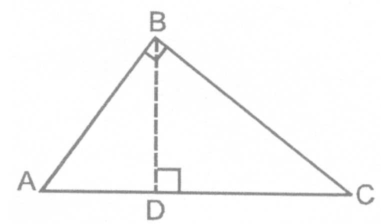
TRIANGLES

PYTHAGOREAN THEOREM

PYTHAGOREOUS THEOREM :

Statement : In a right triangle, the square of the hypotenuse is equal to the sum of the square of the other two sides.



Given : A right triangle ABC, right angled at B.

To prove : $AC^2 = AB^2 + BC^2$

Construction : $BD \perp AC$

Proof : Δ ADB & ΔABC

$\angle DAB = \angle CAB$ [Common]

$\angle BDA = \angle CBA$ [90^0 each]

So, $\Delta ADB \sim \Delta ABC$ [By AA similarity]

$\frac{AD}{AB} = \frac{AB}{AC}$ [Sides are proportional]

or, $AD \cdot AC = AB^2$ .....(i)

Similarly $\Delta BDC \sim \Delta ABC$

So, $\frac{CD}{BC} = \frac{BC}{AC}$

or $CD \cdot AC = BC^2$ .....(ii)

Adding (i) and (ii),

$AD \cdot AC + CD \cdot AC = AB^2 + BC^2$

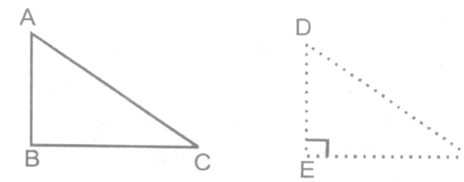
or, $AC (AD + CD) = AB^2 + BC^2$

or $AC.AC = AB^2 + BC^2$

or, $AC^2 = AB^2 + BC^2$ **Hence Proved.**

(A) CONVERSE OF PYTHAGOREANS THEOREM :

Statement : In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.



Given : A triangle ABC such that $AC^2 = AB^2 + BC^2$

Construction : Construct a triangle DEF such that DE = AB, EF = BC and $\angle E = 90^0$

Proof : In order to prove that $\angle B = 90.^0$, it is sufficient to show $\Delta ABC \sim \Delta DEF$. For this we proceed as follows Since ΔDEF is a right - angled triangle with right angle at E. Therefore, by Pythagoras theorem, we have

$DF^2 = DE^2 + EF^2$

$\Rightarrow$ $DF^2 = AB^2 + BC^2$ [$\therefore$ DE = AB and EF = BC (By construction)]

$\Rightarrow$ $DF^23 = AC^2$ [$\therefore AB^2 + BC^2 = AC^2$ (Given)]

$\Rightarrow$ DF = AC .....(i)

Thus, in ΔABC and ΔDEF, we have

AB = DE, BC = EF [By construction]

And AC = DF [From equation (i)]

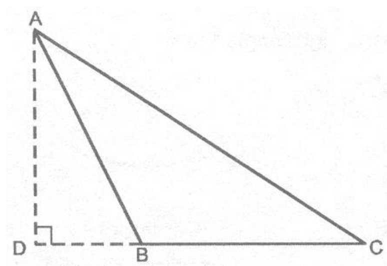
$\therefore \quad \Delta ABC \cong \Delta DEF$ [By SSS criteria of congruency]

$\Rightarrow \quad \angle B = \angle E = 90^0$

Hence, ΔABC is a right triangle, right angled at B.

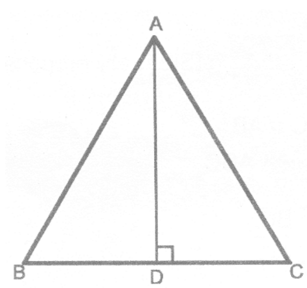
(B) SOME RESULTS DEDUCED FROM PYTHAGOREANS THEOREM :

(i) In the given figure ΔABC is an obtuse triangle, obtuse angled at B. If $AD \perp CD$, then $AC^2 = AB^2 + BC^2 + 2BC \,.\, BC$



(ii) In the given figure, if $\angle B$ of ΔABC is an acute angle and $AD \perp BC$, then

$AC^2 = AB^2 + BC^2 - 2BC \,.\, BD$



(iii) In any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.

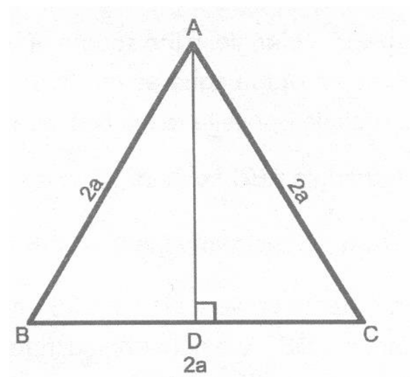
(iv) Three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares o the medians of the triangle.

Ex.1 In a ΔABC, $AB = BC = CA = 2a$ and $AD \perp BC$. Prove that **[CBSE - 2002]**

(i) $AD = a\sqrt{3}$ **(ii)** area $(\Delta ABC) = \sqrt{3}a^2$

Sol. (i) Here, $AD \perp BC$.

Clearly, ΔABC is an equilateral triangle.



Thus, in ΔABD and ΔACD

$AD = AD$ [Common]

$\angle ADB = \angle ADC$ [90^0 each]

And $AB = AC$

by RHS congruency condition

$\Delta ABD \cong \Delta ACD$

$\Rightarrow$ $BD = DC = a$

Now, ΔABD is a right angled triangle

$\therefore$ $AD = \sqrt{AB^2 - BD^2}$ [Using Pythagoreans Theorem]

$AD = \sqrt{4a^2 - a^2} = \sqrt{3}a$ or $a\sqrt{3}$

(ii) Area $(\Delta ABC) = \frac{1}{2} \times BC \times AD$

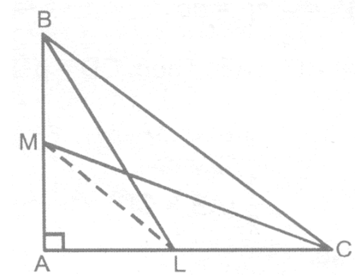
$= \frac{1}{2} \times 2a \times a\sqrt{3}$

$= a^2\sqrt{3}$

Ex.2 BL and Cm are medians of ΔABC right angled at A. Prove that $4(BL^2 + CM^2) = 5\,BC^2$ **[CBSE-2006]**

Sol. In ΔBAL

$BL^2 = AL^2 + AB^2$ ....(i) [Using Pythagoreans theorem]



and In ΔCAM

$CM^2 = AM^2 + AC^2$ .....(ii) [Using Pythagoreans theorem]

Adding (1) and (2) and then multiplying by 4, we get

$4(BL^2 + CM^2) = 4(AL^2 + AB^2 + AM^2 + AC^2)$

$= 4\{AL^2 + AM^2 + (AB^2 + AC^2)\}$ [$\therefore$ Δ ABC is a right triangle]

$= 4(AL^2 + AM^2 + BC^2)$

$= 4(ML^2 + BC^2)$ [$\therefore$ ΔLAM is a right triangle]

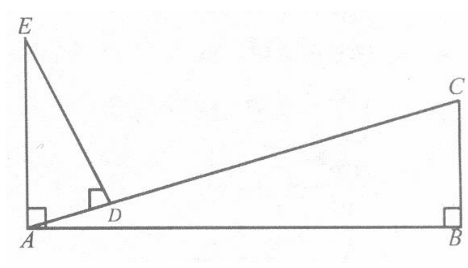
$= 4ML^2 + 4\,BC^2$

[A line joining mid-points of two sides is parallel to third side and is equal to half of it, $ML = BC/2$]

$= BC^2 + 4BC^2 = 5BC^2$ **Hence proved.**

Ex.3 In the given figure, $BC \perp AB$, $AE \perp AB$ and $DE \perp AC$. Prove that DE.BC = AD.AB.

Sol. In ΔABC and ΔEDA, We have



$\angle ABC = \Delta ADE$ [Each equal to 90^0]

$\angle ACB = \angle EAD$ [Alternate angles]

By AA Similarity

$\Delta ABC \sim \Delta EDA$

$\Rightarrow \quad \dfrac{BC}{AB} = \dfrac{AD}{DE}$

$\Rightarrow \quad DE.BC = AD.AB.$ **Hence Proved.**

Ex.4 O is any point inside a rectangle ABCD (shown in the figure). Prove that

$OB^2 + OD^2 = OA^2 + OC^2$ **[CBSE - 2006]**

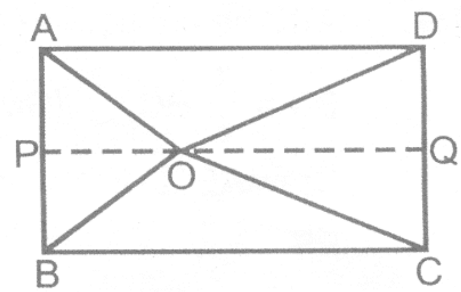
Sol. Through O, draw PQ||BC so that P lies on A and Q lies on DC.

Now, PQ||BC

Therefore, $PQ \perp AB$ and $PQ \perp DC$ [$\angle B = 90^0$ and $\angle C = 90^0$]

So, $\angle BPQ = 90^0$ and $\angle CQP = 90^0$

Therefore, BPQC and APQD are both rectangles.



Now, from Δ OPB,

$OB^2 = BP^2 + OP^2$ ....(i)

Similarly, from Δ ODQ,

$OD^2 = OQ^2 + DQ^2$ ....(ii)

From Δ OQC, we have

$OC^2 = OQ^2 + CQ^2$ ...(iii)

And form Δ OAP, we have

$OA^2 = AP^2 + OP^2$ ....(iv)

Adding (i) and (ii)

$OB^2 + OD^2 = BP^2 + OP^2 + OQ^2 + DQ^2$

$= CQ^2 + OP^2 + OQ^2 + AP^2$

[As BP = CQ and DQ = AP]

$= CQ^2 + OQ^2 + OP^2 + AP^2$

$= OC^2 + OA^2$ [From (iii) and (iv)] **Hence Proved.**