Relations and Functions

1 Chapter

Exercise 1.1

1. Determine whether each of the following relations are reflexive, symmetric and transitive.

i. Relation $\mathbf{R}$ in the set $\mathbf{A} = \{1, 2, 3...13, 14\}$ defined as $\mathbf{R} = \{(x, y): 3x - y = 0\}$

Ans: The given relation is: $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

Since $(1, 1), (2, 2)$ ... and $(14, 14) \notin R$.

We conclude that R is not reflexive.

Since $(1, 3) \in R$, but $(3, 1) \notin R$. [since $3(3)-1 \neq 0$]

We conclude that R does not belong to symmetric.

Since $(1, 3)$ and $(3, 9) \in R$, but $(1, 9) \notin R$. $[3(1)-9 \neq 0]$.

We conclude that R is not transitive.

Therefore, the relation R is not reflexive, symmetric or transitive.

ii. Relation $\mathbf{R}$ in the set $\mathbf{N}$ of natural numbers defined as $\mathbf{R} = \{(x, y): y = x + 5 \text{ and } x<4\}$

Ans: The given relation is: $R = \{(1, 6), (2, 7), (3, 8)\}$.

Since $(1, 1) \notin R$.

We conclude that R is not reflexive.

Since $(1, 6) \in R$ but $(6, 1) \notin R$.

We conclude that R does not belong to symmetric.

In the given relation R there is not any ordered pair such that (x, y) and (y, z) both $\in R$, therefore we can say that (x, z) cannot belong to R.

Therefore R is not transitive.

Hence, the given relation R is not reflexive, symmetric or transitive.

iii. Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(x, y): y$ is divisible by $x\}$

Ans: The given relation is $R = \{(x, y) : y$ is divisible by $x\}$

As we know that any number except 0 is divisible by itself, therefore $(x, x) \in R$

We conclude that R is reflexive.

Since $(2, 4) \in R$ [because 4 is divisible by 2], but $(4, 2) \notin R$ [since 2 is not divisible by 4].

We conclude that R does not belong to symmetric.

Assuming that (x, y) and $(y, z) \in R$, y is divisible by x and z is divisible by y
Hence z is divisible by $x \Rightarrow (x, z) \in R$.

We conclude that R is transitive.

Hence, the given relation R is reflexive and transitive but it does not belong to symmetric.

iv. Relation R in the set Z of all integers defined as $R = \{(x, y): x - y\}$ is as integer

Ans: The given relation is $R = \{(x, y): x - y$ is an integer$\}$

If $x \in Z$, $(x, x) \in R$ because $x - x = 0$ is an integer.

Hence, we conclude that R is reflexive.

For $x, y \in Z$, if $(x, y) \in R$, then $x - y$ is an integer and therefore $(y - x)$ is also an integer.

Therefore, we conclude that $(y, x) \in R$ and hence R is symmetric.

Assuming that (x, y) and $(y, z) \in R$, where $x, y, z \in Z$.

We can say that $(x - y)$ and $(y - z)$ are integers.

Therefore $x - z = (x - y) + (y - z)$ is also an integer, so, $(x, z) \in R$

Hence, we conclude that R is transitive.

Therefore the given relation R is reflexive, symmetric, and transitive.

v. Relation R in the set A of human beings in a town at a particular time given by

a) The relation is: $R = \{(x, y): x \text{ and } y \text{ work at the same place}\}$

Ans: The given relation is: $R = \{(x, y): x \text{ and } y \text{ work at the same place}\}$

This implies that $(x, y) \in R$.

Hence, we conclude that R is reflexive.

Now, $(x, y) \in R$, then x and y work at the same place, which means y and x also work at the same place. Therefore, $(y, x) \in R$.

Hence, we conclude that R is symmetric.

Let us assume that (x, y), $(y, z) \in R$.

Then, we can say that x and y work at the same place and y and z work at the same place. Which means that x and z also work at the same place.

Therefore, $(x, z) \in R$.

Hence, we conclude that R is transitive.

Therefore, the given relation R is reflexive, symmetric and transitive.

b) The relation is: $R = \{(x, y): x \text{ and } y$ live in the same locality$\}$

Ans: The given relation is $R = \{(x, y): x \text{ and } y$ live in the same locality$\}$

Since, $(x, x) \in R$.

Therefore, we conclude that R is reflexive.

Since $(x, y) \in R$, x and y live in the same locality. Therefore, y and x also live in the same locality, so, $(y, x) \in R$.

Hence, R is symmetric.

Let $(x, y) \in R$ and $(y, z) \in R$. Hence x and y live in the same locality and y and z also live in the same locality. Which means that x and z also live in the same locality.

Therefore, $(x, z) \in R$.

Hence, we conclude that R is transitive.

Therefore, the given relation R is reflexive, symmetric and transitive.

c) $R = \{(x, y): x$ is exactly 7 cm taller than $y\}$

Ans: The given relation is:

Since, $(x, x) \notin R$.

$R = \{(x, y): x$ is exactly 7 cm taller than $y\}$

Therefore, we conclude that R not is reflexive.

Let $(x, y) \in R$, Since x is exactly 7 cm taller than y, therefore y is obviously not taller than x, so, $(y, x) \notin R$.

Hence, R is not symmetric.

Assuming that (x, y), $(y, z) \in R$, we can say that x is exactly 7 cm taller than y and y is exactly 7 cm taller than z. Which means that x is exactly 14 cm taller than z. So, $(x, z) \notin R$.

Hence, R is not transitive.

Therefore, the given relation R is not reflexive, symmetric or transitive.

d) $R = \{(x, y): x$ is wife of $y\}$

Ans: The given relation is: $R = \{(x, y): x$ is the wife of $y\}$.

Since, $(x, x) \notin R$

Therefore, we conclude that R not is reflexive.

Let $(x, y) \in R$, Since x is the wife of y, therefore y is obviously not the wife of x, so, $(y, x) \notin R$.

Hence, R is not symmetric.

Assuming that (x, y), $(y, z) \in R$, we can say that x is the wife of y and y is the wife of z, which is not possible. So, $(x, z) \notin R$.

Hence, R is not transitive.

Therefore the given relation R is not reflexive, symmetric or transitive.

e) $R = \{(x, y): x$ is father of $y\}$

Ans: The given relation is: $R = \{(x, y): x \text{ is the father of } y\}$

Since, $(x, x) \notin R$

Therefore, we conclude that R not is reflexive.

Let $(x, y) \in R$, Since x is the father of y, therefore y is obviously not the father of x, so, $(y, x) \notin R$.

Hence, R is not symmetric.

Assuming that (x, y), $(y, z) \in R$, we can say that x is the father of y and y is the father of z, then x is not the father of z. So, $(x, z) \notin R$.

Hence, R is not transitive.

Therefore the given relation R is not reflexive, symmetric or transitive.

2. Show that the relation R in the set R of real numbers, defined $R = \{(a, b): a \le b^2\}$ is neither reflexive nor symmetric nor transitive.

Ans: The given relation is: $R = \{(a, b): a \le b^2\}$

Since $\left(\frac{1}{2}, \frac{1}{4}\right) \notin R$. [Since $\frac{1}{2}$ is not less than $\frac{1}{4}$]

Therefore, R is not reflexive.

Since $(1, 4) \in R$ as $1 < 4^2$, but $(4, 1) \notin R$ as 4^2 is not less than 1^2.

Therefore R is not symmetric.

Assuming that $(3, 2)$, $(2, 1.5) \in R$, so, $3 < 2^2 = 4$ and $2 < (1.5)^2 = 2.25$ but $3 > (1.5)^2 = 2.25$.

Hence, R is not transitive.

Therefore, the given relation R is neither reflexive, nor symmetric, nor transitive.

3. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b): b=a+1\}$ is reflexive, symmetric or transitive.

Ans: The given relation is $R = \{(a, b): b=a+1\}$ defined in the set $A=\{1, 2, 3, 4, 5, 6\}$.

So, $R=\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

Since, $(a, a) \notin R, a \in A$.

$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \notin R$

Therefore, R is not reflexive

Since, $(1, 2) \in R$, but $(2, 1) \notin R$.

Therefore R is not symmetric.

Since $(1, 2), (2, 3) \in R$, but $(1, 3) \notin R$.

Hence, R is not transitive.

Therefore, the given relation R is neither reflexive, nor symmetric, nor transitive.

4. Show that the relation R in R defined as $R = \{(a, b): a \leq b\}$ is reflexive and transitive but not symmetric.

Ans: The given relation is $R = \{(a, b): a \leq b\}$.

Since, $(a, a) \in R$.

Therefore, R is reflexive.

Since, $(2, 4) \in R$ (as $2<4$), but $(4, 2) \notin R$(as $4>2$).

Therefore R is not symmetric.

Assuming that $(a, b), (b, c) \in R$, $a \le b$ and $b \le c$, therefore, $a \le c$.

Hence, R is transitive.

Therefore, the given relation R is reflexive and transitive but not symmetric.

5. Check whether the relation R in R defined as $R = \{(a, b): a \le b^3\}$ is reflexive, symmetric or transitive.

Ans: The given relation is: $R = \{(a, b): a \le b^3\}$

Since $\left(\frac{1}{2}, \frac{1}{8}\right) \notin R$. [Since $\frac{1}{2}$ is not less than $\frac{1}{8}$]

Therefore, R is not reflexive.

Since $(1, 4) \in R$ as $1 < 4^3$, but $(4, 1) \notin R$ as 4 is not less than 1^3.

Therefore R is not symmetric.

Assuming that $\left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{6}{5}\right) \in R$, so, $3 < \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} < \left(\frac{6}{5}\right)^3$ but $3 > \left(\frac{6}{5}\right)^3$.

Hence, R is not transitive.

Therefore, the given relation R is neither reflexive, nor symmetric, nor transitive.

6. Show that the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is symmetric but neither reflexive nor transitive.

Ans: The given relation is $R = \{(1, 2), (2, 1)\}$ on the set $A = \{1, 2, 3\}$.

Since $(1, 1), (2, 2), (3, 3) \notin R$

Therefore, R is not reflexive.

Since, $(1, 2) \in R$ and $(2, 1) \in R$.

Therefore R is symmetric.

Since, $(1, 2) \in R$ and $(2, 1) \in R$, but $(1, 1) \notin R$.

Hence, R is not transitive.

Therefore, the given relation R is symmetric but neither reflexive nor transitive.

7. Show that the relation R in the set A of all the books in a library of a college, given by $R = \{(x, y): x$ and y have same number of pages$\}$ is an equivalence relation.

Ans: The given relation is: $R = \{(x, y): x$ and y have the same number of pages$\}$

Since $(x, x) \notin R$ as x and x have same number of pages.

Therefore, R is reflexive.

Let $(x, y) \in R$, so x and y have the same number of pages, therefore y and x will also have the same number of pages.

Therefore R is symmetric.

Assuming $(x, y) \in R$ and $(y, z) \in R$. x and y have the same number of pages and y and z also have the same number of pages. Therefore, x and z will also have the same number of pages. So, $(x, z) \in R$.

Hence, R is transitive.

Therefore, the given relation R is an equivalence relation.

8. Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b): |a-b|$ is even$\}$, is an equivalence relation. Show that all the elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other. But no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

Ans: Let $a \in A$,

So, $|a-a| = 0$ (which is an even number).

Therefore, R is reflexive.

Let $(a, b) \in R$,

Now, $|a-b|$ is even,

Hence $|a-b|$ and $|b-a|$ are both even

Therefore, $(b, a) \in R$

Therefore R is symmetric.

Let $(a, b) \in R$ and $(b, c) \in R$,

$\Rightarrow |a-b|$ is even and $|b-c|$ is even

$\Rightarrow (a-b)$ is even and $(b-c)$ is even

$\Rightarrow (a-c) = (a-b) + (b-c)$ is even

$\Rightarrow |a-c|$ is even.

$\Rightarrow (a, c) \in R$

Therefore, R is transitive.

Therefore, the given relation R is an equivalence relation.

All the elements of set $\{1, 3, 5\}$ are all odd. Hence, the modulus of the difference of any two elements will be an even number. So, all the elements of this set are related to each other.

All elements of $\{2, 4\}$ are even while all the element of $\{1, 3, 5\}$ are odd so no element of $\{1, 3, 5\}$ can be related to any element of $\{2, 4\}$.

Therefore, the absolute value of the difference between the two elements (from each of these two subsets) will not be an even value.

9. Show that each of the relation R in the set $A = \{x \in Z: 0 \leq x \leq 12\}$, is an equivalence relation. Find the set of all elements related to 1 in each case.

i. $R = \{(a, b) : |a - b|$ is a multiple of 4}

Ans: The given set

$A = \{x \in Z: 0 \leq x \leq 12\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

The given relation is: $R = \{(a, b) : |a - b|$ is a multiple of $4\}$.

Let $a \in A$,

$(a, a) \in R$ as $|a\text{-}a| = 0$ is a multiple of 4.

Therefore, R is reflexive.

Let, $(a, b) \in R \Rightarrow |a\text{-}b|$ is a multiple of 4.

$\Rightarrow |-(a\text{-}b)| = |b\text{-}a|$ is a multiple of 4.

$\Rightarrow (b, a) \in R$

Therefore R is symmetric.

$(a, b), (b, c) \in R$.

$\Rightarrow |a\text{-}b|$ is a multiple of 4 and $|b\text{-}c|$ is a multiple of 4.

$\Rightarrow (a\text{-}b)$ is a multiple of 4 and $(b\text{-}c)$ is a multiple of 4.

$\Rightarrow (a\text{-}c) = (a\text{-}b) + (b\text{-}c)$ is a multiple of 4.

$\Rightarrow |a\text{-}c|$ is a multiple of 4.

$\Rightarrow (a,\ c) \in R$

Therefore, R is transitive.

Therefore, the given relation R is an equivalence relation.

The set of elements related to 1 is $\{1, 5, 9\}$ as

$|1\text{-}1| = 0$ is a multiple of 4..

$|5\text{-}1| = 0$ is a multiple of 4..

$|9\text{-}1| = 0$ is a multiple of 4.

ii. $R = \{(a, b)\ : a = b\}$

Ans: The given relation is: $R = \{(a, b)\ : a = b\}$.

$a \in A, (a,\ a) \in R$, since $a = a$.

Therefore, R is reflexive.

Let $(a,\ b) \in R \Rightarrow a=b.$

$\Rightarrow b=a \Rightarrow (b,\ a) \in R$

Therefore R is symmetric.

$(a, b),\ (b,\ c) \in R$

$\Rightarrow a=b$ and $b=c$

$\Rightarrow a=c$

$\Rightarrow (a,\ c) \in R$

Therefore, R is transitive.

Therefore, the given relation R is an equivalence relation.

The set of elements related to 1 is $\{1\}$.

10. Given an example of a relation. Which is

i. Symmetric but neither reflexive nor transitive.

Ans: Let us assume the relation $R=\{(5, 6), (6, 5)\}$ in set $A=\{5, 6, 7\}$.

So, the relation R is not reflexive as $(5, 5), (6, 6), (7, 7) \notin R$.

The relation R is symmetric as $(5, 6) \in R$ and $(6, 5) \in R$.

The relation R is not transitive as $(5, 6), (6, 5) \in R$, but $(5, 5) \notin R$.

Therefore, the given relation R is symmetric but not reflexive or transitive.

ii. Transitive but neither reflexive nor symmetric.

Ans: Let us assume the relation $R=\{(a, b) : a < b\}$

So, the relation R is not reflexive because for $a \in R$, $(a, a) \notin R$ since a cannot be strictly less than itself.

Let $(1, 2) \in R$ (as $1 < 2$)

Since 2 is not less than 1, $(2, 1) \notin R$.

Therefore R not is symmetric.

Let $(a, b), (b, c) \in R$.

$\Rightarrow a<b$ and $b<c$

$\Rightarrow a<c$

$\Rightarrow (a,\ c) \in R$

Therefore, R is transitive.

So, the relation R is transitive but not reflexive and symmetric.

iii. Reflexive and symmetric but not transitive.

Ans: Let us assume the relation

$R = \{(4,\ 4),\ (6,\ 6),\ (8,\ 8),\ (4,\ 6),\ (6,\ 4),\ (6,\ 8),\ (8,\ 6)\}$ in set $A = \{4,\ 6,\ 8\}$.

The relation R is reflexive since for $a \in R$, $(a,\ a) \in R$.

The relation R is symmetric since $(a,\ b) \in R \Rightarrow (b,\ a) \in R$ for $a, b \in R$.

The relation R is not transitive since $(4,\ 6),\ (6,\ 8) \in R$, but $(4,\ 8) \notin R$.

Therefore the relation R is reflexive and symmetric but not transitive.

iv. Reflexive and transitive but not symmetric.

Ans: Let us take the relation $R = \{(a, b)\ :\ a^3 \geq b^3\}$.

Since $(a,\ b) \in R$.

Therefore R is reflexive.

Since $(2,\ 1) \in R$, but $(1,\ 2) \notin R$,

Therefore R not is symmetric.

Let $(a,\ b),\ (b,\ c) \in R$.

$\Rightarrow a^3 \geq b^3$ and $b^3 \geq c^3$

$\Rightarrow a^3 \geq c^3$

$\Rightarrow (a,\ c) \in R$

Therefore R is transitive.

Therefore the relation R is reflexive and transitive but not symmetric.

v. Symmetric and transitive but not reflexive.

Ans: Let us take a relation $R = \{(-5,\ -6),\ (-6,\ -5),\ (-5,\ -5)\}$ in set $A = \{-5,\ -6\}$.

The relation R is not reflexive as $(-6,\ -6) \notin R$.

Since $(-5,\ -6) \in R$ and $(-6,\ -5) \in R$.

Therefore R is symmetric.

Since $(-5,\ -6),\ (-6,\ -5) \in R$ and $(-5,\ -5) \in R$.

Therefore R is transitive.

Therefore the relation R is symmetric and transitive but not reflexive.

11. Show that the relation R in the set A of points in a plane given by $R = \{(P,\ Q)$: Distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all point related to a point $P \neq (0,\ 0)$ is the circle passing through P with origin as centre.

Ans: The given relation is $R = \{(P,\ Q)$: Distance of P from the origin is the same as the distance of Q from the origin}

Since, $(P,\ P) \in R$.

The relation R is reflexive.

Let $(P, Q) \in R$, distance of P from the origin is the same as the distance of Q from the origin similarly distance of Q from the origin will be the same as the distance of P from the origin. So, $(Q, P) \in R$.

Therefore R is symmetric.

Let $(P, Q),(Q, S) \in R$.

Distance of P from the origin is the same as the distance of Q from the origin and distance of Q from the origin is the same as the distance of S from the origin. So, distance of S from the origin will be same as the distance of P from the origin. So, $(P, S) \in R$.

Therefore R is transitive.

Therefore the relation R is an equivalence relation.

The set of points related to $P \neq (0, 0)$ will be those points whose distance from origin is same as distance of P from the origin and will form a circle with the centre as origin and this circle passes through P.

12. Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides $3, 4, 5$ and T_2 with sides $5, 12, 13$ and T_3 with sides $6, 8, 10$. Which triangles among T_1, T_2 and T_3 are related?

Ans: The given relation is $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$.

The relation R is reflexive since every triangle is similar to itself.

If $(T_1, T_2) \in R$, then T_1 is similar to T_2.

$\Rightarrow T_2$ is similar to T_1.

$\Rightarrow (T_2, T_1) \in R$

Therefore R is symmetric.

Let $(T_1, T_2), (T_2, T_3) \in R$.

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3.

$\Rightarrow T_1$ is similar to T_3.

$\Rightarrow (T_1, T_3) \in R$

Therefore R is transitive.

Therefore the relation R is an equivalence relation.

$$\frac{3}{6} = \frac{4}{8} = \frac{5}{10}\left(= \frac{1}{2} \right)$$

Since, the corresponding sides of triangles T_1 and T_3 are in the same ratio, therefore triangle T_1 is similar to triangle T_3.

Hence, T_1 is related to T_3.

13. Show that the relation R defined in the set A of all polygons as $R = \{(P_1, P_2): P_1$ and P_2 have same number of sides$\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides $3, 4$ and 5?

Ans: $R = \{(P_1, P_2): P_1$ and P_2 have same number of sides$\}$.

Since $(P_1, P_1) \in R$, as same polygon has same number of sides.

The relation R is reflexive.

Let $(P_1, P_2) \in R$.

$\Rightarrow P_1$ and P_2 have same number of sides.

$\Rightarrow P_2$ and P_1 have same number of sides.

$\Rightarrow (P_2, P_1) \in R$

Therefore R is symmetric.

Let $(P_1, P_2),(P_2, P_3) \in R$.

$\Rightarrow P_1$ and P_2 have same number of sides.

$\Rightarrow P_2$ and P_3 have same number of sides.

$\Rightarrow P_1$ and P_3 have same number of sides.

$\Rightarrow (P_1, P_3) \in R$

Therefore R is transitive.

Therefore the relation R is an equivalence relation.

The elements in A related to right-angled triangle (T) with sides $3, 4$ and 5 are the polygons having 3 sides.

14. Let L be the set of all lines in XY plane and R be the relation in L defined as $R = \{(L_1, L_2): L_1$ is parallel to $L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y=2x+4$.

Ans: $R = \{(L_1, L_2): L_1$ is parallel to $L_2\}$.

The relation R is reflexive as any line L_1 is parallel to itself, so, $(L_1, L_1) \in R$.

Let $(L_1, L_2) \in R$.

$\Rightarrow L_1$ is parallel to L_2, therefore L_2 is parallel to L_1.

$\Rightarrow (L_2, L_1) \in R$

Therefore R is symmetric.

Let $(L_1, L_2),(L_2, L_1) \in R$.

$\Rightarrow L_1$ is parallel to L_2

$\Rightarrow L_2$ is parallel to L_3

$\Rightarrow L_1$ is parallel to L_3

$\Rightarrow (L_1, L_3) \in R$

Therefore R is transitive.

Therefore the relation R is an equivalence relation.

Set of all lines related to line $y=2x+4$ is set of all lines that are parallel to the line $y=2x+4$.

Slope of line $y=2x+4$ is $m=2$. Therefore, lines parallel to the given line is of the form $y=2x+c$, where $c \in R$.

15. Let R be the relation in the set $\{1, 2, 3, 4\}$ given by $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation

Ans: $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$.

Since $(a, a) \in R$, for every $a \in \{1, 2, 3, 4\}$

The relation R is reflexive.

Since $(1, 2) \in R$, but $(2, 1) \notin R$.

Therefore R is not symmetric.

$(a, b),(b, c)\in R \Rightarrow (a, c)\in R$ for all $a, b, c\in\{1, 2, 3, 4\}$.

Therefore R is transitive.

Therefore the relation R is reflexive and transitive but not symmetric.

The correct answer is (B) R is reflexive and transitive but not symmetric.

16. Let R be the relation in the set N given by $R=\{(a,b): a=b-2, b>6\}$

Choose the correct answer.

(A) $(2, 4)\in R$

(B) $(3, 8)\in R$

(C) $(6, 8)\in R$

(D) $(8, 7)\in R$

Ans: The given relation is $R=\{(a,b): a=b-2, b>6\}$

Now,

Considering $(2, 4)\in R$.

Since, $b>6$, so, $(2, 4)\notin R$.

Considering $(3, 8)\in R$.

Since $3 \neq 8-2$, so $(3, 8)\notin R$.

Considering $(6, 8)\in R$.

Since 8>6 and 6=8-2, so $(6, 8)\in R$.

Therefore, the correct answer is (C) $(6, 8)\in R$.

Exercise 1.2

1. Show that the function $f: R_* \to R_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_*?

Ans: The function $f: R_* \to R_*$ is defined by $f(x) = \frac{1}{x}$.

For f to be one – one:

$x, y \in R_*$ such that $f(x) = f(y)$

$\Rightarrow \frac{1}{x} = \frac{1}{y}$

$\Rightarrow x = y$

Therefore, the given function f is one – one.

For f to be onto:

For $y \in R_*$ there exists $x = \frac{1}{y} \in R_*$ [as $y \neq 0$] such that

$f(x) = \frac{1}{\left(\frac{1}{y}\right)} = y$

Therefore, the given function f is onto.

Hence the given function f is one – one and onto.

Consider a function $g: N \to R_*$ defined by $g(x) = \frac{1}{x}$

We have, $g(x_1) = g(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

Therefore the function g is one – one.

The function g is not onto as for $1.2 \in = R_*$ there does not exit any x in N such that $g(x) = \frac{1}{1.2}$.

Therefore, function g is one-one but not onto.

2. Check the injectivity and surjectivity of the following functions:

i. $f: N \to N$ given by $f(x)=x^2$

Ans: The given function $f: N \to N$ is defined by $f(x)=x^2$.

For $x, y \in N$,

$f(x) = f(y)$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$

Therefore function f is injective.

Since $2 \in N$, but, there does not exist any x in N such that $f(x)=2$.

Therefore function f is not surjective.

Hence, function f is injective but not surjective.

ii. $f: Z \to Z$ given by $f(x)=x^2$

Ans: The given function $f: Z \to Z$ is defined by $f(x)=x^2$.

Since,

$f(-1) = f(1)$

$= 1$

But $-1 \neq 1$

Therefore function f is not injective.

Since $-2 \in Z$, but, there does not exist any element $x \in Z$ such that

$f(x) = -2$

Therefore function f is not surjective.

Hence, function f is neither injective nor surjective.

iii. f: $R \rightarrow R$ given by $f(x) = x^2$

Ans: The given function $f: R \rightarrow R$ is given by $f(x) = x^2$

Now,

$f(-1) = f(1)$

$= 1$

But $-1 \neq 1$.

Therefore function f is not injective.

Since $-2 \in R$, but, there does not exist any element $x \in R$ such that

$f(x) = -2$.

Therefore function f is not surjective.

Hence, function f is neither injective nor surjective.

iv. f: $N \rightarrow N$ given by $f(x) = x^3$

Ans: The given function $f: N \rightarrow N$ is given by $f(x) = x^3$

For $x, y \in N$,

$f(x) = f(y)$

$\Rightarrow x^3 = y^3$

$\Rightarrow x = y$

Therefore function f is injective.

Since $2 \in N$, but, there does not exist any x in N such that $f(x)=2$.

Therefore function f is not surjective.

Hence, function f is injective but not surjective.

v. f: Z $\rightarrow$ Zgiven by $f(x)=x^3$

Ans: The given function $f: Z \rightarrow Z$ is given by $f(x) = x^3$

For $x, y \in Z$,

$f(x) = f(y)$

$\Rightarrow x^3 = y^3$

$\Rightarrow x = y$

Therefore function f is injective.

Since $2 \in Z$, but, there does not exist any x in Z such that $f(x)=2$.

Therefore function f is not surjective.

Hence, function f is injective but not surjective.

3. Prove that the Greatest Integer Function $f: R \rightarrow R$ given by $f(x) = [x]$, is neither one – one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Ans: The function $f: R \rightarrow R$ is defined by $f(x) = [x]$.

Now,

$f(1.2) = [1.2]$

$= 1$

$f(1.9) = [1.9]$

$= 1$

Therefore, $f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

Hence function f is not one – one.

Taking $0.7 \in R$, $f(x) = [x]$ is an integer. There does not exist any element $x \in R$ such that $f(x) = 0.7$.

Therefore, function f is not onto.

Hence, the greatest integer function is neither one – one nor onto.

4. Show that the Modulus Function $f: R \rightarrow R$ given by $(x) = |x|$ is neither one – one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Ans: $f: R \rightarrow R$ is $f(x) = |x|$

$$= \begin{Bmatrix} x; \ x>0 \\ -x; \ x<0 \end{Bmatrix}$$

Now,

$f(-1) = |-1|$

$= 1$

$f(1) = |1|$

$= 1$

Therefore, $f(1) = f(-1)$, but $-1 \neq 1$.

Hence function f is not one – one.

Taking $-1 \in R$, $f(x) = |x|$ is non-negative. Hence, there does not exist any element $x \in R$ such that $f(x) = -1$.

Therefore, function f is not onto.

Therefore, the modulus function is neither one-one nor onto.

5. Show that the Signum Function $f: R \rightarrow R$, given by $f(x) = \begin{cases} 1 \text{ if } x>0 \\ 0, \text{ if } x=0 \\ -1 \text{ if } x<0 \end{cases}$ is neither one-one nor onto.

Ans: The function $f: R \rightarrow R$ is given by $f(x) = \begin{cases} 1 \text{ if } x>0 \\ 0, \text{ if } x=0 \\ -1 \text{ if } x<0 \end{cases}$

Now,

$f(1) = f(2)$

$= 1$

But $1 \neq 2$

Hence function f is not one – one.

Since $f(x)$ takes only 3 values $(1, 0, \text{or } -1)$, for the element -2 in co-domain R, there does not exist any x in domain R such that $f(x) = -2$.

Therefore, function f is not onto.

Therefore, the Signum function is neither one-one nor onto.

6. Let $A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one – one.

Ans: The function $f: A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$. Where $A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}$

Since,

$f(1) = 4$

$f(2) = 5$

$f(3) = 6$

Hence the images of distinct elements of A under f are distinct.

Therefore, the function f is one – one.

7. In each of the following cases, state whether the function is one – one, onto or bijective.

Justify your answer.

i. $f: R \to R$ defined by $f(x) = 3 - 4x$.

Ans: The function $f: R \to R$ is defined by $f(x) = 3 - 4x$.

Taking $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$,

$\Rightarrow 3- 4x_1 = 3- 4x_2$

$\Rightarrow -4x_1 = -4x_2$

$\Rightarrow x_1 = x_2$

Hence function f is one – one.

For any real number (y) in R, there exists $\frac{3-y}{4}$ in R such that $f\left(\frac{3-y}{4}\right)=3-4\left(\frac{3-y}{4}\right)$

$=y$

So, function f is onto.

Therefore, function f is bijective.

ii. f: R $\rightarrow$ R defined by $f(x) = 1 + x^2$

Ans: The function $f: R \rightarrow R$ defined as $f(x) = 1 + x^2$

Taking $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 1+x_1^2=1+x_2^2$

$\Rightarrow x_1^2=x_2^2$

$\Rightarrow x_1=\pm x_2$

Hence function f is not one – one because $f(x_1) = f(x_2)$ does not mean that $x_1=x_2$.

Taking $-2\in R$. Since $f(x) = 1 + x^2$ is positive for all $x \in R$, so there does not exist any x in domain R such that $f(x) = -2$.

Therefore, function f is not onto.

Hence, the function f is neither one – one nor onto.

8. Let A and B be sets. Show that $f: A \times B \to B \times A$ such that $(a, b) = (b, a)$ is bijective function.

Ans: The function $f: A \times B \to B \times A$ is defined as $f(a, b) = (b, a)$.

$(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

Hence function f is one – one.

For $(b, a) \in B \times A$,

There exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$

So, function f is onto.

Therefore, function f is bijective.

9. Let $f: N \to N$ be defined by $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if n is odd} \\ \frac{n}{2}, \text{ if n is even} \end{cases}$ for all $n \in N$. State whether the function f is bijective. Justify your answer.

Ans: The function $f: N \to N$ is defined by $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if n is odd} \\ \frac{n}{2}, \text{ if n is even} \end{cases}$ for all $n \in N$.

Now,

$f(1)=\frac{1+1}{2}$

$=1$

$f(2)=\frac{2}{2}$

$=1$

Here, $f(1) = f(2)$, but $1 \neq 2$.

Hence function f not is one – one.

Taking $n \in N$;

Case I: n is odd

Hence $n = 2r + 1$, for some $r \in N$ there exists $4r + 1 \in N$ such that

$f(4r + 1) = \frac{4r+1+1}{2}$

$=2r+1$

Case II: n is even

Hence, $n = 2r$ for some $r \in N$ there exists $4r \in N$ such that

$f(4r) = \frac{4r}{2}$

$= 2r$.

So, function f is onto.

Therefore, function f is not bijective.

10. Let $A=R-\{3\}$ and $B=R-\{1\}$. Consider the function $f:A\to B$ defined by $f(x)=\left(\frac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

Ans: The function $f:A\to B$ is defined by $f(x)=\left(\frac{x-2}{x-3}\right)$, where $A=R-\{3\}$ and $B=R-\{1\}$.

For $x, y\in A$ such that $f(x)=f(y)$

$\Rightarrow \frac{x-2}{x-3}=\frac{y-2}{y-3}$

$\Rightarrow (x-2)(y-3)=(y-2)(x-3)$

$\Rightarrow xy-3x-2y+6=xy-2x-3y+6$

$\Rightarrow -3x-2y=-2x-3y$

$\Rightarrow x=y$

Hence function f is one – one.

If $y\in B=R-\{1\}$, then $y\neq 1$.

The function f is onto if there exists $x\in A$ such that $f(x)=y$.

$\Rightarrow \frac{x-2}{x-3}=y$

$\Rightarrow x-2=xy-3y$

$\Rightarrow x(1-y)=-3y+2$

$\Rightarrow x=\dfrac{2-3y}{1-y}\in A\ [y\neq 1]$

for any $y\in B$, there exists $\dfrac{2-3y}{1-y}\in A$ such that,

$$f\left(\frac{2-3y}{1-y}\right)=\frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3}$$

$$=\frac{2-3y-2+2y}{2-3y-3+3y}$$

$$=\frac{-y}{-1}$$

$$=y$$

So, function f is onto.

Hence, function f is one – one and onto.

11. Let $f: R\rightarrow R$ be defined as $f(x)=x^4$. Choose the correct answer.

(A) f is one-one onto

(B) f is many-one onto

(C) f is one-one but not onto

(D) f is neither one-one nor onto

Ans: The function $f: R\rightarrow R$ is defined as $f(x)=x^4$.

Taking $x, y\in A$ such that $f(x)=f(y)$

$\Rightarrow x^4=y^4$

$\Rightarrow x=\pm y$

Therefore, $f(x)=f(y)$ does not necessarily mean that $x=y$.

Hence function f is not one – one.

For $2\in R$, there does not exist any x in domain R such that $f(x)=2$.

So, function f is not onto.

Hence, The correct answer is (D) function f is neither one – one nor onto.

12. Let f: R $\rightarrow$ R be defined as $f(x)=3x$. Choose the correct answer.

(A) f is one – one onto

(B) f is many – one onto

(C) f is one – one but not onto

(D) f is neither one – one nor onto

Ans: The function f: R $\rightarrow$ R is defined as $f(x)=3x$.

Taking $x, y\in A$ such that $f(x)=f(y)$

$\Rightarrow 3x=3y$

$\Rightarrow x=y$

Hence function f is one – one.

For $y\in R$, there exists $\frac{y}{3}$ in R such that;

$$f\left(\frac{y}{3}\right)=3\left(\frac{y}{3}\right)$$

$$=y$$

So, function f is onto.

Therefore, the correct answer is (A) function f is one – one and onto.

Miscellaneous Exercise

1. Show that function $f: R \to \{x \in R: -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}, x \in R$ is one – one and onto function.

Ans: The function $f: R \to \{x \in R: -1 < x < 1\}$ is defined as $f(x) = \frac{x}{1+|x|}, x \in R$.

For the function f to be one – one:

$f(x) = f(y)$, where $x, y \in R$.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

Assuming that x is positive and y is negative:

$$\frac{x}{1+x} = \frac{y}{1+y}$$

$\Rightarrow 2xy = x - y$

Since, $x > y \Rightarrow x - y > 0$.

But $2xy$ is negative.

Therefore, $2xy \neq x - y$.

Hence, x being positive and y being negative is not possible. Similarly x being negative and y being positive can also be ruled out.

So, x and y have to be either positive or negative.

Assuming that both x and y are positive:

$f(x) = f(y)$

$$\Rightarrow \frac{x}{1+x} = \frac{y}{1+y}$$

$\Rightarrow x + xy = y + xy$

$\Rightarrow x=y$

Assuming that both x and y are negative:

$f(x)=f(y)$

$\Rightarrow \frac{x}{1+x}=\frac{y}{1+y}$

$\Rightarrow x+xy=y+xy$

$\Rightarrow x=y$

Therefore, the function f is one – one.

For onto:

$y\in R$ such that $-1<y<1$.

If y is negative, then, there exists $x=\frac{y}{1+y}\in R$ such that

$$f\left(\frac{y}{1+y}\right)=\frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|}$$

$$=\frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)}$$

$$=\frac{y}{1+y-y}$$

$=y$

If y is positive, then, there exists $x=\frac{y}{1-y}\in R$ such that

$$f\left(\frac{y}{1-y}\right)=\frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|}$$

$$=\frac{\frac{y}{1-y}}{1+\left(\frac{-y}{1-y}\right)}$$

$$=\frac{y}{1-y+y}$$

$$=y$$

Therefore, the function f is onto.

Hence the given function f is both one – one and onto.

2. Show that the function $f: R \to R$ given by $f(x) = x^3$ is injective.

Ans: The given function $f: R \to R$ is given as $f(x) = x^3$.

For the function f to be one – one:

$f(x) = f(y)$ where $x, y \in R$.

$\Rightarrow x^3=y^3$ …… (1)

We need to show that $x=y$.

Assuming that $x \neq y$, then,

$\Rightarrow x^3 \neq y^3$

Since this is a contradiction to (1), therefore, $x=y$.

Hence, the function f is injective.

3. Given a non-empty set X, consider $P(X)$ which is the set of all subsets of X. Define the relation R in $P(X)$ as follows:

For subsets A,B in $P(X)$, ARB if and only if $A \subset B$. Is R an equivalence relation on $P(X)$? Justify you answer.

Ans: We know that every set is a subset of itself, ARA for all $A \in P(X)$

Therefore R is reflexive.

Let $ARB \Rightarrow A \subset B$.

This does not mean that $B \subset A$.

If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A.

Therefore R is not symmetric.

If ARB and BRC, then;

$A \subset B$ and $B \subset C$

$\Rightarrow A \subset C$

$\Rightarrow ARC$

Therefore R is transitive.

Hence, R is not an equivalence relation as it is not symmetric.

4. Find the number of all onto functions from the set $\{1, 2, 3, ..., n\}$ to itself.

Ans: The total number of onto maps from $\{1, 2, 3, ..., n\}$ to itself will be same as the total number of permutations on n symbols $1, 2, 3, ..., n$.

Since the total number of permutations on n symbols $1, 2, 3, ..., n$ is n, thus total number of onto maps from $\{1, 2, 3, ..., n\}$ to itself are n.

5. Let $A=\{-1, 0, 1, 2\}$, $B=\{-4, -2, 0, 2\}$ and $f,g:A \rightarrow B$ be functions defined by $f(x)=x^2-x, x \in A$ and $g(x)=2\left|x-\frac{1}{2}\right|-1, x \in A$. Are f and g equal?

Justify your answer. (Hint: One may note that two function $f:A \to B$ and $g:A \to B$ such that $f(a)=g(a) \forall a \in A$, are called equal functions)

Ans: Let $A=\{-1, 0, 1, 2\}, B=\{-4, -2, 0, 2\}$ and $f,g:A \to B$ are defined by $f(x)=x^2-x, x \in A$ and $g(x)=2\left|x-\frac{1}{2}\right|-1, x \in A$.

$f(-1)=(-1)^2-(-1)$

$=1+1$

$=2$

And,

$g(-1)=2\left|(-1)-\frac{1}{2}\right|-1$

$=2\left(\frac{3}{2}\right)-1$

$=3-1$

$=2$

$\Rightarrow f(-1)=g(-1)$

$f(0)=(0)^2-(0)$

$=0$

And,

$g(0)=2\left|(0)-\frac{1}{2}\right|-1$

$=1-1$

$=0$

$\Rightarrow f(0)=g(0)$

$f(1)=(1)^2-(1)$

$=1-1$

$=0$

And,

$g(1)=2\left|(1)-\frac{1}{2}\right|-1$

$=2\left(\frac{1}{2}\right)-1$

$=1-1$

$=0$

$\Rightarrow f(1)=g(1)$

$f(2)=(2)^2-(2)$

$=4-2$

$=2$

And,

$g(2)=2\left|(2)-\frac{1}{2}\right|-1$

$=2\left(\frac{3}{2}\right)-1$

$=3-1$

$=2$

$\Rightarrow f(2)=g(2)$

Therefore, $f(a)=g(a)\ \forall a\in A$. Hence functions f and g are equal.

6. Let $A=\{1,2,3\}$ Then number of relations containing $(1,2)$ and $(1,3)$ which are reflexive and symmetric but not transitive is

(A) 1

(B) 2

(C) 3

(D) 4

Ans: We are given a set $A=\{1,2,3\}$.

Let us take the relation R, containing $(1,2)$ and $(1,3)$, as $R=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(3,1)\}$.

As we can see that $(1,1),(2,2),(3,3)\in R$, therefore relation R is reflexive.

Since $(1,2),(1,3),(2,1)\in R$, the relation R is symmetric.

The relation Relation R is not transitive because $(1,2),(3,1)\in R$, but $(3,2)\notin R$

The relation Relation R will become transitive on adding and two pairs $(3,2),(2,3)$.

Therefore the total number of desired relations is one.

The correct answer is option (A) 1.

7. Let $A=\{1,2,3\}$ Then number of equivalence relations containing $(1,2)$ is

(A) 1

(B) 2

(C) 3

(D) 4

Ans: We are given a set $A=\{1, 2, 3\}$.

Let us take the relation R, containing $(1, 2)$ as

$R=\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$.

Now the pairs left are $(2, 3), (3, 2), (1, 3), (3, 1)$

On order to add one pair, say $(2, 3)$, we must add $(3, 2)$ for symmetry. And we are required to add $(1, 3), (3, 1)$ for transitivity.

So, only equivalence relation (bigger than R) is the universal relation.

Therefore, the total number of equivalence relations containing $(1, 2)$ are two.

Hence, the correct answer is (B) 2.