limits and derivatives

12
Chapter

1. **Evaluate the Given limit:** $\lim\limits_{x \to 3} x+3$

Ans: $\lim\limits_{x \to 3} x+3 = 3+3 = 6$

2. **Evaluate the Given limit:** $\lim\limits_{x \to z} \left(x - \frac{22}{7} \right)$

Ans: $\lim\limits_{x \to z} \left(x - \frac{22}{7} \right) = \left(\pi - \frac{22}{7} \right)$

3. **Evaluate the Given limit:** $\lim\limits_{r \to 1} \pi r^2$

Ans: $\lim \pi r^2 = \pi \left(1^2 \right) = \pi$

4. **Evaluate the Given limit:** $\lim\limits_{x \to 1} \frac{4x+3}{x-2}$

Ans: $\lim\limits_{x \to 1} \frac{4x+3}{x-2} = \frac{4(4)+3}{4-2} = \frac{16+3}{2} = \frac{19}{2}$

5. **Evaluate the Given limit:** $\lim\limits_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$

Ans: $\lim\limits_{x \to -1} \frac{x^{10}+x^5+1}{x-1} = \frac{(-1)^{10}+(-1)^5+1}{-1-1} = \frac{1-1+1}{-2} = -\frac{1}{2}$

6. **Evaluate the Given limit:** $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

Ans: $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

Put $x+1 = yso$ that $y \to 1$ as $x \to 0$

Accordingly, $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x} = \lim_{x \to 1} \frac{(y)^5 - 1}{y-1}$

$5.\ 1^{5-1} \quad \left[\lim_{x \to a} \frac{x^n - a^n}{x-a} = nd^{n-1} \right] = 5$

$\therefore \lim_{x \to 0} \frac{(x+1)^5 - 1}{x} = 5$

7. **Evaluate the Given limit:** $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4}$

Ans: At $x - 2$, the value of the given rational function takes the form $\frac{0}{0}$

$\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$

$\lim_{x \to 2} \frac{3x+5}{x+2}$

$= \frac{3(2)+5}{2+2}$

$- \frac{11}{4}$

8. **Evaluate the Given limit:** $\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

Ans: At $x = 2$, the value of the given rational function takes the form $\frac{0}{0}$

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)\left(x^2+9\right)}{(x-3)(2x+1)}$$

$$\lim_{x \to 3} \frac{(x+3)\left(x^2+9\right)}{(2x+1)}$$

$$= \frac{(3+3)\left(3^2+9\right)}{2(3)+1}$$

$$= \frac{6 \times 18}{7}$$

$$= \frac{108}{7}$$

9. **Evaluate the Given limit:** $\lim_{x \to 0} \frac{ax+b}{cx+1}$

Ans:

$$\lim_{x \to 0} \frac{ax+b}{cx+1} = \frac{a(0)+b}{d(0)+1} = b$$

10. **Evaluate the Given limit:** $\lim_{z \to 1} \frac{\frac{1}{3^3} - 1}{\frac{1}{2^6} - 1}$

Ans: $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{\frac{1}{z^6} - 1}$

At $z = 1$, the value of the given function takes the form $\frac{0}{0}$ Put $z^{\frac{1}{6}} = x$ so that $z \to 1$ as $x \to 1$.

Accordingly, $\lim_{x\to 1}\frac{\frac{1}{\vec{e}}-1}{z^{\frac{1}{t}}-1}=\lim_{x\to 1}\frac{x^2-1}{x-1}$

$=\lim_{x\to 1}\frac{x^2-1}{x-1}$

$=2.1^{2\cdot 1}\quad\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=nd^{n-1}\right]$

$=2$

$\lim_{x\to 1}\frac{z^{\frac{1}{3}}-1}{\frac{1}{t^6}-1}=2$

11. Evaluate the Given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

Ans: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}=\frac{a(1)^2+b(1)+c}{c(1)^2+b(1)+a}$

$=\frac{a+b+c}{a+b+c}$

$=1$

$[a+b+c\neq 0]$

12. Evaluate the given limit: $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Ans: $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

At $x=-2$, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\lim_{x\to -2}\frac{\left(\frac{2+x}{2x}\right)}{x+2}$

$\lim_{x\to -2}\frac{1}{2x}$

$\frac{1}{2(-2)}=\frac{-1}{4}$

13. Evaluate the given limit: $\lim_{x\to 0}\frac{\sin ax}{bx}$

Ans: $\lim_{x\to 0}\frac{\sin ax}{bx}$

At $x=0$, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to 0}\frac{\sin ax}{bx}=\lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{ax}{bx}$

$=\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\times\frac{a}{b}$

$\frac{a}{b}\lim_{\text{ax}\to 0}\left(\frac{\sin ax}{ax}\right)$ $[x\to 0\Rightarrow ax\to 0]$

$=\frac{a}{b}\times 1$

$=\left[\lim_{x\to 0}\left(\frac{\sin y}{y}\right)\right]$

$=\frac{a}{b}$

14. Evaluate the given limit: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}, a, b\neq 0$

Ans: $\lim_{x \to 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \lim_{x \to 0} \left(\frac{\left(\frac{\sin ax}{ax} \right) \times ax}{\left(\frac{\sin bx}{ax} \right) \times bx} \right)$

$$= \frac{a}{b} \times \frac{\lim_{ax \to 0} \left(\frac{\sin ax}{ax} \right)}{\lim_{bx \to 0} \left(\frac{\sin bx}{ax} \right)}$$

$x \to 0 \Rightarrow ax \to 0$

and $x \to 0 \Rightarrow bx \to 0$

$\frac{a}{b} \times \frac{1}{1}$

$\left[\lim_{x \to 0} \left(\frac{\sin y}{y} \right) = 1 \right]$

$= \frac{a}{b}$

15. Evaluate the given limit: $\lim_{x \to z} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

Ans: $\lim_{x \to z} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

It is seen that $x \to \pi \Rightarrow (\pi \cdot x) \to 0$

$$= \lim_{x \to 0} \left(\frac{\sin ax}{ax} \right) \times \frac{a}{b}$$

$$\frac{a}{b}\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right) \quad [x\to 0 \Rightarrow ax\to 0]$$

$$=\frac{a}{b}\times 1$$

$$\left[\lim_{x\to 0}\left(\frac{\sin y}{y}\right)\right]$$

$$=\frac{a}{b}$$

16. Evaluate the given limit: $\lim_{x\to 0}\frac{\cos x}{\pi - x}$

Ans: $\lim_{x\to 0}\frac{\cos x}{\pi - x}=\frac{\cos 0}{\pi - 0}=\frac{1}{\pi}$

17. Evaluate the given limit: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

Ans: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

At $x=0$, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1}$

$$\left[\cos x = 1-2\sin^2\frac{x}{2}\right]$$

$$-\lim_{x\to 0}\frac{\sin^2 x}{\sin^2 \frac{x}{2}} = \lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times \frac{x^2}{4}}$$

$$= 4\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$= 4\frac{\left(\lim_{x\to 0}\frac{\sin^2 x}{x^2}\right)^2}{\left(\lim_{x\to 0}\frac{\sin \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)^2}$$

$$\left[x \to 0 \Rightarrow \frac{x}{2} \to 0\right]$$

$$= 4\frac{1^2}{1^2} \quad \left[\lim_{y\to 0}\frac{\sin y}{y} = 1\right]$$

$$= 8$$

18. Evaluate the given limit: $\lim_{x\to 0}\frac{ax + x\cos x}{b\sin x}$

Ans: $\lim_{x\to 0}\frac{ax + x\cos x}{b\sin x}$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$

$$\lim_{b\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)$$

$$\frac{1}{b}\left(\frac{1}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)}\right)\times\lim_{x\to 0}(a+\cos x)$$

$$\frac{1}{b}\times(a+\cos 0)\quad\left[\lim_{y\to 0}\frac{\sin x}{x}=1\right]$$

$$-\frac{a+1}{b}$$

19. Evaluate the given limit: $\lim_{x\to 0}x\sec x$

Ans: $\lim_{x\to 0}x\sec x=\lim_{x\to 0}\frac{x}{\cos x}=\frac{0}{\cos 0}=\frac{0}{1}=0$

20. Evaluate the given limit: $\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}\, a,b,a+b\neq 0$

Ans: At $x=0$, the value of the given function takes the form $\frac{0}{0}$ Now, $\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax+bx}{ax+bx\left(\frac{\sin bx}{bx}\right)}$$

$$=\frac{\left(\lim_{x\to 0}\frac{\sin ax}{ax}\right)\times\lim_{x\to 0}(ax)+\lim_{x\to 0}(bx)}{\lim_{x\to 0}ax+\lim_{x\to 0}bx\left(\lim_{x\to 0}\frac{\sin bx}{bx}\right)}\quad [\text{ As } x\to 0\Rightarrow \text{ax}\to 0 \text{ and } bx\to 0]$$

$$= \frac{\lim_{x\to 0}(ax) + \lim_{x\to 0} bx}{\lim_{x\to 0} ax + \lim_{x\to 0} bx}$$

$$\left[\lim_{y\to 0} \frac{\sin x}{x} = 1\right]$$

$$= \frac{\lim_{x\to 0}(ax + bx)}{\lim_{x\to 0}(ax + bx)}$$

$$= \lim_{x\to 0}(1)$$

$$= 1$$

21. Evaluate the given limit: $\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$

Ans: At $x = 0$, the value of the given function takes the form $\infty - \infty$ Now, $\lim_{x\to 0}(\operatorname{cosec} x \quad \cot x)$

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0} \frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{\lim_{x\to 0} \frac{1-\cos x}{x}}{\lim_{x\to 0} \frac{\sin x}{x}}$$

$$- \frac{0}{1}$$

$$\left[\lim_{y\to 0}\frac{1-\cos x}{x}=0 \text{ and } \lim_{y\to 0}\frac{\sin x}{x}=1\right]$$

$$=0$$

22. **Evaluate the given limit:** $\lim_{x\to\frac{x}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$

Ans: $\lim_{x\to\frac{z}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$

At $x=\frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$ Now, put So that $x-\frac{\pi}{2}-y$ so that $x\to\frac{\pi}{2}, y\to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}=\lim_{y\to 0}\frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$\lim_{y\to 0}\frac{\tan(\pi+2y)}{y}$$

$$-\lim_{y\to 0}\frac{\tan 2y}{y}$$

$$[\tan(\pi+2y)=\tan 2y]$$

$$-\lim_{y\to 0}\frac{\sin 2y}{y\cos 2y}$$

$$\lim_{y\to 0}\left(\frac{\sin 2y}{2y}\times\frac{2}{\cos 2y}\right)$$

$$-\left(\lim_{y\to 0}\frac{\sin 2y}{2y}\right)\times\lim_{y\to 0}\left(\times\frac{2}{\cos 2y}\right)$$

$$[y\to 0\Rightarrow 2y\to 0]$$

$$-1\times\frac{2}{\cos 0}\quad\left[\lim_{\leftrightarrow\infty 0}\frac{\sin x}{x}=1\right]$$

$$=1\times\frac{2}{1}$$

$$-2$$

23. **Find** $\lim_{x\to 0} f(x)$ **and** $\lim_{x\to 1} f(x)$**, where** $f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1), & x>0\end{cases}$

Ans: The given function is

$$f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1), & x>0\end{cases}$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}[2x+3]=2(0)+3-3$$

$$\lim_{v\to 0^+} f(x)=\lim_{x\to 0}3(x+1)-3(0+1)-3$$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=\lim_{x\to 0} f(x)=3$$

$$\lim_{x\to \pi} f(x)=\lim_{x\to 1}(x+1)=3(1+1)=6$$

$$\therefore \lim_{x\to +} f(x)=\lim_{x\to 1^-} f(x)=\lim_{x\to 1} f(x)=6$$

24. **Find** $\lim_{x\to 1} f(x)$**, when** $f(x)=\begin{cases}x^2-1, & x\le 1\\ -x-1, & x>1\end{cases}$

Ans: The given function is

$$f(x)=\begin{cases}x^2-1, & x\le 1\\ -x-1, & x>1\end{cases}$$

$$\therefore \lim_{x\to T} f(x)=\lim_{x\to 1}\left[x^2-1\right]-1^2-1-1-1=0$$

It is observed that $\lim_{x\to\pi} f(x) \neq \lim_{x\to 1^+} f(x)$.

Hence, lim $f(x)$ does not exist.

25. **Evaluate** $\lim_{x\to 0} f(x)$**, where** $f(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Ans: The given function is $f(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left[\frac{|x|}{x}\right]$

$-\lim_{x\to 0}\left(\frac{-x}{x}\right)$

[When x is negative, $|x|=\cdot x$]

$-\lim_{x\to 0}(-1)$

$-\cdot 1$

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left[\frac{|x|}{x}\right]$

$-\lim_{x\to 0}\left(\frac{x}{x}\right)$

[When x is positive, $|x|-x$]

$=\lim_{x\to 0}(1)$

-1

It is observed that $\lim_{x\to\sigma} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

26. **Find** $\lim_{x\to 0} f(x) - \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Ans: The given function is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left[\frac{x}{|x|}\right]$$

$$-\lim_{x\to 0}\left(\frac{x}{-x}\right)$$

[When $\mathrm{x} < 0, |x| - \cdot \mathrm{x}$]

$$-\lim_{x\to 0}(-1)$$

$$= \cdot 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{x}{|x|}\right]$$

$$-\lim_{x\to 0}\left(\frac{x}{x}\right)$$

[When $x > 0, |x| = x$]

$$-\lim_{x\to 0}(1)$$

$$-1$$

It is observed that $\lim_{x\to \sigma} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

27. **Find** $\lim_{x\to 5} f(x)$**, where** $f(x)=|x|-5$

Ans: The given function is $f(x)=|x|-5$

$\lim_{y\to 5^-} f(x)=\lim_{x\to 5}(|x|-5)$

$=\lim_{x\to 5}(x-5)$

[When $x>0,|x|-x]$

$-5\cdot 5$

-0

$\lim_{x\to^+} f(x)=\lim_{x\to 5}(|x|-5)$

$=\lim_{x\to 5}(x-5)$

[When $x>0,|x|-x]$

$-5-5$

$=0$

$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=0$

Hence, $\lim_{x\to 5} f(x)=0$

28. **Suppose** $f(x)=\begin{cases} a+bx, & x<0 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$ **and if** $\lim_{x\to 1} f(x)=f(1)$ **what are possible values of a and b?**

Ans: The given function is

$$f(x)=\begin{cases} a+bx, & x<0 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$$

$\lim\limits_{-\pi} f(x) = \lim\limits_{x\to 1}(a+bx) = a+b$

$\lim\limits_{x,1-} f(x) = \lim\limits_{x\to 1}(b-ax) = b-a$

$f(1) = 4$

It is given that $\lim\limits_{x\to 1} f(x) = f(1)$.

$\therefore \lim\limits_{x\to \pi} f(x) = \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} f(x) = f(1)$

$\Rightarrow a+b-4$ and $b-a=4$

On solving these two equations, we obtain $a=0$ and $b=4$. Thus, the respective possible values of a and b are 0 and 4 .

29. **Let** $a_1, a_2, \ldots, a_n$ **be fixed real numbers and define a function**

$f(x) = (x-a_1)(x-a_2)\ldots..(x-a_n)$

What is $\lim f(x)$? **For some** $a \neq a_1, a_2, \ldots, a_n$. **Compute** $\lim\limits_{x\to a} f(x)$.

Ans: The given function is $f(x) = (x-a_1)(x-a_2)\ldots..(x-a)$

$\lim\limits_{x\to +3} f(x) = \lim\limits_{x\to a}\left[(x-a_1)(x-a_2)\ldots..(x-a_n)\right]$

$= (a_1-a_1)(a_1-a_2)\ldots..(a_1-a_n) = 0$

$\therefore \lim\limits_{x\to 3} f(x) = 0$

Now, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a}\left[(x-a_1)(x-a_2)\ldots..(x-a_n)\right]$

$-(a-a_1)(a-a_2)\ldots..(a-a)$

$\therefore \lim f(x) = (a-a_1)(a-a_2)\ldots\ldots(a-a)$

30. **If** $f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>1\end{cases}$ **For what value (s) does** $\lim_{x\to 3} f(x)$ **exists?**

Ans: The given function is

If $f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\end{cases}$.

When $\mathrm{a}=0$

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(|x|+1)$

$=\lim_{x\to 0}(-x+1)$

[If $x<0,|x|=-x$]

$=0+1$

-1

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(|x|+1)$

$=\lim_{x\to 0}(x-1)$

[If $x>0,|x|=-x$]

$=0-1$

$=-1$

Here, it is observed that $\lim_{x\to \sigma} f(x)\neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When $a<0$ $\lim_{x\to \tau} f(x)=\lim_{x\to a^-}(|x|+1)$

$=\lim_{x\to a}(-x+1)$

$[x<a<0 \Rightarrow |x|--x]$

$= -a + 1$

$\lim_{x \to t} f(x) = \lim_{x \to \Delta}(|x| + 1)$

$= \lim_{x \to a}(-x + 1)$

$[a < x < 0 \Rightarrow |x| - -x]$

$= -a$

$+1$

$\therefore \lim_{x \to d^-} f(x) = \lim_{x \to a^+} f(x) = -a + 1$

Thus, limit of $f(x)$ exists at $x - a$, where $a < 0$. When $a > 0$

$\lim_{x \to \mathbb{Z}} f(x) = \lim_{x \to a}(|x| + 1)$

$= \lim_{x \to a}(-x - 1)$

$[0 < x < a \Rightarrow |x| - x]$

$= a - 1$

$\lim_{x \to ^2} f(x) = \lim_{x \to \Delta}(|x| - 1)$

$= \lim_{x \to a}(-x - 1)$

$[0 < x < a \Rightarrow |x| = x]$

$= a - 1$

$\therefore \lim_{x \to \pi} f(x) = \lim_{x \to ^+} f(x) = a - 1$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$ Thus, $\lim_{x \to a} f(x)$ exists for all $a \neq 0$.

31. If the function $f(x)$ satisfies, $\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \to 1} f(x)$

Ans: $\lim_{x\to 1}\frac{f(x)-2}{x^2-1}=\pi$

$\Rightarrow \frac{\lim_{x\to 1}(f(x)-2)}{\lim_{x\to 1}\left(x^2-1\right)}=\pi$

$\Rightarrow \lim_{x\to 1}(f(x)-2)=\pi\lim_{x\to 1}\left(x^2-1\right)$

$\Rightarrow \lim_{x\to 1}(f(x)-2)=\pi\left(1^2-1\right)$

$\Rightarrow \lim_{x\to 1}(f(x)-2)=0$

$\Rightarrow \lim_{x\to 1}f(x)-\lim_{x\to 1}2=0$

$\Rightarrow \lim_{x\to 1}f(x)-2=0$

$\therefore \lim_{x\to 1}f(x)=2$

32. **If** $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$

For what integers m **and** n **does** $\lim_{x\to 0}f(x)$ **and** $\lim_{x\to 1}f(x)$ **exist?**

Ans: $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1. \\ nx^3+m, & x>1 \end{cases}$

$\lim_{x\to 0}f(x)=\lim_{x\to 0}\left(mx^2+n\right)$

$=m(0)^2+n$

$=n$

$=n(0)+m$

$= m$

Thus, $\lim_{x \to 0^+} f(x)$ exists if $\mathrm{m} = \mathrm{n}$.

$\lim_{x \to \sqrt{-}} f(x) = \lim_{x \to 1} (nx + m)$

$= n(1) + m$

$= m + n$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} \left(nx^3 + m \right)$

$= n(1)^3 + m$

$= m + n$

$\therefore \lim_{x \to \sqrt{}} f(x) = \lim_{x \to +} f(x) = \lim_{x \to 1} f(x).$

Thus, $\lim_{u \to 1} f(x)$ exists for any internal value of m and n.

Exercise 12.2

1. Find the derivative of $x^2 - 2$ at $x = 10$.

Ans: Let $f(x) = x^2 - 2.$ Accordingly. $f'(10) = \lim_{h \to 0} \dfrac{f(10+h) - f(10)}{h}$

$$= \lim_{h \to 0} \frac{\left[(10+h)^2 - 2 \right] - \left(10^2 - 2 \right)}{h}$$

$$= \lim_{h \to 0} \frac{10^2 + 2 \cdot 10 \cdot h + h^2 - 2 - 10^2 + 2}{h}$$

$$= \lim_{h \to 0} \frac{20h + h^2}{h}$$

$$= \lim_{h \to 0} (20 + h) = 20 + 0 = 20$$

Thus, the derivative of $x^2 - 2$ at $x - 10$ is 20 .

2. Find the derivative of 99x **at** $\text{x}-100$**.**

Ans: Let $f(x)=99x$. Accordingly,

$$f^{'}(100)=\lim_{h\to 0}\frac{f(100+h)-f(100)}{h}$$

$$=\lim_{n\to 0}\frac{99(100+h)-99(100)}{h}$$

$$=\lim_{h\to 0}\frac{99\times 100+99h-99\times 100}{h}$$

$$=\lim_{h\to 0}\frac{99h}{h}$$

$$=\lim_{h\to 0}(99)=99$$

Thus, the derivative of 99 x at $x=100$ is 99 .

3. Find the derivative of x **at** $x=1$**.**

Ans: Let $f(x)=x.$ Accordingly.

$$f^{'}(1)=\lim_{n\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)-1}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}$$

$$=\lim_{h\to 0}(1)=1$$

Thus, the derivative of x at $x=1$ is 1 .

4: Find the derivative of the following functions from first principles.

(i) x^3-27

(ii) $(x-1)(x-2)$

(iii) $\frac{1}{x^2}$

(iv) $\frac{x+1}{x-1}$

Ans: (i) Let $f(x)=x^3-27$. Aocordingly, from the first principle,

$$f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left[(x+h)^3-27\right]-\left(x^3-27\right)}{h}$$

$$=\lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xt^2-x^3}{h}$$

$$=\lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h}$$

$$=\lim_{h\to 0}\left(h^3+3x^2h+3xh^2\right)$$

$$=0+3x^2+0=3x^2$$

(ii) Let $f(x)=(x-1)(x-2)$. Accordingly, from the first principle,

$$f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}$$

$$=\lim_{n\to 0}\frac{\left(x^2+hx-2x+hx+t^2-2h-x-h+2\right)-\left(x^2-2x-x+2\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left(hx+hx+h^2-2h-h\right)}{h}$$

$$=\lim_{h\to 0}\frac{2hx+h^2-3h}{h}$$

$$=\lim_{n\to 0}(2x+h-3)$$

$$-2x-3$$

(iii) Let $\mathrm{f}(\mathrm{x})=\frac{1}{x^2}$. Accordingly, from the first principle,

$$f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-2hx-h^2}{x^2(x+h)^2}\right]$$

$$=\lim_{n\to 0}\frac{1}{h}\left[\frac{-A-2hx}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\left[\frac{-h^2-2x}{x^2(x+h)^2}\right]$$

$$=\frac{0-2x}{x^2(x+0)^2}=\frac{-2}{x^3}$$

(iv) Let $f(x)=\frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$

$$=\lim_{n\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[\frac{\left(x^2+hx+x-x-h-1\right)-\left(x^2+hx-x+x+h-1\right)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$

$$= \lim_{n \to 0} \left[\frac{-2}{(x-1)(x+h-1)}\right]$$

$$= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

5. For the function $F(x) = \frac{x^{10}}{100} + \frac{x^{59}}{99} + \cdots + \frac{x^2}{2} + x + 1$

Prove that $f(1) = 100 f^{'}(0)$

Ans: The given function is

$$F(x) = \frac{x^{\mathrm{m}}}{100} + \frac{x^{59}}{99} + \cdots + \frac{x^2}{2} + x + 1$$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[\frac{x^{1\mathbf{m}}}{100} + \frac{x^{m}}{99} + \cdots + \frac{x^2}{2} + x + 1\right]$$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{x^{100}}{100}\right) + \frac{d}{dx}\left(\frac{x^{99}}{99}\right) + \cdots + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}(x) + \frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$$\frac{d}{dx} f(x) = \frac{100x^{99}}{100} + \frac{99x^{88}}{99} + \cdots + \frac{2x}{2} + 1 + 0$$

$$= x^{99} + x^{88} + \cdots + x + 1$$

$$\therefore f^{'}(x) = x^{99} + x^{18} + \cdots + x + 1$$

At $x = 0$

$f'(0)=1$

At $x=1$,

Thus, $\mathrm{f}(1)=100\mathrm{f}(0)$

6. Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+\cdots+a^{n-1}\chi+a^n$ for some fixed real number a.

Ans: Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+\cdots+a^{n-1}x+a^n$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+\cdots+a^{n-1}x+d^n\right)$$

$$=\frac{d}{dx}\left(x^n\right)+a\frac{d}{dx}\left(x^{n-1}\right)+a^2\frac{d}{dx}\left(x^{n-2}\right)+\cdots+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}\left(x^n\right)=n^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+\cdots+a^{n-1}+a^n(0)$$

$$\therefore f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+\cdots+a^{n-1}$$

7. For some constants a and b, find the derivative of

(i) $(x-a)(x-b)$

(ii) $\left(ax^2+b\right)^2$

(iii) $\frac{x-a}{x-b}$

Ans: (i) Let $f(x)=(x-a)(x-b)$

$$\Rightarrow f(x)=x^2-(a+b)x+ab$$

$$\therefore f'(x)=\frac{d}{dx}\left(x^2-(a+b)x+ab\right)$$

$$= \frac{d}{dx}\left(x^2\right) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a+b) + 0$$

$$= 2x - a - b$$

(ii) Let $f(x) - \left(ax^2 + b\right)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f(x) = \frac{d}{dx}\left(a^2x^4 + 2abx^2 + b^2\right)$$

$$= a^2\frac{d}{dx}\left(x^4\right) + 2ab\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}b^2$$

On using theorem $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$$f'(x) = a^2\left(4x^3\right) + 2ab(2x) + b^2(0)$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax\left(ax^2 + b\right)$$

(iii) Let $f(x) = \frac{x-a}{x-b}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

8. **Find the derivative of $\frac{x^n - a^n}{x-a}$ for some constant a.**

Ans: Let $f(x) = \frac{x^n - d^n}{x-a}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x-a}\right)$$

By quotient rule, $f'(x) = \frac{(x-a)\frac{d}{dx}\left(x^n - a^n\right) - \left(x^n - a^n\right)\frac{d}{dx}(x-a)}{(x-a)^2}$

$$= \frac{(x-a)\left(nx^{n-1} - 0\right) - \left(x^n - a^n\right)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

9: **Find the derivative of**

(i) $2x - \frac{3}{4}$

(ii) $\left(5x^3 + 3x - 1\right)(x-1)$

(iii) $x^{-3}(5+3x)$

(iv) $x^5\left(3 - 6x^{-9}\right)$

(v) $x^{-4}\left(3 - 4x^{-5}\right)$

(vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Ans: (i) Let $f(x)=2x-\frac{3}{4}$

$$f'(x)=\frac{d}{dx}\left(2x-\frac{3}{4}\right)$$

$$=2\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{3}{4}\right)$$

$$-2-0$$

(ii) Let .$f(x)=\left(5x^3+3x-1\right)(x-1)$

By Leibnitz product rule,

$$f'(x)=\left(5x^3+3x-1\right)\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}\left(5x^3+3x-1\right)$$

$$-\left(5x^3+3x-1\right)(1)+(x-1)\left(5.3x^2+3-0\right)$$

$$-\left(5x^3+3x-1\right)+(x-1)\left(15x^2+3\right)$$

$$-5x^3+3x-1+15x^3+3x-15x^2-3$$

$$=20x^3-15x^2+6x-4$$

(iii) Let $f(x)=x^{-3}(5+3x)$

By Leibnitz product rule,

$$f'(x)=x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}\left(x^{-3}\right)$$

$$=x^{-3}(0+3)+(5+3x)\left(3x^{-3-1}\right)$$

$$=x^{-3}(3)+(5+3x)\left(3x^{-4}\right)$$

$$=3x^{-3}-15x^{-4}-9x^{-3}$$

$= -6x^3 - 15x^4$

$= -3x^{-3}\left(2 + \frac{5}{x}\right)$

$= \frac{-3x^{-3}}{x}(2x+5)$

$= \frac{-3}{x^4}(5+2x)$

(iv) Let $f(x) = x^5\left(3-6x^{-9}\right)$

By Leibnitz product rule,

$f'(x) = x^5 \frac{d}{dx}\left(3-6x^9\right) + \left(3-6x^{-9}\right)\frac{d}{dx}\left(x^5\right)$

$= x^5\left\{0-6(-9)x^{-2-1}\right\} + \left(3-6x^9\right)\left(5x^4\right)$

$= x^5\left(54x^{-10}\right) + 15x^4 - 30x^{-5}$

$= 54x^5 + 15x^4 - 30x^5$

$= 24x^5 + 15x^4$

$= 15x^4 + \frac{24}{x^5}$

(v) Let $f(x) = x^{-4}\left(3-4x^5\right)$

By Leibnitz product rule,

$f'(x) = x^{-4}\frac{d}{dx}\left(3-4x^{-5}\right) + \left(3-4x^{-5}\right)\frac{d}{dx}\left(x^{-4}\right)$

$= x^{-4}\left\{0-4(-5)x^{-5-1}\right\} + \left(3-4x^5\right)(-4)x^{-4-1}$

$= x^{-1}\left(20x^6\right) + \left(3-4x^5\right)\left(-4x^{-5}\right)$

$= 20x^{10} - 12x^{-5} + 16x^{-0}$

$= 36x^{-10} - 12x^{-6}$

$= \frac{12}{x^5} + \frac{36}{x^{20}}$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}\left(x^2\right) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left[\frac{(x+1)(0) - 2(0)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - x^2(3)}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

10. Find the derivative of cos x from first principle.

Ans: Let $\mathrm{f(x) - \cos x}$. Accordingly, from the first principle, $f'(x) = \lim_{n\to 0}\frac{f(x+h) - f(x)}{h}$

$$= \lim_{n\to 0}\left[\frac{\cos(x+h) - \cos(x)}{h}\right]$$

$$= \lim_{n\to 0}\left[\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}\right]$$

$$= \lim_{n \to 0} \left[\frac{-\cos x(1-\cos h)}{h} \frac{-\sin x \sin h}{h} \right]$$

$$= -\cos x \left[\lim_{n \to 0} \left(\frac{1-\cos h}{h} \right) \right] - \sin x \left[\lim_{n \to 0} \left(\frac{\sin h}{h} \right) \right]$$

$$= -\cos x(0) - \sin x(1) \quad \left[\lim_{n \to 0} \frac{1-\cos h}{h} = 0 \text{ and } \lim_{n \to 0} \frac{\sin h}{h} = 1 \right]$$

$$\therefore f'(x) = -\sin x$$

11: **Find the derivative of the following functions:**

(i) $\sin x \cos x$

(ii) $\sec x$

(iii) $5\sec x + 4\cos x$

(iv) $\text{cosec}\, x$

(v) $3\cot x + 5\,\text{cosec}\, x$

(vi) $5\sin x - 6\cos x + 7$

(vii) $2\tan x - 7\sec x$

Ans: (i) Let $f(x) = \sin x \cos x.$ Accordingly, from the first principle,

$$f'(x) = \lim_{n \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{n \to 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h}$$

$$= \lim_{h \to 0} \frac{1}{2h} [2\sin(x+h)\cos(x+h) - 2\sin x \cos x]$$

$$= \lim_{n \to 0} \frac{1}{2h} [\sin 2(x+h) - \sin 2x]$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}\cdot\sin\frac{2x+2h-2x}{2}\right]$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[2\cos\frac{4x+2h}{2}\cdot\sin\frac{2h}{2}\right]$$

$$= \lim_{n \to 0} \frac{1}{2h}[\cos(2x+h)\sin h]$$

$$= \lim_{n \to 0} \cos(2x+h)\cdot\lim_{n \to 0}\frac{\sin h}{h}$$

$$= \cos(2x+h)\cdot 1$$

$$= \cos 2x$$

(ii) Let $f(x) = \sec x.$ Accordingly, from the first principle,

$$f'(x) = \lim_{n \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{n \to 0} \frac{\sec(x+h)-\sec x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$= \frac{1}{\cos x}\cdot\lim_{n \to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x}\cdot\lim_{h \to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{2h} \frac{\left[-2\sin\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)} \right]}{\cos(x+h)}$$

$$= \frac{1}{\cos x} \cdot \lim_{i \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{i \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$

$$= \frac{1}{\cos x} \cdot 1 \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \tan x$$

(iii) Let $f(x) = 5\sec x + 4\cos x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{n \to 0} \frac{5\sec(x+h) + 4\cos(x+h) - [5\sec x + 4\cos x]}{h}$$

$$= 5\lim_{n \to 0} \frac{[\sec(x+h) - \sec x]}{h} + 4\lim_{n \to 0} \frac{[\cos(x+h) - \cos x]}{h}$$

$$= 5\lim_{n \to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right] + 4\lim_{n \to 0} \frac{1}{h}[\cos(x+h) - \cos x]$$

$$= 5\lim_{n \to 0} \frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)}\right] + 4\lim_{n \to 0} \frac{1}{h}[\cos x \cos h - \sin x \sin h - \cos x]$$

$$= \frac{5}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right] + 4\left[-\cos x \lim_{h \to 0} \frac{(1-\cos x)}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h}\right]$$

$$= \frac{5}{\cos x} \cdot \lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\cos(x+h)} + 4[-\cos x(0) - \sin x(1)]$$

$$= \frac{5}{\cos x} \cdot \left[\lim_{n \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{n \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \right] - 4\sin x$$

$$= \frac{5}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot 1 - 4\sin x$$

$$= 5\sec x \tan x - 4\sin x$$

(iv) Let $f(x) = \operatorname{cosec} x$. Accordingly, from the first principle.

$$f'(x) = \lim_{n \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{n \to 0} \frac{1}{h}[\operatorname{cosec}(x+h) - \operatorname{cosec} x]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{-\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}\Bigg]}{\sin x\sin(x+h)}$$

$$= \lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin x\sin(x+h)}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x\sin x}\right)\cdot 1$$

$$= -\operatorname{cosec} x\cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$. Accordingly, from the first principle, $f'(x) = \lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{n\to 0}\frac{1}{h}[3\cot(x+h) + 5\operatorname{cosec}(x+h) - 3\cot x - 5\operatorname{cosec} x]$$

$$= 3\lim_{n\to 0}\frac{1}{h}[\cot(x+h) - \cot x] + 5\lim_{n\to 0}\frac{1}{h}[\operatorname{cosec}(x+h) - \operatorname{cosec} x]$$

…..

Now, $\lim_{n\to 0}\frac{1}{h}[\cot(x+h)-\cot x] = \lim_{n\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right]$

$$= \lim_{n\to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x - \cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{n\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{n\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{n\to 0}\left[\frac{1}{\sin x\sin(x+h)}\right]$$

$$= -1\cdot\frac{1}{\sin x\sin(x+h)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \quad ....(2)$$

$$\lim_{n\to 0}\frac{1}{h}[\operatorname{cosec}(x+h)-\operatorname{cosec} x] = \lim_{n\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$= \lim_{n\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin x\sin(x+h)}$$

$$= \lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin x\sin(x+h)}\right)\cdot\lim_{n\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x\sin x}\right)\cdot 1$$

$$= -\operatorname{cosec} x\cot x$$

From (1), (2), and (3), we obtain

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x\cot x$$

(vi) Let $\mathrm{f(x)=5\sin x-6\cos x+7}$. Accordingly, from the first principle, $f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{1}{h}[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7]$$

$$=5\lim_{n\to 0}\frac{1}{h}[\sin(x+h)-\sin x]-6\lim_{n\to 0}\frac{1}{h}[\cos(x+h)-\cos x]$$

$$=5\lim_{n\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{n\to 0}\frac{\cos x\cos h-\sin x\sin\mathrm{h}-\cos x}{h}$$

$$=5\lim_{n\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)\right]-6\lim_{n\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$=5\lim_{n\to 0}\frac{1}{h}\left[\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]-6\lim_{n\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=5\left[\lim_{n\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]-6\left[-\cos x\left(\lim_{n\to 0}\frac{1-\cos h}{h}\right)-\sin x\left(\lim_{n\to 0}\frac{\sin h}{h}\right)\right]$$

$$=5\cos x\cdot 1-6[(-\cos x)\cdot(0)-\sin x\cdot 1]$$

$$=5\cos x+6\sin x$$

(vii) Let $f(x)=2\tan x-7\sec x$. Accordingly, from the first principle, $f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{n\to 0}\frac{1}{h}[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x]$$

$$=2\lim_{n\to 0}\frac{1}{h}[\tan(x+h)-\tan x]-7\lim_{n\to 0}\frac{1}{h}[\sec(x+h)-\sec x]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{n\to 0}\frac{1}{h}\left[\frac{1}{\mathrm{cosec}(x+h)}-\frac{1}{\mathrm{cosec}\,x}\right]$$

$$= 2\lim_{n\to 0}\frac{1}{h}\left[\frac{\cos x\sin(x+h)-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{n\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x+h-x}{\cos x\cos(x+h)}\right]-7\lim_{n\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$= 2\left[\lim_{n\to 0}\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{n\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$= 2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left[\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right]-7\left(\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$= 2\cdot 1\cdot 1\frac{1}{\cos x\cos x}-7\cdot 1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$= 2\sec^2 x-7\sec x\tan x$$

Miscellaneous Exercise

1: Find the derivative of the following functions from first principle:

(i) -x

(ii) $(-x)^{-1}$

(iii) $\sin(x+1)$

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Ans: (i) Let $f(x)=-x$. Accordingly, $f(x+h)=-(x+h)$

By first principle, $f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \frac{-(x+h)-(-x)}{h}$$

$$= \lim_{n \to 0} \frac{-x-h+x}{h}$$

$$= \lim_{n \to 0} \frac{-h}{h}$$

$$= \lim_{h \to 0}(-1) = -1$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$f'(x) = \lim_{n \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-1}{(x+h)} - \left(\frac{-1}{x}\right)\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{h}{x(x+h)}\right]$$

$$= \lim_{n \to 0} \frac{1}{x(x+h)}$$

$$= \frac{1}{x \cdot X} = \frac{1}{x^2}$$

(iii) Let $f(x) = \sin(x+1)$. Accordingly, $f(x+h) = \sin(x+h+1)$

By first principle,

$$f'(x) = \lim_{n \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{n \to 0} \frac{1}{h}[\sin(x+h+1) - \sin(x+1)]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{n \to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{b}{2}\to}\frac{\sin\left(\frac{h}{2}\right)}{h}\frac{h}{\left.\frac{h}{2}\right)} \quad \left[\text{As } h \to 0 \Rightarrow \frac{h}{2} \to 0\right]$$

$$= \cos\left(\frac{2x+0+2}{2}\right)\cdot 1 \quad \left[\lim_{h \to 0}\frac{\sin x}{x} = 1\right]$$

$$= \cos(x+1)$$

(iv) Let $f(x) = \cos\left(x - \frac{\pi}{8}\right)$. Accordingly, $f(x+h) - \cos\left(x + h - \frac{\pi}{8}\right)$

By first principle,

$$f'(x) = \lim_{n \to 0} \frac{f(x+f) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right) - \cos\left(x-\frac{\pi}{8}\right)\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{n\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{n\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right]\cdot\lim_{\frac{\pi}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\quad\left[\text{Ash}\to 0\Rightarrow\frac{\text{h}}{2}\to 0\right]$$

$$=-\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right)\cdot 1$$

$$=-\sin\left(x-\frac{\pi}{8}\right)$$

2. Find the derivative of the following functions (it is to be understood that a, b, c, d. p, q, r and s are fixed non-zero constants and m and n are integers): $(x+a)$

Ans: Let $f(x)=x+a$. Accordingly. $f(x+h)-x+h+a$ By first principle,

$$f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{n\to 0}\frac{x+h+a-x-a}{h}$$

$$\lim_{n\to 0}\left(\frac{h}{h}\right)$$

$$-\lim_{n\to 0}(1)$$

$$=1$$

3. **Find the derivative of the following functions (it is to be understood that a, b, c. d, p, q, r and s are fixed non- zero constants and m and n are integers):**

$(px+q)\left(\frac{r}{x}+s\right)$

Ans: Let $f(x)=(px+q)\left(\frac{r}{x}+s\right)$

By Leibnitz product rule.

$$f'(x)=(px+q)\left(\frac{r}{x}+s\right)'+\left(\frac{r}{x}+s\right)(px+q)'$$

$$-(px+q)\left(rx^{-1}+s\right)'+\left(\frac{r}{x}+s\right)(p)$$

$$-(px+q)\left(-nx^{2}\right)+\left(\frac{r}{x}+s\right)p$$

$$=(px+q)\left(\frac{-r}{x^2}\right)+\left(\frac{r}{x}+s\right)p$$

$$=\frac{-px}{x}-\frac{qr}{x^2}+\frac{pr}{x}+ps$$

$$ps-\frac{qr}{x^2}$$

4. **Find the derivatives of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):**

$(ax+b)(cx+d)^2$

Ans: Let $f'(x)=(ax+b)(cx+d)^2$

By Leibnitz product rule,

$$f'(x)=(ax+b)\frac{d}{dx}(cx+d)^2\frac{d}{dx}(ax+b)$$

$$(ax+b)\frac{d}{dx}\left(c^2x^2+2cdx^2\right)+(cx+d)^2\frac{d}{dx}(ax+b)$$

$$(ax+b)\left[\frac{d}{dx}\left(c^2x^2\right)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right]+(cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right]$$

$$=(ax+b)\left(2c^2x+2cd\right)+(cx+d)^2a$$

$$-2c(ax+b)(cx+d)+a(cx+d)^2$$

5. Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers):

$$\boldsymbol{\frac{ax+b}{cx+d}}$$

Ans: Let $f(x)=\frac{ax+b}{cx+d}$

By quotient rule,

$$f(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$=\frac{(cx+d)(a)-(ax+d)(c)}{(cx+d)^2}$$

$$\frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$\frac{ad-bc}{(cx+d)^2}$$

6. Find the derivative of the following functions (it is to be understood that a , b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

Ans: Let $f(x)=\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}=\dfrac{\frac{x+1}{x}}{\frac{x-1}{x}}=\dfrac{x+1}{x-1}$, where $x\neq 0$

By quotient rule, $f'(x)=\dfrac{(x-1)\frac{d}{dx}(x-1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, x\neq 0,1$

$=\dfrac{(x-1)(1)-(x+1)(1)}{(x-1)^2}, x\neq 0,1$

$\dfrac{x-1-x-1}{(x-1)^2}, x\neq 0,1$

$\dfrac{-2}{(x-1)^2}, x\neq 0,1$

7. Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and.m and n are integers)

$: \dfrac{1}{ax^2+bx+c}$

Ans: Let $f(x)=\dfrac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x)=\frac{\left(ax^2+bx+c\right)\frac{d}{dx}(1)-\frac{d}{dx}\left(ax^2+bx+c\right)}{\left(ax^2+bx+c\right)^2}$$

$$\frac{\left(ax^2+bx+c\right)(0)-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

$$\frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

8: **Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, rand s are fixed non-zero constants and m and.n are integers**

$$\frac{ax+b}{px^2+qx+r}$$

Ans: Let $f(x)=\dfrac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f^{'}(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{apx^2+aqx+ar-aqx+2npx+bq}{\left(px^2+qx+r\right)^2}$$

$$\frac{-apx^2+2bpx+ar-bq}{\left(px^2+qx+r\right)^2}$$

9: **Find the derivative of the following functions (it is to be understood that a, b, c. d, p, q, r and s are fixed non-zero constants and m and n are integers):**

$$\frac{px^2+qx+r}{ax+b}$$

Ans: Let $f(x)=\dfrac{px^2+qx+r}{ax+b}$

By quotient rule,

$$\dot{f}(x)=\frac{(ax+b)\frac{d}{dx}\left(px^2+qx+\eta\right)-\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$\frac{(ax+b)(2px+q)-\left(px^2+qx+r\right)(a)}{(ax+b)^2}$$

$$=\frac{2apx^2+aqx+2bpx+bq-aqx^2-aqx-ar}{(ax+b)^2}$$

$$\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

10: **Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):** $\boldsymbol{\frac{a}{x^4}-\frac{b}{x^2}+\cos x}$

Ans: Let $f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{a}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(\cos x)$$

$$-a\left(-4x^{-5}\right)-b\left(-2x^3\right)+(-\sin x)\quad\left[\frac{d}{dx}\left(x^n\right)=nx^{n-1}\text{ and }\frac{d}{dx}(\cos x)=-\sin x\right]$$

$$\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

11: **Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed nonzero constants and m and n are integers):** $\boldsymbol{4\sqrt{x}-2}$

Ans: Let $f(x)=4\sqrt{x}-2$

$$f'(x)=\frac{d}{dx}(4\sqrt{x}-2)=\frac{d}{dx}(4\sqrt{x})-\frac{d}{dx}(2)$$

$$= 4\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 0 = 4\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= \left(2x^{-\frac{1}{2}}\right) = \frac{2}{\sqrt{x}}$$

12. Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):
$(ax+b)^n$

Ans: Let $f(x) = (ax+b)^n$. Accordingly, $f(x+h) - \{a(x+h)+b\}^n - (ax+ah+b)^n$

By first principle,

$$f'(x) = \lim_{n\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(ax+ah+b) - (ax+b)^n}{h}$$

$$= \lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n - (ax+b)^n}{h}$$

$$= (ax+b)^n \lim_{n\to 0}\frac{1}{h}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{2}\left(\frac{ah}{ax+b}\right)^2+\cdots\right\}-1\right]$$ (using binomial theorem)

$$= (ax+b)^n \lim_{n\to 0}\frac{1}{h}\left[\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{2(ax+b)^2}+\cdots \text{ (Terms containing higher degrees of } h)\right]$$

$$= (ax+b)^n \lim_{n\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h^2}{2(ax+b)^2}+\cdots\right]$$

$$= (ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$= na\frac{(ax+b)^n}{ax+b}$$

$-na(ax+b)^{n-1}$

13. **Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):**
$(ax+b)^n(cx+d)^m$

Ans: Let $f(x)=(ax+b)^n(cx+d)^m$

By Leibnitz product rule,

$$f'(x)=(ax+b)^n\frac{d}{dx}(cx+d)^m+(cx+d)^m\frac{d}{dx}(ax+b)^n$$

Now let $f_1(x)=(cx+d)^m$

$$f_1(x+h)=(cx+ch+d)^m$$

$$f_1'(x)=\lim_{n\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$

$$=\lim_{n\to 0}\frac{(cx+ch+d)^m-(cx+d)^m}{h}$$

$$=(cx+d)^m\lim_{n\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]$$

$$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{c^2h^2}{(cx+d)^2}+\cdots\right)^m-1\right]$$

$$=(cx+d)^m\lim_{n\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+\cdots\text{ (Terms containing higher degree oh } h)\right]$$

$$=(cx+d)^m\lim_{h\to 0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+\cdots\right]$$

$$=(Cx+a)^m\left[\frac{mch}{(cx+d)}+0\right]$$

$= \frac{mc(cx+d)^m}{(cx+d)}$

$= mc(cx+d)^{m-1}$

$\frac{d}{dx}(cx+d)^m = md(x+d)^{m-1}$

Similarly, $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

... (3)

Therefore, from (1), (2), and (3), we obtain

$f'(x) = (ax+b)^n \left\{mc(cx+d)^{m-1}\right\} + (c+d)^m \left\{na(ax+b)^{n-1}\right\}$

$= (ax+b)^{n-1}(cx+d)^{m-1}[mc(ax+b) + na(cx+d)]$

14. Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):
sin(*x*+ *a*)

Ans: Let, $f(x) = \sin(x+a)$

$f(x+h) = \sin(x+h+a)$

By first principle, $f'(x) = \lim_{n\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{n\to 0} \frac{\sin(x+h+a) - \sin(x+a)}{h}$

$= \lim_{n\to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$

$= \lim_{n\to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$

$$= \lim_{h \to 0} \left[\cos\left(\frac{2x + 2a + h}{2} \right) \left[\frac{\sin\left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right] \right]$$

$$= \lim_{h \to 0} \cos\left(\frac{2x + 2a + h}{2} \right) \cdot \lim_{\frac{h}{2} \to 0} \left[\frac{\sin\left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right] \quad \left[\text{ As } h \to 0 \Rightarrow \frac{h}{2} \to 0 \right]$$

$$= \cos\left(\frac{2x + 2a}{2} \right) \times 1 \quad \left[\lim_{n \to 0} \frac{\sin x}{x} = 1 \right]$$

$$= \cos(x + a)$$

15: **Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):**

$\operatorname{cosec} x \cot x$

Ans: Let $f(x) = \operatorname{cosec} x \cot x$

By Leibnitz product rule,

$$f'(x) = \operatorname{cosec} x(\cot x)' + \cot x(\operatorname{cosec} x)' \ldots .(1)$$

Let $f_1(x) = \cot x.$ Accordingly, $f_1(x + h) = \cot(x + h)$

By first principle,

$$f''(x) = \lim_{n \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cot(x + h) - \cot(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{\cos(x + h)}{\sin(x + h)} - \frac{\cos(x)}{\sin x} \right)$$

$$= \lim_{n \to 0} \frac{1}{h} \left(\frac{\sin x \cos(x + h) - \cos x \sin(x + h)}{\sin x \sin(x + h)} \right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sin(x - x + h)}{\sin x \sin(x+h)} \right)$$

$$= \frac{1}{\sin x^{n\to 0}} \lim_{h} \frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)} \right]$$

$$= \frac{-1}{\sin x}\left(\lim_{n\to 0} \frac{\sin h}{h} \right)\left(\lim_{n\to 0} \frac{1}{\sin(x+h)} \right)$$

$$= \frac{-1}{\sin x} \cdot 1 \cdot \left(\lim_{n\to 0} \frac{1}{\sin(x+0)} \right)$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \quad \ldots \text{ (2)}$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle, $f_2(x) = \lim_{n\to 0} \frac{f_2(x+h) - f_2(x)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}[\operatorname{cosec}(x+h) - \operatorname{cosec}(x)]$$

$$= \lim_{n\to 0} \frac{1}{h}\left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right)$$

$$= \lim_{n\to 0} \frac{1}{h}\left(\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right)$$

$$= \frac{1}{\sin x} \cdot \lim_{h\to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x + x + h}{2} \right)\sin\left(\frac{x - x - h}{2} \right)}{\sin(x+h)} \right]$$

$$= \frac{1}{\sin x} \cdot \lim_{n \to 0} \frac{1}{h} \left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)} \right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \right]$$

$$= \frac{-1}{\sin x} \cdot \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{n \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cdot \cot x$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x$$

From (1), (2), and (3), we obtain

$$f'(x) = \operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right) + \cot x(-\operatorname{cosec} x \cot x)$$

$$= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

16. Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):
$\dfrac{\cos x}{1+\sin x}$

Ans: Let $f(x) = \dfrac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x)=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$

$$=\frac{-(1-\sin x)}{(1+\sin x)^2}$$

$$=\frac{-1}{(1+\sin x)^2}$$

17. Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers):

$$\frac{\sin x+\cos x}{\sin x-\cos x}$$

Ans: Let $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$

By quotient rule,

$$f''(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x+\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x+\cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x + \cos x)^2}$$

$$= \frac{-\left[\sin^2 x + \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x\right]}{(\sin x + \cos x)^2}$$

$$= \frac{-[1+1]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

18. Find the derivative of the following functions (it is to be understood that a, b, c, d. p, q, r and s are fixed non-zero constants and m and n are integers):
$\frac{\sec x - 1}{\sec x + 1}$

Ans: Let $f(x) = \frac{\sec x - 1}{\sec x + 1}$

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By quotient rule,

$$f'(x) = \frac{(1+\cos x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2} = \frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$= \frac{2\sin x \sec^2 x}{(\sec x+1)^2}$$

$$= \frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$= \frac{2\sec x \tan x}{(\sec x+1)^2}$$

19. Find the derivative of the following functions (it is to be understood that a, b, c, d. p, q, r and s are fixed non-zero constants and m and n are integers):
$\sin^n x$

Ans: Let $y = \sin^n x$

Accordingly, for $n=1, y=\sin x$

$\therefore \frac{dy}{dx} = \cos x$, i.e., $\frac{d}{dx}\sin x = \cos x$

For $n=2, y=\sin^2 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$$

$= (\sin x)' \left(\sin x + \sin x(\sin x)'\right.$ [By Leibnitz product rule]

$= \cos x \sin x + \sin x \cos x$

$= 2\sin x \cos x$

….(1)

For $n=3, y=\sin^3 x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\sin x \sin^2 x\right)$

$= (\sin x)' \sin^2 x + \sin x(\sin x)'$

[By Leibnitz product rule]

$-\cos x \sin^2 x + \sin x(2\sin x \cos x)$ [Using (1)]

$= \cos x \sin^2 x + \sin^2 x \cos x$

$= 3\sin^2 x \cos x$

We assert that $\frac{d}{dx}\left(\sin^n x\right) = n\sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$.

i.e., $\frac{d}{dx}\left(\sin^k x\right) = k\sin^{(k-1)} x \cos x$ …. (2)

Consider

$\frac{d}{dx}\left(\sin^{k+1} x\right) = \frac{d}{dx}\left(\sin x \sin^{(k)} x\right)$

$= (\sin x)' \sin^k x + \sin x\left(\sin^k x\right)^n$

[By Leibnitz product rule]

$= \cos x \sin^k x + \sin x\left(k\sin^{k-1} \cos x\right)$ [Using (2)]

$= \cos x \sin^k x + 2\sin^k x \cos x$

$-(k+1)\sin^k x \cos x$

Thus, our assertion is true for $n = k+1$.

Hence, by mathematical induction, $\frac{d}{dx}\left(\sin^n x\right) = n\sin^{(n-1)} x \cos x$

20. Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a+b\sin x}{c+d\cos x}$

Ans: Let $f(x)=\frac{a+b\sin x}{c+d\cos x}$

By quotient rule,

$$f'(x)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd\left(\cos^2 x+\sin^2 x\right)}{(C+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

21: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sin(x+a)}{\cos x}$

Ans: Let $f(x)=\frac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x)=\frac{\cos x\frac{d}{dx}[\sin(x+a)]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x) = \frac{\cos x \frac{d}{dx}[\sin(x+a)] - \sin(x+a)\frac{d}{dx}(-\sin x)}{\cos^2 x}$$

Let $g(x) - \sin(x+a)$. Accordingly, $g(x+h) = \sin(x+h+a)$

By first principle,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{n\to 0}\frac{1}{h}[\sin(x+h+a) - \sin(x+a)]$$

$$= \lim_{n\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{n\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{n\to 0}\left[\cos\left(\frac{2x+2a+h}{h}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

$$= \lim_{n\to 0}\cos\left(\frac{2x+2a+h}{h}\right)\cdot\lim_{n\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \quad \left[\text{ As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= \left(\cos\frac{2x+2a}{2}\right)\times 1 \quad \left[\lim_{n\to 0}\frac{\sin h}{h} = 1\right]$$

$$= \cos(x+a) \quad \ldots \text{(ii)}$$

From (i) and (ii), we obtain $f'(x) = \dfrac{\cos x\cos(x+a) + \sin x\sin(x+a)}{\cos^2 x}$

$$= \frac{\cos(x+a-x)}{\cos^2 x}$$

$$= \frac{\cos a}{\cos^2 x}$$

22: **Find the derivative of the following functions (it is to be understood that a, b, c, d. p, q, r and s are fixed non-zero constants and m and n are integers) : $x^4(5\sin x - 3\cos x)$**

Ans: Let $f(x) = x^4(5\sin x - 3\cos x)$

By product rule.

$$f'(x) = x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}\left(x^4\right)$$

$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}\left(x^4\right)$$

$$= x^4[5\cos x - 3(-\sin x)] + (5\sin x - 3\cos x)\left(4x^3\right)$$

$$= x^3[5x\cos x + 3x\sin x + 20\sin x - 12\cos x]$$

23. **Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\left(x^2+1\right)\cos x$**

Ans: Let $f(x) = \left(x^2+1\right)\cos x$

By product rule.

$$f'(x) = \left(x^2+1\right)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}\left(x^2+1\right)$$

$$= \left(x^2+1\right)(-\sin x) + \cos x(2x)$$

$$= -x^2\sin x - \sin x + 2x\cos x$$

24 Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):
$\left(ax^2+\sin x\right)(p+q)$ **cosX**)

Ans: Let $f(x)=\left(ax^2+\sin x\right)(p+q\cos x)$

By product rule.

$$f\ (x)=\left(ax^2+\sin x\right)\frac{d}{dx}(p+q\cos x)+(p+q\cos x)\frac{d}{dx}\left(ax^2+\sin x\right)$$

$$=\left(ax^2+\sin x\right)(-q\sin x)+(p+q\cos x)(2ax+\cos x)$$

$$=-q\sin x\left(ax^2+\sin x\right)+(p+q\cos x)(2ax+\cos x)$$

25 Find the derivative of the following functions (it is to be understood that a, b, c, d. p, q, r and s are fixed non-zero constants and m and n are integers):
$(x+\cos x)(x-\tan x)$

Ans: Let $f(x)=(x+\cos x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$$

$$=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$

$$=(x+\cos x)\left[1-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan(x+h)$

By first principle,

$$g''(x)=\lim_{n\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\tan(x+h) - \tan(x)}{h}$$

$$= \lim_{n \to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{n \to 0} \frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)}\right]$$

$$= \frac{1}{\cos x^{n \to 0}} \lim_{h} \frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x^{h \to 0}} \lim_{h} \frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x}\left(\lim_{n \to 0} \frac{\sin h}{h}\right)\left(\lim_{n \to 0} \frac{1}{\cos(x+h)}\right)$$

$$= \frac{1}{\cos x} \cdot \cdot \left(\frac{1}{\cos(x+0)}\right)$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \quad \text{... (ii)}$$

Therefore, from (i) and (ii). We obtain

$$f'(x) = (x + \cos x)\left(1 - \sec^2 x\right) + (x - \tan x)(1 - \sin x)$$

$$= (x + \cos x)\left(-\tan^2 x\right) + (x - \tan x)(1 - \sin x)$$

$$= -\tan^2 x(x + \cos x) + (x - \tan x)(1 - \sin x)$$

26: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

$$\frac{4x + 5\sin x}{3x + 7\cos x}$$

Ans: Let $f(x)=\dfrac{4x+5\sin x}{3x+7\cos x}$

Quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}(x)+7\frac{d}{dx}(\cos x)\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)[4x+5\cos x]-(4x+5\sin x)[3-7\sin x]}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28x\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\left(\cos^2 x+\sin^2 x\right)}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35\left(\cos^2 x+\sin^2 x\right)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

27: **Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):**

$$\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$$

Ans: Let $\mathrm{f}(\mathrm{x})=\dfrac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule, $f'(x)=\cos\left(\frac{\pi}{4}\right)\left[\dfrac{\sin x\frac{d}{dx}\left(x^2\right)-x^2\frac{d}{dx}(\sin x)}{\sin^2 x}\right]$

$$= \cos\left(\frac{\pi}{4}\right)\left[\frac{\sin x(2x) - x^2(\cos x)}{\sin^2 x}\right]$$

$$= \frac{x\cos\frac{\pi}{4}[2\sin x - x\cos x]}{\sin^2 x}$$

28. Find the derivative of the following functions (it is to be understood that a, b, c. d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{1+\tan x}$

Ans: Let $f(x) = \frac{x}{1+\tan x}$

$$f(x) = \frac{(1+\tan x)\frac{d}{dx}(x) - (x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x) = \frac{(1+\tan x) - x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

Let $g(x) = 1+\tan x.$ Accordingly $g(x+h) = 1+\tan(x+h)$.

By first principle, $\dot{g}(x) = \lim_{n\to 0}\frac{g(x+h)-g(x)}{h}$

$$\lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-\tan(x)}{h}\right]$$

$$\lim_{n\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$\lim_{n\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos x(x+h)}{\cos(x+h)\cos x}\right]$$

$$\lim_{n\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$\lim_{h\to 0}\frac{1}{h}\left[\frac{\sinh}{\cos(x+h)\cos x}\right]$$

$$-\left(\lim_{n\to 0}\frac{\sinh}{h}\right)\cdot\left(\lim_{n\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$-1\times\frac{1}{\cos^2}=\sec^2 x$$

$$\Rightarrow\frac{d}{dx}\left(1+\tan^2 x\right)=\sec^2 x$$

From (i) and (ii), we obtain

$$\dot{f}(x)=\frac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$$

29. Find the derivative of the following functions (it is to be understood that a, b, c **, d, p, q, r and s are fixed non-zero constants and m and n are integers):**
$\mathbf{(x+\sec x)(x-\tan x)}$

Ans: Let $f(x)=(x+\sec x)(x-\tan x)$

By product rule.

$$f(x)=(x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$

$$-(x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\sec x\right]$$

$$-f(x+\sec x)\left[1-\frac{d}{dx}\tan x\right)\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right]$$

…(i)

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)\cdot\tan(x+h)$ and $f_2(x+h)-\sec(x+h)$

$$f_1^{'}(x) = \lim_{n\to 0}\left(\frac{f_1(x+h) - f_1(x)}{h}\right)$$

$$= \lim_{h\to 0}\left[\frac{\tan(x+h) - \tan(x)}{h}\right]$$

$$- \lim_{n\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$- \lim_{n\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x \cos x(x+h)}{\cos(x+h)\cos x}\right]$$

$$- \lim_{n\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$- \lim_{n\to 0}\frac{1}{h}\left[\frac{\text{sinh}}{\cos(x+h)\cos x}\right]$$

$$- \left(\lim_{n\to 0}\frac{\text{sinh}}{h}\right)\cdot\left(\lim_{n\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$-1 \times \frac{1}{\cos^2} = \sec^2 x$$

$$\Rightarrow \frac{d}{dx}\left(1 + \tan^2 x\right) = \sec^2 x$$

$$f_2^{'}(x) = \lim_{n\to 0}\left(\frac{f_2 + (x+h) - f_2(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sec(x+h) - \sec(x)}{h}\right)$$

$$= \lim_{n\to 0}\frac{1}{h}\left(\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right)$$

$$= \lim_{n\to 0}\frac{1}{h}\left(\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x}\right)$$

$$= \frac{1}{\cos x} \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}}{\cos(x+h)} \right]$$

$$= \sec x \frac{\left\{\lim_{n \to 0} \sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}}{\lim_{n \to 0} \cos(x+h)}$$

$$= \sec x \cdot \frac{\sin x \cdot 1}{\cos x}$$

$$\Rightarrow \frac{d}{dx} \sec x = \sec x \tan x$$

From (i). (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)\left(1 - \sec^2 x\right) + (x - \tan x)(1 + \sec x \tan x)$$

30: **Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):** $\dfrac{x}{\sin^n x}$

Ans: Let $f(x)=\dfrac{x}{\sin^n x}$

By quotient rule, $f'(x)=\dfrac{\sin^n x\dfrac{d}{dx}x - x\dfrac{d}{dx}\sin^n x}{\sin^{2n} x}$

It can be easily shown that $\dfrac{d}{dx}\sin^n x = n\sin^{n-1} x\cos x$

Therefore,

$$f'(x)=\frac{\sin^n\times\dfrac{d}{dx}x - x\dfrac{d}{dx}\sin^n x}{\sin^{2n} x}$$

$$=\frac{\sin^n x\cdot 1 - x\left(\text{nin}^{n-1} x\cos x\right)}{\sin^{2n} x}$$

$$=\frac{\sin^{n-1} x(\sin x - nx\cos x)}{\sin^{2n} x}$$

$$=\frac{\sin x - nx\cos x}{\sin^{n+1} x}$$