A|$

B. $k^2|A|$

C. $k^3|A|$

D. $3k|A|$

Ans: Since, A is a square matrix of order 3×3.

Let us suppose $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Thus, $kA=\begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}$

$\therefore |kA|=\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$

Taking out (k) common from each row, we have:

$\Rightarrow |kA|=k^3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\therefore |kA|=k^3|A|$

Hence, **B.** $k^3|A|$ is the correct option.

16. **Which of the following is correct?**

A. Determinant is a square matrix.

B. Determinant is a number associated to a matrix.
C. Determinant is a number associated to a square matrix.
D. None of these.

Ans: For every square matrix, $A=\left[a_{ij}\right]$ of order n, we can determine or associate a value which is termed as determinant of square matrix A, where $a_{ij}=(i,j)^{th}$ element of A .

Thus, the determinant is a number associated to a square matrix.

Hence, **C.** Determinant is a number associated to a square matrix is the correct option.

Exercise 4.2

1. Find area of the triangle with vertices at the point given in each of the following:
i. $\mathbf{(1,0),(6,0),(4,3)}$

Ans: Given vertices, $(1,0),(6,0),(4,3)$

We know, if we have three points (x_1,y_1), (x_2,y_2) and (x_3,y_3), then the area of the triangle is given by,

$$\Delta=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus, the area of the triangle is given by,

$$\Rightarrow \Delta=\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{2}\left[1(0\text{-}3)\text{-}0(6\text{-}4)+1(18\text{-}0)\right]$$

$$\Rightarrow \Delta=\frac{1}{2}[\text{-}3+18]$$

$$\Rightarrow \Delta=\frac{15}{2} \text{ square units}$$

$\therefore$ Area of the triangle with vertices $(1,0),(6,0),(4,3)$ is $\frac{15}{2}$ square units.

ii. $(2,7),(1,1),(10,8)$

Ans: Given vertices, $(2,7),(1,1),(10,8)$

We know, if we have three points (x_1,y_1), (x_2,y_2) and (x_3,y_3), then the area of the triangle is given by,

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus, the area of the triangle is given by,

$$\Rightarrow \Delta=\frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta=\frac{1}{2}\left[2(1\text{-}8)\text{-}7(1\text{-}10)+1(8\text{-}10)\right]$$

$$\Rightarrow \Delta=\frac{1}{2}\left[2(\text{-}7)\text{-}7(\text{-}9)+1(\text{-}2)\right]$$

$$\Rightarrow \Delta=\frac{1}{2}\left[-16+63\right]$$

$$\Rightarrow \Delta=\frac{47}{2} \text{ square units}$$

$\therefore$ Area of the triangle with vertices $(2,7),(1,1),(10,8)$ is $\frac{47}{2}$ square units.

iii. $(-2,-3),(3,2),(-1,-8)$

Ans: Given vertices, $(-2,-3),(3,2),(-1,-8)$

We know, if we have three points (x_1,y_1), (x_2,y_2) and (x_3,y_3), then the area of the triangle is given by,

$$\Delta=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus, the area of the triangle with vertices $(-2,-3),(3,2),(-1,-8)$ is given by,

$$\Rightarrow \Delta=\frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta=\frac{1}{2}\left[-2(2+8)+3(3+1)+1(-24+2)\right]$$

$$\Rightarrow \Delta=\frac{1}{2}\left[-20+12-22\right]$$

$$\Rightarrow \Delta=-\frac{30}{2}$$

$$\Rightarrow \Delta=-15$$

$\therefore$The area of the triangle with vertices $(2,7),(1,1),(10,8)$ is $|-15|=15$ square units.

2. Show that points $A(a,b+c),B(b,c+a),C(c,a+b)$ are collinear.

Ans: To show that the points $A(a,b+c),B(b,c+a),C(c,a+b)$ are collinear, the area of the triangle formed by these points as vertices should be zero.

$\therefore$ Area of ΔABC is given by,

$$\Rightarrow \Delta=\frac{1}{2}\begin{vmatrix} a & b+a & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

Applying the row operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b-a & a-b & 0 \\ c-a & a-c & 0 \end{vmatrix}$$

Taking out $(a-b)$ and $(c-a)$ common from R_2 and R_3 respectively,

$$\Rightarrow \Delta=\frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

Applying the row operation $R_3 \to R_3+R_2$

$$\Rightarrow \Delta=\frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

Since all the elements of the last row of the matrix are zero then the value of the determinant will be 0.

$\therefore \Delta=0$

Thus, the area of the triangle formed by points A , B and C is zero.

Hence, the points $A(a,b+c),B(b,c+a),C(c,a+b)$ are collinear.

3. Find values of k if area of triangle is 4 square units and vertices are

i. $(k,0),(4,0),(0,2)$

Ans: Given vertices are $(k,0),(4,0),(0,2)$.

We know, if we have three points (x_1,y_1), (x_2,y_2) and (x_3,y_3), then the area of the triangle is given by,

$$\Delta=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus, the area of the triangle is given by,

$$\Rightarrow \Delta=\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta=\frac{1}{2}\left[k(0-2)-0(4-0)+1(8-0)\right]$$

$$\Rightarrow \Delta=\frac{1}{2}[-2k+8]$$

$\therefore \Delta=-k+4$

Since the area is given to be 4 square units, thus

$-k+4=\pm 4$

When $-k+4=-4$

$\therefore k = 8$.

When $-k + 4 = 4$

$\therefore k = 0$.

Hence, $k=0,8$.

ii. $(-2,0),(0,4),(0,k)$

Ans: Given vertices are $(-2,0),(0,4),(0,k)$.

We know, if we have three points (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by,

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The area of the triangle is given by,

$$\Rightarrow \Delta = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{2}\left[-2(4-k)\right]$$

$$\Rightarrow \Delta = k - 4$$

Since the area is given to be 4 square units, thus

$k - 4 = \pm 4$

When $k - 4 = -4$

$\therefore k = 0$.

When $k - 4 = 4$

$\therefore k = 8$.

Hence, $k = 0,8$.

4. Determine the following:

i. Find equation of line joining $(1,2)$ and $(3,6)$ using determinants.

Ans: Let us assume a point, $P(x, y)$ on the line joining points $A(1,2)$ and $B(3,6)$

Then, the point A, B and P are collinear.

Thus, the area of triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[1(6-y)-2(3-x)+1(3y-6x)\right]=0$$

$$\Rightarrow 6-y-6+2x+3y-6x=0$$

$$\Rightarrow 2y-4x=0$$

$$\Rightarrow y=2x$$

$\therefore$ The equation of the line joining the given points is $y = 2x$.

ii. Find equation of line joining $(3,1)$ and $(9,3)$ using determinants.

Ans: Let us assume a point, $P(x, y)$ on the line joining points $A(3,1)$ and $B(9,3)$.

Then, the point A, B and P are collinear.

Thus, the area of the triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix}=0$$

$$\Rightarrow \frac{1}{2}\left[3(3-y)-1(9-x)+1(9y-3x)\right]=0$$

$$\Rightarrow 9-3y-9+x+9y-3x=0$$

$$\Rightarrow 6y-2x=0$$

$$\Rightarrow x-3y=0$$

$\therefore$ The equation of the line joining the given points is $x-3y=0$.

5. If the area of triangle is 35 square units with vertices $(2,-6)$, $(5,4)$ and $(k,4)$. Then k is

A. 12
B. -2
C. -12,-2
D. 12,-2

Ans: Given vertices, $(2,-6)$, $(5,4)$ and $(k,4)$

We know, if we have three points (x_1,y_1), (x_2,y_2) and (x_3,y_3), then the area of the triangle is given by,

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The area of the triangle is given by,

$$\Rightarrow \Delta=\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta=\frac{1}{2}\left[2(4-4)+6(5-k)+1(20-4k)\right]$$

$$\Rightarrow \Delta=\frac{1}{2}[50-10k]$$

$$\Rightarrow \Delta=25-5k$$

Given, the area of the triangle is 35 square units .

Thus, we have:

$$\Rightarrow 25-5k=\pm 35$$

$$\Rightarrow 5(5-k)=\pm 35$$

$$\Rightarrow 5-k=\pm 7.$$

When $5-k=7$

$\therefore k=-2$.

When $5-k=-7$

$\therefore k=12$.

Hence, $k=12,-2$.

Thus, D. $12,-2$ is the correct option.

Exercise 4.3

1. Write Minors and Cofactors of the elements of following determinants:

i. $\begin{vmatrix} \mathbf{2} & \mathbf{-4} \\ \mathbf{0} & \mathbf{3} \end{vmatrix}$

Ans: Given, $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

Minor of an element is termed as the determinant obtained by removing the row and the column in which that element is present.

Minor of element a_{ij} is denoted by M_{ij},where

i and j denotes the row and the column of the determinant respectively.

$\therefore M_{11}=3$

$M_{12}=0$

$M_{21}=-4$

$M_{22}=2$

Cofactor of an element is termed as the determinant obtained by removing the row and the column in which that element is present preceded by a negative or a positive sign based on the position of the element.

Thus,

Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$

$\Rightarrow A_{11}=(-1)^{1+1}M_{11}$

$\Rightarrow A_{11}=(-1)^{2}(3)$

$\therefore A_{11}=3$

Similarly,

$\Rightarrow A_{12}=(-1)^{1+2}M_{12}$

$\Rightarrow A_{12}=(-1)^{3}(0)$

$\therefore A_{12}=0$

$\Rightarrow A_{21}=(-1)^{2+1} M_{21}$

$\Rightarrow A_{21}=(-1)^{3}(-4)$

$\therefore A_{21}=4$

$\Rightarrow A_{22}=(-1)^{2+2} M_{22}$

$\Rightarrow A_{22}=(-1)^{4}(2)$

$\therefore A_{22}=2$

ii. $\begin{vmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{vmatrix}$

Ans: Given, $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Minor of element a_{ij} is denoted by M_{ij}.

$\therefore M_{11}=d$

$M_{12}=b$

$M_{21}=c$

$M_{22}=a$

Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j} M_{ij}$

$\Rightarrow A_{11}=(-1)^{1+1} M_{11}$

$\Rightarrow A_{11}=(-1)^{2}(d)$

$\therefore A_{11} = d$

Similarly,

$\Rightarrow A_{12} = (-1)^{1+2} M_{12}$

$\Rightarrow A_{12} = (-1)^{3}(b)$

$\therefore A_{12} = -b$

$\Rightarrow A_{21} = (-1)^{2+1} M_{21}$

$\Rightarrow A_{21} = (-1)^{3}(c)$

$\therefore A_{21} = -c$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22}$

$\Rightarrow A_{22} = (-1)^{4}(a)$

$\therefore A_{22} = a$

2. Write Minors and Cofactors of the elements of following determinants:

i. $\begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{vmatrix}$

Ans: Given determinant, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$.

Minor of element a_{ij} is denoted by M_{ij}.

$$\therefore M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\Rightarrow M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow M_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\Rightarrow M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow M_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$$\Rightarrow A_{11} = (-1)^{1+1} M_{11} = 1$$

$$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = 0$$

$$\Rightarrow A_{13} = (-1)^{1+3} M_{13} = 0$$

$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = 0$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = 1$

$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = 0$

$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = 0$

$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = 0$

$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = 1$

ii. $\begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{vmatrix}$

Ans: Given determinant, $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Minor of element a_{ij} is denoted by M_{ij}.

$\Rightarrow M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10+1=11$

$\Rightarrow M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6-0=6$

$\Rightarrow M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3-0=3$

$$\Rightarrow M_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0-4 = -4$$

$$\Rightarrow M_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2-0 = 2$$

$$\Rightarrow M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1-0 = 1$$

$$\Rightarrow M_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0-20 = -20$$

$$\Rightarrow M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1-12 = -13$$

$$\Rightarrow M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5-0 = 5$$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$$\Rightarrow A_{11} = (-1)^{1+1} M_{11} = 11$$

$$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = 6$$

$$\Rightarrow A_{13} = (-1)^{1+3} M_{13} = 3$$

$$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = -4$$

$$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = 2$$

$$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = 1$$

$$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = -20$$

$$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = -13$$

$$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = 5$$

3. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Ans: Given determinant, $\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Determining the minors and cofactors, we get:

$$\Rightarrow M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$$

$$\therefore A_{21} = (-1)^{2+1} M_{21} = 7$$

$$\Rightarrow M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$$

$$\therefore A_{22} = (-1)^{2+2} M_{22} = 7$$

$$\Rightarrow M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$$

$$\therefore A_{23} = (-1)^{2+1} M_{23} = -7$$

Since, Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$\Rightarrow \Delta=2(7)+0(7)+1(7)$

Hence, $\Delta=21$.

4. Using Cofactors of elements of third column, evaluate $\Delta=\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

Ans: Given determinant, $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Determining the minors and cofactors, we get:

$\Rightarrow M_{13}=\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix}=z-y$

$\therefore A_{13}=(-1)^{1+3}M_{13}=(z-y)$

$\Rightarrow M_{23}=\begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix}=z-x$

$\therefore A_{23}=(-1)^{2+3}M_{23}=(x-z)$

$\Rightarrow M_{33}=\begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix}=y-x$

$\therefore A_{33}=(-1)^{3+3}M_{33}=(y-x)$

Since, Δ is equal to the sum of the product of the elements of the first row with their corresponding cofactors.

$\therefore \Delta=a_{13}A_{13}+a_{23}A_{23}+a_{33}A_{33}$

$\Rightarrow \Delta = yz(z - y) + zx(x - z) + xy(y - x)$

$\Rightarrow \Delta = yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$

$\Rightarrow \Delta = (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y)$

$\Rightarrow \Delta = (x - y)[zx + zy - z^2 - xy]$

$\Rightarrow \Delta = (x - y)[z(x - z) + y(z - x)]$

Thus, $\Delta = (x-y)(y-z)(z-x)$.

5. **If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij}, then value of Δ is given by**

a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$
b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$
d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Ans: It is given that $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

The value of Δ by expanding along first column is obtained as,

$a_{11}(a_{22}.a_{33} - a_{23}.a_{32}) - a_{21}(a_{12}.a_{33} - a_{13}.a_{32}) + a_{31}(a_{12}.a_{23} - a_{13}.a_{22})$ ….. 1

Now, the cofactor A_{ij} of element a_{ij} is given by $(-1)^{i+j}M_{ij}$, where M_{ij} is the minor. Minor is the determinant obtained by cancelling the ith row and jth column of the original matrix.

Now for element a_{11}, the minor is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}.a_{33} - a_{23}.a_{32}$ and the cofactor is $A_{11} = (-1)^{1+1}(a_{22}.a_{33} - a_{23}.a_{32}) \Rightarrow A_{11} = (a_{22}.a_{33} - a_{23}.a_{32})$.

Next for element a_{21}, the minor is $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}.a_{33} - a_{13}.a_{32}$ and the cofactor is $A_{21} = (-1)^{2+1}(a_{12}.a_{33} - a_{13}.a_{32}) \Rightarrow A_{21} = -(a_{12}.a_{33} - a_{13}.a_{32})$.

Next for element a_{31}, the minor is $M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}.a_{23} - a_{13}.a_{22}$ and the cofactor is $A_{31} = (-1)^{3+1}(a_{12}.a_{23} - a_{13}.a_{22}) \Rightarrow A_{31} = (a_{12}.a_{23} - a_{13}.a_{22})$.

Now substituting the terms as obtained from above computation in equation 1,

$a_{11}A_{11} - a_{21}(-A_{21}) + a_{31}A_{31}$

$a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

This matches with option d.

Exercise 4.4

1. Find the adjoint of each of the matrices. $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$

Ans: Let $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$.

Thus,

$\Rightarrow A_{11}=(-1)^{1+1}M_{11} = (-1)^2 M_{11}$

$\Rightarrow A_{11} = 4$

$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12}$

$\Rightarrow A_{12} = -3$

$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21}$

$\Rightarrow A_{21} = -2$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22}$

$\Rightarrow A_{22} = 1$

We know that adjoint of a matrix is the transpose of its cofactor matrix.

Thus, $\text{adj}A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T$

$\therefore \text{adj}A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

2. Find adjoint of each of the matrices $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

Ans: Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Since, Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

Thus,

$$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^{2}M_{11}$$

$$\Rightarrow A_{11}=\begin{vmatrix}3 & 5\\ 0 & 1\end{vmatrix}=3-0=3$$

$$\Rightarrow A_{12}=(-1)^{1+2}M_{12}=(-1)^{3}M_{12}$$

$$\Rightarrow A_{12}=\begin{vmatrix}2 & 5\\ -2 & 1\end{vmatrix}=-(2+10)=-12$$

$$\Rightarrow A_{13}=(-1)^{1+3}M_{13}=(-1)^{4}M_{13}$$

$$\Rightarrow A_{13}=\begin{vmatrix}2 & 3\\ -2 & 0\end{vmatrix}=0+6=6$$

Similarly,

$$\Rightarrow A_{21}=(-1)^{2+1}M_{21}=(-1)^{3}M_{21}$$

$$\Rightarrow A_{21}=\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=-(-1-0)=1$$

$$\Rightarrow A_{22}=(-1)^{2+2}M_{22}=(-1)^{4}M_{22}$$

$$\Rightarrow A_{22}=\begin{vmatrix}1 & 2\\ -2 & 1\end{vmatrix}=1+4=5$$

$$\Rightarrow A_{23}=(-1)^{2+3}M_{23}=(-1)^{5}M_{23}$$

$$\Rightarrow A_{23}=\begin{vmatrix}1 & -1\\ -2 & 0\end{vmatrix}=(0-2)=2$$

and

$$\Rightarrow A_{31}=(-1)^{3+1}M_{31}=(-1)^{4}M_{31}$$

$$\Rightarrow A_{31} = \begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix} = -5-4 = -9$$

$$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^5 M_{32}$$

$$\Rightarrow A_{32} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^6 M_{33}$$

$$\Rightarrow A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2=5 .$$

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\text{Thus, } \operatorname{adj}A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -9 & -1 & 5 \end{bmatrix}$$

3. **Verify $A(\operatorname{adj}A) = (\operatorname{adj}A)A = |A|I$. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$**

Ans: Given, $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$$\therefore |A| = -12 - (-12)$$

$$\Rightarrow |A| = 0$$

$$\text{Hence, } |A|I = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow |A|I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since, Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

Then,

$$\Rightarrow A_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11}$$

$$\Rightarrow A_{11} = -6$$

$$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12}$$

$$\Rightarrow A_{12} = 4$$

$$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21}$$

$$\Rightarrow A_{21} = -3$$

$$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22}$$

$$\Rightarrow A_{22} = 2$$

Cofactor matrix is $\begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

Thus, $\text{adj}A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

Now, multiplying A with its adjoint, we have:

$$\Rightarrow A(\text{adj}A) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$\Rightarrow A(\text{adj}A)=\begin{bmatrix}-12+12 & -6+6\\ 24-24 & 12-12\end{bmatrix}$$

$$\therefore A(\text{adj}A)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Similarly, multiplying $(\text{adj}A)$ with A, we get:

$$\Rightarrow(\text{adj}A)A=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$$

$$\Rightarrow(\text{adj}A)A=\begin{bmatrix}-12+12 & -18+18\\ 8-8 & 12-12\end{bmatrix}$$

$$\therefore(\text{adj}A)A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Thus, $A(\text{adj}A)=(\text{adj}A)A=|A|I$

Hence verified.

4. Verify $\mathbf{A(adjA)=(adjA)A=|A|I}$. $\begin{bmatrix}\mathbf{1} & \mathbf{-1} & \mathbf{2}\\ \mathbf{3} & \mathbf{0} & \mathbf{-2}\\ \mathbf{1} & \mathbf{0} & \mathbf{3}\end{bmatrix}$.

Ans: Let $A=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$

$$\Rightarrow|A|=1(0-0)+1(9+2)+2(0-0)$$

$$\therefore|A|=11$$

$$|A|I=11\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}$$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$.

Thus,

$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^2 M_{11}$

$\Rightarrow A_{11}=0$

$\Rightarrow A_{12}=(-1)^{1+2}M_{12}=(-1)^3 M_{12}$

$\Rightarrow A_{12}=-(9+2)=-11$

$\Rightarrow A_{13}=(-1)^{1+3}M_{13}=(-1)^4 M_{13}$

$\Rightarrow A_{13}=0$

Similarly,

$\Rightarrow A_{21}=(-1)^{2+1}M_{21}=(-1)^3 M_{21}$

$\Rightarrow A_{21}=-(-3+0)=3$

$\Rightarrow A_{22}=(-1)^{2+2}M_{22}=(-1)^4 M_{22}$

$\Rightarrow A_{22}=3-2=1$

$\Rightarrow A_{23}=(-1)^{2+3}M_{23}=(-1)^5 M_{23}$

$\Rightarrow A_{23}=-(0+1)=-1$

and

$\Rightarrow A_{31}=(-1)^{3+1}M_{31}=(-1)^4 M_{31}$

$\Rightarrow A_{31} = 2 - 0 = 2$

$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^{5} M_{32}$

$\Rightarrow A_{32} = -(-2-6) = 8$

$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$

$\Rightarrow A_{33} = 0 + 3 = 3.$

Cofactor matrix is $\begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}^{T}$$

$$\therefore \text{adj}A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now, multiplying A with its adjoint, we have:

$$\Rightarrow A(\text{adj}A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\Rightarrow A(\text{adj}A) = \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$\therefore A(\text{adj}A)=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Similarly, multiplying $(\text{adj}A)$ with A, we get:

$$\Rightarrow (\text{adj}A)A=\begin{bmatrix}0 & 3 & 2\\ -11 & 1 & 8\\ 0 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

$$\Rightarrow (\text{adj}A)A=\begin{bmatrix}0+9+2 & 0+0+0 & 0-6+6\\ -11+3+8 & 11+0+0 & -22-2+24\\ 0-3+3 & 0+0+0 & 0+2+9\end{bmatrix}$$

$$\therefore (\text{adj}A)A=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Thus, $A(\text{adj}A)=(\text{adj}A)A=|A|I$

Hence verified.

5. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix}\mathbf{2} & \mathbf{-2}\\ \mathbf{4} & \mathbf{3}\end{bmatrix}$

Ans: Let $A=\begin{bmatrix}2 & -2\\ 4 & 3\end{bmatrix}$

$$\Rightarrow |A|=6+8$$

$$\therefore |A|=14$$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$.

Then,

$\Rightarrow A_{11} = (-1)^{1+1} M_{11} = (-1)^{2} M_{11}$

$\Rightarrow A_{11} = 3$

$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = (-1)^{3} M_{12}$

$\Rightarrow A_{12} = -4$

$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^{3} M_{21}$

$\Rightarrow A_{21} = 2$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^{4} M_{22}$

$\Rightarrow A_{22} = 2$

Cofactor matrix is $\begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$\therefore \text{adj}A = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

Hence, the inverse of the matrix A is given by,

$\Rightarrow A^{-1} = \frac{1}{|A|} \text{adj}A$

$\therefore A^{-1} = \frac{1}{14} \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$.

6. **Find the inverse of each of the matrices (if it exists).** $\begin{bmatrix} \mathbf{-1} & \mathbf{5} \\ \mathbf{-3} & \mathbf{2} \end{bmatrix}$

Ans: Let $A=\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

$\Rightarrow |A| = -2+15$

$\therefore |A| = 13$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j} M_{ij}$.

Then,

$\Rightarrow A_{11}=(-1)^{1+1} M_{11} = (-1)^{2} M_{11}$

$\Rightarrow A_{11}=2$

$\Rightarrow A_{12}=(-1)^{1+2} M_{12} = (-1)^{3} M_{12}$

$\Rightarrow A_{12}=3$

$\Rightarrow A_{21}=(-1)^{2+1} M_{21} = (-1)^{3} M_{21}$

$\Rightarrow A_{21} = -5$

$\Rightarrow A_{22}=(-1)^{2+2} M_{22} = (-1)^{4} M_{22}$

$\Rightarrow A_{22} = -1$

Cofactor matrix is $\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$\therefore \text{adj}A=\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

Hence, the inverse of the matrix A is given by,

$$\Rightarrow A^{-1}=\frac{1}{|A|}\text{adj}A$$

$$\therefore A^{-1}=\frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}.$$

7. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{bmatrix}$

Ans: Let $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Then,

$$\Rightarrow |A|=1(10-0)-2(0-0)+3(0-0)$$

$$\therefore |A|=10$$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$.

Thus,

$$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^{2}M_{11}$$

$$\Rightarrow A_{11}=10-0=10$$

$$\Rightarrow A_{12}=(-1)^{1+2}M_{12}=(-1)^{3}M_{12}$$

$$\Rightarrow A_{12}=-(0+0)=0$$

$\Rightarrow A_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13}$

$\Rightarrow A_{13} = 0$

Similarly,

$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21}$

$\Rightarrow A_{21} = -(10-0) = -10$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22}$

$\Rightarrow A_{22} = 5-0 = 5$

$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^5 M_{23}$

$\Rightarrow A_{23} = -(0-0) = 0$

And

$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^4 M_{31}$

$\Rightarrow A_{31} = 8-6 = 2$

$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^5 M_{32}$

$\Rightarrow A_{32} = -(4-0) = -4$

$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^6 M_{33}$

$\Rightarrow A_{33} = 2-0 = 2.$

Cofactor matrix is $\begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\therefore \text{adj}A=\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the inverse of the matrix A is given by,

$$\Rightarrow A^{-1}=\frac{1}{|A|}\text{adj}A$$

$$\therefore A^{-1}=\frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

8. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{3} & \mathbf{0} \\ \mathbf{5} & \mathbf{2} & \mathbf{-1} \end{bmatrix}$

Ans: Let $A=\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Then,

$$\Rightarrow |A|=1(-3-0)-0+0$$

$$\therefore |A|=-3$$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$.

Thus,

$$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^{2}M_{11}$$

$$\Rightarrow A_{11}=-3-0=-3$$

$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12}$

$\Rightarrow A_{12} = -(-3-0) = 3$

$\Rightarrow A_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13}$

$\Rightarrow A_{13} = 6-15 = -9$

Similarly,

$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21}$

$\Rightarrow A_{21} = -(0+0) = 0$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22}$

$\Rightarrow A_{22} = -1-0 = -1$

$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^5 M_{23}$

$\Rightarrow A_{23} = -(2-0) = -2$

and

$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^4 M_{31}$

$\Rightarrow A_{31} = 0-0 = 0$

$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^5 M_{32}$

$\Rightarrow A_{32} = -(0-0) = 0$

$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^6 M_{33}$

$\Rightarrow A_{33} = 3-0 = 3.$

Cofactor matrix is $\begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}^T$$

$$\therefore \text{adj}A = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

Hence, the inverse of the matrix A is given by,

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj}A$$

$$\therefore A^{-1} = \frac{1}{10}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

9. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{4} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-7} & \mathbf{2} & \mathbf{1} \end{bmatrix}$

Ans: Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Thus,

$\Rightarrow |A| = 2(-1-0) - 1(4-0) + 3(8-7)$

$\Rightarrow |A| = 2(-1) - 1(4) + 3(1)$

$\therefore |A| = -3$

Since, Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

Thus,

$\Rightarrow A_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11}$

$\Rightarrow A_{11} = -1 - 0 = -1$

$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12}$

$\Rightarrow A_{12} = -(4-0) = -4$

$\Rightarrow A_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13}$

$\Rightarrow A_{13} = 8 - 7 = 1$

Similarly,

$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21}$

$\Rightarrow A_{21} = -(1-6) = 5$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22}$

$\Rightarrow A_{22} = 2 + 21 = 23$

$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^5 M_{23}$

$\Rightarrow A_{23} = -(4+7) = -11$

and

$$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^4 M_{31}$$

$$\Rightarrow A_{31} = 0 + 3 = 3$$

$$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^5 M_{32}$$

$$\Rightarrow A_{32} = -(0 - 12) = 12$$

$$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^6 M_{33}$$

$$\Rightarrow A_{33} = -2 - 4 = -6.$$

Cofactor matrix is $\begin{bmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{bmatrix}^T$$

$$\therefore \text{adj}A = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

Hence, the inverse of the matrix A is given by,

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{adj}A$$

$$\therefore A^{-1} = \frac{1}{10} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

10. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{-3} \\ \mathbf{3} & \mathbf{-2} & \mathbf{4} \end{bmatrix}$

Ans: Let $A=\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Expanding along column C_1,

$\Rightarrow |A|=1(8-6)\text{-}0+3(3-4)$

$\therefore |A|=-1$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$.

Thus,

$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^2 M_{11}$

$\Rightarrow A_{11}=8\text{-}6=2$

$\Rightarrow A_{12}=(-1)^{1+2}M_{12}=(-1)^3 M_{12}$

$\Rightarrow A_{12}=-(0+9)=-9$

$\Rightarrow A_{13}=(-1)^{1+3}M_{13}=(-1)^4 M_{13}$

$\Rightarrow A_{13}=0-6=-6$

Similarly,

$\Rightarrow A_{21}=(-1)^{2+1}M_{21}=(-1)^3 M_{21}$

$\Rightarrow A_{21}=-(-4+4)=0$

$$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22}$$

$$\Rightarrow A_{22} = 4 - 6 = -2$$

$$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^5 M_{23}$$

$$\Rightarrow A_{23} = -(-2 + 3) = -1$$

and

$$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^4 M_{31}$$

$$\Rightarrow A_{31} = 3 - 4 = -1$$

$$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^5 M_{32}$$

$$\Rightarrow A_{32} = -(-3 - 0) = 3$$

$$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^6 M_{33}$$

$$\Rightarrow A_{33} = 2 - 0 = 2.$$

Cofactor matrix is $\begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^T$$

$$\Rightarrow \text{adj}A = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

The inverse of the matrix A is given by,

$$\Rightarrow A^{-1}=\frac{1}{|A|}\text{adj}A$$

$$\therefore A^{-1}=-1\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\text{Hence, } A^{-1}=\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}.$$

11. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\cos a} & \mathbf{\sin a} \\ \mathbf{0} & \mathbf{\sin a} & \mathbf{-\cos a} \end{bmatrix}$

Ans: Let $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & \sin a & -\cos a \end{bmatrix}$

Expanding along column, C_1

$$\Rightarrow |A|=1\left(-\cos^2 a-\sin^2 a\right)$$

$$\Rightarrow |A|=-\left(\cos^2 a+\sin^2 a\right)$$

$$\therefore |A|=-1$$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$.

Thus,

$$\Rightarrow A_{11} = (-1)^{1+1} M_{11} = (-1)^{2} M_{11}$$

$$\Rightarrow A_{11} = -\cos^2 a - \sin^2 a = -1$$

$$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = (-1)^{3} M_{12}$$

$$\Rightarrow A_{12} = 0$$

$$\Rightarrow A_{13} = (-1)^{1+3} M_{13} = (-1)^{4} M_{13}$$

$$\Rightarrow A_{13} = 0$$

Similarly,

$$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^{3} M_{21}$$

$$\Rightarrow A_{21} = 0$$

$$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^{4} M_{22}$$

$$\Rightarrow A_{22} = -\cos a$$

$$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^{5} M_{23}$$

$$\Rightarrow A_{23} = -\sin a$$

and

$$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^{4} M_{31}$$

$$\Rightarrow A_{31} = 0$$

$$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^{5} M_{32}$$

$$\Rightarrow A_{32} = -\sin a$$

$$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$$

$\Rightarrow A_{33} = \cos a$.

Cofactor matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}^T$$

$$\therefore \text{adj}A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$$

The inverse of the matrix A is given by,

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj}A$$

$$\therefore A^{-1} = -1\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$$

Hence, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & \sin a & -\cos a \end{bmatrix}$.

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Ans: Let $A=\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

Thus, determining the value of $|A|$,

$\Rightarrow |A|=15-14$

$\therefore |A|=1$

Since, Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j} M_{ij}$.

Thus,

$\Rightarrow A_{11}=(-1)^{1+1} M_{11}=(-1)^{2} M_{11}$

$\Rightarrow A_{11}=5$

$\Rightarrow A_{12}=(-1)^{1+2} M_{12}=(-1)^{3} M_{12}$

$\Rightarrow A_{12}=-2$

$\Rightarrow A_{21}=(-1)^{2+1} M_{21}=(-1)^{3} M_{21}$

$\Rightarrow A_{21}=-7$

$\Rightarrow A_{22}=(-1)^{2+2} M_{22}=(-1)^{4} M_{22}$

$\Rightarrow A_{22}=3$

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$\therefore adjA=\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}^{T}$

$\Rightarrow adjA=\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

The inverse of a matrix is given by, $A^{-1}=\frac{1}{|A|}\text{adj}A$

Hence, $A^{-1}=\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

For $B=\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

$\Rightarrow |B|=54-56$

$\therefore |B|=-2$

Thus,

$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^2 M_{11}$

$\Rightarrow A_{11}=9$

$\Rightarrow A_{12}=(-1)^{1+2}M_{12}=(-1)^3 M_{12}$

$\Rightarrow A_{12}=-7$

$\Rightarrow A_{21}=(-1)^{2+1}M_{21}=(-1)^3 M_{21}$

$\Rightarrow A_{21}=-8$

$\Rightarrow A_{22}=(-1)^{2+2}M_{22}=(-1)^4 M_{22}$

$\Rightarrow A_{22}=6$

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$\therefore \text{adj}A=\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}^T$

$$\Rightarrow \text{adj}A=\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

Hence, $\text{adj}B=\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$$\therefore B^{-1}=\frac{1}{|B|}\text{adj}B=-\frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

Thus, $B^{-1}=\begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}$.

Now, multiplying B^{-1} and A^{-1}, we get:

$$\Rightarrow B^{-1}A^{-1}=\begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$\Rightarrow B^{-1}A^{-1}=\begin{bmatrix} -\frac{45}{2}-8 & \frac{63}{2}+12 \\ \frac{35}{2}+6 & -\frac{49}{2}-9 \end{bmatrix}$$

$$\therefore B^{-1}A^{-1}=\begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \quad \ldots\ldots(1)$$

Similarly, multiplying the matrices A and B, we get:

$$\Rightarrow AB=\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$$

$$\Rightarrow AB=\begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$$

$$\therefore AB=\begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

The value of $|AB|$ is

$$\Rightarrow |AB|=67\times61-87\times47$$

$$\Rightarrow |AB|=4087-4089$$

$$\therefore |AB|=-2$$

The adjoint of (AB) is given by,

$$\Rightarrow adj(AB)=\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

Thus, the inverse is,

$$\Rightarrow (AB)^{-1}=\frac{1}{|AB|}adj(AB)$$

$$\Rightarrow (AB)^{-1}-\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$\therefore (AB)^{-1}=\begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \quad \ldots\ldots. (2)$$

From (1) and (2), we have:

$$(AB)^{-1}=B^{-1}A^{-1}$$

Hence proved.

13. **If $A=\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$,show that $A^2-5A+7I=O$. Hence find A^{-1}.**

Ans: Given, $A=\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

We can write, $A^2=A.A$

$\Rightarrow A^2=\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$\therefore$ The value of $A^2-5A+7I$ is:

$\Rightarrow A^2-5A+7I=\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}-5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}+7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A^2-5A+7I=\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}+\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$\therefore A^2-5A+7I=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence, $A^2-5A+7I=0$

$\Rightarrow A^2-5A=-7I$

Multiplying by A^{-1} on both the sides, we have:

$$\Rightarrow AA\left(A^{-1}\right)-5AA^{-1}=-7IA^{-1}$$

$$\Rightarrow A\left(AA^{-1}\right)-5I=-7IA^{-1}$$

$$\Rightarrow AI-5I=-7IA^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{7}\left(A-5I\right)$$

$$\Rightarrow A^{-1}=\frac{1}{7}\left(5I-A\right)$$

$$\Rightarrow A^{-1}=\frac{1}{7}\left(\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}-\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}\right)$$

$$\therefore A^{-1}=\frac{1}{7}\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$$

14. For the matrix $A=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$ .find the number a and b such that.

$A^2+aA+bI=0$.

Ans: Given $A=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$

We can write, $A^2=A.A$

$$\therefore A^2=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$$

$$\Rightarrow A^2=\begin{bmatrix}9+2 & 6+2\\3+1 & 2+1\end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Solving $A^2 + aA + bI = 0$ by multiplying the whole equation by A^{-1}.

$$\Rightarrow (AA)A^{-1} + aAA^{-1} + bIA^{-1} = 0$$

$$\Rightarrow A\left(AA^{-1}\right) + aI + b\left(IA^{-1}\right) = 0$$

$$\Rightarrow AI + aI + bA^{-1} = 0$$

$$\Rightarrow A + aI = -bA^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{b}(A + aI)$$

Now, determining the value of A^{-1}.

We know that the adjoint of a square matrix is the transpose of its cofactor matrix.

Hence, the adjoint of matrix A is:

$$\therefore \text{adj}A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

The inverse is given by, $A^{-1} = \frac{1}{|A|}\text{adj}A$.

$$\Rightarrow A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Thus,

$$\Rightarrow\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=-\frac{1}{b}\left(\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}+\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}\right)$$

$$\Rightarrow\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=-\frac{1}{b}\begin{bmatrix}3+a & 2\\ 1 & 1+a\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}\frac{-3-a}{b} & -\frac{2}{b}\\ -\frac{1}{b} & \frac{-1-a}{b}\end{bmatrix}$$

Equating the corresponding elements of the two matrices, we get:

$$\Rightarrow -\frac{1}{b}=-1$$

$$\therefore b=1$$

$$\Rightarrow \frac{-3-a}{b}=1$$

$$\therefore a=-4$$

Thus, -4 and 1 are the required values of a and b respectively.

15. For the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$ show that $A^3-6A^2+5A+11I=0$. Hence, A^{-1}.

Ans: Given, $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$

$$\Rightarrow A^2=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$\Rightarrow A^2=\begin{bmatrix}1+1+1 & 1+2-1 & 1-3+3\\ 1+2-6 & 1+4+3 & 1-6-9\\ 2-1+6 & 2-2-3 & 2+3+9\end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}$$

$$\Rightarrow A^3=A^2.A=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$\Rightarrow A^3=\begin{bmatrix}4+2+2 & 4+4-1 & 4-6+3\\ -3+8-28 & -3+16+14 & -3-24-42\\ 7-3+28 & 7-6-14 & 7+9+42\end{bmatrix}$$

$$\therefore A^3=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}$$

Substituting the values for A^3, A^2 and A in $A^3\text{-}6A^2+5A+11I$, we have:

$$\Rightarrow A^3\text{-}6A^2+5A+11I=$$

$$\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}-6\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}+5\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}+11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow A^3\text{-}6A^2+5A+11I=$$

$$\begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 2 & -5 & 15 \end{bmatrix} + 11 \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$\Rightarrow A^3 - 6A^2 + 5A + 11I = \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 5A + 11I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Thus, $A^3 - 6A^2 + 5A + 11I = 0$

Since, $A^3 - 6A^2 + 5A + 11I = 0$.

Multiplying the whole equation by A^{-1}, we have:

$$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0$$

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) = 11(IA^{-1})$$

$$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I) \quad \ldots.\ (1)$$

Now, $A^2 - 6A + 5I$ is given by:

$$\Rightarrow A^2 - 6A + 5I = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^2-6A+5I=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}-\begin{bmatrix}6 & 6 & 6\\ 6 & 12 & -18\\ 12 & 6 & 18\end{bmatrix}+\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$\Rightarrow A^2-6A+5I=\begin{bmatrix}4 & 2 & 1\\ -3 & 13 & -14\\ 7 & -3 & 19\end{bmatrix}-\begin{bmatrix}6 & 6 & 6\\ 6 & 12 & -18\\ 12 & -6 & 18\end{bmatrix}$$

$$\therefore A^2-6A+5I=\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}$$

Substituting for $A^2-6A+5I$ equation (1), we get

$$\Rightarrow A^{-1}=-\frac{1}{11}\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{11}\begin{bmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{bmatrix}$$

16. **If $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ verify that $A^3-6A^2+9A+4I=0$ and hence find A^{-1}.**

Ans: Given, $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

$$\Rightarrow A^2=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^2=\begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

Similarly,

$$\Rightarrow A^3=A^2A=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^3=\begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$\therefore A^3=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3-6A^2+9A-4I$ is given by:

$$\Rightarrow A^3-6A^2+9A-4I=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}-6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}+9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}-4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^3\text{-}6A^2+9A\text{-}4I=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-\begin{bmatrix}36 & -30 & 30\\ -30 & 36 & -30\\ 30 & -30 & 36\end{bmatrix}+\begin{bmatrix}18 & -9 & 9\\ -9 & 18 & -9\\ 9 & -9 & 18\end{bmatrix}-\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$\Rightarrow A^3\text{-}6A^2+9A\text{-}4I=\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}-\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore A^3\text{-}6A^2+9A\text{-}4I=0$

Since, $A^3\text{-}6A^2+9A\text{-}4I=0$.

Multiplying the whole equation by A^{-1}, we have:

$\Rightarrow (AAA)A^{-1}\text{-}6(AA)A^{-1}+9AA^{-1}\text{-}4IA^{-1}=0$

$\Rightarrow AA(AA^{-1})\text{-}6A(AA^{-1})+9(AA^{-1})=4(IA^{-1})$

$\Rightarrow AAI\text{-}6AI+9I=4A^{-1}$

$\Rightarrow A^2\text{-}6A+9I=4A^{-1}$

$\Rightarrow A^{-1}=\frac{1}{4}(A^2\text{-}6A+9I)$ …... (1)

Now, $A^2\text{-}6A+9I$ is given by:

$$A^2\text{-}6A+9I=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-6\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}+9\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$A^2\text{-}6A+9I=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-\begin{bmatrix}12 & -6 & 6\\ -6 & 12 & -6\\ 6 & -6 & 12\end{bmatrix}+\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$\therefore A^2\text{-}6A+9I=\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 3 & 3 \end{bmatrix}$$

Substituting for $A^2\text{-}6A+9I$ equation (1), we get

$$A^{-1}=\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 3 & 3 \end{bmatrix}.$$

17. Let A be nonsingular square matrix of order 3×3 . Then $adjA|$ is equal to

A. $|A|$

B. $|A|^2$

C. $|A|^3$

D. $3|A|$

Ans: Given A is a nonsingular square matrix, i.e., it is a square matrix whose determinant is not equal to zero.

The inverse of a matrix is given as $A^{-1}=\frac{1}{|A|}adjA$.

$$\Rightarrow A^{-1}A=\frac{1}{|A|}adjA$$

$$\Rightarrow |A|I=adjA$$

The adjoint of matrix A is given by,

$$\Rightarrow (adjA)=A=|A|I=\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow \left|(\text{adj}A)A\right| = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow |\text{adj}A||A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A|^3 (I)$$

$\therefore |\text{adj}A| = |A|^3$

Hence, **B.** $|A|^2$ is the correct answer.

18. If A is an invertible matrix of order 2, then $\det\left(A^{-1}\right)$ is equal to

A. $\det(A)$

B. $\dfrac{1}{\det(A)}$

C. 1

D. 0

Ans: Since A is an invertible matrix, thus A^{-1} exists and it is given by:

$A^{-1} = \dfrac{1}{|A|}\text{adj}A$.

As matrix A is of order 2,

$\therefore$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Hence, $|A| = ad - bc$.

The adjoint of A would be, $\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Now, the inverse of the matrix is given by:

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj}A$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A^2|}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A^2|}(ad\text{-}bc)$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A^2|}.|A|$$

$$\therefore |A^{-1}| = \frac{1}{|A|}$$

Thus, $\det(A^{-1}) = \frac{1}{\det(A)}$.

Hence, **B.** $\frac{1}{\det(A)}$ is the correct answer.

Exercise 4.5

1. Examine the consistency of the system of equations.

x+2y=2

2x+3y=3

Ans: Given equations ,

x+2y=2

2x+3y=3

Let us suppose $A=\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ such that, the given system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=1(3)-2(2)=3-4$

$\therefore |A|=-1\neq 0$

Hence, A is non-singular.

Thus, the inverse of A, i.e., A^{-1} exists.

$\therefore$ The given system of equations is consistent.

2. Examine the consistency of the system of equations.

2x-y=5

x+y=4

Ans: Given equations ,

2x-y=5

$x+y=4$

Let us suppose $A=\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ such that, the given system of equation can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=2(1)-(-1)(1)=2+1$

$\therefore |A|=3\neq 0$

Hence, A is non-singular.

Thus, A^{-1} exists.

$\therefore$ The given system of equations is consistent.

3. **Examine the consistency of the system of equations.**

$x+3y=5$

$2x+6y=8$

Ans: Given equations ,

$x+3y=5$

$2x+6y=8$

We know, that a given system of equations is consistent if it has at least one solution.

Let $A=\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 5 \\ 8 \end{bmatrix}$ such that, the given system of equation can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A| = 1(6) - 3(2)$

$\therefore |A| = 0$

Hence, A is a singular matrix.

We know that the adjoint of a square matrix is the transpose of its cofactor matrix.

Determining the adjoint of the matrix A, we have:

$$\Rightarrow (adjA) = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\Rightarrow (adjA)B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30-24 \\ -10+8 \end{bmatrix}$$

$$\therefore (adjA)B = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$$

Thus, the solution of the given system of equations does not exists.

$\therefore$ The given system of equations is inconsistent.

4. Examine the consistency of the system of equations.

$x+y+z=1$

$2x+3y+2z=2$

$ax+ay+2az=4$

Ans: Givenequations

$x+y+z=1$

$2x+3y+2z=2$

$ax+ay+2az=4$

We know, that a given system of equations is consistent if it has at least one solution.

Let $A=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ a & a & 2a\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}$ such that, the given system of equation can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=1(6a-2a)-1(4a-2a)+1(2a-3a)$

$\Rightarrow |A|=4a-2a-a$

$\therefore |A|=a \neq 0$

Hence A is non-singular matrix.

Thus, A^{-1} exists.

$\therefore$ The given system of equation is consistent.

5. **Examine the consistency of the system of equations.**

3x-y-2z=2

2y-z=-1

3x-5y=3

Ans: Given equations ,

$3x-y-2z=2$

$2y-z=-1$

$3x-5y=3$

We know, that a given system of equations is consistent if it has at least one solution.

Let $A=\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=3(-5)-0+3(1+4)$

$\Rightarrow |A|=-15+15$

$\Rightarrow |A|=0$

$\therefore A$ is a singular matrix.

Determining the adjoint of matrix A.

Writing the cofactors,

$$A_{11}=(-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix}=-5$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix}=-3$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}=-6$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix}=10$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix}=6$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = -[-15+3] = 12$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} = [1+4] = 5$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = 3$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6$$

Cofactor matrix is $\begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}$

Taking transpose,

$$\Rightarrow (\text{adj}A) = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\Rightarrow (\text{adj}A)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix}$$

$$\therefore (\text{adj}A)B = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq 0$$

Thus, the solution of the given system of equation does not exist.

$\therefore$ The system of equations is inconsistent.

6. **Examine the consistency of the system of equations.**

5x-y+4z=5

2x+3y+5z=2

5x-2y+6z=-1

Ans: Given equations ,

5x-y+4z=5

2x+3y+5z=2

5x-2y+6z=-1

Let $A=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 3 & -2 & 6 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ such that, the system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=5(18+10)+1(12-25)+4(-4-15)$

$\Rightarrow |A|=5(28)+1(-13)+4(-19)$

$\Rightarrow |A|=140-13-76$

$\therefore |A|=51\neq 0$

Hence, A is a non-singular matrix.

Thus, A^{-1} exists.

$\therefore$ The given system of equations is consistent.

7. **Solve system of linear equations, using matrix method.**

5x+2y=4

7x+3y=5

Ans: Given equations ,

5x+2y=4

7x+3y=5

Let $A=\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=15-14$

$\therefore |A|=1\neq 0$

Thus, A is a non-singular matrix.

Hence, its inverse exists.

We know that the adjoint of a square matrix is the transpose of its cofactor matrix.

$$\therefore \text{adj}A=\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

Thus,

$$\Rightarrow A^{-1}=\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\Rightarrow X=A^{-1}B=\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12+10 \\ -28+52 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Thus, $x=2$ and $y=-3$.

8. **Solve system of linear equations, using matrix method.**

2x-y=-2

3x+4y=3

Ans: Given equations ,

2x-y=-2

3x+4y=3

Let $A=\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=8+3$

$\therefore |A|=11\neq 0$

Thus, A is non-singular.

$\therefore$ It's inverse exists.

We know that the adjoint of a square matrix is the transpose of its cofactor matrix.

$$\therefore \text{adj}A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

Now,

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$\Rightarrow A^{-1} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Thus, $x = -\frac{5}{11}$ and $y = \frac{12}{11}$.

9. Solve system of linear equations, using matrix method.

4x-3y=3

3x-5y=7

Ans: Given equations,

$4x-3y=3$

$3x-5y=7$.

Let $A=\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=-20+9$

$\therefore |A|=-11\neq 0$

Thus, A is a non-singular matrix.

Hence, its inverse exists.

Formula for inverse is $A^{-1}=\frac{adjA}{|A|}$.

Finding cofactors,

$A_{11}=(-1)^{1+1}(-5)=-5$

$A_{12}=(-1)^{1+2}(3)=-3$

$A_{21}=(-1)^{2+1}(-3)=3$

$A_{22}=(-1)^{2+2}(4)=4$

Cofactor matrix is $\begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$.

Taking its transpose to get adjoint matrix as $\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}$.

Therefore inverse is

$\Rightarrow A^{-1}=-\frac{1}{11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}$

$$\therefore X=A^{-1}B=-\frac{1}{11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}15-21\\ 9-28\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-\frac{6}{11}\\ -\frac{19}{11}\end{bmatrix}$$

Thus, $x=-\frac{6}{11}$ and $y=-\frac{19}{11}$.

10. Solve system of linear equations, using matrix method.

5x+2y=3

3x+2y=5

Ans: Given equations ,

5x+2y=3

5x+2y=3

Let $A=\begin{bmatrix}5 & 2\\ 3 & 2\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 5\end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=10-6$

$\therefore |A| = 4 \neq 0$

Thus A is non-singular,

Therefore, its inverse exists.

Formula for inverse is $A^{-1} = \dfrac{\text{adj}A}{|A|}$.

Finding cofactors,

$A_{11} = (-1)^{1+1}(2) = 2$

$A_{12} = (-1)^{1+2}(3) = -3$

$A_{21} = (-1)^{2+1}(2) = -2$

$A_{22} = (-1)^{2+2}(5) = 5$

Cofactor matrix is $\begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}$.

Taking its transpose to get adjoint matrix as $\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$.

Therefore inverse is

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{4}{4} \\ \frac{16}{4} \end{bmatrix}$$

Thus, $x=-1$ and $y=4$.

11. Solve system of linear equations, using matrix method.

2x+y+z=1

x-2y-z=$\frac{3}{2}$

3y-5z=9

Ans: Given equations ,

2x+y+z=1

x-2y-z=$\frac{3}{2}$

3y-5z=9

Let $A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

The determinant of A is found by expanding along the first column,

$\Rightarrow \left|A\right|=2\left(10+3\right)-1\left(-5-3\right)+0$

$\Rightarrow \left|A\right|=2\left(13\right)-1\left(-8\right)$

$\Rightarrow \left|A\right|=34\neq 0$

Thus, A is a non-singular matrix.

$\therefore$ Its inverse exists.

Hence,

$\Rightarrow A_{11}=\left(-1\right)^{1+1}M_{11}=\left(-1\right)^{2}M_{11}$

$\Rightarrow A_{11}=13$

$\Rightarrow A_{12}=\left(-1\right)^{1+2}M_{12}=\left(-1\right)^{3}M_{12}$

$\Rightarrow A_{12}=5$

$\Rightarrow A_{13}=\left(-1\right)^{1+3}M_{13}=\left(-1\right)^{4}M_{13}$

$\Rightarrow A_{13}=3$

Similarly,

$\Rightarrow A_{21}=\left(-1\right)^{2+1}M_{21}=\left(-1\right)^{3}M_{21}$

$\Rightarrow A_{21}=8$

$\Rightarrow A_{22}=\left(-1\right)^{2+2}M_{22}=\left(-1\right)^{4}M_{22}$

$\Rightarrow A_{22}=-10$

$\Rightarrow A_{23}=\left(-1\right)^{2+3}M_{23}=\left(-1\right)^{5}M_{23}$

$\Rightarrow A_{23}=-6$

and

$$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^{4} M_{31}$$

$$\Rightarrow A_{31} = 1$$

$$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^{5} M_{32}$$

$$\Rightarrow A_{32} = 3$$

$$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$$

$$\Rightarrow A_{33} = -5.$$

Cofactor matrix is $\begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}^{T}$$

$$\Rightarrow \text{adj}A = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -16 & -5 \end{bmatrix}$$

The inverse of a matrix is given by:

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$\therefore A^{-1} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -16 & -5 \end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -16 & -5\end{bmatrix}\begin{bmatrix}1\\ \frac{3}{2}\\ 9\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{34}\begin{bmatrix}13+12+9\\ 5-15+27\\ 3-9-45\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{34}\begin{bmatrix}34\\ 17\\ -51\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ \frac{1}{2}\\ -\frac{3}{2}\end{bmatrix}$$

Thus, $x=1$ and $y=\frac{1}{2}$ and $z=-\frac{3}{2}$.

12. Solve system of linear equations, using matrix method.

x-y+z=4

2x+y-3z=0

x+y+z=2

Ans: Given equations ,

x-y+z=4

$2x+y-3z=0$

$x+y+z=2$

Let $A=\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=1(1+3)+1(2+3)+1(2-1)$

$\Rightarrow |A|=4+5+1$

$\therefore |A|=10 \neq 0$

Thus A is non-singular.

$\therefore$ Its inverse exists.

Hence,

$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^{2}M_{11}$

$\Rightarrow A_{11}=4$

$\Rightarrow A_{12}=(-1)^{1+2}M_{12}=(-1)^{3}M_{12}$

$\Rightarrow A_{12}=-5$

$\Rightarrow A_{13}=(-1)^{1+3}M_{13}=(-1)^{4}M_{13}$

$\Rightarrow A_{13}=1$

Similarly,

$\Rightarrow A_{21}=(-1)^{2+1}M_{21}=(-1)^{3}M_{21}$

$\Rightarrow A_{21}=2$

$\Rightarrow A_{22}=(-1)^{2+2}M_{22}=(-1)^{4}M_{22}$

$\Rightarrow A_{22}=0$

$\Rightarrow A_{23}=(-1)^{2+3}M_{23}=(-1)^{5}M_{23}$

$\Rightarrow A_{23}=-2$

and

$\Rightarrow A_{31}=(-1)^{3+1}M_{31}=(-1)^{4}M_{31}$

$\Rightarrow A_{31}=2$

$\Rightarrow A_{32}=(-1)^{3+2}M_{32}=(-1)^{5}M_{32}$

$\Rightarrow A_{32}=5$

$\Rightarrow A_{33}=(-1)^{3+3}M_{33}=(-1)^{6}M_{33}$

$\Rightarrow A_{33}=3.$

Cofactor matrix is $\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A=\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^{T}$$

The inverse of a matrix is given by:

$$\Rightarrow A^{-1}=\frac{1}{|A|}(\text{adj}A)$$

$$\therefore A^{-1}=\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore X=A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Thus, $x=2$, $y=-1$ and $z=1$.

13. Solve system of linear equations, using matrix method.

2x+3y+3z=5

x-2y+z=-4

3x-y-2z=3

Ans: Given equations,

$2x+3y+3z=5$

$x-2y+z=-4$

$3x-y-2z=3$

Let $A=\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$\Rightarrow |A|=2(4+1)-3(2-3)+3(-1+6)$

$\Rightarrow |A|=2(5)-3(-5)+3(5)$

$\Rightarrow |A|=10+15+15$

$\Rightarrow |A|=40 \neq 0$

Thus, A is a non-singular matrix.

$\therefore$ It's inverse exists.

Hence,

$\Rightarrow A_{11}=(-1)^{1+1} M_{11} = (-1)^2 M_{11}$

$\Rightarrow A_{11}=5$

$\Rightarrow A_{12}=(-1)^{1+2} M_{12} = (-1)^3 M_{12}$

$\Rightarrow A_{12} = 5$

$\Rightarrow A_{13}=(-1)^{1+3} M_{13} = (-1)^4 M_{13}$

$\Rightarrow A_{13} = 5$

Similarly,

$\Rightarrow A_{21} = \left(-1\right)^{2+1} M_{21} = \left(-1\right)^{3} M_{21}$

$\Rightarrow A_{21} = 3$

$\Rightarrow A_{22} = \left(-1\right)^{2+2} M_{22} = \left(-1\right)^{4} M_{22}$

$\Rightarrow A_{22} = -13$

$\Rightarrow A_{23} = \left(-1\right)^{2+3} M_{23} = \left(-1\right)^{5} M_{23}$

$\Rightarrow A_{23} = 11$

and

$\Rightarrow A_{31} = \left(-1\right)^{3+1} M_{31} = \left(-1\right)^{4} M_{31}$

$\Rightarrow A_{31} = 9$

$\Rightarrow A_{32} = \left(-1\right)^{3+2} M_{32} = \left(-1\right)^{5} M_{32}$

$\Rightarrow A_{32} = 1$

$\Rightarrow A_{33} = \left(-1\right)^{3+3} M_{33} = \left(-1\right)^{6} M_{33}$

$\Rightarrow A_{33} = -7.$

Cofactor matrix is $\begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T$$

The inverse of a matrix is given by:

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$\therefore A^{-1} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Thus, $x = 1$, $y = 2$ and $z = -1$.

14. Solve system of linear equations, using matrix method.

x-y+2z=7

3x+4y-5z=-5

2x-y+3z=12

Ans: Given equations ,

x-y+2z=7

3x+4y-5z=-5

2x-y+3z=12

Let $A=\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$$\Rightarrow |A|=1(12-5)+1(9+10)+2(-3-8)$$

$$\Rightarrow |A|=7+19-22$$

$$\therefore |A|=4 \neq 0$$

Thus, A is a non-singular matrix.

$\therefore$ It's inverse exists.

Now,

$$\Rightarrow A_{11}=(-1)^{1+1} M_{11}=(-1)^{2} M_{11}$$

$$\Rightarrow A_{11}=7$$

$$\Rightarrow A_{12}=(-1)^{1+2} M_{12}=(-1)^{3} M_{12}$$

$\Rightarrow A_{12} = -19$

$\Rightarrow A_{13} = (-1)^{1+3} M_{13} = (-1)^{4} M_{13}$

$\Rightarrow A_{13} = -11$

Similarly,

$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^{3} M_{21}$

$\Rightarrow A_{21} = 1$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^{4} M_{22}$

$\Rightarrow A_{22} = -1$

$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^{5} M_{23}$

$\Rightarrow A_{23} = -1$

and

$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^{4} M_{31}$

$\Rightarrow A_{31} = -3$

$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^{5} M_{32}$

$\Rightarrow A_{32} = 11$

$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$

$\Rightarrow A_{33} = 7.$

Cofactor matrix is $\begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^T$$

$$\Rightarrow \text{adj}A = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

The inverse of a matrix is given by:

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\3\end{bmatrix}$$

Thus $x=2$, $y=1$ and $z=3$

15. **If $A=\begin{bmatrix}2 & -3 & 5\\3 & 2 & -4\\1 & 1 & -2\end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations**

$$\mathbf{2x-3y+5z=11}$$

$$\mathbf{3x+2y-4z=-5}$$

$$\mathbf{x+y-2z=-3}$$

Ans: Given equations,

$$2x-3y+5z=11$$

$$3x+2y-4z=-5$$

$$x+y-2z=-3$$

Let $A=\begin{bmatrix}2 & -3 & 5\\3 & 2 & -4\\1 & 1 & -2\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}11\\-5\\-3\end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

Determining the value of A, we have:

$$\Rightarrow|A|=2(-4+4)+3(-6+4)+5(3-2)$$

$$\Rightarrow|A|=0-6+5$$

$$\therefore|A|=-1\neq 0$$

Thus, A is a non-singular matrix.

$\therefore$ It's inverse exists.

Now,

$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^{2}M_{11}$

$\Rightarrow A_{11}=0$

$\Rightarrow A_{12}=(-1)^{1+2}M_{12}=(-1)^{3}M_{12}$

$\Rightarrow A_{12}=2$

$\Rightarrow A_{13}=(-1)^{1+3}M_{13}=(-1)^{4}M_{13}$

$\Rightarrow A_{13}=1$

Similarly,

$\Rightarrow A_{21}=(-1)^{2+1}M_{21}=(-1)^{3}M_{21}$

$\Rightarrow A_{21}=-1$

$\Rightarrow A_{22}=(-1)^{2+2}M_{22}=(-1)^{4}M_{22}$

$\Rightarrow A_{22}=-9$

$\Rightarrow A_{23}=(-1)^{2+3}M_{23}=(-1)^{5}M_{23}$

$\Rightarrow A_{23}=-5$

and

$\Rightarrow A_{31}=(-1)^{3+1}M_{31}=(-1)^{4}M_{31}$

$\Rightarrow A_{31}=2$

$\Rightarrow A_{32}=(-1)^{3+2}M_{32}=(-1)^{5}M_{32}$

$\Rightarrow A_{32} = 23$

$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$

$\Rightarrow A_{33} = 13.$

Cofactor matrix is $\begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^{T}$$

$$\Rightarrow \text{adj}A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

The inverse of a matrix is given by:

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$\Rightarrow A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$\therefore X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1 \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1 \begin{bmatrix} 5-6 \\ 22+45-69 \\ 11+25-39 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1 \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus $x=1$, $y=2$ and $z=3$.

16. The cost of $4kg$ onion, $3kg$ wheat and $2kg$ rice is Rs 60. The cost of $2kg$ onion, $4kg$ wheat and $6kg$ rice is Rs 90. The cost of $6kg$ onion $2kg$ wheat and $3kg$ rice is Rs 70.

Find cost of each item per kg by matrix method.

Ans: Let us suppose that the cost of onions, wheat and rice per kg be Rs x , Rs y and Rs z respectively.

Then, the given situation can be represented by a system of equations as:

$\Rightarrow 4x+3y+2z=60$

$\Rightarrow 2x+4y+6z=90$

$\Rightarrow 6x+2y+3z=70$

Let $A=\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$ such that, this system of equations can be written in the form of $AX=B$.

$\Rightarrow |A|=4(12\text{-}12)\text{-}3(6\text{-}36)+2(4\text{-}24)$

$\Rightarrow |A|=0+90\text{-}40$

$\Rightarrow |A|=50 \neq 0$

Thus,

$\Rightarrow A_{11}=(\text{-}1)^{1+1} M_{11}=(\text{-}1)^{2} M_{11}$

$\Rightarrow A_{11}=0$

$\Rightarrow A_{12}=(\text{-}1)^{1+2} M_{12}=(\text{-}1)^{3} M_{12}$

$\Rightarrow A_{12}=30$

$\Rightarrow A_{13}=(\text{-}1)^{1+3} M_{13}=(\text{-}1)^{4} M_{13}$

$\Rightarrow A_{13}=-20$

Similarly,

$\Rightarrow A_{21}=(\text{-}1)^{2+1} M_{21}=(\text{-}1)^{3} M_{21}$

$\Rightarrow A_{21}=-5$

$\Rightarrow A_{22}=(\text{-}1)^{2+2} M_{22}=(\text{-}1)^{4} M_{22}$

$\Rightarrow A_{22}=0$

$\Rightarrow A_{23}=(\text{-}1)^{2+3} M_{23}=(\text{-}1)^{5} M_{23}$

$\Rightarrow A_{23} = 10$

And

$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^{4} M_{31}$

$\Rightarrow A_{31} = 10$

$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^{5} M_{32}$

$\Rightarrow A_{32} = -20$

$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$

$\Rightarrow A_{33} = 10.$

Cofactor matrix is $\begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}^{T}$$

$$\therefore (\text{adj}A) = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$\therefore A^{-1} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Since, $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0+450+700 \\ 1800+0-1400 \\ -1200+900+700 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Thus $x=5$, $y=8$, and $z=8$

Hence, the cost of onions is Rs 5 per kg, the cost of wheat is Rs 8 per kg, and the cost of rice is Rs 8 per kg.

Miscellaneous Solutions

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ **is independent of** θ**.**

Ans: Let $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

Solving it, we have:

$\Rightarrow \Delta = x\left(x^2 - 1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$

$\Rightarrow \Delta = x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$\Rightarrow \Delta = x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right)$

$\Rightarrow \Delta = x^3 - x + x$

$\therefore \Delta = x^3$

Hence, Δ is independent of θ .

2. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Ans: Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along column C_3

$\Rightarrow \Delta = -\sin\alpha\left(-\sin\alpha\sin^2\beta + \cos^2\beta\sin\alpha\right) + \cos\alpha\left(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta\right)$

$\Rightarrow \Delta = \sin^2\alpha\left(s\sin^2\beta + \cos^2\beta\right) + \cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right)$

$\Rightarrow \Delta = \sin^2\alpha(1) + \cos^2\alpha(1)$

$\therefore \Delta = 1$

3. **If** $\mathbf{A^{-1}=\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}}$ **and** $\mathbf{B=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}}$**, find** $\mathbf{(AB)^{-1}}$**.**

Ans: The below result will be used for simplification,

$$(AB)^{-1}=B^{-1}A^{-1}$$

Finding inverse of matrix B. so, the determinant is

$$|B|=1(3-0)-2(-1-0)-2(2-0)$$

$$\Rightarrow |B|=3+2-4$$

$$\therefore |B|=1$$

Now finding the cofactors,

$$B_{11}=(-1)^{1+1}\begin{vmatrix}3 & 0\\ -2 & 1\end{vmatrix}\Rightarrow B_{11}=3$$

$$B_{12}=(-1)^{1+2}\begin{vmatrix}-1 & 0\\ 0 & 1\end{vmatrix}\Rightarrow B_{12}=1$$

$$B_{13}=(-1)^{1+3}\begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix}\Rightarrow B_{13}=2$$

$$B_{21}=(-1)^{2+1}\begin{vmatrix}2 & -2\\ -2 & 1\end{vmatrix}\Rightarrow B_{21}=-(2-4)=2$$

$$B_{22}=(-1)^{2+2}\begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix}\Rightarrow B_{22}=1$$

$$B_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 0 & -2\end{vmatrix}\Rightarrow B_{23}=2$$

$$B_{31}=(-1)^{3+1}\begin{vmatrix}2 & -2\\ 3 & 0\end{vmatrix}\Rightarrow B_{31}=6$$

$$B_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} \Rightarrow B_{32} = 2$$

$$B_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \Rightarrow B_{33} = (3+2) = 5$$

The cofactor matrix is $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$. The adjoint will be the transpose of cofactor matrix.

$$\text{adj}(B) = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

The inverse is given by $B^{-1} = \dfrac{\text{adj}(B)}{|B|}$. So,

$$B^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now, it is already given that $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

So, we can compute $(AB)^{-1} = B^{-1}A^{-1}$ as below,

$$(AB)^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\Rightarrow (AB)^{-1} = \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

4. **Let** $\mathbf{A}=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{5} \end{bmatrix}$ **verify that**

i. $\mathbf{[adjA]^{-1} = adj(A^{-1})}$

Ans: Given, $A=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$$\Rightarrow |A|=1(15-1)-2(10-1)+1(2-3)$$

$$\Rightarrow |A|=14-18-1$$

$$\therefore |A|=-5$$

Thus,

$$\Rightarrow A_{11}=(-1)^{1+1} M_{11} = (-1)^{2} M_{11}$$

$$\Rightarrow A_{11}=14$$

$$\Rightarrow A_{12}=(-1)^{1+2} M_{12} = (-1)^{3} M_{12}$$

$$\Rightarrow A_{12} = -9$$

$$\Rightarrow A_{13}=(-1)^{1+3} M_{13} = (-1)^{4} M_{13}$$

$$\Rightarrow A_{13} = -1$$

Similarly,

$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21}$

$\Rightarrow A_{21} = -9$

$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22}$

$\Rightarrow A_{22} = 4$

$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^5 M_{23}$

$\Rightarrow A_{23} = 1$

And

$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^4 M_{31}$

$\Rightarrow A_{31} = -1$

$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^5 M_{32}$

$\Rightarrow A_{32} = 1$

$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^6 M_{33}$

$\Rightarrow A_{33} = -1.$

Cofactor matrix is $\begin{bmatrix} 14 & -9 & -1 \\ -9 & 4 & 1 \\ -1 & 1 & -1 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} 14 & -9 & -1 \\ -9 & 4 & 1 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\therefore \text{adj}A=\begin{bmatrix}14 & -9 & -1\\ -9 & 4 & 1\\ -1 & 1 & -1\end{bmatrix}$$

Let us denote the adjoint of A as B. So, $B=\begin{bmatrix}14 & -9 & -1\\ -9 & 4 & 1\\ -1 & 1 & -1\end{bmatrix}$.

The inverse of A is given by

$$A^{-1}=\frac{1}{|A|}(\text{adj}A)$$

$$\Rightarrow A^{-1}=-\frac{1}{5}\begin{bmatrix}14 & -9 & -1\\ -9 & 4 & 1\\ -1 & 1 & -1\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{5}\begin{bmatrix}-14 & 9 & 1\\ 9 & -4 & -1\\ 1 & -1 & 1\end{bmatrix}$$

Now, we have to verify $[\text{adj}A]^{-1}=\text{adj}(A^{-1})$.

Let us compute the RHS first, i.e. the adjoint of $A^{-1}=\frac{1}{5}\begin{bmatrix}-14 & 9 & 1\\ 9 & -4 & -1\\ 1 & -1 & 1\end{bmatrix}$ or

$$A^{-1}=\begin{bmatrix}-\frac{14}{5} & \frac{9}{5} & \frac{1}{5}\\ \frac{9}{5} & -\frac{4}{5} & -\frac{1}{5}\\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5}\end{bmatrix}.$$

Thus,

$$\Rightarrow A_{11}=(-1)^{1+1}M_{11}=(-1)^{2}M_{11}$$

$$\Rightarrow A_{11}=-\frac{1}{5}$$

$$\Rightarrow A_{12}=(-1)^{1+2}M_{12}=(-1)^{3}M_{12}$$

$$\Rightarrow A_{12}=-\frac{2}{5}$$

$$\Rightarrow A_{13}=(-1)^{1+3}M_{13}=(-1)^{4}M_{13}$$

$$\Rightarrow A_{13}=-\frac{1}{5}$$

Similarly,

$$\Rightarrow A_{21}=(-1)^{2+1}M_{21}=(-1)^{3}M_{21}$$

$$\Rightarrow A_{21}=-\frac{2}{5}$$

$$\Rightarrow A_{22}=(-1)^{2+2}M_{22}=(-1)^{4}M_{22}$$

$$\Rightarrow A_{22}=-\frac{3}{5}$$

$$\Rightarrow A_{23}=(-1)^{2+3}M_{23}=(-1)^{5}M_{23}$$

$$\Rightarrow A_{23}=-\frac{1}{5}$$

And

$$\Rightarrow A_{31}=(-1)^{3+1}M_{31}=(-1)^{4}M_{31}$$

$$\Rightarrow A_{31} = -\frac{1}{5}$$

$$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^{5} M_{32}$$

$$\Rightarrow A_{32} = -\frac{1}{5}$$

$$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$$

$$\Rightarrow A_{33} = -1.$$

Cofactor matrix is $\begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -1 \end{bmatrix}$. We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -1 \end{bmatrix}^{T}$$

$$\therefore \text{adj}A^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -1 \end{bmatrix}$$

Next, moving to the LHS, i.e. the inverse of adjoint of A. We have adjoint of A as matrix B,

$$B=\begin{bmatrix} 14 & -9 & -1 \\ -9 & 4 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

Determinant of B is

$$|B| = \left[14(-4-1)+9(9+1)-1(-9+4)\right]$$

$$\Rightarrow |B| = [-70+90+5]$$

$$\therefore |B| = 25$$

Now the cofactors,

Thus,

$$\Rightarrow B_{11}=(-1)^{1+1}M_{11}=(-1)^{2}M_{11}$$

$$\Rightarrow B_{11}=-5$$

$$\Rightarrow B_{12}=(-1)^{1+2}M_{12}=(-1)^{3}M_{12}$$

$$\Rightarrow B_{12}=-10$$

$$\Rightarrow B_{13}=(-1)^{1+3}M_{13}=(-1)^{4}M_{13}$$

$$\Rightarrow B_{13}=-5$$

Similarly,

$$\Rightarrow B_{21}=(-1)^{2+1}M_{21}=(-1)^{3}M_{21}$$

$$\Rightarrow B_{21}=-10$$

$$\Rightarrow B_{22}=(-1)^{2+2}M_{22}=(-1)^{4}M_{22}$$

$\Rightarrow B_{22} = -15$

$\Rightarrow B_{23} = (-1)^{2+3} M_{23} = (-1)^{5} M_{23}$

$\Rightarrow B_{23} = -5$

And

$\Rightarrow B_{31} = (-1)^{3+1} M_{31} = (-1)^{4} M_{31}$

$\Rightarrow B_{31} = -5$

$\Rightarrow B_{32} = (-1)^{3+2} M_{32} = (-1)^{5} M_{32}$

$\Rightarrow B_{32} = -5$

$\Rightarrow B_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$

$\Rightarrow B_{33} = -25.$

Cofactor matrix is $\begin{bmatrix} -5 & -10 & -5 \\ -10 & -15 & -5 \\ -5 & -5 & -25 \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}B = \begin{bmatrix} -5 & -10 & -5 \\ -10 & -15 & -5 \\ -5 & -5 & -25 \end{bmatrix}^{T}$$

$$\therefore \text{adj}B = \begin{bmatrix} -5 & -10 & -5 \\ -10 & -15 & -5 \\ -5 & -5 & -25 \end{bmatrix}$$

Now the inverse is found as

$$B^{-1}=\frac{1}{|B|}(\text{adj}B)$$

$$\Rightarrow B^{-1}=\frac{1}{25}\begin{bmatrix} -5 & -10 & -5 \\ -10 & -15 & -5 \\ -5 & -5 & -25 \end{bmatrix}$$

$$\therefore B^{-1}=\begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -1 \end{bmatrix}$$

So, LHS is $[\text{adj}A]^{-1}=\begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -1 \end{bmatrix}$.

Since LHS=RHS, it is verified that $[\text{adj}A]^{-1}=\text{adj}(A^{-1})$.

ii. $\mathbf{(A^{-1})^{-1}=A}$

Ans:

Since $A^{-1}=\begin{bmatrix} -\frac{14}{5} & \frac{9}{5} & \frac{1}{5} \\ \frac{9}{5} & -\frac{4}{5} & -\frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$ and $\text{adj}A^{-1}=\begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -1 \end{bmatrix}$, thus,

$$\Rightarrow |A^{-1}|=\left[-\frac{14}{5}\left(-\frac{4}{25}-\frac{1}{25}\right)-\frac{9}{5}\left(\frac{9}{25}+\frac{1}{25}\right)+\frac{1}{5}\left(-\frac{9}{25}+\frac{4}{25}\right)\right]$$

$$\Rightarrow |A^{-1}|=\left[\frac{70}{125}-\frac{90}{125}-\frac{5}{125}\right]$$

$$\therefore |A^{-1}|=-\frac{1}{5}$$

Also, we know that an inverse of a matrix is given by:

$$\Rightarrow \left(A^{-1}\right)^{-1}=\frac{\text{adj}A^{-1}}{|A|}$$

$$\Rightarrow \left(A^{-1}\right)^{-1}=-5\begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -1 \end{bmatrix}$$

$$\Rightarrow \left(A^{-1}\right)^{-1}=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \left(A^{-1}\right)^{-1}=A$$

Hence verified that $\left(A^{-1}\right)^{-1}=A$.

5. **Evaluate** $\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{x+y} \\ \mathbf{y} & \mathbf{x+y} & \mathbf{x} \\ \mathbf{x+y} & \mathbf{x} & \mathbf{y} \end{vmatrix}$

Ans: Given, $\Delta=\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Applying $R_1 \rightarrow R_1+R_2+R_3$

$$\Delta=\begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Taking $2(x+y)$ common from R_1

$$\Rightarrow \Delta=2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying the row operations $C_2 \rightarrow C_2-C_1$ and $C_3 \rightarrow C_3-C_1$

$$\Delta=2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x\text{-}y \\ x+y & \text{-}y & \text{-}x \end{vmatrix}$$

Expanding along R_1

$$\Rightarrow \Delta=2(x+y)\left[1\left(\left(x\times(\text{-}x)\right)-\left(-y(x-y)\right)\right)+0+0\right]$$

$$\Rightarrow \Delta=2(x+y)\left[1\left((-x^2)-(-yx+y^2)\right)\right]$$

$$\Rightarrow \Delta=2(x+y)\left[-x^2+xy-y^2\right]$$

$$\Rightarrow \Delta=-2(x+y)\left[x^2-xy+y^2\right]$$

Applying the identity,

$$\therefore \Delta=-2\left[x^3+y^3\right]$$

6. **Evaluate** $\begin{vmatrix} \mathbf{1} & \mathbf{x} & \mathbf{y} \\ \mathbf{1} & \mathbf{x+y} & \mathbf{y} \\ \mathbf{1} & \mathbf{x} & \mathbf{x+y} \end{vmatrix}$

Ans: Given, $\Delta=\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Applying the row operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta=\begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding along C_1

$$\Rightarrow \Delta=1(xy-0)$$

$$\therefore \Delta = xy$$

7. **Solve the system of the following equations**

$$\mathbf{\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4}$$

$$\mathbf{\frac{4}{x}+\frac{6}{y}+\frac{5}{z}=1}$$

$$\mathbf{\frac{6}{x}+\frac{9}{y}+\frac{20}{z}=2}$$

Ans: Given equations ,

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}+\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}+\frac{20}{z}=2$$

Let $\frac{1}{x}=p$, $\frac{1}{y}=q$ and $\frac{1}{z}=r$

Then the given system of equations becomes:

$2p+3q+10r=4$

$4p-6q+5r=1$

$6p+9q+20r=2$

Let $A=\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $X=\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ such that, this system can be written in the form of $AX=B$.

Now,

$\Rightarrow |A|=2(120-45)-3(-80-30)+10(36+36)$

$\Rightarrow |A|=150+330+720$

$\Rightarrow |A| = 1200$

Thus, A is a non-singular matrix.

$\therefore$ Its inverse exists.

So,

$$\Rightarrow A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix}$$

$$\Rightarrow A_{11} = 75$$

$$\Rightarrow A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix}$$

$$\Rightarrow A_{12} = 110$$

$$\Rightarrow A_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}$$

$$\Rightarrow A_{13} = 72$$

Similarly,

$$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^3 \begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix}$$

$$\Rightarrow A_{21} = 150$$

$$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix}$$

$$\Rightarrow A_{22} = 100$$

$$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix}$$

$\Rightarrow A_{23}=0$

and

$$\Rightarrow A_{31}=(-1)^{3+1}M_{31}=(-1)^{4}\begin{vmatrix}3 & 10\\ -6 & 5\end{vmatrix}$$

$\Rightarrow A_{31}=75$

$$\Rightarrow A_{32}=(-1)^{3+2}M_{32}=(-1)^{5}\begin{vmatrix}2 & 10\\ 4 & 5\end{vmatrix}$$

$\Rightarrow A_{32}=30$

$$\Rightarrow A_{33}=(-1)^{3+3}M_{33}=(-1)^{6}\begin{vmatrix}2 & 3\\ 4 & -6\end{vmatrix}$$

$\Rightarrow A_{33}=-24.$

The inverse of a matrix is given by:

$$\Rightarrow A^{-1}=\frac{1}{|A|}(\text{adj}A)$$

$$\therefore A^{-1}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$

Now, $X=A^{-1}B$

$$\Rightarrow \begin{bmatrix}p\\ q\\ r\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$\therefore P=\frac{1}{2},\ q=\frac{1}{3}\ \text{and}\ r=\frac{1}{5}$

Thus $x=2$, $y=3$ and $z=5$.

8. Choose the correct answer.

If X,Y,Z are nonzero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

A. $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

B. $\mathbf{xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}}$

C. $\mathbf{\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}}$

D. $\mathbf{\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

Ans: Given, $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$\Rightarrow |A|=x(yz-0)$

$\therefore |A|=xyz \neq 0$

Thus,

$\Rightarrow A_{11}=(-1)^{1+1} M_{11}=(-1)^{2} M_{11}$

$\Rightarrow A_{11}=yz$

$\Rightarrow A_{12}=(-1)^{1+2} M_{12}=(-1)^{3} M_{12}$

$\Rightarrow A_{12}=0$

$\Rightarrow A_{13}=(-1)^{1+3} M_{13}=(-1)^{4} M_{13}$

$\Rightarrow A_{13}=0$

Similarly,

$$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = (-1)^{3} M_{21}$$

$$\Rightarrow A_{21} = 0$$

$$\Rightarrow A_{22} = (-1)^{2+2} M_{22} = (-1)^{4} M_{22}$$

$$\Rightarrow A_{22} = xz$$

$$\Rightarrow A_{23} = (-1)^{2+3} M_{23} = (-1)^{5} M_{23}$$

$$\Rightarrow A_{23} = 0$$

and

$$\Rightarrow A_{31} = (-1)^{3+1} M_{31} = (-1)^{4} M_{31}$$

$$\Rightarrow A_{31} = 0$$

$$\Rightarrow A_{32} = (-1)^{3+2} M_{32} = (-1)^{5} M_{32}$$

$$\Rightarrow A_{32} = 0$$

$$\Rightarrow A_{33} = (-1)^{3+3} M_{33} = (-1)^{6} M_{33}$$

$$\Rightarrow A_{33} = xy.$$

Cofactor matrix is $\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$.

We know that adjoint of a matrix is the transpose of its cofactor matrix.

$$\Rightarrow \text{adj}A = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}^{T}$$

$$\therefore \text{adj}A=\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

The inverse of a matrix is given by:

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$\Rightarrow A^{-1}=\frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\Rightarrow A^{-1}=\begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix}$$

$$\Rightarrow A^{-1}=\begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

$$\therefore A^{-1}=\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

Thus, **A.** $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ is the correct answer.

9. Choose the correct answer.

Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$**, where** $0 \le \theta \le 2n$ **, then**

A. $\text{Det}(A) = 0$

B. $\text{Det}(A) \in (2, \infty)$

C. $\text{Det}(A) \in (2, 4)$

D. $\text{Det}(A)[2, 4]$

Ans: Given, $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$\therefore |A| = 1(1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$

$\Rightarrow |A| = 1 + \sin^2\theta + \sin^2\theta + 1$

$\Rightarrow |A| = 2 + 2\sin^2\theta$

$\Rightarrow |A| = 2(1 + \sin^2\theta)$

We know, $0 \le \sin\theta \le 1$

$\Rightarrow 1 \le 1 + \sin^2\theta \le 2$

$\Rightarrow 2 \le 2(1 + \sin^2\theta) \le 4$

Thus, **D.** $\text{Det}(A)[2, 4]$ is the correct answer.