binomial theorem

7 Chapter

Exercise 7.1

Question 1:

Expand the expression $(1-2x)^5$

Solution 1:

By using Binomial Theorem, the expression $(1-2x)^5$ can be expanded as $(1-2x)^5$

$$={}^5C_0(1)^5-{}^5C_1(1)^4(2x)+{}^5C_2(1)^3(2x)^2-{}^5C_3(1)^2(2x)^3$$
$$+{}^5C_4(1)^1(2x)^3+{}^5C_4(1)^1(2x^4)-{}^5C_5(2x)^5$$
$$=1-5(2x)+10(4x)^2-10(8x^3)+5(16x^4)-(32x^5)$$
$$=1-10x+40x^2-80x^3+80x^4-32x^5$$

Question 2:

Expand the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

Solution 2:

By using Binomial Theorem, the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ can be expanded as

$$\left(\frac{2}{x}-\frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2$$

$$-{}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)$$

$$=\frac{32}{x^3} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) -$$

$$10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$$

$$=\frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$$

Question 3:

Expand the expression $(2x-3)^6$

Solution 3:

By using Binomial Theorem, the expression $(2x-3)^6$ can be expanded as

$$(2x-3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2$$

$$-{}^6C_3(2x)^3(3)^3 - {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)^2(3)^4 - {}^6C_6(3)^6$$

$$= 64x^6 - (32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27)$$

$$+15(4x^2)(81) - 6(2x)(243) + 729$$

$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

Question 4:

Expand the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Solution 4:

By using Binomial Theorem, the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ can be expanded as

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{2}\right)^2$$

$$+{}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$$

$$=\frac{x^5}{243}+5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right)+10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right)+10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right)$$

$$+5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right)+\frac{1}{x^5}$$

$$=\frac{x^5}{243}+\frac{5x^3}{81}+\frac{10x}{9x}+\frac{5}{3x^3}+\frac{1}{x^5}$$

Question 5:

Expand $\left(x+\frac{1}{x}\right)^6$

Solution 5:

By using Binomial Theorem, the expression $\left(x+\frac{1}{x}\right)^6$ can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2$$

$$+{}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right)$$

$$+15(x)\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Question 6:

Using Binomial Theorem, evaluate $(96)^3$

Solution 6:

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 96 = 100-4

$=(96)^3 = (100-4)^3$

$$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - {}^3C_3(4)^3$$

$$= (100)^3 - 3(100)^2(4) + (100)(4)^2 - (4)^3$$

$=1000000-120000+4800-64$
$=884736$

Question 7:

Using Binomial Theorem, evaluate $(102)^5$

Solution 7:

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 102 =100+2

$\therefore (102)^5 = (100+2)^5$

$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2$

$+ {}^5C_3(100)^2(2)^2 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$

=10000000000+1000000000+40000000+800000+8000 +32

=11040808032

Question 8:

Using Binomial Theorem, evaluate $(101)^4$

Solution 8:

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 101=100+1

$\therefore (101)^4 = (100+1)^4$

$$={}^{4}C_{0}(100)^{4}+{}^{4}C_{1}(100)^{3}(1)+{}^{4}C_{2}(100)^{2}(1)^{2}$$
$$+{}^{4}C_{3}(100)(1)^{3}+{}^{4}C_{4}(1)^{4}$$
$$=(100)^{4}+4(100)^{3}+6(100)^{2}+4(100)+(1)^{4}$$
=100000000+4000000+60000++400+1
=104060401

Question 9:

Using Binomial Theorem, evaluate $(99)^{5}$

Solution 9:

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 99 =100+1

$$\therefore (99)^{5}=(100-1)^{5}$$
$$={}^{5}C_{0}(100)^{5}-{}^{5}C_{1}(100)^{4}+{}^{5}C_{2}(100)^{3}(1)^{2}$$
$$+{}^{5}C_{3}(100)^{2}(1)^{3}+{}^{5}C_{4}(100)(1)^{4}-{}^{5}C_{5}(1)^{5}$$
$$=(100)^{5}-5(100)^{4}+10(100)^{3}-10(100)^{2}+5(100)-1$$
=10000000000-500000000+10000000-100000+500-1
=10010000500-500100001
=9509900499

Question 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution 10:

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ be obtained as

$(1.1)^{10000} = (1+0.1)^{10000}$

$= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms}$

$=1+10000\times1.1+$Other positive terms

=1+11000+Otherpositive terms

>1000

Hence, $(1.1)^{10000} > 1000.$

Question 11:

Find $(a+b)^4 - (a-b)^4$ Hence, evaluate.

$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$

Solution 11:

Using Binomial Theorem, the expressions,

$(a+b)^4$ *and* $(a-b)^4$ can be expanded as

$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$

$(a-b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$

$\therefore (a+b)^4 - (a-b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$

$-\left[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4\right]$

$2\left({}^4C_1a^3b + {}^4C_3ab^3\right) = 2\left(4a^3b + 4ab^3\right)$

$= 8ab\left(a^2 + b^2\right)$

By putting $a=\sqrt{3}$ and $b=\sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^4-\left(\sqrt{3}-\sqrt{2}\right)^4=8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left\{\left(\sqrt{3}\right)^2+\left(\sqrt{2}\right)^2\right\}$$

$$=8\left(\sqrt{6}\right)\{3+2\}=40\sqrt{6}$$

Question 12:

Find $(x+1)^6+(x-1)^6$. Hence or otherwise evaluate.

$$\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$$

Solution 12:

Using Binomial Theorem, the expression,

$(x+1)^6$ *and* $(x-1)^6$ can be expanded as

$$(x+1)^6={}^6C_0x^6+{}^6C_1x^5+{}^6C_2x^4+{}^6C_3x^3+{}^6C_4x^2+{}^6C_5x+{}^6C_6$$

$$(x-1)^6={}^6C_0x^6-{}^6C_1x^5+{}^6C_2x^4-{}^6C_3x^4-{}^6C_3x^3+{}^6C_4x^2-{}^6C_5x+{}^6C_6$$

$$\therefore (x-1)^6+(x-1)^6=2\left[{}^6C_0x^6+{}^6C_2x^4+{}^6C_4x^2+{}^6C_6\right]$$

$$=2\left[x^6+15x^4+15x^2+1\right]$$

By putting $x=\sqrt{2}$ we obtain

$$\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6=2\left[\left(\sqrt{2}\right)^6+15\left(\sqrt{2}\right)^4+15\left(\sqrt{2}\right)^2+1\right]$$

$$=2\left(8+15\times4+15\times2+1\right)$$

$$=\left(8+60+30+1\right)$$

$$=2(99)=198$$

Question 13:

Show that $9^{n+1}-8n-9$ is divisible by 64, whenever n is a positive integer.

Solution 13:

In order to show that $9^{n+1}-8n-9$ is divisible by 64, it has to be prove that, $9^{n+1}-8n-9=64k$,

where k is some natural number

By Binomial Theorem,

$(1+a)^m = {}^mC_0 + {}^mC_1a + {}^mC_2a^2 + \ldots + {}^mC_ma^m$

For a =8 and m= n+1, we obtain

$\Rightarrow 9^{n+1} = 1+(n+1)(8)+8^2\left[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$

$\Rightarrow 9^{n+1} = 9+8n+64\left[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$

$\Rightarrow 9^{n+1}-8n-9=64k$, where

$k = {}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + .. + {}^{n+1}C_{n+1}(8)^{n-1}$

is a natural number Thus, $9^{n+1}-8n-9$ is divisible by 64, whenever n is a positive integer.

Question 14:

Prove that $\sum_{r=0}^{n} 3^r \, {}^nC_r = 4^n$

Solution 14:

By Binomial Theorem,

$\sum_{r=0}^{n} {}^nC_r a^{n-r} b^r = (a+n)^n$

By putting b =3 and a =1 in the above equation, we obtain

$$\sum_{r=0}^{n} {}^nC_r (1)^{n-r} (3)^r = (1+3)^n$$

$$\Rightarrow \sum_{r=0}^{n} 3^r \ {}^nC_r = 4^n$$

Hence proved.

Miscellaneous Exercise

Question 1:

If a and b are distinct integers, prove that *a*-*b* is a factor of a^n - b^n , whenever *n* is a positive integer. [**Hint:** write $a^n = (a-b+b)^n$ and expand]

Solution 1:

In order to prove that (*a* -*b*) is a factor of (a^n -b^n), it has to be proved that

$a^n - b^n = k(a-b)$, where k is some natural number

It can be written that, *a* = *a*-*b*+*b*

$$\therefore = (a-b+b)^n = [(a-b)+b]^n$$

$$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + ... + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_nb^n$$

$$= (a-b)^n + {}^nC_1(a-b)^{n-1}b + ... + {}^nC_{n-1}(a-b)b^{n-1} + b^n$$

$\Rightarrow a^n - b^n = (a-b)$

$\left[(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + \ldots + {}^nC_{n-1} b^{n-1}\right]$

$\Rightarrow a^n - b^n = k(a-b)$

Where, $k = \left[(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + \ldots + {}^nC_{n-1} b^{n-1}\right]$ is a natural number This shows that (a -b) is a factor of $\left(a^n - b^n\right)$, where n is a positive integer.

Question 2:

Evaluate$\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$

Solution 2:

Firstly, the expression$(a+b)^6 - (a-b)^6$is simplified by using Binomial Theorem. This can be done as

$(a+b)^6 = {}^6C_0 a^6 + {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 + {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 + {}^6C_5 a^1 b^5 + {}^6C_6 b^6$

$= a^6 + 6a^5 b + 15a^4 b^2 + 20a^2 b^3 + 15a^2 b^4 + 6ab^5 + b^6$

$(a-b)^6 = {}^6C_0 a^6 - {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 - {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 - {}^6C_5 a^1 b^5 + {}^6C_6 b^6$

$= a^6 - 6a^5 b + 15a^4 b^2 - 20a^2 b^3 + 15a^2 b^4 - 6ab^5 + b^6$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$

$= 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right) + 20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3 + 6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$

$= 2\left[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}\right]$

$= 2 \times 198\sqrt{6}$

$= 396\sqrt{6}$

Question 3:

Find the value of $\left(a^2 + \sqrt{a^2 - 1}\right)^4 + \left(a^2 - \sqrt{a^2 - 1}\right)^4$

Solution 3:

Firstly, the expression is simplified by using Binomial Theorem. $(x + y)^4 + (x - y)^4$

This can be done as

$(x + y)^4 = {}^4C_0 x^4 + {}^4C_1 x^3 y + {}^4C_2 x^2 y^2 + {}^4C_3 x y^3 + {}^4C_4 y^4$

$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$

$(x - y)^4 = {}^4C_0 x^4 - {}^4C_1 x^3 y + {}^4C_2 x^2 y^2 - {}^4C_3 x y^3 + {}^4C_4 y^4$

$= x^4 - 4x^3 y + 6x^2 y^2 - 4xy^3 + y^4$

Putting $x = a^2$ and $y = \sqrt{a^2 - 1}$, we obtain

$\left(a^2 + \sqrt{a^2 - 1}\right)^4 + \left(a^2 - \sqrt{a^2 - 1}\right)^4$

$= 2\left[\left(a^2\right)^4 + 6\left(a^2\right)^2\left(\sqrt{a^2 - 1}\right)^2 + \left(\sqrt{a^2 - 1}\right)^4\right]$

$= 2\left[a^8 + 6a^4\left(a^2 - 1\right) + \left(a^2 - 1\right)^2\right]$

$= 2\left[a^8 + 6a^4 - 6a^4 + a^4 - 2a^2 + 1\right]$

$= 2\left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1\right]$

$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

Question 4:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution 4:

0.99 = 1-0.01

$(0.99)^5 = (1-0.01)^5$

$= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2$

[Approximately]

$= 1 - 0.05 + 0.001$

$= 1.001 - 0.05$

$= 0.951$

Thus, the value of $(0.99)^5$ is approximately 0.951.

Question 5:

Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$

Solution 5:

$$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$$

$$= {}^nC_0\left(1+\frac{x}{2}\right)^4 - {}^nC_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^nC_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^nC_4\left(\frac{2}{x}\right)^4$$

$$= \left(1+\frac{x}{2}\right)^4 - 4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4-\frac{8}{x}\left(1+\frac{x}{2}\right)^3+\frac{24}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}-\frac{16}{x^2}+\frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4-\frac{8}{x}\left(1+\frac{x}{2}\right)^3+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}..(1)$$

Again by using Binomial Theorem, we obtain

$$\left(1+\frac{x}{2}\right)^4={}^4C_0(1)^4+{}^4C_1(1)^3\left(\frac{x}{2}\right)+{}^4C_2(1)^2\left(\frac{x^2}{2}\right)$$

$$+{}^4C_3(1)^3\left(\frac{x}{2}\right)^3+{}^4C_4\left(\frac{x}{2}\right)^4$$

$$=1+4\times\frac{x}{2}+6\times\frac{x^4}{4}+4\times\frac{x^3}{8}+\frac{x^3}{16}$$

$$=1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}...(2)$$

$$\left(1+\frac{x}{2}\right)^3={}^3C_0(1)^3+{}^3C_2(1)^2\left(\frac{x}{2}\right)+{}^3C_3(1)\left(\frac{x}{2}\right)+{}^3C_3\left(\frac{x}{2}\right)^3$$

$$=1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}+\frac{x^3}{8}...(3)$$

From (1), (2) and (3), we obtain

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$=1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-\frac{8}{x}\left(1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}\right)+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}$$

$$=1+2x+\frac{3}{2}x^2+\frac{x^3}{2}+\frac{x^4}{16}-\frac{8}{x}-12-6x+x^2-\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}$$

$$=\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-5$$

Question 6:

Find the expansion of $\left(3x^2-2ax+3a^2\right)^3$ using binomial theorem.

Solution 6:

Using Binomial Theorem, the given expression $\left(3x^2-2ax+3a^2\right)^3$ can be expanded as $\left[3x^2-2ax+3a^2\right]^3$

$$={}^3C_0\left(3x^2-2ax\right)^3+{}^3C_1\left(3x^2-2ax\right)^2\left(3a^2\right)+{}^3C_2\left(3x^2-2ax\right)\left(3a^2\right)^2+{}^3C_3\left(3a^2\right)^3$$

$$=\left(3x^2-2ax\right)^3+3\left(9x^4-12ax^3+4a^2x^2\right)\left(3a^2\right)+3\left(3x^2-2ax\right)\left(9a^4\right)+2a^6$$

$$=\left(3x^2-2ax\right)^3+81a^2x^4-108a^3x^3+36a^4x^2+81a^4x^2-54a^5x+27a^6$$

$$=\left(3x^2-2ax\right)^3+81a^2x^4-108a^3x^3+117a^4x^2-54a^5x+27a^6 ...(1)$$

Again by using Binomial Theorem, we obtain

$$\left(3x^2-2ax\right)^3$$

$$={}^3C_0\left(3x^2\right)^3-{}^3C_1\left(3x^2\right)^2\left(2ax\right)+{}^3C_2\left(3x^2\right)\left(2ax\right)^2-{}^3C_3\left(2ax\right)^3$$

$$=27x^6-3\left(9x^4\right)\left(2ax\right)+3\left(3x^2\right)\left(4a^2x^2\right)-8a^3x^3$$

$$=27x^6-54ax^5+36a^2x^4-8a^3x^3 ...(2)$$

From (1) and (2), we obtain

$$\left(3x^2 - 2ax + 3a^2\right)^3$$

$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$

$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$