# principle of mathematical induction

4
Chapter

**1. Prove that following by using the principle of mathematical induction for all $n \in N$:**

$$1+3+3^2+.....+3^{n-1}=\frac{\left(3^n-1\right)}{2}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P\left(n\right):1+3+3^2+.....+3^{n-1}=\frac{\left(3^n-1\right)}{2}$$

For $n=1$,

$$L.H.S.=3^{1-1}=1$$

$$R.H.S.=\frac{\left(3^1-1\right)}{2}=\frac{3-1}{2}=\frac{2}{2}=1$$

Therefore, $P\left(n\right)$ is true for $n=1$.

Let us assume that $P\left(k\right)$ is true for some positive integer $k$, i.e.,

$$1+3+3^2+.....+3^{k-1}=\frac{\left(3^k-1\right)}{2} \quad .....(i)$$

Now, we have to prove that $P\left(k+1\right)$ is also true.

Consider

$$1+3+3^2+.....+3^{k-1}+3^{(k+1)-1}$$

$$=\left(1+3+3^2+....+3^{k-1}\right)+3^k$$

$$=\frac{\left(3^k-1\right)}{2}+3^k \qquad [\text{Using(i)}]$$

$$=\frac{\left(3^k-1\right)+2.3^k}{2}$$

$$=\frac{\left(1+2\right)3^k-1}{2}$$

$$=\frac{3.3^k-1}{2}$$

$$=\frac{3^{k+1}-1}{2}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**2.** **Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\mathbf{1^3+2^3+3^3+.....+n^3=\left(\frac{n(n+1)}{2}\right)^2}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(1):1^3+2^3+3^3+.....+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

For $n=1$,

$$\text{L.H.S.}=1^3=1$$

$$\text{R.H.S.}=\left(\frac{1(1+1)}{2}\right)^2=(1)^2=1$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$1^3+2^3+3^3+.....+k^3=\left(\frac{k(k+1)}{2}\right)^2 \quad .....(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$1^3+2^3+3^3+.....+k^3+(k+1)^3$$

$$=\left(1^3+2^3+3^3+.....+k^3\right)+(k+1)^3$$

$$=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \qquad \text{[Using (i)]}$$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3$$
$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$
$$=\frac{(k+1)^2\{k^2+4(k+1)\}}{4}$$
$$=\frac{(k+1)^2\{k^2+4k+4\}}{4}$$
$$=\frac{(k+1)^2(k+2)^2}{4}$$
$$=\frac{(k+1)^2(k+1+1)^2}{4}$$
$$=\left(\frac{(k+1)(k+1+1)}{2}\right)^2$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**3. Prove the following by using the principle of mathematical induction for all $n \in N$ :**

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+.....+\frac{1}{(1+2+3+...n)}=\frac{2n}{(n+1)}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{1}{1+2+3+...n}=\frac{2n}{n+1}$$

For $n=1$,

$$\text{L.H.S.}=\frac{1}{1}=1$$

$$\text{R.H.S.}=\frac{2\cdot 1}{1+1}=1$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+.....+\frac{1}{(1+2+3+...+k)}=\frac{2k}{(k+1)}......(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+....+\frac{1}{1+2+3+...+k}+\frac{1}{1+2+3+...+k+(k+1)}$$

$$=\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k}\right)+\frac{1}{1+2+3+...+k+(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+...+k+(k+1)} \qquad \text{[Using (i)]}$$

$$=\frac{2k}{k+1}+\frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)} \qquad \left[1+2+3...+n=\frac{n(n+1)}{2}\right]$$

$$=\frac{2k}{k+1}+\frac{2}{(k+1)(k+2)}$$

$$=\frac{2}{(k+1)}\left(k+\frac{1}{k+2}\right)$$

$$=\frac{2}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$=\frac{2}{(k+1)}\left[\frac{(k+1)^2}{k+2}\right]$$

$$=\frac{2(k+1)}{(k+2)}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**4. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\mathbf{1.2.3+2.3.4+...+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):1.2.3+2.3.4+...+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$$

For $n=1$,

$$\text{L.H.S.}=1.2.3=6$$

$$\text{R.H.S.}=\frac{1(1+1)(1+2)(1+3)}{4}=\frac{1.2.3.4}{4}=6$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$1.2.3+2.3.4+...+k(k+1)(k+2)=\frac{k(k+1)(k+2)(k+3)}{4}\ ...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$1.2.3+2.3.4+...+k(k+1)(k+2)+(k+1)(k+2)(k+3)$$

$$=\{1.2.3+2.3.4+...+k(k+1)(k+2)\}+(k+1)(k+2)+(k+3)$$

$$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3) \quad \text{[Using (i)]}$$

$$=(k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

$$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$=\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**5. Prove the following by using the principle of mathematical induction for all $n\in N$:**

$$\mathbf{1.3+2.3^2+3.3^3+...+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):1.3+2.3^2+3.3^3+...+n3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

For $n=1$,

$$\text{L.H.S.}=1.3^1=3$$

$$\text{R.H.S.}=\frac{(2.1-1)3^{1+1}+3}{4}=\frac{3^2+3}{4}=\frac{12}{4}=3$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$1.3+2.3^2+3.3^3+...+k3^k=\frac{(2k-1)3^{k+1}+3}{4}...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$1.3+2.3^2+3.3^3+...+k.3^k+(k+1).3^{k+1}$$

$$=(1.3+2.3^2+3.3^3+...+k.3^k)+(k+1).3^{k+1}$$

$$=\frac{(2k-1)3^{k+1}+3}{4}+(k+1)3^{k-1} \qquad \text{[Using (i)]}$$

$$=\frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

$$=\frac{3^{k+1}\{2k-1+4(k+1)\}+3}{4}$$

$$=\frac{3^{k+1}\{6k+3\}+3}{4}$$

$$=\frac{3^{k+1}.3\{2k+1\}+3}{4}$$

$$=\frac{3^{(k+1)+1}\{2k+1\}+3}{4}$$

$$=\frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**6.** **Prove the following by using the principle of mathematical induction for all $\mathbf{n \in N}$:**

$$\mathbf{1.2+2.3+3.4+...+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):1.2+2.3+3.4+...+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

For $n=1$,

$$L.H.S.=1.2=2$$

$$R.H.S.=\frac{1(1+1)(1+2)}{3}=\frac{1.2.3}{3}=2$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$1.2+2.3+3.4+...+k.(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right]...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$1.2+2.3+3.4+...+k.(k+1)+(k+1).(k+2)$$

$$=\left[1.2+2.3+3.4+...+k.(k+1)\right]+(k+1).(k+2)$$

$$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \qquad \text{[Using (i)]}$$

$$=(k+1)(k+2)\left(\frac{k}{3}+1\right)$$

$$=\frac{(k+1)(k+2)(k+3)}{3}$$

$$=\frac{(k+1)(k+1+1)(k+1+2)}{3}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**7.** **Prove the following by using the principle of mathematical induction for all $\mathbf{n \in N}$:**

$$\mathbf{1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n): 1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

For $n=1$,

$$\text{L.H.S.}=1.3=3$$

$$\text{R.H.S.}=\frac{1(4.1^2+6.1-1)}{3}=\frac{4+6-1}{3}=\frac{9}{3}=3$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$1.3+3.5+5.7+...+(2k-1)(2k+1)=\frac{k(4k^2+6k-1)}{3} \quad ...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$(1.3+3.5+5.7+...+(2k-1)(2k+1))+\{(k+1)-1\}\{2(k+1)+1\}$$

$$=\frac{k(4k^2+6k-1)}{3}+(2k+2-1)(2k+2+1) \quad \text{[Using (i)]}$$

$$=\frac{k(4k^2+6k-1)}{3}+(2k+1)(2k+3)$$

$$=\frac{k(4k^2+6k-1)}{3}+(4k^2+8k+3)$$

$$=\frac{k(4k^2+6k-1)+3(4k^2+8k+3)}{3}$$

$$=\frac{4k^3+6k^2-k+12k^2+24k+9}{3}$$

$$=\frac{4k^3+18k^2+23k+9}{3}$$

$$=\frac{4k^3+14k^2+9k+4k^2+14k+9}{3}$$

$$=\frac{k\left(4k^2+14k+9\right)+1\left(4k^2+14k+9\right)}{3}$$

$$=\frac{(k+1)\left(4k^2+14k+9\right)}{3}$$

$$=\frac{(k+1)\left\{4k^2+8k+4+6k+6-1\right\}}{3}$$

$$=\frac{(k+1)\left\{4\left(k^2+2k+1\right)+6(k+1)-1\right\}}{3}$$

$$=\frac{(k+1)\left\{4(k+1)^2+6(k+1)-1\right\}}{3}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**8. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\mathbf{1.2+2.2^2+3.2^2+....+n.2^n=(n-1)2^{n+1}+2}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$P(n): 1.2+2.2^2+3.2^2+....+n.2^n=(n-1)2^{n+1}+2$

For $n=1$,

$L.H.S.=1.2=2$

$R.H.S.=(1-1)2^{1+1}+2=0+2=2$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$1.2+2.2^2+3.2^2+...+k.2^k=(k-1)2^{k+1}+2...(i)$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$\left\{1.2+2.2^2+3.2^2+...+k.2^k\right\}+(k+1).2^{k+1}$

$=(k-1)2^{k+1}+2+(k+1)2^{k+1}$

$=2^{k+1}\{(k-1)+(k+1)\}+2$

$=2^{k+1}.2k+2$

$=k.2^{(k+1)+1}+2$

$=\{(k+1)-1\}2^{(k+1)+1}+2$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**9. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}=1-\frac{1}{2^n}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n): \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}=1-\frac{1}{2^n}$$

For $n=1$,

$$L.H.S.=\frac{1}{2}$$

$$R.H.S.=1-\frac{1}{2^1}=\frac{1}{2}$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^k}=1-\frac{1}{2^k} ...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

$$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}} \qquad \text{[Using (i)]}$$

$$1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$$

$$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**10. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\mathbf{\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

For $n=1$,

$$\text{L.H.S.}=\frac{1}{2.5}=\frac{1}{10}$$

$$\text{R.H.S.}=\frac{1}{6.1+4}=\frac{1}{10}$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3k+1)(3k+2)}=\frac{k}{(6k+4)} \quad ...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3k+1)(3k+2)}+\frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

$$=\frac{k}{6k+4}+\frac{1}{(3k+3-1)(3k+3+2)} \qquad \text{[Using (i)]}$$

$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{1}{(3k+2)}\left(\frac{k}{2}+\frac{1}{3k+5}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{2(3k+5)}\right)$$

$$=\frac{(k+1)}{6k+10}$$

$$=\frac{(k+1)}{6(k+1)+4}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**11. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+...+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+...+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

For $n=1$,

$$\text{L.H.S.}=\frac{1}{1.2.3}=\frac{1}{6}$$

$$\text{R.H.S.}=\frac{1.(1+3)}{4(1+1)(1+2)}=\frac{1.4}{4.2.3}=\frac{1}{6}$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+...+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$\left[\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+...+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \qquad \text{[Using (i)]}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+6k+9)+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+2k+1)+4(k^2+2k+1)}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$$

$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**12.** **Prove the following by using the principle of mathematical induction for all $\mathbf{n \in N}$:**

$$\mathbf{a+ar+ar^2+...+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P\left(n\right):a+ar+ar^2+...+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

For $n=1$,

$L.H.S.=a$

$$R.H.S.=\frac{a\left(r^1-1\right)}{\left(r-1\right)}=a$$

Therefore, $P\left(n\right)$ is true for $n=1$.

Let us assume that $P\left(k\right)$ is true for some positive integer $k$, i.e.,

$$a+ar+ar^2+...+ar^{k-1}=\frac{a\left(r^k-1\right)}{r-1}...(i)$$

Now, we have to prove that $P\left(k+1\right)$ is also true.

Consider

$$\left\{a+ar+ar^2+...+ar^{k-1}\right\}+ar^{(k+1)-1}$$

$$=\frac{a\left(r^k-1\right)}{r-1}+ar^k \qquad \text{[Using (i)]}$$

$$=\frac{a\left(r^k-1\right)+ar^k\left(r-1\right)}{r-1}$$

$$=\frac{a\left(r^k-1\right)+ar^{k+1}-ar^k}{r-1}$$

$$=\frac{ar^k-a+ar^{k+1}-ar^k}{r-1}$$

$$=\frac{ar^{k+1}-a}{r-1}$$

$$=\frac{a\left(r^{k+1}-1\right)}{r-1}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**13. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

For $n=1$,

$$\text{L.H.S.}=\left(1+\frac{3}{1}\right)=4$$

$$\text{R.H.S.}=(1+1)^2=2^2=4$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)=(k+1)^2 \ldots\text{(i)}$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$$

$$=(k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right) \qquad \text{[Using (i)]}$$

$$=(k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$$

$=(k+1)^2+2(k+1)+1$

$=\{(k+1)+1\}^2$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**14. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right)=(n+1)$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right)=(n+1)$$

For $n=1$,

$$\text{L.H.S.}=\left(1+\frac{1}{1}\right)=2$$

$$\text{R.H.S.}=(1+1)=2$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$P(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right)=(k+1)...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$\left[P(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$$=(k+1)\left(1+\frac{1}{k+1}\right) \qquad \text{[Using (i)]}$$

$$=(k+1)\left[\frac{(k+1)+1}{(k+1)}\right]$$

$$=(k+1)+1$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**15. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\mathbf{1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n):1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

For $n=1$,

$$\text{L.H.S.}=1^2=1$$

$$\text{R.H.S.}=\frac{1(2.1-1)(2.1+1)}{3}=\frac{1.1.3}{3}=1$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$P(k):1^2+3^2+5^2+...+(2n-1)^2=\frac{k(2k-1)(2k+1)}{3}...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$\left\{1^2+3^2+5^2+...+(2k-1)^2\right\}+\left\{2(k+1)-1\right\}^2$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+2-1)^2 \qquad \text{[Using (i)]}$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

$$=\frac{2(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$=\frac{(2k+1)\{k(2k-1)+3(2k+1)\}}{3}$$

$$=\frac{(2k+1)\{2k^2-k+6k+3\}}{3}$$

$$=\frac{(2k+1)\left\{2k^2+5k+3\right\}}{3}$$

$$=\frac{(2k+1)\left\{2k^2+2k+3k+3\right\}}{3}$$

$$=\frac{(2k+1)\left\{2k(k+1)+3(k+1)\right\}}{3}$$

$$=\frac{(2k+1)(k+1)(2k+3)}{3}$$

$$=\frac{(k+1)\left\{2(k+1)-1\right\}\left\{2(k+1)+1\right\}}{3}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**16. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\mathbf{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n): \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

For $n=1$,

$$\text{L.H.S.}=\frac{1}{1.4}=\frac{1}{4}$$

$$\text{R.H.S.}=\frac{1}{3.1+1}=\frac{1}{4}$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$P(k): \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3k-2)(3k+1)}=\frac{k}{(3k+1)} \quad ...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)} \qquad \text{[Using (i)]}$$

$$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)}{3(k+1)+1}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**17. Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\mathbf{\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{2(2n+3)}}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{2(2n+3)}$$

For $n=1$,

$$\text{L.H.S.}=\frac{1}{3.5}$$

$$\text{R.H.S.}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$P(k):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2(2k+3)} \quad ...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

$$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \qquad \text{[Using (i)]}$$

$$=\frac{1}{(2k+3)}\left[\frac{k}{3}+\frac{1}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$$

$$=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)}{3\{2(k+1)+3\}}$$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**18.** **Prove the following by using the principle of mathematical induction for all $n \in N$:**

$$\mathbf{1+2+3+...+n<\frac{1}{8}(2n+1)^2}$$

**Ans:** Let us denote the given equality by $P(n)$, i.e.,

$$P(n): 1+2+3+...+n<\frac{1}{8}(2n+1)^2$$

For $n=1$,

$$1<\frac{1}{8}(2.1+1)^2=\frac{9}{8}$$

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some positive integer $k$, i.e.,

$$1+2+3+...+k<\frac{1}{8}(2k+1)^2 \quad ...(i)$$

Now, we have to prove that $P(k+1)$ is also true.

Consider

$$1+2+3+...+k+(k+1)<\frac{1}{8}(2k+1)^2+(k+1) \qquad \text{[Using (i)]}$$

$$<\frac{1}{8}\left\{(2k+1)^2+8(k+1)\right\}$$

$$<\frac{1}{8}\left\{4k^2+4k+1+8k+8\right\}$$

$$<\frac{1}{8}\left\{4k^2+12k+9\right\}$$

$$<\frac{1}{8}(2k+3)^2$$

$$<\frac{1}{8}\left\{2(k+1)+1\right\}^2$$

Hence, $1+2+3+...+k+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**19. Prove that following by using the principle of mathematical induction for all $\mathbf{n \in N}$:**

**$\mathbf{n(n+1)(n+5)}$ is a multiple of 3**

**Ans:** Let us denote the given statement by $P(n)$, i.e.,

$P(n): n(n+1)(n+5)$, which is a multiple of $3$.

For $n=1$,

$1(1+1)(1+5)=12$, which is a multiple of $3$.

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some natural number $k$, i.e.,

$k(k+1)(k+5)$ is a multiple of $3$.

$\therefore k(k+1)(k+5)=3m$, where $m \in N$ ...(i)

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true.

Consider

$(k+1)\{(k+1)+1\}\{(k+1)+5\}$

$=(k+1)(k+2)\{(k+1)+5\}$

$=(k+1)(k+2)(k+5)+(k+1)(k+2)$

$=\{k(k+1)(k+5)+2(k+1)(k+5)\}+(k+1)(k+2)$

$=3m+(k+1)\{2(k+5)+(k+2)\}$

$=3m+(k+1)\{2k+10+k+2\}$

$=3m+(k+1)\{3k+12\}$

$=3m+3(k+1)\{k+4\}$

$=3\{m+(k+1)(k+4)\}=3\times q$, where $q=\{m+(k+1)(k+4)\}$ is some natural number.

Hence, $(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of $3$.

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**20. Prove that following by using the principle of mathematical induction for all $\mathbf{n \in N}$:**

**$\mathbf{10^{2n-1}+1}$ is divisible by 11**

**Ans:** Let us denote the given statement by $P(n)$, i.e.,

$P(n): 10^{2n-1}+1$ is divisible by $11$.

For $n=1$,

$P(1)=10^{2n-1}+1=11$ and $P(1)$ is divisible by $11$.

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some natural number $k$, i.e.,

i.e., $10^{2k-1}+1$ is divisible by $11$.

$\therefore 10^{2k-1}+1=11m$, where $m \in N \ldots$(i)

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true.

Consider

$10^{2(k+1)-1}+1$

$=10^{2k+2-1}+1$

$=10^{2k+1}+1$

$=10^2\left(10^{2k-1}+1-1\right)+1$

$=10^2\left(10^{2k-1}+1\right)-10^2+1$

$=10^2.11m-100+1$ [Using (i)]

$=100\times 11m-99$

$=11(100m-9)$

$=11r$, where $r=(100m-9)$ is some natural number

Therefore, $10^{2(k+1)-1}+1$ is divisible by $11$.

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**21. Prove that following by using the principle of mathematical induction for all $n \in N$:**

**$x^{2n}-y^{2n}$ is divisible by $x+y$.**

**Ans:** Let us denote the given statement by $P(n)$, i.e.,

$P(n): x^{2n}-y^{2n}$ is divisible by $x+y$.

For $n=1$,

$P(1)=x^{2\times 1}-y^{2\times 1}=x^2-y^2=(x+y)(x-y)$, which is clearly divisible by $x+y$.

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some natural number $k$, i.e.,

$x^{2k}-y^{2k}$ is divisible by $x+y$.

$\therefore$ Let $x^{2k}-y^{2k}=m(x+y)$, where $m\in N \ldots$(i)

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true.

Consider

$x^{2(k+1)-y^{2(k+1)}}$

$=x^{2k}.x^2-y^{2k}.y^2$

$=x^2\left(x^{2k}-y^{2k}+y^{2k}\right)-y^{2k}.y^2$

$=x^2\left\{m(x+y)+y^{2k}\right\}-y^{2k}.y^2$

$=m(x+y)x^2+y^{2k}.x^2-y^{2k}.y^2$

$=m(x+y)x^2+y^{2k}\left(x^2-y^2\right)$

$=m(x+y)x^2+y^{2k}(x+y)(x-y)$

$=(x+y)\left\{mx^2+y^{2k}(x-y)\right\}$, which is a factor of $(x+y)$.

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**22. Prove that following by using the principle of mathematical induction for all $n\in N$:**
**$3^{2n+2}-8n-9$ is divisible by $8$.**

**Ans:** Let us denote the given statement by $P(n)$, i.e.,

$P(n):3^{2n+2}-8n-9$ is divisible by $8$.

For $n=1$,

$P(n)=3^{2\times1+2}-8\times1-9=64$, which is divisible by $8$.

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some natural number $k$, i.e.,

$3^{2k+2}-8k-9$ is divisible by $8$.

$\therefore 3^{2k+2}-8k-9=8m$; where $m\in N \ldots$(i)

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true.

Consider

$3^{2(k+1)+2}-8(k+1)-9$

$3^{2k+2}.3^2-8k-8-9$

$=3^2\left(3^{2k+2}-8k-9+8k+9\right)-8k-17$

$=3^2\left(3^{2k+2}-8k-9\right)+3^2(8k+9)-8k-17$

$=9.8m+9(8k+9)-8k-17$

$=9.8m+72k+81-8k-17$

$=9.8m+64k+64$

$=8(9m+8k+8)$

$=8r$, where $r=(9m+8k+8)$ is a natural number

Therefore, $3^{2(k+1)+2}-8(k+1)-9$ is divisible by $8$.

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**23. Prove that following by using the principle of mathematical induction for all $n \in N$:**
**$41^n-14^n$ is a multiple of $27$.**

**Ans:** Let us denote the given statement by $P(n)$, i.e.,

$P(n):41^n-14^n$ is a multiple of $27$.

It can be observed that $P(n)$ is true for $n=1$

For $n=1$,

$P(1)=41^1-14^1=27$, which is a multiple of $27$.

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some natural number $k$, i.e.,

$41^k-14^k$ is a multiple of $27$.

$\therefore 41^k-14^k=27m,\ m \in N \quad \ldots(i)$

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true.

Consider

$41^{k+1}-14^{k+1}$

$=41^k.41-14^k.14$

$=41\left(41^k-14^k+14^k\right)-14^k.14$

$=41.27m+14^k(41-14)$

$=41.27m+27.14^k$

$=27(41m-14^k)$

$=27\times r$, where $r=(41m-14^k)$ is a natural number.

Therefore, $41^{k+1}-14^{k+1}$ is a multiple of $27$.

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.

**24. Prove that following by using the principle of mathematical induction for all $n\in N$:**

$$(2n+7)<(n+3)^2$$

**Ans:** Let us denote the given statement by $P(n)$, i.e.,

$P(n):(2n+7)<(n+3)^2$

For $n=1$,

$P(1)=2.1+7=9<(1+3)^2=16$, which is true because $9<16$.

Therefore, $P(n)$ is true for $n=1$.

Let us assume that $P(k)$ is true for some natural number $k$, i.e.,

$(2k+7)<(k+3)^2$ …(i)

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true.

Consider

$\{2(k+1)+7\}=(2k+7)+2$

$\therefore \{2(k+1)+7\}=(2k+7)+2<(k+3)^2+2$ [Using (i)]

$2(k+1)+7<k^2+6k+9+2$

$2(k+1)+7<k^2+6k+11$

Now, $k^2+6k+11<k^2+8k+16$

$\therefore 2(k+1)+7<(k+4)^2$

$2(k+1)+7<\{(k+1)+3\}^2$

Therefore, $P(k+1)$ holds whenever $P(k)$ holds.

Hence, the given equality is true for all natural numbers i.e., $N$ by the principle of mathematical induction.