three dimensional geometry

11 Chapter

Exercise 11.1

1. Find the direction cosines if the line makes angles $90^o, 135^o, 45^o$ with x, y and z axes respectively.

Ans: Let usconsider l, m and n be the direction cosines of line

Then,

$l = \cos 90^o = 0$,

$m = \cos 135^o$

$= \cos\left(90^o + 45^o\right)$

$= \sin 45^o$

$= -\frac{1}{\sqrt{2}}$

And,

$n = \cos 45^o = \frac{1}{\sqrt{2}}$

Therefore, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

2. Find the direction cosines if the line makes equal angles with the coordinate axes.

Ans: Let us consider that the line makes an angle α with coordinate axes

Which means $l = \cos\alpha, m = \cos\alpha, n = \cos\alpha$

Now, we know that

$l^2 + m^2 + n^2 \Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha$

$$\Rightarrow 3\cos^2\alpha=1$$

$$\Rightarrow \cos^2\alpha=\frac{1}{3} \Rightarrow \cos\alpha=\pm\frac{1}{\sqrt{3}}$$

Therefore, the direction cosines of the line are $\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}$.

3. Find the direction cosines of a line having direction ratios $-18,12,-4$

Ans: We have the direction ratios as $-18,12,-4$,

Now, the direction cosines will be as

$$l=\frac{-18}{\sqrt{(-18)^2+(12)^2+(-4)^2}},m=\frac{12}{\sqrt{(-18)^2+(12)^2+(-4)^2}},n=\frac{-4}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$$

$$\frac{-18}{22},\frac{12}{22},\frac{-4}{22} \Rightarrow \frac{-9}{11},\frac{6}{11},\frac{-2}{11}$$

Therefore, direction cosines of the line are $\frac{-9}{11},\frac{6}{11}$ and $\frac{-2}{11}$.

4. Show that $(2,3,4),(-1,-2,1),(5,8,7)$ are collinear.

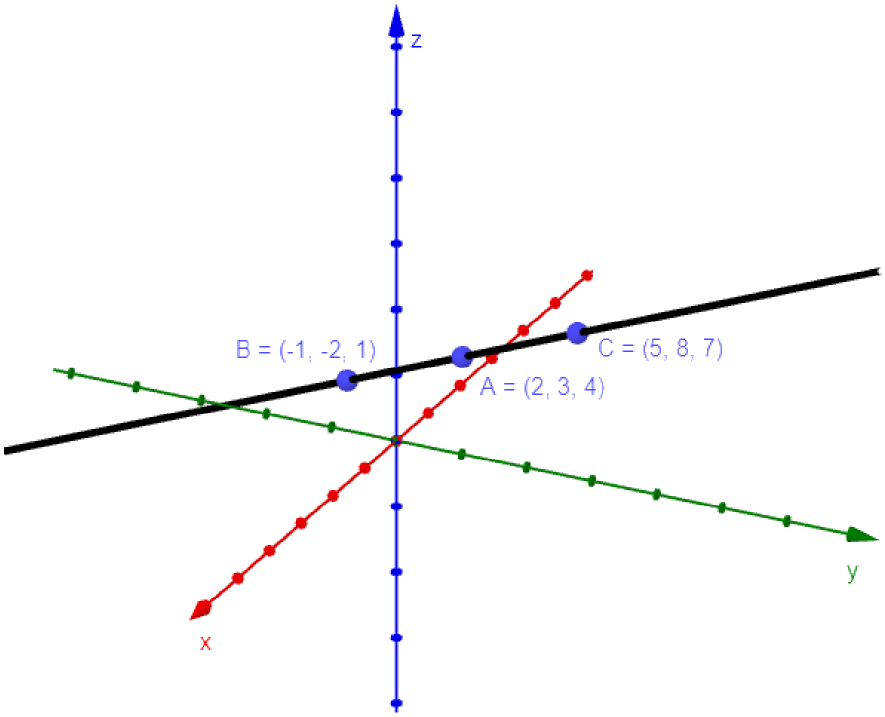
Ans: Let us consider the points be $A(2,3,4),B(-1,-2,1)$ and $C(5,8,7)$.

Now, as we know that direction cosines can be found by $(x_2-x_1),(y_2-y_1)$, and (z_2-z_1)

Therefore,

Direction ratios of AB and BC be $-3,-5,-3$ and $6,10,6$ respectively.

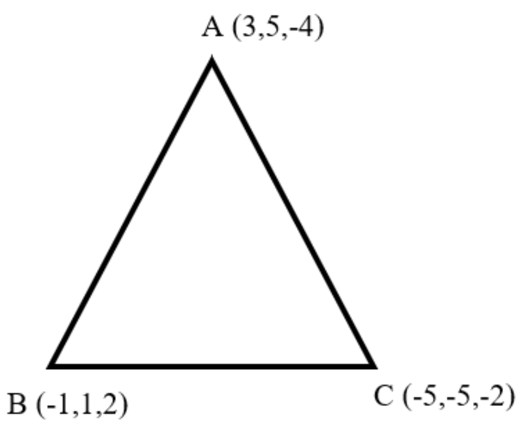
As we can see that AB and BC are proportional, we get that AB is parallel to BC.



Therefore, the points are collinear.

5. If the vertices of a triangle are $(3,5,-4),(-1,1,2),(-5,-5,-2)$, find its direction cosines.

Ans: Let us consider the points be $A(3,5,-4), B(-1,1,2)$ and $C(-5,-5,-2)$.



Now, the direction ratios of AB will be $-4,-4$ and 6,

We get

$$\sqrt{(-4)^2+(-4)^2+(6)^2}=\sqrt{68} \Rightarrow 2\sqrt{17}$$

Now,

$$l=\frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}},m=\frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}},n=\frac{6}{\sqrt{(-4)^2+(-4)^2+(6)^2}}$$

$$\Rightarrow l=\frac{-2}{\sqrt{17}},m=\frac{-2}{\sqrt{17}},n=\frac{3}{\sqrt{17}}$$

Therefore, the direction cosines of AB are $\frac{-2}{\sqrt{17}},\frac{-2}{\sqrt{17}},\frac{3}{\sqrt{17}}$

Similarly, the direction ratios of side BC will be $-4,-6$ and -4.

Now,

$$l=\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}},m=\frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}},n=\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$$

$$l=\frac{-4}{2\sqrt{17}},m=\frac{-6}{2\sqrt{17}},n=\frac{-4}{2\sqrt{17}}$$

Therefore, the direction cosines of BC is $\frac{-2}{\sqrt{17}},\frac{-3}{\sqrt{17}},\frac{-2}{\sqrt{17}}$

Similarly, the direction ratios of CA will be $-8,-10$ and 2.

Now,

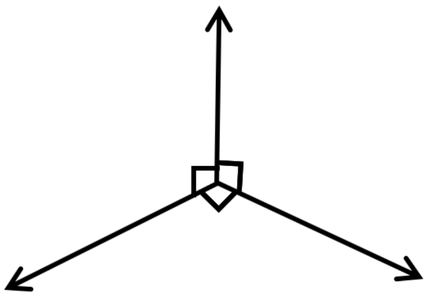
$$l=\frac{-8}{\sqrt{(-8)^2+(10)^2+(2)^2}},m=\frac{10}{\sqrt{(-8)^2+(10)^2+(2)^2}},n=\frac{2}{\sqrt{(-8)^2+(10)^2+(2)^2}}$$

$$l=\frac{-8}{2\sqrt{42}},m=\frac{-10}{2\sqrt{42}},n=\frac{2}{2\sqrt{42}}.$$

Therefore, the direction cosines of CA is $\frac{-4}{\sqrt{42}},\frac{-5}{\sqrt{42}},\frac{1}{\sqrt{42}}$

Exercise- 11.2

1. Show that the three lines are mutually perpendicular if they have direction cosines be $\frac{12}{13},\frac{-3}{13},\frac{-4}{13};\frac{4}{13},\frac{12}{13},\frac{3}{13};\frac{3}{13},\frac{-4}{13},\frac{12}{13}$



Ans: As we know, if $l_1l_2+m_1m_2+n_1n_2=0$, the lines are perpendicular

i. Now, from direction cosines $\frac{12}{13},\frac{-3}{13},\frac{-4}{13}$ and $\frac{4}{13},\frac{12}{13},\frac{3}{13}$, we get

$$l_1l_2+m_1m_2+n_1n_2=\frac{12}{13}\times\frac{4}{13}+\left(\frac{-3}{13}\right)\times\frac{12}{13}+\left(\frac{-4}{13}\right)\times\frac{3}{13}$$

$$\Rightarrow\frac{48}{169}-\frac{36}{169}-\frac{12}{169}$$

$$\Rightarrow 0$$

Therefore, the lines are perpendicular.

ii. Similarly, if we take $\frac{4}{13},\frac{12}{13},\frac{3}{13}$ and $\frac{12}{13},\frac{-3}{13},\frac{-4}{13}$, we get

$$l_1l_2+m_1m_2+n_1n_2=\frac{4}{13}\times\frac{3}{13}+\left(\frac{12}{13}\right)\times\left(\frac{-4}{13}\right)+\frac{3}{13}\times\left(\frac{12}{13}\right)$$

$$\Rightarrow\frac{12}{169}-\frac{48}{169}-\frac{36}{169}=0$$

Therefore, the lines are perpendicular.

iii. Again, if we consider $\frac{-3}{13},\frac{-4}{13},\frac{12}{13}$ and $\frac{12}{13},\frac{-3}{13},\frac{-4}{13}$, we get

$$l_1l_2+m_1m_2+n_1n_2=\frac{3}{13}\times\frac{12}{13}+\left(\frac{-4}{13}\right)\times\left(\frac{-4}{13}\right)+\frac{12}{13}\times\left(\frac{-4}{13}\right)$$

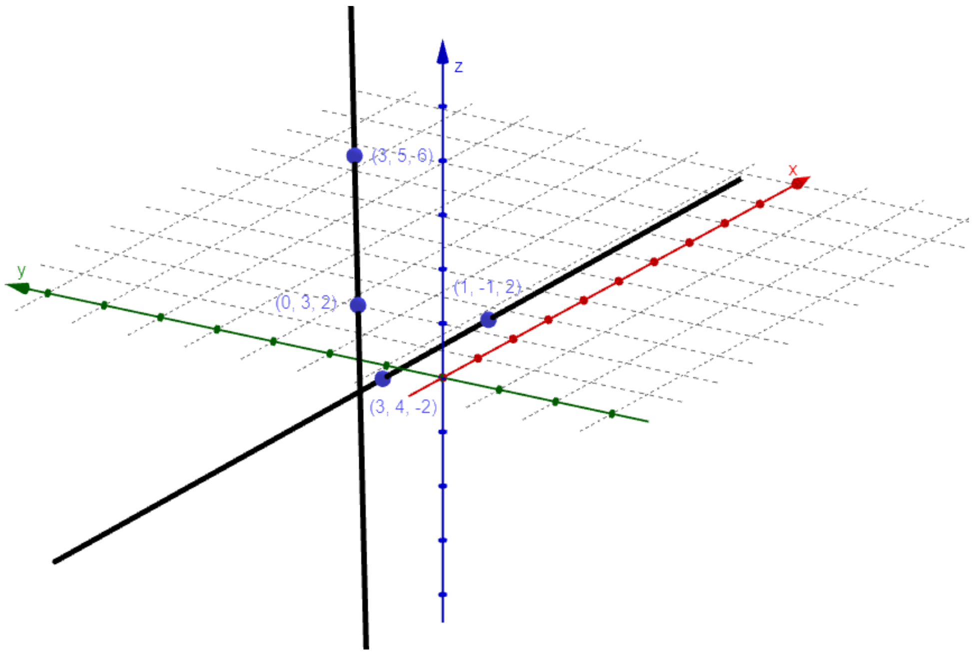
$$\Rightarrow \frac{36}{169}-\frac{12}{169}-\frac{48}{169}=0$$

Therefore, the lines are perpendicular.

Therefore, we can say that all the lines are mutually perpendicular.

2. How can you show that the line passing through the points $(1,-1,2)(3,4,-2)$ is perpendicular to the line through the points $(0,3,2)$ and $(3,5,6)$?

Ans: Let us consider that AB and CD are the lines that pass through the points, $(1,-1,2)$, $(3,4,-2)$ and $(0,3,2)$, $(3,5,6)$, respectively,



Now, we have $a_1=(2),b_1=(5),c_1=(-4)$ and $a_2=(3),b_2=(2),c_2=(4)$

As we know that if $AB\perp CD$ then $a_1a_2+b_1b_2+c_1c_2=0$

Now,

$a_1a_2+b_1b_2+c_1c_2=2\times3+5\times2+(-4)\times4$

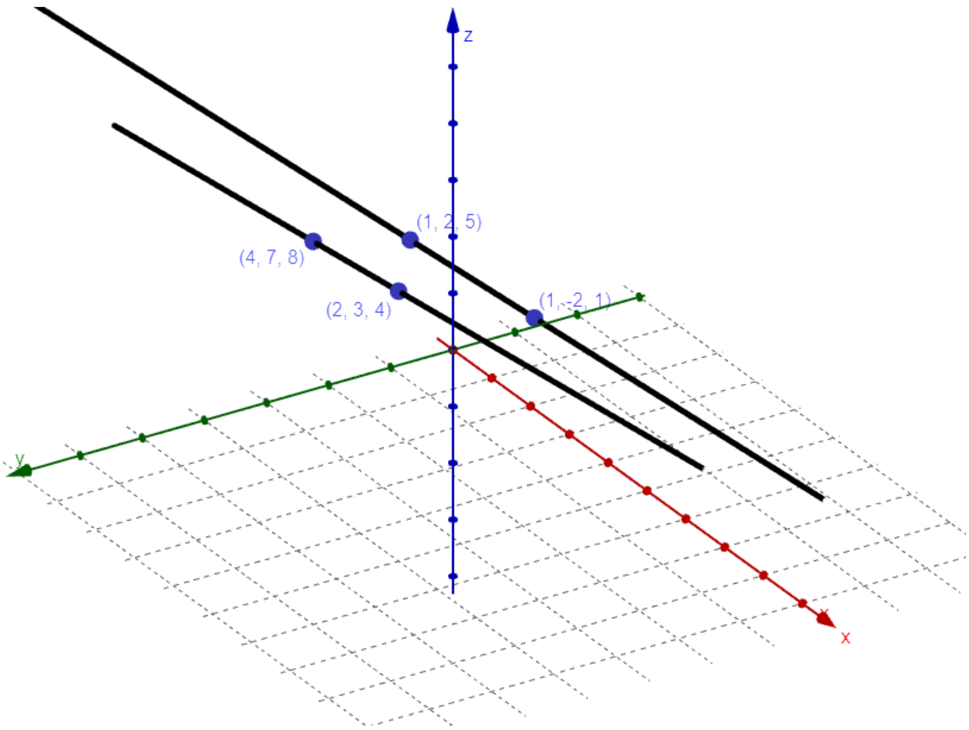
$\Rightarrow 2\times3+5\times2-4\times4=6+10-16$

$\Rightarrow 0$

Therefore, AB and CD are perpendicular to each other.

3. **Show that the line through the points $(4,7,8)(2,3,4)$ is parallel to the line through the points $(1,-2,1)(1,2,5)$.**

Ans: Let us consider the lines AB and CD that pass through points $(4,7,8)$ $(2,3,4)$, and $(-1,-2,1)$, $(1,2,5)$ respectively.



Now, we get

$a_1=(2-4), b_1=(3-7), c_1=(4-8)$ and $a_2=(1+1), b_2=(2+2), c_2=(5-1)$

$a_1=(-2), b_1=(-4), c_1=(-4)$ and $a_2=(2), b_2=(4), c_2=(4)$

Now, we know that if $AB \| CD$ then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$,

Now,

$$\frac{a_1}{a_2}=\frac{-2}{2} \Rightarrow -1, \frac{b_1}{b_2}=\frac{-4}{4} \Rightarrow -1, \frac{c_1}{c_2}=\frac{-4}{4}=-1$$

We got $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Therefore, AB is parallel to CD.

4. **Find the equation of the line if it is parallel to vector $3i+2j-2k$ and which passes through point $(1,2,3)$.**

Ans: Now, let us consider the position vector A be $a=i+2j+3k$ and let $\hat{b}=3\hat{i}+2\hat{j}-2k$

Now, we know that the line passes through A and is parallel to $\vec{b}$,

As we know $\vec{r}=\vec{a}+\lambda\vec{b}$ where λ is a constant

$\Rightarrow \vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(3\hat{i}+2\hat{j}-2\hat{k}\right)$

Therefore, the equation of the line is $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(3\hat{i}+2\hat{j}-2\hat{k}\right)$

5. **If the line passes through the point with positive vector $2\hat{i}-\hat{j}-4\hat{k}$ and is in the direction $\hat{i}+2\hat{j}-\hat{k}$. Find the equation of the line in vector and in Cartesian form.**

Ans: We know that the line passes through the point with positive vector

Now, let us consider $\vec{a}=2\hat{i}-\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$

Now, line passes through point A and parallel to $\vec{b}$, we get

$\vec{r}=2\hat{i}-\hat{j}+4\hat{k}+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$

Therefore, the equation of the line in vector form is $\vec{r}=2\hat{i}-\hat{j}+4\hat{k}+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$.

Now, we know

$\vec{r}=x\hat{i}-y\hat{j}+z\hat{k} \Rightarrow x\hat{i}-y\hat{j}+z\hat{k}=(\lambda+2)\hat{i}+(2\lambda-1)\hat{j}+(-\lambda+4)\hat{k}$

Therefore, the equation of the line in cartesian form will be $\dfrac{x-2}{1}=\dfrac{y+1}{2}=\dfrac{z-4}{-1}$.

6. **If the line passes through the point $(-2,4,-5)$ and parallel to the line given**

by $\dfrac{x+3}{3}=\dfrac{y-4}{5}=\dfrac{z+8}{6}$, find the Cartesian equation of the line.

Ans: We know that the line passes through point $(-2,4,-5)$ and also parallel to $\dfrac{x+3}{3}=\dfrac{y-4}{5}=\dfrac{z+8}{6}$

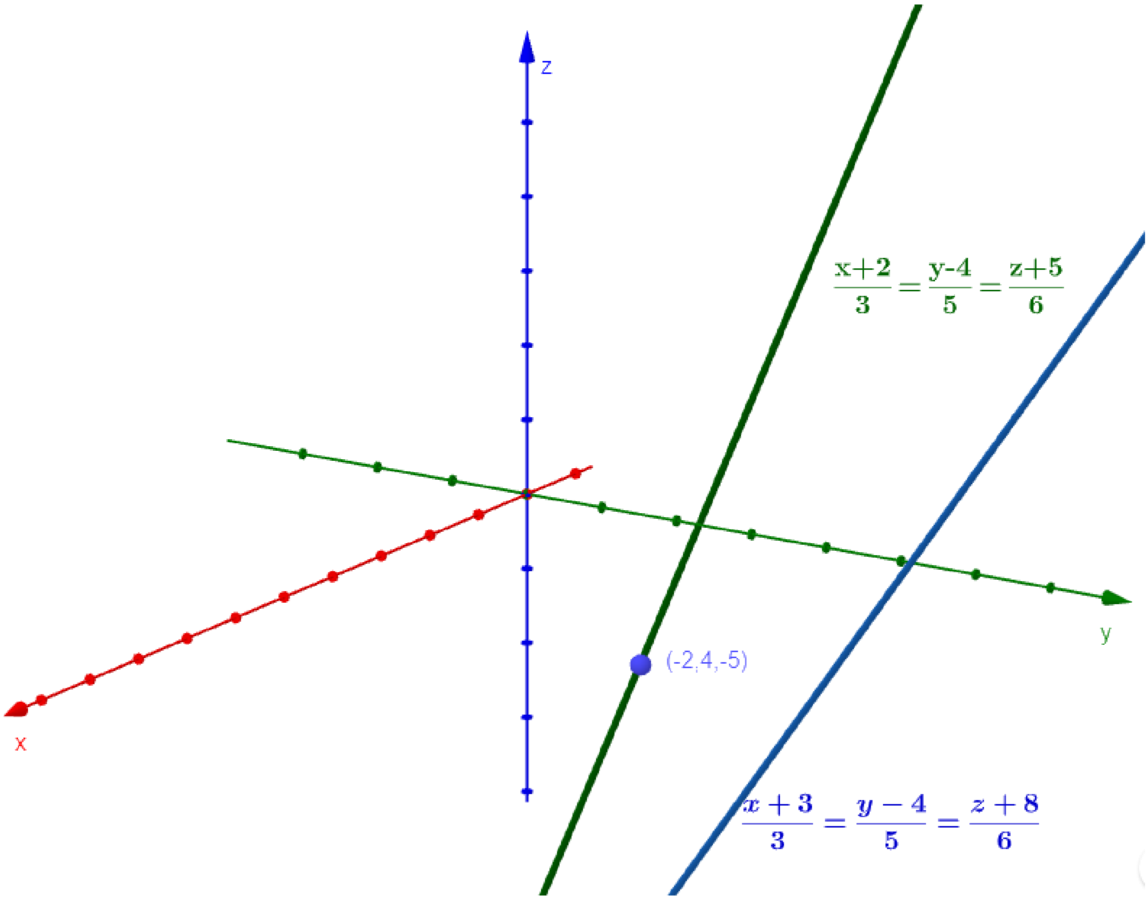
Now, as we can see the direction ratios of the line are $3,5$ and 6.

As we know the required line is parallel to $\dfrac{x+3}{3}=\dfrac{y-4}{5}=\dfrac{z+8}{6}$

Therefore, the direction ratios will be $3k,5k$ and $6k$

As we know that the equation of the line through the point and with direction ratio is shown in form $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$

Therefore, the equation of the line $\dfrac{x+2}{3}=\dfrac{y-4}{5}=\dfrac{z+5}{6}$.



7. **Write the vector form of the line if the Cartesian equation of a line is** $\mathbf{\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}}$ **.**

Ans: Aswe can see the cartesian equation of the line, we can tell that the line is passing through $(5,4,-6)$, and he direction ratios are $3,7$ and 2.

Now, we got the position vector $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

From this we got the direction of the vector be $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

Therefore, the vector form of the line will be $\vec{r}=5\hat{i}-4\hat{j}+6\hat{k}+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$

8. **Find the angle between the lines**

$\mathbf{(i)\ \vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda\left(3\hat{i}-2\hat{j}+6\hat{k}\right)}$ **and** $\mathbf{\vec{r}=7\hat{i}-6\hat{k}+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)}$

Ans: let us consider the angle be θ,

As we know that the angle between the lines can be found by $\cos\theta=\left|\frac{\vec{b}_1\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

As the lines are parallel to $\vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$ and $\vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$, we got

$\left|\vec{b}_1\right|=\sqrt{3^2+2^2+6^2}=7$, $\left|\vec{b}_2\right|=\sqrt{1^2+2^2+2^2}=3$ and $\vec{b}_1\vec{b}_2=\left(3\hat{i}+2\hat{j}+6\hat{k}\right)\left(\hat{i}+2\hat{j}+2\hat{k}\right)=19$

Therefore, the angle between the lines will be

$$\cos\theta=\frac{19}{7\times3}$$

$$\Rightarrow\theta=\cos^{-1}\frac{19}{21}$$

(ii) $\vec{r}=3\hat{i}+\hat{j}-2\hat{k}+\lambda\left(\hat{i}-\hat{j}-2\hat{k}\right)$ **and** $\vec{r}=3\hat{i}-\hat{j}-56\hat{k}+\mu\left(3\hat{i}-5\hat{j}-4\hat{k}\right)$

Ans: As the lines are parallel to the vectors $b_1=\hat{i}-\hat{j}-2\hat{k}$ and $b_2=3\hat{i}-5\hat{j}+-4\hat{k}$, we get

$\left|\vec{b}_1\right|=\sqrt{1^2+(-1)^2+(-2)^2}=\sqrt{6}$, $\left|\vec{b}_2\right|=\sqrt{3^2+(-5)^2+(-2)^2}=5\sqrt{2}$ and

$\vec{b}_1\vec{b}_2=\left(\hat{i}-\hat{j}-2\hat{k}\right)\left(3\hat{i}-5\hat{j}+-4\hat{k}\right)=16$

Therefore, the angle between them will be,

$$\cos\theta=\frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos\theta=\frac{8}{5\sqrt{3}}$$

$$\Rightarrow \theta=\cos^{-1}\frac{8}{5\sqrt{3}}$$

9. **Find the angle between the lines**

(i) $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ **and** $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$

Ans: Let us take $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the lines, we get

$\vec{b}_1=2\hat{i}+5\hat{j}-3\hat{k}$ and $\vec{b}_2=-\hat{i}+8\hat{j}+4\hat{k}$

Now $\left|\vec{b}_1\right|=\sqrt{2^2+5^2+(-3)^2}=\sqrt{38}$, $\left|\vec{b}_2\right|=\sqrt{(-1)^2+8^2+4^2}=9$

And,

$\vec{b}_1\vec{b}_2=\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\left(-\hat{i}+8\hat{j}+4\hat{k}\right)$

$=2(-1)+5(8)+4(-3)$

$=26$

We can find the angle by using $\cos\theta=\left|\frac{\vec{b}_1\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

Therefore,

$$\cos\theta=\frac{26}{9\sqrt{38}}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

Therefore, the angle will be $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$.

(ii) $\mathbf{\frac{x}{2}=\frac{y}{2}=\frac{z}{1}}$ **and** $\mathbf{\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}}$

Ans: Similarly let us consider $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to lines, we get

$\vec{b}_1=2\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}_2=4\hat{i}+\hat{j}+8\hat{k}$

Now, $\left|\vec{b}_1\right|=\sqrt{2^2+2^2+(1)^2}=3$, $\left|\vec{b}_2\right|=\sqrt{4^2+1^2+8^2}=9$ and

$$\vec{b}_1\vec{b}_2=\left(2\hat{i}+2\hat{j}+1\hat{k}\right).\left(4\hat{i}+\hat{j}+8\hat{k}\right)$$

$$=2(4)+2(1)+1(8)$$

$$=18$$

As we know the angle can be found by $\cos\theta=\left|\frac{\vec{b}_1\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

Therefore,

$$\cos\theta=\frac{18}{27}=\frac{2}{3}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{2}{3}\right)$$

Therefore, the angle is $\cos^{-1}\left(\frac{2}{3}\right)$.

10. We needed to find the values of p so the line $\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{2}$ are at right angles.

Ans: As we know that the correct form of the equation is as follows,

$$\frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2} \text{and} \frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$$

From this we get the direction ratios as

$$a_1=-3, b_1=\frac{2p}{7}, c_1=2 \text{ and } a_2=\frac{-3p}{7}, b_2=1, c_2=-5$$

As we know the lines are perpendicular, we get

$$a_1a_2+b_1b_2+c_1c_2=0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}=10$$

$$\Rightarrow 11p=70$$

$$\Rightarrow p=\frac{70}{11}$$

Therefore, the value of p is $\frac{70}{11}$.

11. We needed to show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

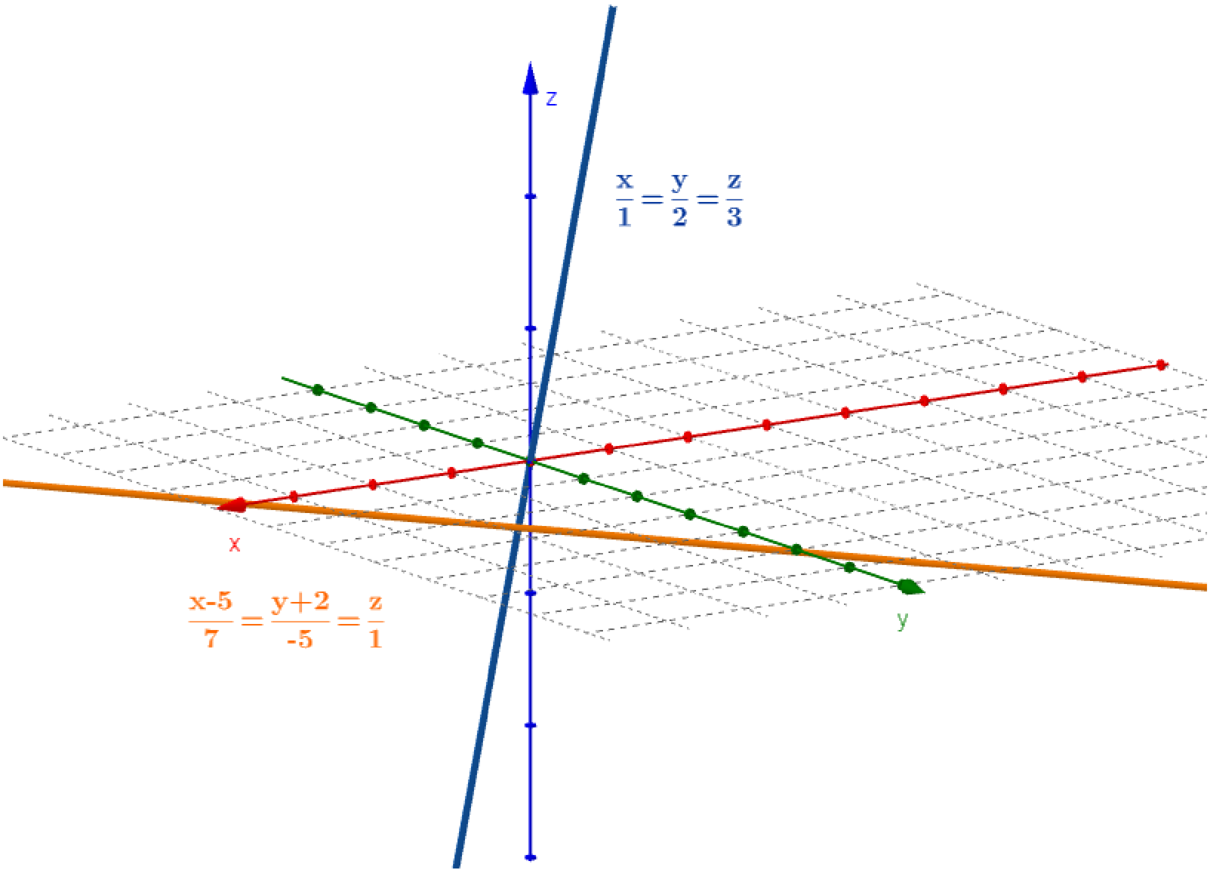
Ans: From the given equation, we get the direction ratios as,

$$a_1=7, b_1=-5, c_1=1, a_2=1, b_2=2, c_2=3$$

As we know, if $a_1a_2+b_1b_2+c_1c_2=0$, the lines are perpendicular to each other

Now,

$$7(1)+(-5)2+1(3) \Rightarrow 7-10+3=0$$



Therefore, the lines are perpendicular.

12. If the lines are $\vec{r}=\hat{i}+2\hat{j}+\hat{k}+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$, find the shortest distance between them.

Ans: We have been given lines, $r=\hat{i}+2\hat{j}+\hat{k}+\lambda\left(\hat{i}-\hat{j}-\hat{k}\right)$ and $r=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

As we know that the shortest distance can be found as $d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$

Now, from the given lines we get that

$\vec{a}_1=\hat{i}+2\hat{j}+\hat{k},$

$\vec{b}_1=\hat{i}-\hat{j}-\hat{k}$

$\vec{a}_2=2\hat{i}-\hat{j}-\hat{k},$

$\vec{b}_2=2\hat{i}+\hat{j}+2\hat{k}$

$$\vec{a}_2-\vec{a}_1=\left(2\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+\hat{k}\right)$$

$$=\hat{i}-3\hat{j}-2\hat{k},$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & -1 & 2\end{vmatrix}$$

$$\Rightarrow \vec{b}_1\times\vec{b}_2=-3\hat{i}+3\hat{k}$$

Then, $\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{\left(-3\right)^2+3^2}=3\sqrt{2}$

Now, if we put all the values in theirs places, we get

$$d=\left|\frac{\left(-3\hat{i}+3\hat{k}\right)\left(\hat{i}-3\hat{j}-2\hat{k}\right)}{3\sqrt{2}}\right|\Rightarrow d=\left|\frac{-3\left(1\right)+3\left(2\right)}{3\sqrt{2}}\right|$$

$$d=\left|\frac{-9}{3\sqrt{2}}\right|\Rightarrow d=\frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the lines is $\frac{3\sqrt{2}}{2}$ units.

13. Find the shortest distance between the lines

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

Ans: As we know that the shortest distance can be found by,

$$d=\frac{\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\sqrt{\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2+\left(a_1b_2-a_2b_1\right)^2}}$$

Now, from the given lines we got that

$$x_1=-1,y_1=-1,z_1=-1,a_1=7,b_1=-6,c_1=1$$

$x_2=3, y_2=5, z_2=7, a_2=1, b_2=-2, c_2=1.$

And,

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$=4(-6+2)-6(1+7)+8(-14+6)$$

$$=-16-36-64$$

$$=-116$$

And,

$$\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}=\sqrt{(-6+2)^2+(1+7)^2+(-14+6)^2}$$

$$\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}=2\sqrt{29}$$

Putting all the values, we get

$$d=\frac{-116}{2\sqrt{29}}$$

$$d=\frac{-58}{\sqrt{29}} \Rightarrow \frac{-58\sqrt{29}}{29}$$

$$d=\frac{-58}{\sqrt{29}} \Rightarrow |d|=2\sqrt{29}$$

Therefore, the distance the distance between the lines is $2\sqrt{29}$ units.

14. Find the shortest distance between the lines whose vector equations are $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$ and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$

Ans: We have been given lines $r=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$ and

$\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$

As we know that the shortest distance between the lines can be found by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

Now, from the given lines, we got

$\vec{a}_1=\hat{i}+2\hat{j}+3\hat{k},\ \vec{b}_1=\hat{i}-3\hat{j}+2\hat{k}$

$\vec{a}_2=4\hat{i}+5\hat{j}+6\hat{k},\ \vec{b}_2=2\hat{i}+3\hat{j}+\hat{k}$

$\vec{a}_2-\vec{a}_1=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)$

$=3\hat{i}+3\hat{j}+3\hat{k}$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ 2 & 3 & 1\end{vmatrix}$$

$\Rightarrow \vec{b}_1\times\vec{b}_2=-9\hat{i}+3\hat{j}+9\hat{k}$

$\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(-9)^2+3^2+9^2}=3\sqrt{19}$

Now, putting all the values, we get

$$d=\left|\frac{9}{3\sqrt{19}}\right|=\frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the lines is $\frac{3}{\sqrt{19}}$ units.

15. We needed to find the shortest distance between the lines whose vector equations are $\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and $\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$.

Ans: We have been given lines $\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and $\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$

$\Rightarrow \vec{r}=\hat{i}-2\hat{j}+3\hat{k}+t\left(-\hat{i}+\hat{j}-2\hat{k}\right)$ and $\vec{r}=\hat{i}-\hat{j}+\hat{k}+s\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

Now, the shortest distance can be found by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

Now, from the given lines we got,

$\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k},\ \vec{b}_1=-\hat{i}+\hat{j}-2\hat{k}$,

$\vec{a}_2=\hat{i}-\hat{j}-\hat{k},\ \vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$

$\vec{a}_2-\vec{a}_1=\left(\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=\hat{j}-4\hat{k}$,

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & -2\\ 1 & 2 & -2\end{vmatrix}$$

$\Rightarrow \vec{b}_1\times\vec{b}_2=2\hat{i}-4\hat{j}-3\hat{k}$,

$\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(2)^2+(-4)^2+(-3)^2}=\sqrt{29}$

$\left(\vec{b}_1\times\vec{b}_2\right)\times\left(\vec{a}_2-\vec{a}_1\right)=\left(2\hat{i}-4\hat{j}-3\hat{k}\right)\left(\hat{j}-4\hat{k}\right)$

$=-4+12$

$=8$

Putting all the values, we get

$$d=\left|\frac{8}{\sqrt{29}}\right|=\frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Miscellaneous Exercise

1. The direction ratios are a,b,c and $b\text{-}c,\ c\text{-}a,\ a\text{-}b,$ find the angle between the lines

Ans: As we know that, for any angle θ, with direction cosines, a,b,c and $b\text{-}c,\ c\text{-}a,\ a\text{-}b$ can be found by,

$$\cos\theta=\left|\frac{a\left(b\text{-}c\right)+b\left(b\text{-}c\right)+c\left(c\text{-}a\right)}{\sqrt{a^2+b^2+c^2+\sqrt{\left(b\text{-}c\right)^2+\left(c\text{-}a\right)^2+\left(a\text{-}b\right)^2}}}\right|$$

Solving this we get, $\cos\theta=0$

$\theta=\cos^{-1}0$

$\Rightarrow \theta=90^\circ$

Therefore, the angle between the two lines will be 90°.

2. Find the equation of a line passing through the origin and line parallel to x-axis

Ans: As it is given that the line is passing through the origin and is also parallel to x-axis is x-axis,

Now,

Let us consider a point on x-axis be A

So, the coordinates of A will be $\left(a,0,0\right)$

Now, the direction ratios of OA will be,

$\Rightarrow \left(a\text{-}0\right)=a,0,0$

The equation of $OA \Rightarrow \frac{x-0}{a}=\frac{y-0}{0}=\frac{z-0}{0} \Rightarrow \frac{x}{1}=\frac{y}{0}=\frac{z}{0}=a$

Therefore, the equation of the line passing through origin and parallel to x-axis is $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$.

3. Find the value of k if the lines $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$ and $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ are perpendicular.

Ans: From the given equation we can say that $a_1=-3, b_1=2k, c_1=2$ and $a_2=3k, b_2=1, c_2=-5$.

We know that the two lines are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$

$$-3(3k)+2k\times1+2(-5)=0$$

$$\Rightarrow -9k+2k-10=0$$

$$\Rightarrow 7k=-10$$

$$\Rightarrow k=\frac{-10}{7}$$

Therefore, the value of k is $-\frac{10}{7}$

4. What is the shortest distance between these two lines

$$\vec{\mathbf{r}}=\mathbf{6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right)}$$

$$\vec{\mathbf{r}}=\mathbf{-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right)}$$

Ans: According to the question, we need to find the distance between the lines,

$$\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right)$$

$$\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right)$$

As we know we can find the shortest distance by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).(\vec{a}_1-\vec{a}_2)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

Now, from the equation of lines we get

$\overrightarrow{a_1}=6\hat{i}+2\hat{j}+2\hat{k}$

$\overrightarrow{b_1}=\hat{i}-2\hat{j}+2\hat{k}$

$\overrightarrow{a_2}=-4\hat{i}-\hat{k}$

$\overrightarrow{b_2}=3\hat{i}-2\hat{j}-2\hat{k}$

$\Rightarrow \overrightarrow{a_2}-\overrightarrow{a_1}=\left(-4\hat{i}-\hat{k}\right)-\left(6\hat{i}+2\hat{j}+2\hat{k}\right)=-10\hat{i}-2\hat{j}-3\hat{k}$

$$\Rightarrow \overrightarrow{b_1}\times\overrightarrow{b_2}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix}=\left(4+4\right)\hat{i}-\left(-2-6\right)\hat{j}+\left(-2+6\right)\hat{k}$$

$=8\hat{i}+8\hat{j}+4\hat{k}$

$\left(\vec{b}_1\times\vec{b}_2\right).\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)=\left(8\hat{i}+8\hat{j}+4\hat{k}\right).\left(-10\hat{i}-2\hat{j}-3\hat{k}\right)$

$=-80-16-12$

$=-108$

Now, putting these values in $d=\left|\dfrac{\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_1-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$, we get

$$d=\left|\frac{-108}{12}\right|=9$$

Therefore, shortest distance between the above two lines is of 9 units.

5. Find the vector equation of the line passing through the points $\left(1,2,-4\right)$ and perpendicular to the two lines $\dfrac{x-8}{3}=\dfrac{y+19}{-16}=\dfrac{z-10}{7}$ and $\dfrac{x-15}{3}=\dfrac{y-29}{8}=\dfrac{z-5}{-5}$

Ans: According to the question, we get that $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{a}=\hat{i}+2\hat{j}-4\hat{k}$

We know that the equation of the line passing through point and also parallel to vector, we get

$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$ … (1)

Now, the equation of the two lines will be

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \ \ldots (2)$$

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \ \ldots (3)$$

As we know that line (1) and (2) are perpendicular to each other, we get

$$3b_1-16b_2+7b_3=0 \ \ldots (4)$$

Also, we know that the line (1) and (3) are perpendicular to each other, we get

$$3b_1+8b_2-5b_3=0 \ \ldots (5)$$

Now, from equation (4) and (5) we get that

$$\frac{b_1}{(-16)(-5)-8(7)}=\frac{b_2}{7(3)-3(-5)}=\frac{b_3}{3(8)-3(-16)}$$

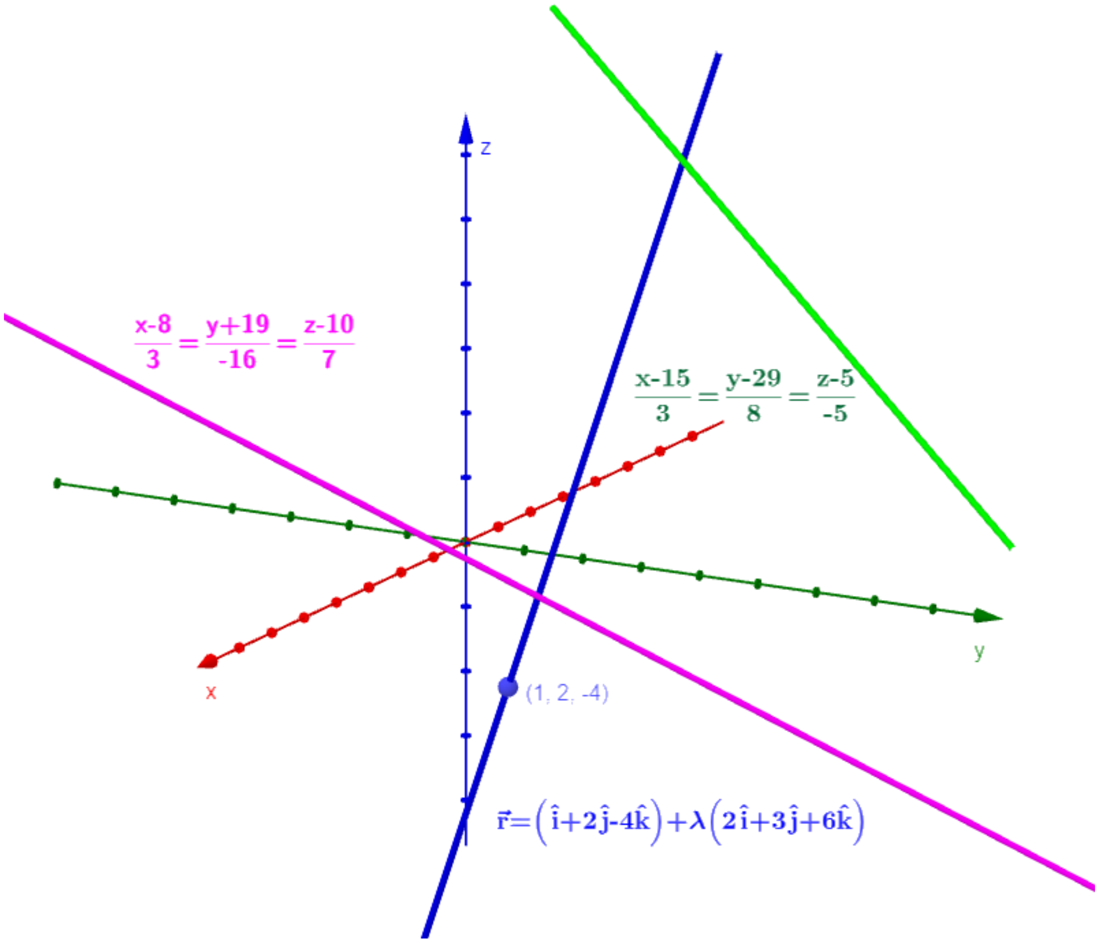
$$\Rightarrow \frac{b_1}{24}=\frac{b_2}{36}=\frac{b_3}{72} \Rightarrow \frac{b_1}{2}=\frac{b_2}{3}=\frac{b_3}{6}$$

Therefore, direction ratios of $\vec{b}$ are $2,3,6$

Which means $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

Putting $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ in equation (1), we get

$$\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$$



Therefore, the vector equation will be $\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$.