Outline

Diagonalizability

Consider $\mathbf{A} \in R^{n \times n}$, having n eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$; with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$.

$$\begin{aligned}
\mathbf{AS} &= \mathbf{A}[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n] \\
&= [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \mathbf{S}\Lambda
\end{aligned}$$

$$\Rightarrow \mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^{-1} \quad \text{and} \quad \mathbf{S}^{-1}\mathbf{AS} = \Lambda$$

Diagonalization: The process of changing the basis of a linear transformation so that its new matrix representation is diagonal, i.e. so that it is decoupled among its coordinates.

Diagonalizability

Diagonalizability:

> *A matrix having a complete set of n linearly independent eigenvectors is diagonalizable.*

Existence of a complete set of eigenvectors:

> *A diagonalizable matrix possesses a complete set of n linearly independent eigenvectors.*

- All distinct eigenvalues implies *diagonalizability*.
- But, diagonalizability does **not** imply distinct eigenvalues!
- However, a *lack* of diagonalizability certainly implies a *multiplicity mismatch*.

Canonical Forms

- *Jordan canonical form (JCF)*
- *Diagonal (canonical) form*
- *Triangular (canonical) form*

Other convenient forms

- *Tridiagonal form*
- *Hessenberg form*

Canonical Forms

Jordan canonical form (JCF): composed of Jordan blocks

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_k \end{bmatrix}, \quad \mathbf{J}_r = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$$

The key equation $\mathbf{AS} = \mathbf{SJ}$ in extended form gives

$$\mathbf{A}[\cdots \quad \mathbf{S}_r \quad \cdots] = [\cdots \quad \mathbf{S}_r \quad \cdots] \begin{bmatrix} \ddots & & \\ & \mathbf{J}_r & \\ & & \ddots \end{bmatrix},$$

where Jordan block $\mathbf{J}_r$ is associated with the subspace of

$$\mathbf{S}_r = [\mathbf{v} \quad \mathbf{w}_2 \quad \mathbf{w}_3 \quad \cdots]$$

Canonical Forms

Equating blocks as $\mathbf{AS}_r = \mathbf{S}_r\mathbf{J}_r$ gives

$$[\mathbf{Av} \quad \mathbf{Aw}_2 \quad \mathbf{Aw}_3 \quad \cdots] = [\mathbf{v} \quad \mathbf{w}_2 \quad \mathbf{w}_3 \quad \cdots]\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & \ddots \\ & & & \ddots \end{bmatrix}$$

Columnwise equality leads to

$$\mathbf{Av} = \lambda\mathbf{v}, \quad \mathbf{Aw}_2 = \mathbf{v} + \lambda\mathbf{w}_2, \quad \mathbf{Aw}_3 = \mathbf{w}_2 + \lambda\mathbf{w}_3, \quad \cdots$$

Generalized eigenvectors $\mathbf{w}_2$, $\mathbf{w}_3$ etc:

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} &= \mathbf{0}, \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 &= \mathbf{v} \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{w}_2 = \mathbf{0}, \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_3 &= \mathbf{w}_2 \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})^3\mathbf{w}_3 = \mathbf{0}, \quad \cdots \end{aligned}$$

Canonical Forms

Diagonal form

- Special case of Jordan form, with each Jordan block of 1×1 size
- Matrix is diagonalizable
- Similarity transformation matrix $\mathbf{S}$ is composed of n linearly independent eigenvectors as columns
- None of the eigenvectors admits any *generalized eigenvector*
- Equal geometric and algebraic multiplicities for every eigenvalue

Canonical Forms

Triangular form

Triangularization: Change of basis of a linear tranformation so as to get its matrix in the triangular form

- For real eigenvalues, always possible to accomplish with orthogonal similarity transformation
- Always possible to accomplish with unitary similarity transformation, with complex arithmetic
- Determination of eigenvalues

Note: The case of complex eigenvalues: 2×2 real diagonal block

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \sim \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix}$$

Canonical Forms

Forms that can be obtained with pre-determined number of arithmetic operations (without iteration):

Tridiagonal form: non-zero entries only in the (leading) diagonal, sub-diagonal and super-diagonal

- useful for symmetric matrices

Hessenberg form: A slight generalization of a triangular matrix

$$\mathbf{H}_u = \begin{bmatrix} * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ & * & * & \cdots & * & * \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & * & * \end{bmatrix}$$

Note: Tridiagonal and Hessenberg forms do not fall in the category of canonical forms.

Symmetric Matrices

A real symmetric matrix has all real eigenvalues and is diagonalizable through an orthogonal similarity transformation.

- Eigenvalues must be real.
- A complete set of eigenvectors exists.
- Eigenvectors corresponding to distinct eigenvalues are necessarily orthogonal.
- Corresponding to repeated eigenvalues, orthogonal eigenvectors are available.

In all cases of a symmetric matrix, we can form an orthogonal matrix $\mathbf{V}$*, such that* $\mathbf{V}^T\mathbf{A}\mathbf{V} = \Lambda$ *is a real diagonal matrix.*

Further, $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$.

Similar results for complex *Hermitian* matrices.

Symmetric Matrices

Proposition: Eigenvalues of a real symmetric matrix must be real.

Take $\mathbf{A} \in R^{n\times n}$ such that $\mathbf{A} = \mathbf{A}^T$, with eigenvalue $\lambda = h + ik$.

Since $\lambda\mathbf{I} - \mathbf{A}$ is singular, so is

$$\begin{aligned}\mathbf{B} &= (\lambda\mathbf{I} - \mathbf{A})\,(\bar{\lambda}\mathbf{I} - \mathbf{A}) = (h\mathbf{I} - \mathbf{A} + ik\mathbf{I})(h\mathbf{I} - \mathbf{A} - ik\mathbf{I}) \\ &= (h\mathbf{I} - \mathbf{A})^2 + k^2 I\end{aligned}$$

For some $\mathbf{x} \neq \mathbf{0}$, $\mathbf{Bx} = \mathbf{0}$, and

$$\mathbf{x}^T\mathbf{Bx} = 0 \Rightarrow \mathbf{x}^T(h\mathbf{I} - \mathbf{A})^T(h\mathbf{I} - \mathbf{A})\mathbf{x} + k^2\mathbf{x}^T\mathbf{x} = 0$$

Thus, $\|(h\mathbf{I} - \mathbf{A})\mathbf{x}\|^2 + \|k\mathbf{x}\|^2 = 0$

$$\boxed{k = 0 \text{ and } \lambda = h}$$

Symmetric Matrices

Proposition: A symmetric matrix possesses a complete set of eigenvectors.

Consider a repeated real eigenvalue λ of $\mathbf{A}$ and examine its Jordan block(s).

Suppose $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.
The first generalized eigenvector $\mathbf{w}$ satisfies $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$, giving

$$\begin{aligned} \mathbf{v}^T(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}^T\mathbf{v} \quad &\Rightarrow \quad \mathbf{v}^T\mathbf{A}^T\mathbf{w} - \lambda\mathbf{v}^T\mathbf{w} = \mathbf{v}^T\mathbf{v} \\ &\Rightarrow \quad (\mathbf{A}\mathbf{v})^T\mathbf{w} - \lambda\mathbf{v}^T\mathbf{w} = \|\mathbf{v}\|^2 \\ &\Rightarrow \quad \|\mathbf{v}\|^2 = 0 \end{aligned}$$

which is absurd.

An eigenvector will not admit a generalized eigenvector.

All Jordan blocks will be of 1×1 size.

Symmetric Matrices

Proposition: Eigenvectors of a symmetric matrix corresponding to distinct eigenvalues are necessarily orthogonal.

Take two eigenpairs $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$, with $\lambda_1 \neq \lambda_2$.

$$\begin{aligned} \mathbf{v}_1^T \mathbf{A} \mathbf{v}_2 &= \mathbf{v}_1^T (\lambda_2 \mathbf{v}_2) = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \\ \mathbf{v}_1^T \mathbf{A} \mathbf{v}_2 &= \mathbf{v}_1^T \mathbf{A}^T \mathbf{v}_2 = (\mathbf{A}\mathbf{v}_1)^T \mathbf{v}_2 = (\lambda_1 \mathbf{v}_1)^T \mathbf{v}_2 = \lambda_1 \mathbf{v}_1^T \mathbf{v}_2 \end{aligned}$$

From the two expressions, $(\lambda_1 - \lambda_2)\mathbf{v}_1^T \mathbf{v}_2 = 0$

$$\boxed{\mathbf{v}_1^T \mathbf{v}_2 = 0}$$

Proposition: Corresponding to a repeated eigenvalue of a symmetric matrix, an appropriate number of orthogonal eigenvectors can be selected.

If $\lambda_1 = \lambda_2$, then the entire subspace $< \mathbf{v}_1, \mathbf{v}_2 >$ is an eigenspace. Select any two mutually orthogonal eigenvectors for the basis.

Symmetric Matrices

Facilities with the 'omnipresent' symmetric matrices:

- Expression

$$\begin{aligned}\mathbf{A} &= \mathbf{V}\Lambda\mathbf{V}^T \\ &= [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\ &= \lambda_1\mathbf{v}_1\mathbf{v}_1^T + \lambda_2\mathbf{v}_2\mathbf{v}_2^T + \cdots + \lambda_n\mathbf{v}_n\mathbf{v}_n^T = \sum_{i=1}^{n} \lambda_i\mathbf{v}_i\mathbf{v}_i^T\end{aligned}$$

- Reconstruction from a sum of rank-one components
- Efficient storage with only large eigenvalues and corresponding eigenvectors
- Deflation technique
- Stable and effective methods: easier to solve the eigenvalue problem

Similarity Transformations

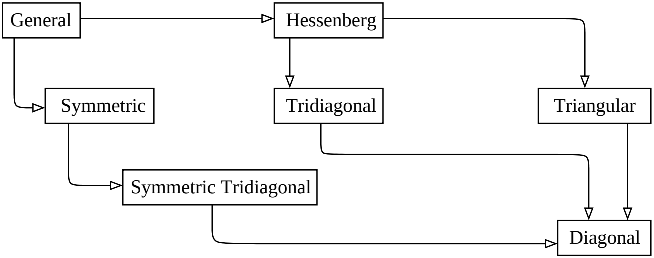

Figure: Eigenvalue problem: forms and steps

How to find suitable similarity transformations?

1. rotation
2. reflection
3. matrix decomposition or factorization
4. elementary transformation

Points to note

- Generally possible reduction: Jordan canonical form
- Condition of diagonalizability and the diagonal form
- Possible with orthogonal similarity transformations: triangular form
- Useful non-canonical forms: tridiagonal and Hessenberg
- *Orthogonal diagonalization of symmetric matrices*

Caution: Each step in this context to be effected through similarity transformations

Necessary Exercises: **1,2,4**