QUADRATIC EQUATION

Nature or Characteristics of Roots of Quadratic Equation

ROOTS OF A QUADRATIC EQUATION

The value of x which satisfies the given quadratic equation is known as its root. The roots of the given equation are known as its solution.

General form of a quadratic equation is :

$ax^2 + bx + c = 0$

or $4a^2x^2 + 4abx + 4ac = - 4ac$ [Multiplying by 4a]

or $4a^2x^2 + 4abx = - 4ac$ [By adding b^2 both sides]

or $4a^2x^2 + 4abc + b^2 = b^2 - 4ac$

or $(2ax + b)^2 = b^2 - 4ac$

Taking square root of both the sides

$2ax + b = \pm\sqrt{b^2 - 4ac}$

Or $x = \frac{-b \pm \sqrt{b2 - 4ac}}{2a}$

Hence, roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

REMARK :

A quadratic equation is satisfied by exactly two values of 'a' which may be real or imaginary. The equation, $ax^2 + bx + c = 0$ is :

- ★ A quadratic equation if Two roots
- ★ A linear equation if a = 0, One root
- ★ A contradiction if a = b = 0, No root

- ★ An identify if $a = b = c = 0$ Infinite roots
- ★ A quadratic equation cannot have more than two roots.
- ★ If follows from the above statement that if a quadratic equation is satisfied by more than two values of x, then it is satisfied by every value of x and so it is an identity.

NATURE Of ROOTS :

Consider the quadratic equation, $ax^2 + bx + c = 0$ having $\alpha\beta$ as its roots and $b^2 - 4ac$ is called discriminate of roots of quadratic equation. It is denoted by D or .

Roots of the given quadratic equation may be

(i) Real and unequal (ii) Real and equal (iii) Imaginary and unequal.

Let the roots of the quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0, b, c \in R$) be α and β then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad .....(i) \qquad \text{and} \qquad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad ....(ii)$$

The nature of roots depends upon the value of expression '$b^2 - 4ac$' with in the square root sign. This is known as discriminate of the given quadratic equation.

Case-1 When $b^2 - 4ac > 0$, $(D > 0)$

In this case roots of the given equation are real and distinct and are as follows

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(i) When $a(\neq 0), b, c \in Q$ and $b^2 - 4ac$ is a perfect square

In this case both the roots are rational and distinct.

(ii) When $a(\neq 0), b, c \in Q$ and $b^2 - 4ac$ is not a perfect square

In this case both the roots are irrational and distinct. **[See remarks also]**

Case-2 When $b^2 - 4ac = 0$, $(D = 0)$

In this case both the roots are real and equal to $-\frac{b}{2a}$.

Case-3 When b^2 - 4ac < 0, (D < 0)

In this case b^2 - 4ac < 0, then 4ac - b^2 > 0

$$\therefore \quad \alpha = \frac{-b + \sqrt{-(4ac - b^2)}}{2a} \text{ and } \beta = \frac{-b - \sqrt{(4ac - b^2)}}{2a}$$

or $$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a} \qquad \left[\because \sqrt{-1} = i\right]$$

i.e. in this case both the root are imaginary and distinct

REMARK :

★ If **a, b, c** $\in$ **Q** and **b^2 - 4ac** is positive **(D > 0)** but not a perfect square, then the roots are irrational and they always occur in conjugate pairs like $2+\sqrt{3}$ and $2-\sqrt{3}$. However, if **a , b , c** are irrational number and **b^2 - 4ac** is positive but not a perfect square, then the roots may not occur in conjugate pairs.

★ If **b^2 - 4ac** is negative **(D > 0)**, then the roots are complex conjugate of each other. In fact, complex roots of an equation with real coefficients always occur in conjugate pairs like 2 + 3i and 2 - 3i. However, this may not be true in case of equations with complex coefficients. For example, x^2 - 2ix - 1 = 0 has both roots equal to i.

★ If **a** and **c** are of the same sign and b has a sign opposite to that of a as well as c, then both the roots are positive, the sum as well as the product of roots is positive $(D \geq 0)$.

★ If **a , b,** are of the same sign then both the roots are negative, the sum of the roots is negative

Ex. 1 Without solving, examine the nature of roots of the equations :

(i) $2x^2 + 2x + 3 = 0$

(ii) $2x^2 - 7x + 3 = 0$

(iii) $x^2 - 5x - 2 = 0$

(iv) $4x^2 - 4x + 1 = 0$

Sol.(i) Comparing $2x^2 + 2x + 3 = 0$

with $ax^2 + bx + c = 0$; we get : $a = 2$, $b = 2$ and $c = 3$

$D = b^2 - 4ac = (2)^2 - 4 \times 2 \times 3 = 4 - 24$

$= -20$; which is negative.

$\therefore$ The roots of the given equation are imaginary.

(ii) Comparing $2x^2 - 7x + 3 = 0$

with $ax^2 + bx + c = 0$;

we get : $a = 2$, $b = -7$ and $c = 3$

$D = b^2 - 4ac = (-7)^2 - 4 \times 2 \times 3$

$= 49 - 24 = 25$, which is perfect square.

$\therefore$ The roots of the given equation are rational and unequal.

(iii) Comparing $x^2 - 5x - 2 = 0$

with $ax^2 + bx + c = 0$;

we get : $a = 1$, $b = -5$ and $c = -2$

$D = b^2 - 4ac = (-5)^2 - 4 \times 1 \times -2$

$= 25 + 8 = 33$; which is positive but not a perfect square.

$\therefore$ The roots of the given equation are irrational and unequal.

(iv) Comparing $4x^2 - 4x + 1 = 0$

with $ax^2 + bx + c = 0$;

we get : $a = 4$, $b = -4$, and $c = 1$

$D = b^2 - 4ac = (-4)^2 - 4 \times 4 \times 1$

$= 16 - 16 = 0$

$\therefore$ Roots are real and equal.

Ex. 2 For what value of m, are the roots of the equation $(3m + 1) x^2 + (11 + m) x + 9 = 0$ equal?

Sol. Comparing the given equation

with $ax^2 + bx + c = 0$;

we get : $a = 3m + 1$, $b = 11 + m$ and $c = 9$

$\therefore$ Discriminant, $D = b^2 - 4ac$

$= (11 + m)^2 - 4(3m + 1) \times 9$

$= 121 + 22m + m^2 - 108 m - 36$

$= m^2 - 86m + 85$

$= m^2 - 85m - m + 85$

$= m (m - 85) - 1 (m - 85)$

$= (m - 85) (m - 1)$

Since the roots are equal, $D = 0$

$\Rightarrow (m - 85) (m - 1) = 0$

$\Rightarrow m - 85 = 0$ or $m - 1 = 0$

$\Rightarrow m = 85$ or $m = 1$

SUM AND PRODUCT OF THE ROOTS

Let α and β be the two roots of the quadratic equation $ax^2 + bx + c = 0$.

Since, $ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

then, let : $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$\therefore$ The sum of the roots $= \alpha + \beta$

$$= \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2b}{2a} = -\frac{b}{a}$$

And, the product of the roots = $\alpha . \beta$

$$= \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2}$$

$$= \frac{b^2-b^2+4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$\therefore$ If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$; then :

(i) The sum of the roots

$$= \alpha + \beta = -\frac{b}{a}$$

$$= -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

(ii) The product of the roots

$$= \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

TO CONSTRUCT A QUADRATIC EQUATION WHOSE ROOTS ARE GIVEN

x^2 – (sum of roots) x + product of roots = 0

To get the quadratic equation with given roots :

(i) Find the sum of the roots.

(ii) Find the product of the roots.

(iii) Substitute the values of steps (i) and (ii) in

x^2 – (sum of the roots)x + (product of roots) = 0

and get the required quadratic equation.

Ex. 3 For each quadratic equation given below, find the sum of the roots and the product of the roots :

(i) $x^2 + 3x - 6 = 0$

(ii) $2x^2 + 5\sqrt{3}x + 6 = 0$

(iii) $3x^2 + 2\sqrt{5}x - 5 = 0$

Sol.(i) Comparing $x^2 + 3x - 6 = 0$ with $ax^2 + bx + c = 0$, we get : a = 1, b = 3 and c = – 6

$\therefore$ The sum of the roots $= -\frac{b}{a} = -\frac{3}{2}$

And, the product of the roots $= \frac{c}{a} = \frac{-6}{1} = -6$

(ii) Comparing $2x^2 + 5\sqrt{3}x + 6 = 0$ with $ax^2 + bx + c = 0$; we get : a = 2, b = $5\sqrt{3}$ and c = 6

$\therefore$ The sum of the roots $= -\frac{b}{a} = -\frac{5\sqrt{3}}{2}$

And, the product of the roots $= \frac{c}{a} = \frac{6}{2} = 3$

(iii) Comparing $3x^2 + 2\sqrt{5}x - 5 = 0$ with $ax^2 + bx + c = 0$; we get : a = 3, b = $2\sqrt{5}$ and c= – 5

$\therefore$ The sum of the roots $= -\frac{b}{a} = -\frac{2\sqrt{5}}{3}$

and, the product of the roots $= \frac{c}{a} = \frac{-5}{3}$

Ex. 4 Construct the quadratic equation whose roots are given below -

(i) 3, – 3

(ii) $3 + \sqrt{3}, 3 - \sqrt{3}$

(iii) $\frac{2+\sqrt{5}}{2}, \frac{2-\sqrt{5}}{2}$

Sol.(i) Since, the sum of the roots = (3) + (–3) = 3 – 3 = 0

and, the product of the roots = (3) (–3) = – 9

$\therefore$ The required quadratic equation is :

x^2 – (sum of roots) x + (product of roots) = 0

$\Rightarrow \quad x^2 - (0)x + (-9) = 0$, i.e., $x^2 - 9 = 0$

(ii) Since, the sum of the roots

$= 3 + \sqrt{3} + 3 - \sqrt{3} = 6$

and, the product of the roots

$= (3 + \sqrt{3})(3 - \sqrt{3}) = 9 - 3 = 6$

$\therefore$ The required quadratic equation is :

x^2 – (sum of roots) x + (product of roots) = 0

$\Rightarrow \quad x^2 - 6x + 6 = 0$

(iii) Since, the sum of the roots

$= \frac{2+\sqrt{5}}{2} + \frac{2-\sqrt{5}}{2} = \frac{2+\sqrt{5}+2-\sqrt{5}}{2} = \frac{4}{2} = 2$

and, the product of the roots

$= \left(\frac{2+\sqrt{5}}{2}\right)\left(\frac{2-\sqrt{5}}{2}\right) = \frac{4-5}{4} - \frac{1}{4}$

$\therefore$ The required quadratic equation is :

x^2 – (sum of roots) x + (product of roots) = 0

$\Rightarrow x^2 - 2x + \left(-\frac{1}{4}\right) = 0$

$\Rightarrow x^2 - 2x - \frac{1}{4} = 0$,

i.e., $4x^2 - 8x - 1 = 0$

Ex. 5 If a and c are such that the quadratic equation $ax^2 - 5x + 3 = 0$ has 10 as the sum of the roots and also as the product of the roots, find a and c.

Sol. For $ax^2 - 5x + c = 0$

the sum of roots $= -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2} = -\dfrac{-5}{a} = \dfrac{5}{a}$

and the product of roots $= \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{c}{a}$

Given : The sum of the roots = 10

$\Rightarrow \dfrac{5}{a} = 10$, i.e., $10a = 5 \Rightarrow a = \dfrac{5}{10} = \dfrac{1}{2}$

The product of roots = 10

$\Rightarrow \dfrac{c}{a} = 10 \Rightarrow c = 10a = 10 \times \dfrac{1}{2} = 5$

$\Rightarrow \quad a = \dfrac{1}{2}$ and $c = 5$

Relation between Roots and Coefficients

If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are α and β then :

(i) $(\alpha - \beta) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= \pm \dfrac{\sqrt{b^2 - 4ac}}{a} = \dfrac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \dfrac{b^2 - 2ac}{a^2}$

(iii) $\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= -\dfrac{b\sqrt{b^2 - 4ac}}{a^2} = \dfrac{\pm\sqrt{D}}{a}$

(iv) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$.

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$$= -\frac{b(b^2-3ac)}{a^3}$$

(v) $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \beta^2 + \alpha\beta)$.

$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

$$= \sqrt{(\alpha+\beta)^2-4\alpha\beta}\{(\alpha + \beta)^2 - \alpha\beta\}$$

$$= \frac{(b^2-ac)\sqrt{b^2-4ac}}{a^3}$$

(vi) $\alpha^4 + \beta^4 = \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2$

$$= \left(\frac{b^2-2ac}{a^2}\right)^2 - 2\,\frac{c^2}{a^2}$$

(vii) $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2)$

$$= \frac{-b(b^2-2ac)\sqrt{b^2-4ac}}{a^4}$$

(viii) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta$

(ix) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

(x) $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta)$

(xi) $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2$

$$= \frac{\alpha^4+\beta^4}{\alpha^2\beta^2} = \frac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}$$

Ex. 6 Find the equation whose roots are the reciprocals of the roots of $3x^2 - 5x + 7 = 0$

Sol. The equation whose roots are the reciprocals of the roots of $f(x) = 0$ is $f\left(\frac{1}{2}\right) = 0$

The required equation is $3\left(\frac{1}{x}\right)^2 - 5\left(\frac{1}{x}\right) + 7 = 0$

$$\left(\frac{3}{x^2}\right)-\frac{5}{x}+7=0$$

$7x^2 - 5x + 3 = 0$

Ex. 7 If one root of $x^2 - x - k = 0$ is square that of other, then find the value of k.

Sol. Other root = a^2

Sum of the roots = + 1

$\Rightarrow \quad a + a^2 = 1$ ....(i)

product of roots = – k

$\Rightarrow a \,.\, a^2 = -k \quad \Rightarrow a^3 = -k \quad \Rightarrow a = (-k)^{\frac{1}{3}}$ ....(ii)

$a^2 + a = +1$

substituting a = $(-k)^{\frac{1}{3}}$ from equation (2), we get

$$\left[(-k)^{\frac{1}{3}}\right]^2+(-k)^{\frac{1}{3}}=+1$$

$$k^{\frac{2}{3}}-k^{\frac{1}{3}}=1$$

$$\left[k^{\frac{2}{3}}-k^{\frac{1}{3}}\right]^3=(1)^3 \quad \Rightarrow \quad k^2 - k - 3k = 1$$

$\Rightarrow k^2 - 4k = 1 \quad \Rightarrow \quad k^2 - 4k - 1 = 0$

$$k=\frac{4\pm\sqrt{16+4}}{2} \qquad k=\frac{4\pm\sqrt{20}}{2}$$

$$k=\frac{4\pm2\sqrt{5}}{2}=2\pm\sqrt{5}$$

Ex. 8 Find the condition that the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ may have a common root.

Sol. Let a be a common root of the given equations.

Then $\alpha^2 + a\alpha + b = 0$ and $\alpha^2 + b\alpha + a = 0$

By the method of cross – multiplication, we get $\frac{\alpha^2}{a^2-b^2}=\frac{\alpha}{b-a}=\frac{1}{b-a}$

This gives $\alpha^2=\frac{a^2-b^2}{b-a}=(a+b) and \alpha=1$

$\Rightarrow (1)^2 = -(a+b)$

$\Rightarrow 1 = -a - b$

$\Rightarrow a + b + 1 = 0$ is the required