introduction to three-dimensional geometry

11 Chapter

Exercise 11.1

1. A point is on the x- axis. What are its y-coordinate and z-coordinates?

Ans: When a point is on the x-axis, then the y-coordinate and z-coordinate of that point are both zero.

2. A point is in the xz-plane. What can you say about its y-coordinate?

Ans: When a point is on the xz-plane, then y-coordinate of that point is zero.

3. Name the octant in which the following points lie:

$\mathbf{(1,2,3),(4,-2,3),(4,-2,-5),(4,2,-5),\ (-4,2,-5),(-4,2,5),\ (-3,-1,6),\ (-2,-4,-7)}$.

Ans: Consider the following table.

Octants	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

By following rules given in the above table, we can conclude the following results.

Since, all the three coordinates in the point $(1,2,3)$ are positive, so this point is in the octant I.

Since in the point $(4,-2,3)$, the x and z-coordinate are positive and the y-coordinate is negative, so this point is in the octant IV.

Since in the point $(4,-2,-5)$, the y and -coordinate are negative and the x-coordinate is positive, so this point is in the octant $VIII$.

Since in the point $(4,2,-5)$, the and y-coordinate are positive and the z-coordinate is negative, so this point is in the octant V.

z

Since in the point $(-4,2,-5)$, the and -coordinate are negative and the -coordinate is positive, so this point is in the octant VI.

x

Since in the point $(-3,-1,6)$, the and y-coordinate are negative and the z-coordinate is positive, so this point is in the octant II.

x z y

Since in the point $(-2,-4,-7)$, all the three coordinates in the point are negative, so this point is in the octant VII.

x

4. Fill in the following blanks:

(i) The x-axis and y-axis taken together determine a plane known as ________.

Ans: The x-axis and y-axis taken together determine a plane known as XY plane.

(ii) The coordinates of points in the XY-plane are of the form __________.

Ans: The coordinates of points in the XY-plane are of the form $(x,y,0)$.

(iii) Coordinate planes divided the space into ________ octants.

Ans: Coordinate planes divided the space into eight octants.

Exercise 11.2

1. Find the distance between the following pairs of points:

(i) $(2,3,5)$ and $(4,3,1)$.

Ans: Recall that, distance between any two points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ is $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Therefore, distance between the points $(2,3,5)$ and $(4,3,1)$ is

$=\sqrt{(4-2)^2+(3-3)^2+(1-5)^2}$

$=\sqrt{2^2+0^2+(-4)^2}$

$=\sqrt{4+16}$

$=\sqrt{20}$

$=2\sqrt{5}$ units.

(ii) $(-3,7,2)$ and $(2,4,-1)$.

Ans: The distance between the points $(-3,7,2)$ and $(2,4,-1)$ is

$=\sqrt{(2+3)^2+(4-7)^2+(-1-2)^2}$

$=\sqrt{5^2+(-3)^2+(-3)^2}$

$=\sqrt{25+9+9}$

$=\sqrt{43}$ units

(iii) $(-1,3,-4)$ and $(1,-3,4)$.

Ans: The distance between the points $(-1,3,-4)$ and $(1,-3,4)$ is

$=\sqrt{(1+1)^2+(-3-3)^2+(4+4)^2}$

$=\sqrt{2^2+(-6)^2+8^2}$

$=\sqrt{4+36+64}$

$=\sqrt{104}$

$=2\sqrt{26}$ units

(iv) $(2,-1,3)$ and $(-2,1,3)$.

Ans: The distance between the points $(2,-1,3)$ and $(-2,1,3)$ is

$= \sqrt{(-2-2)^2 + (1+1)^2 + (3-3)^2}$

$= \sqrt{(-4)^2 + 2^2 + 0^2}$

$= \sqrt{16+4}$

$= \sqrt{20}$

$= 2\sqrt{5}$ units

2. Show that the points $(-2,3,5),(1,2,3)$ and $(7,0,-1)$ are collinear.

Ans: Recall that, any points P, Q, R are said to be collinear if they lie on a line.

Now, suppose that the given points are $P(-2,3,5)$, $Q(1,2,3)$, and $R(7,0,-1)$.

Then, $PQ = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2}$

$= \sqrt{3^2 + (-1)^2 + (-2)^2}$

$= \sqrt{9+1+4}$

$= \sqrt{14}$ units

$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2}$

$= \sqrt{6^2 + (-2)^2 + (-4)^2}$

$= \sqrt{36+4+16}$

$= \sqrt{56}$

$= 2\sqrt{14}$ units

Also, $PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2}$

$= \sqrt{9^2 + (-3)^2 + (-6)^2}$

$= \sqrt{81+9+36}$

$= \sqrt{126}$

$= 3\sqrt{14}$ units.

Notice that, $PQ + QR = \sqrt{14} + 2\sqrt{14} = 3\sqrt{14} = PR$.

Thus, the points lie in the same line.

Hence, the given points are collinear.

3. Verify the following statements:

(i) $(0,7,-10),(1,6,-6)$ and $(4,9,-6)$ are the vertices of an isosceles triangle.

Ans: Recall that, in an isosceles triangle any two sides are of equal length.

Now, let the given points are $P(0,7,-10)$, $Q(1,6,-6)$ and $R(4,9,-6)$.

Then, $PQ = \sqrt{(1-0)^2 + (6-7)^2 + (-6+10)^2}$

$= \sqrt{1^2 + (-1)^2 + 4^2}$

$= \sqrt{1+1+16}$

$= \sqrt{18}$

$= 3\sqrt{2}$ units.

$QR = \sqrt{(4-1)^2 + (9-6)^2 + (-6+6)^2}$

$= \sqrt{3^2 + 3^2 + 0^2}$

$= \sqrt{9+9}$

$= \sqrt{18}$

$= 3\sqrt{2}$ units

Also, $RP = \sqrt{(0-4)^2 + (7-9)^2 + (-10+6)^2}$

$= \sqrt{(-4)^2 + (-2)^2 + (-4)^2}$

$= \sqrt{16+4+16}$

$= \sqrt{36}$

$= 6$ units.

Note that, $PQ = QR \neq RP$

Hence, the provided points are the vertices of an isosceles triangle.

(ii) $(0,7,10),(-1,6,6)$ and $(-4,9,6)$ are the vertices of a right-angled triangle.

Ans: Recall that, according to the Pythagorean theorem, a triangle is said to be a right-angled if the sum of the squares of two sides equal to the square of the third side.

Now, let the given points are $P(0,7,-10)$, $Q(1,6,-6)$ and $R(4,9,-6)$.

Then, $PQ=\sqrt{(1-0)^2+(6-7)^2+(-6+10)^2}$

$=\sqrt{1^2+(-1)^2+4^2}$

$=\sqrt{1+1+16}$

$=\sqrt{18}$

$=3\sqrt{2}$ units.

$QR=\sqrt{(4-1)^2+(9-6)^2+(-6+6)^2}$

$=\sqrt{3^2+3^2+0^2}$

$=\sqrt{9+9}$

$=\sqrt{18}$

$=3\sqrt{2}$ units

Also, $RP=\sqrt{(0-4)^2+(7-9)^2+(-10+6)^2}$

$=\sqrt{(-4)^2+(-2)^2+(-4)^2}$

$=\sqrt{16+4+16}$

$=\sqrt{36}$

$=6$ units.

Now, note that,

$PQ^2 + QR^2 = \left(3\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2$

$= 18 + 18$

$= 36$

$= \left(RP\right)^2$

That is, $PQ^2 + QR^2 = RP^2$.

Hence, according to the Pythagorean theorem, the given points form a right-angled triangle.

(iii) $(-1,2,1),(1,-2,5),(4,-7,8)$ and $(2,-3,4)$ are the vertices of a parallelogram.

Ans: Recall that, a quadrilateral is said to be a parallelogram if the opposite sides are equal.

Now, suppose that, the given points are $P(-1,2,1)$, $Q(1,-2,5)$, $R(4,-7,8)$, and $S(2,-3,4)$.

Then, $PQ = \sqrt{(1+1)^2 + (-2-2)^2 + (5-1)^2}$

$= \sqrt{2^2 + (-4)^2 + 4^2}$

$= \sqrt{4+16+16}$

$= \sqrt{36}$

$= 6$ units.

$QR = \sqrt{(4-1)^2 + (-7+2)^2 + (8-5)^2}$

$= \sqrt{3^2 + (-5)^2 + 3^2}$

$= \sqrt{9+25+9}$

$= \sqrt{43}$ units

$RS = \sqrt{(2-4)^2 + (-3+7)^2 + (4-8)^2}$

$= \sqrt{(-2)^2 + 4^2 + (-4)^2}$

$= \sqrt{4+16+16}$

$= \sqrt{36}$

$= 6$ units

Also, $SP=\sqrt{(-1-2)^2+(2+3)^2+(1-4)^2}$

$=\sqrt{(-3)^2+5^2+(-3)^2}$

$=\sqrt{9+25+9}$

$=\sqrt{43}$ units.

Therefore, we have

$PQ=RS=6$ units and $QR=SP=\sqrt{43}$ units.

Thus, in the quadrilateral $PQRS$, the opposite sides are equal.

Hence, $PQRS$ is a parallelogram, that is, the provided points are the vertices of a parallelogram.

4. Find the equation of the set of points which are equidistant from the points $(1,2,3)$ and $(3,2,-1)$.

Ans: Suppose that, the points $A(1,2,3)$ and $B(3,2,-1)$ are equidistant from the point $P(x,y,z)$.

Then, we have, $AP=BP$

$\Rightarrow AP^2=BP^2$

$\Rightarrow (x-1)^2+(y-2)^2+(z-3)^2=(x-3)^2+(y-2)^2+(z+1)^2$

$\Rightarrow x^2-2x+1+y^2-4y+4+z^2-6z+9=x^2-6x+9+y^2-4y+4+z^2+2z+1$

$\Rightarrow -2x-4y-6z+14=-6x-4y+2z+14$

$\Rightarrow -2x-6z+6x-2z=0$

$\Rightarrow 4x-8z=0$

$\Rightarrow x-2z=0$

Hence, the equation of the set of points that are equidistant from the given points is given by
$x-2z=0$.

5. Find the equation of the set of points P, the sum of whose distances from $A(4,0,0)$ and $B(-4,0,0)$ is equal to 10.

Ans: Suppose that, the points $A(4,0,0)$ and $B(-4,0,0)$ are equidistant from the point $P(x,y,z)$.

Then, by the given condition, we have

$AP+BP=10$

$$\Rightarrow \sqrt{(x-4)^2+(y-0)^2+(z-0)^2}+\sqrt{(x+4)^2+(y-0)^2+(z-0)^2}=10$$

$$\Rightarrow \sqrt{(x-4)^2+y^2+z^2}=10-\sqrt{(x+4)^2+y^2+z^2}$$

Squaring both sides of the equation, yields

$$(x-4)^2+y^2+z^2=100-20\sqrt{(x+4)^2+y^2+z^2}+(x+4)^2+y^2+z^2$$

$$\Rightarrow (x-4)^2+y^2+z^2=100-20\sqrt{(x+4)^2+y^2+z^2}+x^2+8x+16+y^2+z^2$$

$$\Rightarrow 20\sqrt{x^2+y^2+z^2+8x+16}=100+16x$$

$$\Rightarrow 5\sqrt{x^2+y^2+z^2+8x+16}=25+4x$$

Again, squaring both sides of the equation, gives

$$25\left(x^2+8x+16+y^2+z^2\right)=625+16x^2+200x$$

$$\Rightarrow 25x^2+200x+400+25y^2+25z^2=625+16x^2+200x$$

$$\Rightarrow 9x^2+25y^2+25z^2-225=0$$

Hence, the equation of the set of points, the sum of whose distances from the given points is equal to 10, is given by

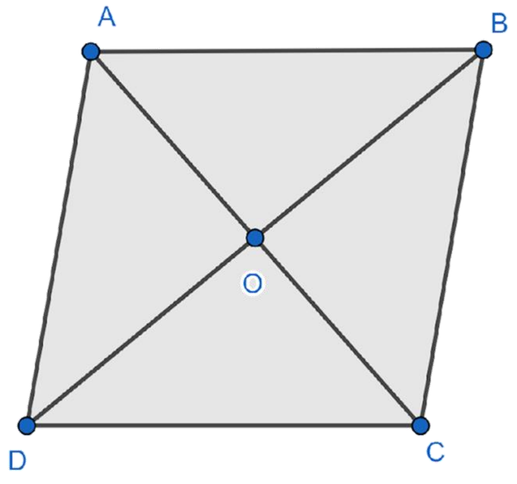
$9x^2+25y^2+25z^2-225=0$.

Miscellaneous Exercise

1. Three vertices of a parallelogram $ABCD$ are $A(3,-1,2)$, $B(1,2,-4)$ and $C(-1,1,2)$. Find the coordinates of the fourth vertex.

Ans: We are given the three vertices of a parallelogram $ABCD$ are $A(3,-1,2)$, $B(1,2,-4)$ and $C(-1,1,2)$.

Let the coordinates of the fourth vertex of the parallelogram $ABCD$ be $D(x,y,z)$.



According to the property of parallelogram, the diagonals of the parallelogram bisect each other.

In this parallelogram $ABCD$, AC and BD at point O.

So,

Mid-point of AC = Mid-point of BD

$$\left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right) = \left(\frac{x+1}{2}, \frac{y+1}{2}, \frac{z-4}{2}\right)$$

$$(1,0,2) = \left(\frac{x+1}{2}, \frac{y+1}{2}, \frac{z-4}{2}\right)$$

$$\frac{x+1}{2} = 1$$

$$\frac{y+2}{2} = 0$$

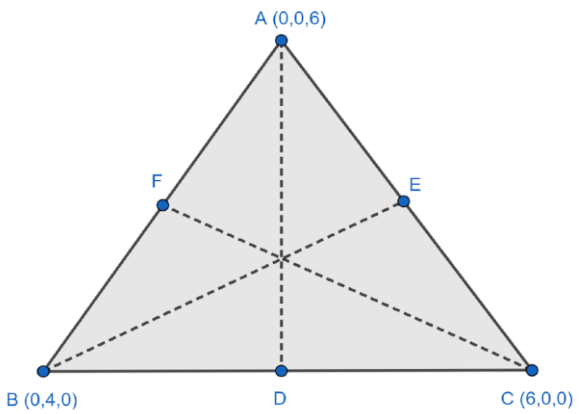
$$\frac{z-4}{2} = 2$$

We get, $x = 1$, $y = 2$ and $z = 8$

Therefore, the coordinates of the fourth vertex of the parallelogram $ABCD$ are $D(1,-2,8)$.

2. Find the lengths of the medians of the triangle with $A(0,0,6)$, $B(0,4,0)$ and $C(6,0,0)$.

Ans: For the given triangle ABC. Let AD, BE and CF are the medians.



We know that, median divides the line segment into two equal parts, so D is the midpoint of BC, therefore,

$$\text{Coordinates of point D} = \frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2})$$

$$\text{Coordinates of point D} = (3,2,0)$$

$$AD = \sqrt{(0-3) + (0-2)^2 + (6-0)}$$

$$AD = \sqrt{9+4+36}$$

$$AD = \sqrt{49}$$

$$AD = 7$$

Similarly, E is the midpoint of AC,

$$\text{Coordinates of point E} = \frac{0+6}{2}, \frac{0+0}{}, \frac{0+6}{})$$

$$\text{Coordinates of point E} = (3,0,3)$$

$$AC = \sqrt{(3-0)^2 + (0-4)^2 + (3-0)^2}$$

$$AC = \sqrt{9+16+9}$$

$$AC = \sqrt{34}$$

Similarly, F is the midpoint of AB,

$$\text{Coordinates of point F} = \frac{0+0}{2}, \frac{0+4}{}, \frac{6+0}{})$$

$$\text{Coordinates of point F} = (0,2,3)$$

$$CF = \sqrt{(6-0)^2 + (0-2)^2 + (0-3)^2}$$

$$CF = \sqrt{36+4+9}$$

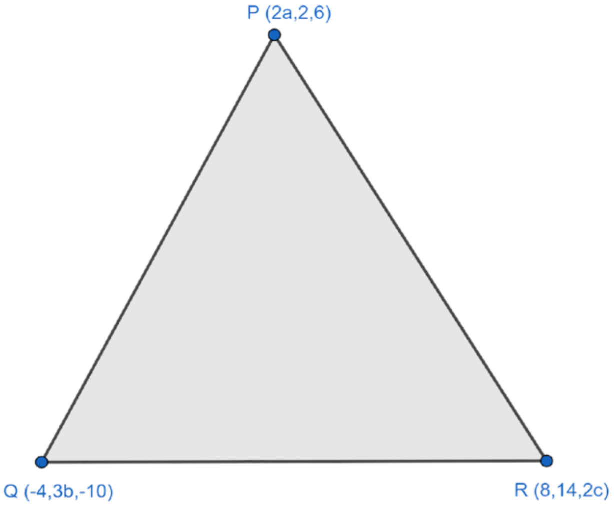
$$CF = \sqrt{49}$$

$$CF = 7$$

Therefore, the lengths of the medians of the triangle ABC we obtain are, $7, \sqrt{34}, 7$

3. **If the origin is the centroid of the triangle PQR with vertices $P(2a,2,6)$, $Q(-4,3b,-10)$ and $R(8,14,2c)$, then find the values of a, b and c**

Ans: The given triangle PQR



We know that the coordinates of the centroid of triangle with the vertices (x_1,y_1,z_1), (x_2,y_2,z_2) and (x_3,y_3,z_3) are,

$$\frac{x_1+x_2+x_3}{3}=\frac{y_1+y_2+y_3}{3}=\frac{z_1+z_2+z_3}{3}$$

For triangle PQR, the coordinates will be,

$$\Delta PQR=\frac{2a-4+8}{3}=\frac{2+3b+14}{3}=\frac{6-10+2c}{3}$$

$$\Delta PQR=\frac{2a+4}{3}=\frac{3b+16}{3}=\frac{2c-4}{3}$$

Now, we are given that centroid is the origin,

$$\frac{2a+4}{3}=0$$

$$\frac{3b+16}{3}=0$$

$$\frac{2c-4}{3} = 0$$

$a = -2$,

$b = -\frac{16}{3}$

$c = 2$

Therefore, we obtain the values as $a = -2$, $b = -\frac{16}{3}$ and $c = 2$.

4. If A and B be the points $(3,4,5)$ and $(-1,3,-7)$ respectively, find the equation of the set of points P such that $PA^2+PB^2 = k^2$, where k is a constant.

Ans: Let the coordinates of point P be (x,y,z).

Using distance formula we get,

$$PA^2=(x-3)^2+(y-4)^2+(z-5)^2$$

$$PA^2 = x^2+9-6x+y^2+16-8y+z^2+25-10z$$

$$PA^2 = x^2-6x+y^2-8y+z^2-10z+50$$

Similarly,

$$PB^2 = (x-1)^2+(y-3)^2+(z-7)^2$$

$$PA^2 = x^2-2x+y^2-6y+z^2-14z+59$$

We are given that,

$$PA^2+PB^2 = k^2$$

So,

$$(x^2-6x+y^2-8y+z^2-10z+50)+(x^2-2x+y^2-6y+z^2-14z+59) = k^2$$

$$2x^2+2y^2+2z^2-4x-14y+14z+109 = k^2$$

$$2(x^2+y^2+z^2-2x-7y+7z) = k^2-109$$

$$x^2+y^2+z^2-2x-7y+7z = \frac{k^2-109}{2}$$

Therefore, the equation is as follows $x^2+y^2+z^2-2x-7y+2z = \frac{k^2-109}{2}$