complex number and quadratic equation

Chapter 4

Exercise 4.1

1. Express the given complex number in the form $a+ib:(5i)\left(-\frac{3}{5}i\right)$

And evaluate

Ans: Evaluate the complex number

$$(5i)\left(-\frac{3}{5}i\right)=-5\times\frac{3}{5}\times i\times i$$

$$(5i)\left(-\frac{3}{5}i\right)=-3i^2\cdots\left[i^2=-1\right]$$

$$(5i)\left(-\frac{3}{5}i\right)=3$$

We get the final answer

2. Express the given complex number in the form $a+ib:i^9+i^{19}$

And evaluate

Ans: Evaluate the complex number

$$i^9+i^{19}=i^{4\times2+1}+i^{4\times4+3}$$

$$i^9+i^{19}=\left(i^4\right)^2.i+\left(i^4\right)^4.i^3\cdots\left[i^4=1,i^3=-1\right]$$

$$i^9+i^{19}=0$$

We get the final answer

3. Express the given complex number in the form $a + ib : i^{-39}$

And evaluate

Ans: Evaluate the complex number

$$i^{-39} = i^{(-4\times 9)-3}$$

$$i^{-39} = \left(i^4\right)^{-9}.i^{-3}$$

$$i^{-39} = (1)^{-9}.i^{-3} \cdots \left[i^4 = 1\right]$$

$$i^{-39} = \frac{1}{i^3} = \frac{1}{-i} = \frac{-i}{i^2} = \frac{-i}{-1} \quad \left[\because i^3 = -i, i^2 = -1\right]$$

$$i^{-39} = i$$

We get the final answer

4. Express the given complex number in the for $a + ib : 3(7 + i7) + i(7 + i7)$

And evaluate

Ans: Evaluate the complex number

$$3(7 + i7) + i(7 + i7) = 21 + 21i + 7i + 7i^2$$

$$3(7 + i7) + i(7 + i7) = 21 + 28i + 7i^2 \cdots \left[i^2 = -1\right]$$

$$3(7 + i7) + i(7 + i7) = 14 + 28i$$

We get the final answer

5. Express the given complex number in the form $a + ib : (1 - i) - (-1 + i6)$

And evaluate

Ans: Evaluate the complex number

$$(1-i)-(-1+6i)=1-i+1-i6$$

$$(1-i)-(-1+6i)=2-7i$$

We get the final answer

6. Express the given complex number in the form

$$\mathbf{a+ib:\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)}$$

And evaluate

Ans: Evaluate the complex number

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)=\frac{1}{5}+i\frac{2}{5}-4-i\frac{5}{2}$$

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)=\frac{-19}{5}+i\left[\frac{-21}{10}\right]$$

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)=\frac{-19}{5}-\frac{21}{10}i$$

We get the final answer

7. Express the given complex number in the form

$$\mathbf{a+ib:\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)-\left(-\frac{4}{3}+i\right)\right]}$$

And evaluate

Ans: Evaluate the complex number

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)-\left(-\frac{4}{3}+i\right)\right]=\frac{1}{3}+i\frac{7}{3}+4+i\frac{1}{3}+\frac{4}{3}-i$$

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)-\left(-\frac{4}{3}+i\right)\right]=\left(\frac{1}{3}+4+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-1\right)$$

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)-\left(-\frac{4}{3}+i\right)\right]=\frac{17}{3}+i\frac{5}{3}$$

We get the final answer

8. Express the given complex number in the form $a+ib:(1-i)^4$

And evaluate

Ans: Evaluate the complex number

$$(1-i)^4=\left[1+i^2-2i\right]^2$$

$$(1-i)^4=[1-1-2i]^2$$

$$(1-i)^4=(-2i)\times(-2i)$$

$$(1-i)^4=-4$$

We get the final answer

9. Express the given complex number in the form $a+ib:\left(\frac{1}{3}+3i\right)^3$

And evaluate

Ans: Evaluate the complex number

$$\left(\frac{1}{3}+3i\right)^3=\left(\frac{1}{3}\right)^3+(3i)^3+\frac{3}{3}3i\left(\frac{1}{3}+3i\right)$$

$$\left(\frac{1}{3}+3i\right)^3=\frac{1}{27}-(27i)+3i\left(\frac{1}{3}+3i\right)$$

$$\left(\frac{1}{3}+3i\right)^3=\frac{-242}{27}-26i$$

We get the final answer

10. Express the given complex number in the form $\mathbf{a+ib}$: $\left(\mathbf{-2-\frac{1}{3}i}\right)^3$
And evaluate

Ans: Evaluate the complex number

$$\left(-2-\frac{1}{3}i\right)^3=(-1)^3\left(2+\frac{1}{3}i\right)^3$$

$$\left(-2-\frac{1}{3}i\right)^3=-\left(2^3+\left(\frac{i}{3}\right)^3+6\frac{i}{3}\left(2+\frac{i}{3}\right)\right)$$

$$\left(-2-\frac{1}{3}i\right)^3=-\frac{22}{3}-\frac{107}{27}i$$

We get the final answer

11. Find the multiplicative inverse of the complex number $\mathbf{4-3i}$
And evaluate

Ans: Let $z=4-3i$

Then,

$$\bar{z}=4+3i \;\&\; |\bar{z}|=4^2+(-3)^2=16+9=25$$

Therefore, the multiplicative inverse of $4-3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

Here we got final answer

12. Find the multiplicative inverse of the complex number $\sqrt{5}$ + 3i

And evaluate

Ans: Let $z = \sqrt{5} + 3i$

Then,

$\bar{z} = \sqrt{5} - 3i$ and $|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$

Therefore, the multiplicative inverse of $\sqrt{5} + 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3i}{14}$$

Here we got final answer

13. Find the multiplicative inverse of the complex number -i

And evaluate

Ans: Let $z = -i$

Then,

$\bar{z} = i$ and $|z|^2 = 1^2 = 1$

Therefore, the multiplicative inverse of -i is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1} = i$$

Here we got final answer

14. **Express the following expression in the form of $a+ib$**

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+i\sqrt{2}\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

Evaluate

Ans: The following expression

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}=\frac{(3)^2-(i\sqrt{5})^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i}$$

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}=\frac{9-5i^2}{2\sqrt{2}i}$$

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}=\frac{9-5\left(-1\right)}{2\sqrt{2}i}$$

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}=\frac{-7\sqrt{2}i}{2}$$

Here we got final answer

Exercise 4.2

1. **Find the modulus and the argument of the complex number $z=-1-i\sqrt{3}$**

Evaluate

Ans: The complex number is

$$z=-1-i\sqrt{3}$$

Let $r\cos\theta=-1$ and $r\sin\theta=-\sqrt{3}$

Squaring and adding

$$(r\cos\theta)^2+(r\sin\theta)^2=(-1)^2+(-\sqrt{3})^2$$

$$r^2\left(\cos^2\theta+\sin^2\theta\right)=1+3$$

$r^2 = 4 \quad \left[\cos^2\theta + \sin^2\theta = 1\right]$

$r = \sqrt{4} = 2 \quad$ [Conventionally, r > 0]

Modulus = 2

$2\cos\theta = -1$ and $2\sin\theta = -\sqrt{3}$

$\cos\theta = \frac{-1}{2}$ and $\sin\theta = \frac{-\sqrt{3}}{2}$

Since both the values of $\sin\theta$ and $\cos\theta$ negative and $\sin\theta$ and $\cos\theta$ are negative in 3rd quadrant,

$$\text{Argument} = -\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$$

Thus, the modulus and argument of the complex number $-1-\sqrt{3}i$ are 2 and $-\frac{2\pi}{3}$

Respectively

2. Find the modulus and the argument of the complex number $z = -\sqrt{3} + i$

Evaluate

Ans: The complex number is

$z = -\sqrt{3} + i$

Let $r\cos\theta = -\sqrt{3}$ and $r\sin\theta = 1$

squaring and adding

$(r\cos\theta)^2 + (r\sin\theta)^2 = (-\sqrt{3})^2 + (-1)^2$

$r^2 = 3 + 1 = 4 LLL\left[\cos^2\theta + \sin^2\theta = 1\right]$

$r = \sqrt{4} = 2$LLL[Conventionally, r > 0]

Modulus = 2

$2\cos\theta = -\sqrt{3}$ and $2\sin\theta = 1$

$\cos\theta = \frac{-\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ LL[As θ lies in the II quadrant]

Thus, the modulus and argument of the complex number $-\sqrt{3} + i$ are 2 and $\frac{5\pi}{6}$

Respectively

3. Convert the given complex number in polar form $1 - i$

And evaluate

Ans: The complex number is

$1 - i$

Let $r\cos\theta = 1$ and $r\sin\theta = -1$

squaring and adding

$r^2\cos^2\theta + r^2\sin^2\theta = 1^2 + (-1)^2$

$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 1 + 1$

$r^2 = 2$

$r = \sqrt{2}$ [Conventionally, $r > 0$]

$\sqrt{2}\cos\theta = 1$ and $\sqrt{2}\sin\theta = -1$

$\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$

$\theta = -\frac{\pi}{4}$ [As θ liesin the IV quadrant]

$$1-i = r\cos\theta + ir\sin\theta = \sqrt{2}\cos\left(-\frac{\pi}{4}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{4}\right) = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

Required polar form

4. Convert the given complex number in polar form $-1+i$

And evaluate

Ans: The complex number is

$-1+i$

Let $r\cos\theta = -1$ and $r\sin\theta = 1$

Squaring and adding

$$r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + 1^2$$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 1+1$$

$$r^2 = 2$$

$$r = \sqrt{2}$$

$\sqrt{2}\cos\theta = -1$ and $\sqrt{2}\sin\theta = 1$

$\text{Þ}\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sqrt{2}\sin\theta = 1$

$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ L[As θ lies in the II quadrant]

It can be written,

$$-1+i = r\cos\theta + ir\sin\theta = \sqrt{2}\cos\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Required polar form

5. Convert the given complex number in polar form $-1-i$

And evaluate

Ans: The complex number is

$-1-i$

Let $r\cos\theta = -1$ and $r\sin\theta = -1$

Squaring and adding

$$r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + (-1)^2$$
$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 1+1$$
$$r^2 = 2$$
$$r = \sqrt{2}$$

$$\sqrt{2}\cos\theta = -1 \text{ and } \sqrt{2}\sin\theta = -1$$
$$\cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = -\frac{1}{\sqrt{2}}$$
$$\theta - -\left(\pi - \frac{\pi}{4}\right) - -\frac{3\pi}{4} \quad [\text{As 0 lies in the III quadrant}]$$

$$-1-i = r\cos\theta + ir\sin\theta = \sqrt{2}\cos\frac{-3\pi}{4} + i\sqrt{2}\sin\frac{-3\pi}{4} = \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$$

Required polar form

6. Convert the given complex number in polar form -3

And evaluate

Ans: The complex number is

-3

Let $r\cos\theta = -3$ and $r\sin\theta = 0$

Squaring and adding

$r^2\cos^2\theta + r^2\sin^2\theta = (-3)^2$

$r^2\left(\cos^2\theta + \sin^2\theta\right) = 9$

$r^2 = 9$

$r = \sqrt{9} = 3$

$3\cos\theta = -3$ and $3\sin\theta = 0$

$\cos\theta = -1$ and $\sin = 0$

$\theta = \pi$

$-3 = r\cos\theta + ir\sin\theta = 3\cos\pi + i3\sin\pi = 3(\cos\pi + i\sin\pi)$

Required polar form

7. Convert the given complex number in polar form $\sqrt{3} + i$

And evaluate

Ans: The complex number is

$\sqrt{3} + i$

Let $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$

Squaring and adding

$r^2\cos^2\theta + r^2\sin^2\theta = (\sqrt{3})^2 + 1^2$

$r^2\left(\cos^2\theta + \sin^2\theta\right) = 3 + 1$

$r^2 = 4$

$r = \sqrt{4} = 2$

$2\cos\theta = \sqrt{3}$ and $2\sin\theta = 1$

$\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ $\left[\text{As } \theta \text{ lies in the I quadrant}\right]$

$$\sqrt{3}+i = r\cos\theta + ir\sin\theta = 2\cos\frac{\pi}{6} + i2\sin\frac{\pi}{6} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Required polar form

8. Convert the given complex number in polar form i

And evaluate

Ans: The complex number is i

Let $r\cos\theta = 0$ and $r\sin\theta = 1$

Squaring and adding

$$r^2\cos^2\theta + r^2\sin^2\theta = 0^2 + 1^2$$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 1$$

$$r^2 = 1$$

$$r = \sqrt{1} = 1 \quad [\text{ Conventionally, } r > 0]$$

$$\cos\theta = 0 \text{ and } \sin\theta = 1$$

$$\theta = \frac{\pi}{2}$$

$$i = r\cos\theta + ir\sin\theta = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Required polar form

Exercise 5.3

1. Solve the equation $x^2 + 3 = 0$

And evaluate

Ans: Quadratic equation $x^2 + 3 = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = 1, b = 0$, and $c = 3$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 0^2 - 4 \times 1 \times 3 = -12$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{\pm\sqrt{-12}}{2 \times 1} = \frac{\pm\sqrt{12}i}{2}$$

$$= \frac{\pm 2\sqrt{3}i}{2} = \pm\sqrt{3}i$$

2. Solve the equation $2x^2 + x + 1 = 0$

And evaluate

Ans: Quadratic equation $2x^2 + x + 1 = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = 2, b = 1$, and $c = 1$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 1^2 - 4 \times 2 \times 1 = -7$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{\pm\sqrt{-7}}{2 \times 2} = \frac{\pm\sqrt{7}i}{4}$$

3. Solve the equation $x^2 + 3x + 9 = 0$

And evaluate

Ans: Quadratic equation $x^2 + 3x + 9 = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = 1, b = 3$, and $c = 9$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = -27$$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2 \times 1} = \frac{-3 \pm 3\sqrt{3}i}{2}$$

4. Solve the equation $-x^2 + x - 2 = 0$

And evaluate

Ans: Quadratic equation $-x^2 + x - 2 = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = -1, b = 1$, and $c = -2$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = 1^2 - 4 \times -1 \times -2 = -7$$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times -1} = \frac{-1 \pm \sqrt{7}i}{-2}$$

5. Solve the equation $x^2 + 3x + 5 = 0$

And evaluate

Ans: Quadratic equation $x^2 + 3x + 5 = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = 1, b = 3$, and $c = 5$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 3^2 - 4\times1\times5 = -11$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-11}}{2\times1} = \frac{-3 \pm \sqrt{11}i}{2}$$

6. Solve the equation $x^2 - x + 2 = 0$

And evaluate

Ans: Quadratic equation $x^2 - x + 2 = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = 1, b = 3 - 1,$ and $c = 2$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = (-1)^2 - 4\times1\times2 = -7$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-1) \pm \sqrt{-7}}{2\times1} = \frac{1 \pm \sqrt{7}i}{2}$$

7. Solve the equation $\sqrt{2}x^2 + x + \sqrt{2} = 0$

And evaluate

Ans: Quadratic equation $\sqrt{2}x^2 + x + \sqrt{2} = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = \sqrt{2}, b = 1,$ and $c = \sqrt{2}$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 1^2 - 4 \times \sqrt{2} \times \sqrt{2} = -7$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times \sqrt{2}} = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$$

8. Solve the equation $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

And evaluate

Ans: Quadratic equation $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = \sqrt{3}, b = -\sqrt{2}$, and $c = 3\sqrt{3}$

$D = b^2 - 4ac = \left(-\sqrt{2}\right)^2 - 4 \times \sqrt{3} \times 3\sqrt{3} = -34$

$D = b^2 - 4ac = \left(-\sqrt{2}\right)^2 - 4 \times \sqrt{3} \times 3\sqrt{3} = -34$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{-\left(-\sqrt{2}\right) \pm \sqrt{-34}}{2 \times \sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}}$$

9. Solve the equation $x^2 + x + \frac{1}{\sqrt{2}} = 0$

And evaluate

Ans: Quadratic equation $x^2 + x + \frac{1}{\sqrt{2}} = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = \sqrt{2}, b = \sqrt{2}$, and $c = 1$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = \left(\sqrt{2}\right)^2 - 4 \times \sqrt{2} \times 1 = 2 - 4\sqrt{2}$$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{-\left(\sqrt{2}\right) \pm \sqrt{2 - 4\sqrt{2}}}{2 \times \sqrt{2}} = \frac{-1 \pm \left(\sqrt{2\sqrt{2} - 1}\right)i}{2}$$

10. **Solve the equation $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$**

And evaluate

Ans: Quadratic equation $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

General form $ax^2 + bx + c = 0$

We obtain $a = \sqrt{2}, b = 1,$ and $c = \sqrt{2}$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (1)^2 - 4 \times \sqrt{2} \times \sqrt{2} = -7$$

Therefore, the required solutions are

$$= \frac{-b \pm \sqrt{D}}{2a} = \frac{-(1) \pm \sqrt{-7}}{2 \times \sqrt{2}} = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$$

Miscellaneous Exercise

1. **Evaluate the expression**

$$\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$$

Ans: Expression

$$\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3=\left[i^{4\times4+2}+\frac{1}{i^{4\times6+1}}\right]^3$$

$$=\left[\left(i^4\right)^4\times i^2+\frac{1}{\left(i^4\right)^6\times i}\right]^3$$

$$=\left[i^2+\frac{1}{i}\right]^3 \quad \left[i^4=1\right]$$

$$=\left[-1+\frac{1}{i}\times\frac{i}{i}\right]^3 \quad \left[i^2=-1\right]$$

$$=\left[-1+\frac{i}{i^2}\right]^3$$

$$=[-1-i]^3$$

$$=(-1)^3[1+i]^3$$

$$=-\left[1^3+i^3+3\times1\times i(1+i)\right]$$

$$=-\left[1+i^3+3i+3i^2\right]$$

$$=-[1-i+3i-3]$$

$$=-[-2+2i]$$

$$=2-2i$$

The expression is evaluated

2. **For any two complex numbers z_1 and z_2, prove that**

$$\mathbf{Re\left(z_1z_2\right)=Rez_1Rez_2-Imz_1Imz_2}$$

Ans: Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$

$$z_1z_2=\left(x_1+iy_1\right)\left(x_2+iy_2\right)$$

$$= x_1\left(x_2 + iy_2\right) + iy_1\left(x_2 + iy_2\right)$$

$$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$

$$= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2$$

$$= \left(x_1x_2 - y_1y_2\right) + i\left(x_1y_2 + y_1x_2\right)$$

$$\text{Re}\left(z_1z_2\right) = x_1x_2 - y_1y_2$$

$$\text{Re}\left(z_1z_2\right) = \text{Re}z_1\text{Re}z_2 - \text{Im}z_1\text{Im}z_2$$

Hence, proved

3. Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form

Ans: Expression

$$\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \left[\frac{(1+i)-2(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right]$$

$$= \left[\frac{1+i-2+8i}{1+i-4i-4i^2}\right]\left[\frac{3-4i}{5+i}\right] = \left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right]$$

$$= \left[\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}\right] = \frac{33+31i}{28-10i} = \frac{33+31i}{2(14-5i)}$$

$$= \frac{(33+31i)}{2(14-5i)} \times \frac{(14+5i)}{(14+5i)} \text{ [On multiplying numerator and denominator by} (14+5i)]$$

$$= \frac{462+165i+434i+155i^2}{2\left[(14)^2-(5i)^2\right]} = \frac{307+599i}{2\left(196-25i^2\right)}$$

$$= \frac{307+599i}{2(221)} = \frac{307+599i}{442} = \frac{307}{442} + \frac{599i}{442}$$

This is the required standard form

4. **If $x - iy = \sqrt{\frac{a - ib}{c - id}}$ prove that $\left(x^2 + y^2\right)^2 = \frac{a^2 + b^2}{c^2 + d^2}$**

Ans: Expression

$$x - iy = \sqrt{\frac{a - ib}{c - id}}$$

$$= \sqrt{\frac{a - ib}{c - id} \times \frac{c + id}{c + id}} \quad \text{[On multiplying numerator and denominator by } (c + id)\text{]}$$

$$= \sqrt{\frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}}$$

$$\backslash (x - iy)^2 = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}$$

$$x^2 - y^2 - 2ixy = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}$$

On comparing

$$x^2 - y^2 = \frac{ac + bd}{c^2 + d^2}, -2xy = \frac{ad - bc}{c^2 + d^2} \ldots\ldots(1)$$

$$\left(x^2 + y^2\right)^2 = \left(x^2 - y^2\right)^2 + 4x^2y^2$$

$$= \left(\frac{ac + bd}{c^2 + d^2}\right)^2 + \left(\frac{ad - bc}{c^2 + d^2}\right)$$

$$= \frac{a^2c^2 + b^2d^2 + 2acbd + a^2d^2 + b^2c^2 - 2adbc}{\left(c^2 + d^2\right)^2}$$

$$= \frac{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}{\left(c^2 + d^2\right)^2}$$

$$= \frac{a^2\left(c^2 + d^2\right) + b^2\left(c^2 + d^2\right)}{\left(c^2 + d^2\right)^2}$$

$$= \frac{\left(c^2 + d^2\right)\left(a^2 + b^2\right)}{\left(c^2 + d^2\right)^2}$$

$$= \frac{a^2 + b^2}{c^2 + d^2}$$

Hence, proved

5. **If $z_1 = 2 - i, z_2 = 1 + i$ Find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$**

Evaluate

Ans: Complex numbers

$$z_1 = 2 - i, z_2 = 1 + i$$

$$\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\frac{(2 - i) + (1 + i) + 1}{(2 - i) - (1 + i) + 1}\right|$$

$$= \left|\frac{4}{2 - 2i}\right| = \left|\frac{4}{2(1 - i)}\right|$$

$$= \left|\frac{2}{1 - i} \times \frac{1 + i}{1 + i}\right| = \left|\frac{2(1 + i)}{\left(1^2 - i^2\right)}\right|$$

$$= \left|\frac{2(1 + i)}{1 + 1}\right| \quad \left[i^2 = -1\right]$$

$$= \left|\frac{2(1 + i)}{2}\right|$$

$$= |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Thus, the value of $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$ is $\sqrt{2}$

6. **If $a+ib=\frac{(x+i)^2}{2x^2+1}$ Prove that $a^2+b^2=\frac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$**

Ans: Expression

$$a+ib=\frac{(x+i)^2}{2x^2+1}$$

$$=\frac{x^2+i^2+2xi}{2x^2+1}$$

$$=\frac{x^2-1+i2x}{2x^2+1}$$

$$=\frac{x^2-1}{2x^2+1}+i\left(\frac{2x}{2x^2+1}\right)$$

On comparing

$$a^2+b^2=\left(\frac{x^2-1}{2x^2+1}\right)^2+\left(\frac{2x}{2x^2+1}\right)^2$$

$$=\frac{x^4+1-2x^2+4x^2}{(2x+1)^2}$$

$$=\frac{x^4+1+2x^2}{\left(2x^2+1\right)^2}$$

$$=\frac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$$

$$a^2+b^2=\frac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$$

Hence, proved

7. **Let $z_1=2-i, z_2=-2+i$ Find**

$$\mathbf{Re}\left(\frac{\mathbf{z_1 z_2}}{\overline{\mathbf{z}}_1}\right)$$

$$\mathbf{Im}\left(\frac{\mathbf{1}}{\mathbf{z_1 \overline{z}_1}}\right)$$

Ans: Complex numbers

$$z_1 = 2 - i, z_2 = -2 + i$$
$$z_1 z_2 = (2 - i)(-2 + i) = -4 + 2i + 2i - i^2 = -4 + 4i - (-1) = -3 + 4i$$
$$\overline{z_1} = 2 + i$$

$$\frac{z_1 z_2}{\overline{z}_1} = \frac{-3 + 4i}{2 + i}$$

On multiplying numerator and denominator by $(2 - i)$, we obtain

$$\frac{z_1 z_2}{\overline{z}_1} = \frac{(-3 + 4i)(2 - i)}{(2 + i)(2 - i)} = \frac{-6 + 3i + 8i - 4i^2}{2^2 + 1^2} = \frac{-6 + 11i - 4(-1)}{2^2 + 1^2}$$
$$= \frac{-2 + 11i}{5} = \frac{-2}{5} + \frac{11}{5}i$$

On comparing real parts, we obtain

$$\text{Re}\left(\frac{z_1 z_2}{\overline{z}_1}\right) = \frac{-2}{5}$$
$$\frac{1}{z_1 \overline{z}_1} = \frac{1}{(2 - i)(2 + i)} = \frac{1}{(2)^2 + (1)^2} = \frac{1}{5}$$

On comparing imaginary parts, we obtain

$$\text{Im}\left(\frac{1}{z_1 \overline{z}_1}\right) = 0$$

Hence, solved

8. Find the real numbers $\mathbf{x \,\&\, y}$ if $\mathbf{(x - iy)(3 + 5i)}$ is the conjugate of $\mathbf{-6 - 24i}$

Ans: Let $z = (x - iy)(3 + 5i)$

$$z = 3x + 5xi - 3yi - 5yi^2 = 3x + 5xi - 3yi + 5y = (3x + 5y) + i(5x - 3y)$$
$$\overline{z} = (3x + 5y) - i(5x - 3y)$$

It is given that, $\bar{z} = -6 - 24i$

$(3x + 5y) - i(5x - 3y) = -6 - 24i$

Equating real and imaginary parts, we obtain

$3x + 5y = -6 \quad \ldots\ldots(i)$

$5x - 3y = 24 \ldots\ldots\ldots(ii)$

On solving we will get

$3(3) + 5y = -6$

$5y = -6 - 9 = -15$

$y = -3$

Thus, the values of x and y are 3 and -3 respectively

9. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$ Evaluate

Ans: Expression

$$\frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$$

$$= \frac{1 + i^2 + 2i - 1 - i^2 + 2i}{1^2 + 1^2}$$

$$= \frac{4i}{2} = 2i$$

$$\left|\frac{1+i}{1-i} - \frac{1-i}{1+i}\right| = |2i| = \sqrt{2^2} = 2$$

Here we get the answer

10. Find the modulus of $(x + iy)^3 = u + iv$ Than show that $\frac{u}{x} + \frac{v}{y} = 4\left(x^2 - y^2\right)$

Ans: $(x + iy)^3 = u + iv$

$$\Rightarrow x^3 + (iy)^3 + 3 \times x \times iy(x + iy) = u + iv$$

$$\Rightarrow x^3 + i^3y^3 + 3x^2yi + 3xy^2i^2 = u + iv$$

$$\Rightarrow x^3 - iy^3 + 3x^2yi - 3xy^2 = u + iv$$

$$\Rightarrow \left(x^3 - 3xy^2\right) + i\left(3x^2y - y^3\right) = u + iv$$

On equating real and imaginary

$$u = x^3 - 3xy^2, v = 3x^2y - y^3$$

$$\frac{u}{x} + \frac{v}{y} = \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y}$$

$$= \frac{x\left(x^2 - 3y^2\right)}{x} + \frac{y\left(3x^2 - y^2\right)}{y}$$

$$= x^2 - 3y^2 + 3x^2 - y^2$$

$$= 4x^2 - 4y^2$$

$$= 4\left(x^2 - y^2\right)$$

$$\frac{u}{x} + \frac{v}{y} = 4\left(x^2 - y^2\right)$$

Hence, proved

11. If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$.

Ans: Let $\alpha = a + ib$ & $\beta = x + iy$

It is given that, $|\beta| = 1$

$\sqrt{x^2+y^2}=1$

$\Rightarrow x^2+y^2=1 \ldots\ldots \left|\frac{\beta-\alpha}{1-\bar{\alpha}}\right| = \left|\frac{(x+iy)-(a+ib)}{1-(a-ib)(x+iy)}\right|$

$= \left|\frac{(x-a)+i(y-b)}{1-(ax+aiy-ibx+by)}\right|$

$= \left|\frac{(x-a)+i(y-b)}{(1-ax-by)+i(bx-ay)}\right|$

$= \left|\frac{(x-a)+i(y-b)}{(1-ax-by)+i(bx-ay)}\right|$

$= \frac{\sqrt{(x-a)^2+(y-b)^2}}{\sqrt{(1-ax-by)^2+(bx-ay)^2}}$

$= \frac{\sqrt{x^2+a^2-2ax+y^2+b^2-2by}}{\sqrt{1+a^2x^2+b^2y^2-2ax+2abxy-2by+b^2x^2+a^2y^2-2abxy}}$

$= \frac{\sqrt{\left(x^2+y^2\right)+a^2+b^2-2ax-2by}}{\sqrt{1+a^2\left(x^2+y^2\right)+b^2\left(y^2+x^2\right)-2ax-2by}}$

$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=1$

12. Find the number of non-zero integral solutions of the equation $|1-i|^x = 2^x$

Ans: Equation

$|1-i|^x = 2^x$

$\left(\sqrt{1^2+(-1)^2}\right)^x = 2^x$

$(\sqrt{2})^x = 2^x$

$2^{x/2} = 2^x$

$$\frac{x}{2} = x$$

$$x = 2x$$

$$x = 0$$

Thus, 0 is the only integral solution of the given equation. Therefore, the number of nonzero integral solutions of the given equation is 0 .

13. **If $(a+ib)(c+id)(e+if)(g+ih) = A+iB$ Then show that**

$$\mathbf{\left(a^2+b^2\right)\left(c^2+d^2\right)\left(e^2+f^2\right)\left(g^2+h^2\right) = A^2+B^2}$$

Ans: Expression

$$(a+ib)(c+id)(e+if)(g+ih) = A+iB$$
$$|(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$$
$$|(a+ib)|\times|(c+id)|\times|(e+if)|\times|(g+ih)| = |A+iB| \quad \because \left[|z_1z_2| = |z_1||z_2|\right]$$
$$\sqrt{a^2+b^2}\times\sqrt{c^2+d^2}\times\sqrt{e^2+f^2}\times\sqrt{g^2+h^2} = \sqrt{A^2+B^2}$$

By squaring

$$\left(a^2+b^2\right)\left(c^2+d^2\right)\left(e^2+f^2\right)\left(g^2+h^2\right) = A^2+B^2$$

Hence, proved

14. **If $\left(\frac{1+i}{1-i}\right)^m = 1$**

Then find the least positive integral value of m

Ans: $\left(\frac{1+i}{1-i}\right)^m = 1$

$$\left(\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right)^m=1$$

$$\left(\frac{(1+i)^2}{1^2+1^2}\right)^m=1$$

$$\left(\frac{1^2+i^2+2i}{2}\right)^m=1$$

$$\left(\frac{1-1+2i}{2}\right)^m=1$$

$$\left(\frac{2i}{2}\right)^m=1$$

$$i^m=1$$

$$i^m=i^{4k}$$

$m=4k$, where k is some integer

Therefore, the least positive is one

Thus, the least positive integral value of m is $4=\left(4\times1\right)$