Quadratic Equations

Quadratic Polynomials

The general form of a quadratic polynomial in x is $ax^2 + bx + c$

where a,b,c, are real numbers and $a \neq 0$.

Zeros of quadratic polynomials

When a quadratic polynomial is equated to zero, we get two values of x which is called zeros of a quadratic polynomial.

Quadratic Equations

A quadratic polynomial equated to zero is called a quadratic equation. The general form of a quadratic equation is $ax^2 + bx + = 0$, where a, b, c are real numbers and $a \neq 0$.

Quadratic formula

Let $ax^2 + bx + c = 0$

Multiplying by 4a we get

$4a^2x^2 + 4abx + 4ac = 0$

Or, $(2ax)^2 + 2.2ax.b + b^2 = b^2 - 4ac$

Or, $(2ax + b)^2 = b^2 - 4ac$

$$2ax - b = \pm\sqrt{b^2 - 4ac}$$

Let $D = b^2 - 4ac$

$$\therefore\ 2ax + b = \pm\sqrt{D}$$

Or, $2ax = -b \pm \sqrt{D}$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a}$$

$$x = \frac{-b + \sqrt{D}}{2a}, \frac{-b - \sqrt{D}}{2a}$$

If $D < 0, \sqrt{D}$ is not real and, therefore, the equation has no real roots.

"D" is called the discriminat of the quadratic equation.

Case-I, When $D = b^2 - 4ac > 0$

i. e. $b^2 > 4ac$ then the equation has two distinct real roots $\alpha \text{ and } \beta$ given by

$$\alpha = \frac{-b + \sqrt{D}}{2a}, \beta = \frac{-b - \sqrt{D}}{2a}$$

Case-II, when D = b^2 - 4ac = 0

$$\alpha = \frac{-b}{2a}, \beta = \frac{-b}{2a}$$

Both the roots are real and equal.

Example 1

Solve: $x^2 - 16 = 0$

Solution

$x^2 - 16 = 0$

(x + 4) (x - 4) = 0

Either x + 4 = 0 which gives x = -4

Or, x - 4= 0, which gives x = 4

∴ x = -4, 4

Example 2.

Solve: $8x^2 - 22x - 21 = 0$

Solution

$8x^2 - 22x - 21 = 0$

$8x^2 - (28 - 6)x - 21 = 0$

$8x^2 - 28x + 6x - 21 = 0$

$4x(2x - 7) + 3(2x - 7) = 0$

$(2x - 7)(4x + 3) = 0$

Either $2x - 7 = 0$ or, $4x + 3 = 0$

Either $2x = 7$ or, $4x = -3$

Either $x = 7/2$ or, $x = -3/4$

$\therefore x = -\frac{3}{4}, \frac{7}{2}$

Example 3

Solve: $\frac{1}{x+1} + \frac{1}{x+2} = \frac{1}{x+4} \quad (x \neq -1, -2, -4)$

Solution:

$$\frac{1}{x+1} + \frac{1}{x+2} = \frac{1}{x+4}$$

$$\frac{(x+2) + (x+1)}{(x+1)(x+2)} = \frac{1}{x+4}$$

$$\frac{(2x+3)}{x^2 + x + 2x + 2} = \frac{1}{x+4}$$

Or, $(2x + 3)(x + 4) = x^2 + 3x + 2$

Or, $2x^2 + 8x + 3x + 12 = x^2 + 3x + 2$

Or, $x^2 + 8x + 10 = 0$

Here a = 1, b = 6, c = 8

$D = b^2 - 4ac$

$= (8)^2 - 4 \times 1 \times 10$

$= 64 - 40$

$= 24$

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-8 \pm \sqrt{24}}{2}$$

$$= \frac{-8 + 2\sqrt{6}}{2}, \frac{-3 - 2\sqrt{6}}{2}$$

$$= -4 \pm \sqrt{6}$$

Example 4.

Find the value of p for which the quadratic equation $2x^2 + 3x + p = 0$ has real roots.

Solution

$2x^2 + 3x + p = 0$

$a = 2, b = 3, c = p$

$D = b^2 - 4ac$

$= 3^2 - 4 \times 2 \times p$

$= 9 - 8p$

For real roots, $D \geq 0$

or, $9 \quad 8p \geq 0$

or, $9 \geq 8p$

or, $8p \leq 9$

. $p \leq \frac{9}{8}$

Example5

Find the value of α such that the quadratic equation $(\alpha-12)x^2+2(\alpha-12)x+2=0$ has equal roots.

Solution:-

$(\alpha-12)x^2+2(\alpha-12)x+2=0$

Here $a=\alpha-12, b=2(\alpha-12), c=2$

D = b^2 - 4ac

$=4(\alpha-12)^2-4(\alpha-12)\times 2$

$=4(\alpha-12)(\alpha-12-2)$

$\therefore 4(\alpha-12)(\alpha-14)$

For equal roots, $D=0$

i.e. $4(\alpha-12)(\alpha-14)=0$

Or, $\alpha=12,14$

But for $\alpha=12,$ the quadratic equation does not exist.

$\therefore \alpha=14$

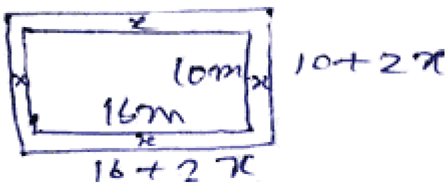
Application of Quadratic Equations

Example 1

A rectangular field is 16m long and 10m wide, there is a path of uniform width all around it having an area of 120sq.m final the width of the path.

Solution

Let width of the path be x m.



Area of the outer rectangle

= (16 + 2x) (10 + 2x)

$= 160 + 32x + 20x + 4x^2$

$= 160 + 52x + 4x^2$

Area of the inner rectangle

= 16m x 10m

$= 160m^2$

Area of the path

$= 160 + 52x + 4x^2 - 160 = 120$

or, $4x^2 + 52x - 120 = 0$

or, $x^2 + 13x - 30 = 0$

or, (x + 15) (x - 2) = 0

or, x = -15, 2

As width cannot be negative ∴ x = 2 m

i. e. width of the path = 2 m

Example 2

The sum of the reciprocals of Rehman's ages (in years), 3 years ago and 5 years from now is 1/3. Find his present age.

Solution :- Let his present age be x

His age 3 years ago = x – 3 and 5 years hence = x + 5

According to the question

$$\frac{1}{x-3}+\frac{1}{x+5}=\frac{1}{3}$$

$$\frac{(x+5)+(x-3)}{(x-3)(x+5)}=\frac{1}{3}$$

$$\frac{2x+2}{x^2-3x+5x-15}=\frac{1}{3}$$

or

$$\frac{2x+2}{x^2-3x+5x-15}=\frac{1}{3}$$

$x^2 + 2x - 15 = 6x + 6$
$x^2 - 4x - 21 = 0$
$(x + 3)(x - 7) = 0$

Either, x + 3 = 0 Or, x – 7 = 0

Thus, x = 7 or – 3; but age cannot be negative.

Hence his age is 7 years.

Example 3.

A plane left 30 minutes later than the scheduled time and in order to reach the destination 1500km away in time, it has to increase the speed by 250 km/hr from the usual speed. Find the usual speed.

Solution

Let the usual speed of plane be x km/hr.

The increased speed of the plane = (x + 250) km/hr.

Distance = 1500 km

$$\frac{1500}{x} - \frac{1500}{x+250} = \frac{1}{2}$$

or, $$\frac{1500x + 375000 - 1500x}{x(x+250)} = \frac{1}{2}$$

or, $x^2 + 250x - 750000 = 0$

or, $x^2 + 1000x - 750x - 750000 = 0$

or, $x(x + 1000) - 750(x + 1000) = 0$

or, $(x + 1000)(x - 750) = 0$

or, $x = -1000, 750$

As speed cannot be negative

$\therefore$ Usual speed = 750 km/hr.

Example 4.

If the list price of a book is reduced by Rs. 5, a person can buy 5 more books for Rs.300. Find the original list price of the book.

Solution

Let the original price of the book be Rs. x.

Then the reduced price of the book = Rs. (x - 5)

Total amount = Rs. 300

According to the question

$$\frac{300}{x-5} - \frac{300}{x} = 5$$

Or, $$\frac{300x - 300(x-5)}{x(x-5)} = 5$$

Or, $\frac{300x - 300x + 1500}{x^2 - 5x} = 5$

Or, $5(x^2 - 5x) = 1500$

Or, $x^2 - 5x = 300$

Or, $x^2 - 5x - 300 = 0$

Or, $x^2 - 20x + 15x - 300 = 0$

Or, $x(x - 20) + 15(x - 20) = 0$

Or, $(x - 20)(x + 15) = 0$

$x = -15, 20$

As price cannot be negative x = Rs. 20.

Example 5

In a class test, the sum of Shefali's marks in Mathematics and English is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of their marks would have been 210. Find her marks in the two subjects.

Solution

Let her marks in Mathematics = x, then marks in English = 30 – x.
As per statements, her marks in Mathematics = x + 2

Her marks in English

$30 - x - 3 = 27 - x$
$(x + 2)(27 - x) = 210$
Or,

$27x - x^2 + 54 - 2x = 210$
Or, $x^2 - 25x + 156 = 0$
Or,

$(x - 12)(x - 13) = 0$
Either, $x - 12 = 0$, Or, $x - 13 = 0$

Thus, x = 12, 13

Hence, either marks in Mathematics = 12 & marks in English = 18, Or, marks in Mathematics = 13 & marks in English = 17.

Example 6.

Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2? If so, find its length and breadth.

Solution

Let breadth = x, then length = 2x

According to question, $l \times b = 2x \times x = 800$

Or, $2x^2 = 800$

Or, $x^2 = 400$

Or, x = 20s

Hence, it is possible to design of length = 2×20 = 40 m and breadth = 20 m.

Example 7.

Is the following situation possible? If so, determine their present ages.

The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48.

Solution

Let present age of one friend be x and that of the other 20 – x.

4 years ago age of first friend = x – 4, and that of the other = 20 – x – 4 = 16 – x.

According to question, $(x - 4)(16 - x) = 48$

Or, $16x - x^2 - 64 + 4x = 48$

Or, $x^2 + 20x - 112 = 0$

Or, $x^2 - 20x + 112 = 0$

Here, a = 1, b = – 20, c = 112

$D = b^2 - 4ac = (-20)^2 - 4 \times 1 \times 112 = -48 < 0.$

Hence, the given situation is not possible.

Example 8.

A number consists of two digits whose product is 18. when 27 is subtracted from the number, the digits change their places. Find the number.

Solution

Let the digit at unit place be x and the digit at ten's place be y.

$\because$ Number 10y + x

Reversed number = 10x + y

According to the question

10y + x - 27 = 10x + y

Or, 9y - 9x = 27

Or, y - x = 3

$\therefore$ y = x + 3 ------------- (i)

xy = 18 ----------------- (ii)

x(x + 3) = 18

Or, $x^2 + 3x - 18 = 0$

Or, $x^2 + 6x - 3x - 18 = 0$

Or, x(x + 6) -3 (x + 6) = 0

Or, x = -6, 3

As digits ae never negative $\therefore$ x = 3

From (i) y = x + 3

= 3 + 3

= 6

$\therefore$ Number = 10y + x = 10 X 6 + 3 = 63