Chapter 1

Introduction To Ordinary Differential Equations

In this chapter we begin by introducing the concept of solvability of initial value problems (IVP) corresponding to first order differential equations (ODE) in $\Re^n$. We also give, along with appropriate examples of practical importance, all related notions like orbit or trajectory, direction field, isocline and solution curve of IVP.
The reader is also familiarized with difference equations and numerical solutions of ODE through the process of discretization. An interesting concept of neural solution of ODE is also touched upon. Towards the end, we recall standard methods of solving some classes of first order ODE.

1.1 First Order Ordinary Differential Equations

Let $\Omega \subset \Re^n$ be open and let $f : \Omega \to \Re^n$ be a mapping (not necessarily linear) given by

$$f(x_1, x_2, x_n) = (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots x_n), \ldots, f_n(x_1, x_2, \ldots, x_n))$$

We shall refer to f as vector field on Ω. A vector field may also depend on an additional parameter t, that is, $f = f(t, \overline{x})$.

The main thrust of our study is the following differential equation

$$\frac{d\bar{x}}{dt} = f(t, \overline{x}) \tag{1.1.1}$$

where $f : I \times \Omega \to \Re^n, I$ a subinterval of $\Re$.

The value $\overline{x}(t) \in \Omega \subset \Re^n$ is sometimes referred as the state of the system described by the ordinary differential equation (ODE) - Eq.(1.1.1). The set Ω is then called the state space or phase space.

Definition 1.1.1 *The Initial Value Problem (IVP) corresponding to Eq. (1.1.1) is given by*

$$\frac{d\overline{x}}{dt} = f(t, \overline{x}(t)) \tag{1.1.1(a)}$$

$$\overline{x}(t_0) = \overline{x}_0 \tag{1.1.1(b)}$$

Definition 1.1.2 *A function $\overline{x}(t)$ is said to be a solution of the IVP - Eq. (1.1.1), if there exists an interval $J \subset I$, containing t_0 such that $\overline{x}(t)$ is differentiable on J with $\overline{x}(t) \in \Omega$ for all $t \in J$ and $\bar{x}(t)$ satisfies Eq. (1.1.1).*

As $\overline{x}(t)$ needs to satisfy Eq. (1.1.1) only on J, this solution is sometimes referred as a local solution. If $t \to f(t, .)$ is defined on the entire line $\Re$ and $\bar{x}(t)$ satisfies Eq. (1.1.1) on $\Re$, then $\bar{x}(t)$ is said to be a global solution of the IVP. In Chapter 3, we shall discuss the local and global solvability of IVP in detail.

At times, it is desirable to indicate the dependence of the solution $\overline{x}(t)$ on its initial value $\overline{x}_0$. Then we use the notation $\overline{x}(t, t_0, \overline{x}_0)$ for $\overline{x}(t)$.

Definition 1.1.3 *The orbit or the trajectory of the ODE- Eq.(1.1.1), is the set $\{\overline{x}(t, t_0, \bar{x}_0) : t \in J\}$ in the state space Ω. Whereas the solution curve is the set $\{(t, \overline{x}(t, t_0, \bar{x}_0)) : t \in J\} \subset I \times \Omega$.*

Definition 1.1.4 *The direction field of ODE - Eq. (1.1.1), is the vector field $(1, f(t, \overline{x}))$.*

It is clear from the above definitions that the orbit of the ODE - Eq. (1.1.1) is tangent to the vector field whereas the solution curve is tangent to the direction field at any point.

The following proposition gives the equivalence of solvability of IVP - Eq. (1.1.1) with the solvability of the corresponding integral equation.

Proposition 1.1.1 *Assume that $f : I \times \Omega \to \Re^n$ is continuous. $\bar{x}(t)$ is a solution the IVP - Eq. (1.1.1) iff $\bar{x}(t)$ is the solution of the integral equation*

$$\bar{x}(t) = \bar{x}_0 + \int_{t_0}^{t} f(s, \bar{x}(s))ds, \quad t \in J \tag{1.1.2}$$

Proof : If $\bar{x}(t)$ is a solution of Eq.(1.1.1), by direct integration on the interval $[t_0, t]$ for a fixed t, we get

$$\int_{t_0}^{t} \left(\frac{d\bar{x}}{ds}\right) ds = \int_{t_0}^{t} f(s, \bar{x}(s))ds$$

This gives

$$\bar{x}(t) - \bar{x}(t_0) = \int_{t_0}^{t} f(s, \bar{x}(s))ds$$

and hence we have Eq. (1.1.2).

Conversely, if Eq.(1.1.2) holds, $\bar{x}(t)$ given by Eq. (1.1.2) is differentiable and differentiating this equation we get

$$\frac{d\bar{x}}{dt} = \frac{d}{dt}\left[\int_{t_0}^{t} f(s,\bar{x}(s))ds\right] = f(t,\overline{x}(t))$$

Also we have $\bar{x}(t_0) = \bar{x}_0$. ∎

Definition 1.1.5 *The set of points in Ω where $f(t,\bar{x}) = 0$, is called the set of equilibrium points of ODE - Eq. (1.1.1). It is clear that at these points the orbit or trajectory is a singleton.*

Example 1.1.1 *(Population Dynamics Model)*
We investigate the variation of the population $N(t)$ of a species in a fixed time span. An important index of such investigation is growth rate per unit time, denoted by

$$R(t) = \left[\frac{1}{N(t)}\right]\frac{dN}{dt} \tag{1.1.3}$$

If we only assume that the population of species changes due to birth and death, then growth rate is constant and is given by,

$$R(t) = R(t_0) = b - d$$

where b and d are birth and death rates, respectively.

If the initial population $N(t_0) = N_0$, we get the following initial value problem

$$\frac{dN}{dt} = R(t_0)N \tag{1.1.3(a)}$$

$$N(t_0) = N_0 \tag{1.1.3(b)}$$

The solution of the above IVP is given by

$$N(t) = N_0 \exp(R(t - t_0)), \quad t \in \Re$$

Although the above model may predict the population $N(t)$ in the initial stages, we realise that no population can grow exponentially for ever. In fact, as population grows sufficiently large, it begins to interact with its environment and also other species and consequently growth rate $R(t)$ diminishes. If we assume the form of $R(N)$ as equal to $a - cN(a, c$ positive constant$)$, we get the following IVP

$$\frac{dN}{dt} = N(a - cN) \tag{1.1.4a}$$

$$N(t_0) = N_0 \tag{1.1.4b}$$

This is known as logistic equation with a as the growth rate without environmental influences and c representing the effect of increase population density.

The equilibrium points are 0 *and* $a/c, C = a/c$ *is called the carrying capacity of the environment.*

One can easily solve the above ODE by the method of separation of variables and get

$$N(t) = \frac{\frac{a}{c}}{\left[1 + \frac{\left(\frac{a}{c} - N_0\right)}{N_0} \exp(-a(t - t_0))\right]}$$

It follows from the above representation that $\lim_{t \to \infty} N(t) = a/c = C$ *(carrying capacity). So the above solution has the representation*

$$N(t) = \frac{C}{\left[1 + \frac{(C - N_0)}{N_0} \exp(-a(t - t_0))\right]} \tag{1.1.5}$$

The solution curve of the above population model is as under.

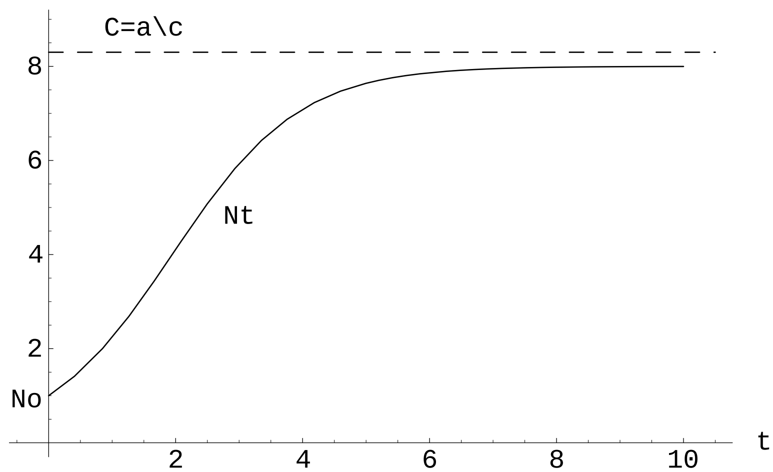

Figure 1.1.1: Growth of population $N(t)$

The parameters a *and* C *are usually not known but can be estimated by minimising the mean square deviation* $\phi(a, C)$ *between the known discrete population data* $N_d(t_m)$ *and theoretical data* $N(t_m)$:

$$\phi(a, C) = \sum_{m=1}^{M} [N(t_m) - N_d(t_m)]^2 \tag{1.1.6}$$

By using one of the algorithms of the unconstrained optimization techniques (refer Algorith 2.3.1), we can actually carry out the computation of a *and* C *from the given data.*

Example 1.1.2 *Consider the following ODE*

$$\frac{dx}{dt} = -tx \tag{1.1.7}$$

A general solution of the above ODE is given by

$$x(t) = ae^{-\frac{1}{2}t^2}, \quad a \text{ is any constant}$$

The solution passing through (t_0, x_0) *in* (t, x) *plane is given by*

$$x(t) = x_0 e^{-\frac{1}{2}(t^2 - t_0^2)} \tag{1.1.8}$$

The following graphics demonstrates the solution curve and the direction field of the above ODE.

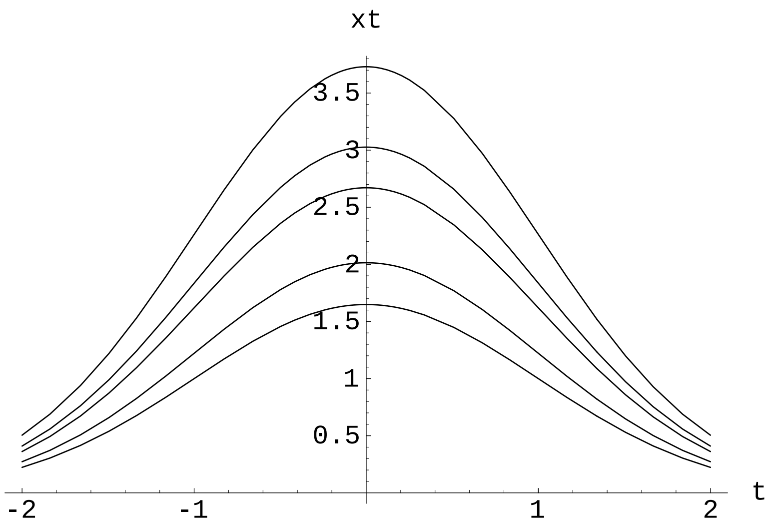

Figure 1.1.2: Solution curves $x(t)$

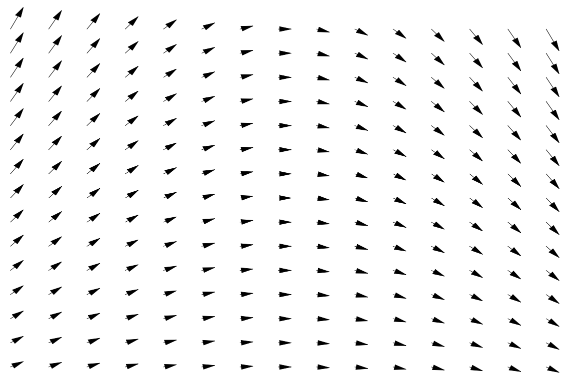

Figure 1.1.3: Direction field of the ODE

If IVP corresponding to Eq. (1.1.1) has a unique solution $\overline{x}(t, t_0, x_0)$ passing through $(t_0, \overline{x}_0)$, no two solution curves of the IVP can intersect. Thus the uniqueness theorem, to be dealt with in the Chapter 3, will be of utmost importance to us.
As we have seen through Example 1.1.2, the direction field for a given differential equation can be an effective way to obtain qualitative information about the behaviour of the system and hence it would be more appropriate to expand on this concept.
To sketch the direction field for a given ODE in 1-dimension, a device called isocline can be used.

Definition 1.1.6 *Isocline is a curve in $t - x$ plane through the direction field along which $p = f(t, x)$ is constant. The family of isoclines is then the family of curves $f(t, x) = p$ in $t - x$ plane.*

Example 1.1.3 *We wish to find the direction field of the following ODE*

$$\frac{1 + \dfrac{dx}{dt}}{1 - \dfrac{dx}{dt}} = 2\sqrt{t + x}$$

Solving for $\dfrac{dx}{dt}$, we get

$$\frac{dx}{dt}(1 + 2\sqrt{t + x}) = 2\sqrt{t + x} - 1$$

which gives

$$\frac{dx}{dt} = \frac{2\sqrt{t + x} - 1}{2\sqrt{t + x} + 1} = p$$

So iscolines are given by

$$\frac{2\sqrt{t+x}-1}{2\sqrt{t+x}+1} = p$$

That is

$$t + x = \frac{1}{2}\left[\frac{1+p}{1-p}\right]^2, -1 \leq p < 1 \tag{1.1.9}$$

Thus isoclines are all lines of slope -1 with $-1 \leq p < 1$. So we get the following graphics. The solution of the ODE passing through (t_0, x_0) is given by

$$(x-t) + \sqrt{t+x} = (x_0 - t_0) + \sqrt{t_0 + x_0} \tag{1.1.10}$$

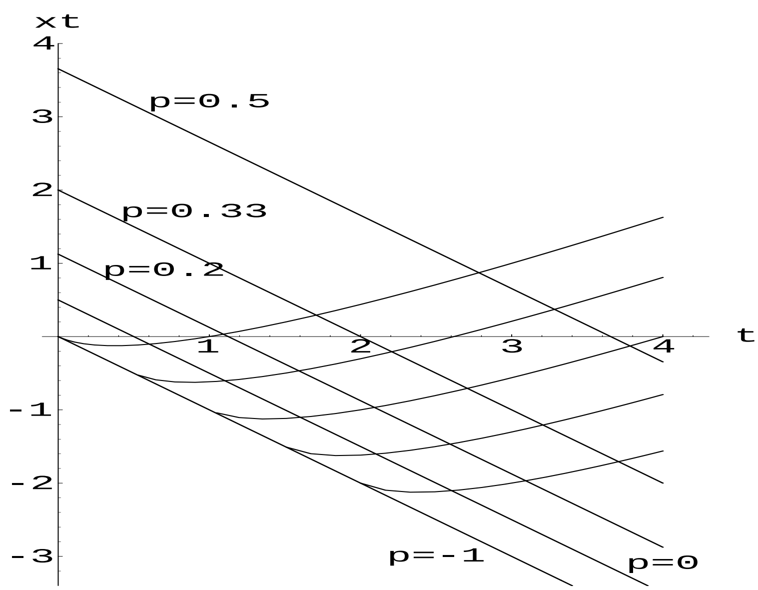

Figure 1.1.4: Isoclines with solution curves

1.2 Classification of ODE

The ODE

$$\frac{d\overline{x}}{dt} = f(t, \overline{x}) \tag{1.2.1}$$

is called linear if f has the form

$$f(t, \overline{x}) = A(t)\overline{x} + \overline{b}(t) \tag{1.2.2}$$

where $A(t) \in \Re^{n\times n}$ and $\overline{b}(t) \in \Re^n$ for all t. A linear ODE is called homogeneous if $\overline{b}(t) \equiv 0$ and if $A(t) = A$ (constant matrix), then we call it a linear ODE with constant coefficients. Linear ODE will be discussed in detail in Chapter 4.

The ODE - Eq. (1.2.1) is called autonomous if f does not depend on t. That is ODE has the form

$$\frac{d\overline{x}}{dt} = f(\overline{x}) \tag{1.2.3}$$

For autonomous ODE we have the following proposition (refer Mattheij and Molenaar [9]).

Proposition 1.2.1 *If $\overline{x}(t)$ is a solution of Eq. (1.2.3) on the interval $I = (a, b)$, then for any $s \in \Re$, $\overline{x}(t+s)$ is a solution of Eq. (1.2.3) on the interval $(a-s, b-s)$. Hence the trajectory $x(t)$ of Eq. (1.2.3) satisfies the following property*

$$\overline{x}(t, t_0, \overline{x}_0) = \overline{x}(t - t_0, 0, \overline{x}_0) \text{ for all } t_0 \in I$$

This implies that the solution is completely determined by the initial state $\bar{x}_0$ at $t = 0$.

A non-autonomous ODE - (1.2.1) is called periodic if

$$f(t+T, \overline{x}) = f(t, \overline{x}) \text{ for some } T > 0 \tag{1.2.4}$$

The smallest of such T is called its period. From Eq. (1.2.4), it follows that the form of the solution is not affected if we shift t_0 to $t_0 \pm nT$, $n \in \aleph$ (set of natural numbers). That is

$$\overline{x}\left(t \pm nT, t_0 \pm nT, \overline{x}_0\right) = \overline{x}(t, t_0, \overline{x}_0)$$

It is not necessary that a solution of periodic ODE is periodic as we see in the following example.

Example 1.2.1

$$\frac{d\overline{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{x} + \begin{bmatrix} 0 \\ 2\cos t \end{bmatrix}$$

$f(t, \overline{x}) = A\overline{x} + \overline{b}(t)$, $\overline{b}(t) = (0, 2\cos t)$. *$f$ is periodic with period 2π but its solution (to be discussed in Chapter 4) is given by*

$$\overline{x}(t) = (t - \sin t, \sin t + t\cos t)$$

However, we have the following theorem regarding the periodic solution of the periodic ODE.

Proposition 1.2.2 *If a solution of a periodic ODE is periodic. Then it has the same period as the vector field.*

Proof : Assume that $f(t, \overline{x})$ has a period T and the solution $\overline{x}(t)$ has period S with $S \neq T$. Because $\bar{x}$ is periodic, its derivative $\dot{\overline{x}}(t) = f(t, \overline{x})$ will also be periodic with period S as well T. That is

$$f(t+S, \overline{x}) = f(t, \overline{x})$$

Also, T periodicity of f implies that

$$f(t+S-nT,\overline{x}) = f(t,\overline{x})$$

We choose n such that $0 < S - nT < T$. It implies that f is periodic with a period smaller than T, a contradiction. Hence $T = S$ and the solution $\bar{x}(t)$ has the same period as the vector field. ∎

Example 1.2.2 *(Predator-Prey Model)*
Predator-prey model represents the interaction between two species in an environment. For example, we shall focus on sharks and small fish in sea.
If the food resource of sharks (fish) is non-existent, sharks population exponentially decays and is increased with the existence of fish. So, the growth rate of sharks $\left(\frac{1}{S}\frac{dS}{dt}\right)$ *is modelled as* $-k$ *without the fish population and* $-k+\lambda F$ *with fish population* F*. Thus*

$$\frac{dS}{dt} = S(-k+\lambda F)$$

For the growth rate of fish, we note that it will decrease with the existence of sharks and will flourish on small plankton (floating organism in sea) without sharks. Thus

$$\frac{dF}{dt} = F(a-cS)$$

Thus, we have, what is called the famous Lotka - Volterra model for the predator-prey system

$$\frac{dF}{dt} = F(a-cS) \qquad (1.2.4(a))$$

$$\frac{dS}{dt} = S(-k+\lambda F) \qquad (1.2.4(b))$$

This reduces to the following autonomous system of ODE in $\Re^2$

$$\frac{d\bar{x}}{dt} = f(\bar{x})$$

where

$$\overline{x} = (F,S), f(\bar{x}) = (F(a-cS), S(-k+\lambda F))$$

The equilibrium points are given by

$$\hat{F} = k/\lambda, \hat{S} = a/c; \hat{F} = 0, \hat{S} = 0.$$

We shall have the discussion of the phase-plane analysis in the Chapter 5. However, if we linearize the system given by Eq.(1.2.4) around the equilibrium point

$\hat{F} = k/\lambda, \hat{S} = a/c$, we get the following linear system (refer Section 2.3 for linearization)

$$\begin{aligned} \frac{d\overline{x}}{dt} &= A\overline{x}(t) \\ \overline{x} &= (F - \hat{F}, S - \hat{S}) \\ A &= \begin{bmatrix} 0 & -\frac{kc}{\lambda} \\ \frac{a\lambda}{c} & 0 \end{bmatrix} \end{aligned} \tag{1.2.5}$$

The solution of the IVP corresponding to Eq. (1.2.5) is given by

$$S = \hat{S} + S_0 \cos wt + \frac{a\lambda}{cw} F_0 \sin wt \tag{1.2.6a}$$

$$F = \hat{F} + F_0 \sin wt - \frac{cw}{a} S_0 \cos wt \tag{1.2.6b}$$

(Refer Section 4.3 for solution analysis of this linear system).

Thus, the solution is periodic with period $\frac{2\pi}{w} = 2\pi(ak)^{-1/2}$. The following graphics depict the fish and shark population in $t - F/S$ plane.

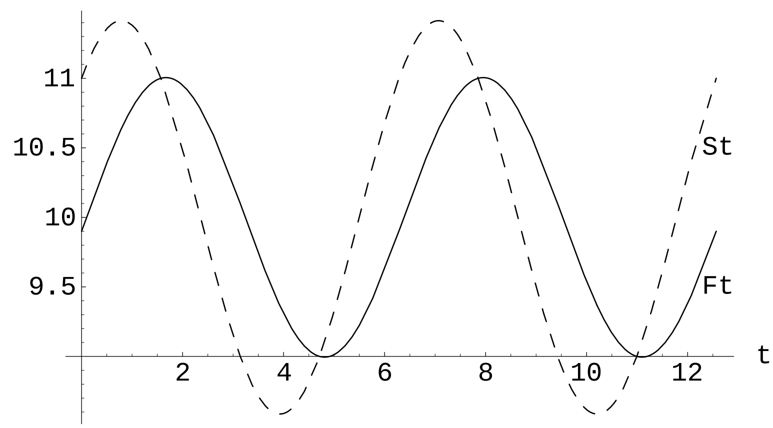

Figure 1.2.1: Fish and shark population with time

The average population of the general predator-prey system given by Eq. (1.2.4) is obtained as follows.

We have

$$\frac{1}{S}\frac{dS}{dt} = -k + \lambda F$$

This gives

$$ln\left[\frac{S(t)}{S(t_0)}\right] = -k(t - t_0) + \lambda \int_{t_0}^{t} F(\tau)d\tau \tag{1.2.7}$$

In view of the information obtained from the linearized model, we can assume that both S and F are periodic of period $T = t_f - f_0$. That is, $S(t_f) = S(t_0)$ and $F(t_f) = F(t_0)$ and hence Eq. (1.2.7) gives

$$0 = -kT + \lambda \int_{t_0}^{t_0+T} F(\tau)d\tau$$

This implies that

$$\frac{1}{T}\left[\int_{t_0}^{t_0+T} F(\tau)d\tau\right] = k/\lambda \qquad (1.2.8(a))$$

and similarly

$$\frac{1}{T}\left[\int_{t_0}^{t_0+T} S(\tau)d\tau\right] = a/c \qquad (1.2.8(b))$$

So in predator-prey system, no matter what the solution trajectory is, the average population remains around the equilibrium point.
It is also interesting to analyse the men's influence on the above ecosystem if we fish both predator and prey. Then, we have

$$\begin{aligned} \frac{dF}{dt} &= F(a - cS) - \sigma_1 F \\ \frac{dS}{dt} &= S(-k + \lambda F) - \sigma_2 S \end{aligned}$$

This is equivalent to

$$\begin{aligned} \frac{dF}{dt} &= F(a' - cS) \\ \frac{dS}{dt} &= S(-k' + \lambda F) \\ a' = a - \sigma_1, \qquad k' &= k + \sigma_2 \end{aligned}$$

Hence it follows that the average population of predator is $\frac{a'}{c} = \frac{a - \sigma_1}{c}$ *(decreases) and that of prey is* $\frac{k'}{c} = \frac{k + \sigma_2}{c}$ *(increases).*

1.3 Higher Order ODE

Assume that $x(t)$ is a scalar valued $n-$times continuously differentiable function on an interval $I \subset \Re$. Let us denote the derivative $\frac{d^k x}{dt^k}$ by $x^{(k)}(t)$.

We shall be interested in the following higher order ordinary differential equation

$$x^{(n)}(t) = g(t, x(t), x^{(1)}(t), \ldots, x^{(n-1)}(t)) \qquad (1.3.1)$$

where $g : I \times \Re \times \Re \times \ldots \times \Re \to \Re$ is a given mapping. We define a vector $\overline{x}(t)$ with components $x_i(t), i = 1, \ldots n$ by

$$x_i(t) = x^{(i-1)}(t),\ x_1(t) = x(t);\ \ 2 \le i \le n \qquad (1.3.2)$$

and the vector field $f(t, \bar{x})$ by

$$f(t, \bar{x}) = (x_2, x_3, \ldots, g(t, x_1, \ldots, x_n))$$

Then the higher order ODE - Eq. (1.3.1) is equivalent to the following first order ODE

$$\frac{d\overline{x}}{dt} = f(t, \overline{x})$$

A linear higher order ODE has the form

$$x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_0(t)x(t) = b(t) \tag{1.3.3}$$

where $a_i(t)(0 \leq i \leq n-1)$ and $b(t)$ are given functions. Then in view of Eq. (1.3.2) - Eq. (1.3.3) we have

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2(t) \\
\frac{dx_2}{dt} &= x_3(t) \\
&\vdots \\
\frac{dx_n}{dt} &= b(t) - a_0(t)x_1(t) - a_1(t)x_2(t) - \cdots - a_{n-1}(t)x_n(t)
\end{aligned}$$

This is equivalent to the following first order system

$$\frac{d\overline{x}}{dt} = A(t)\overline{x}(t) + \overline{b}(t) \tag{1.3.4}$$

where

$$\begin{aligned}
A(t) &= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \\
\overline{b}(t) &= (0, 0, \ldots, b(t))
\end{aligned}$$

The matrix $A(t)$ is called the companion matrix.

Example 1.3.1 *(Mechanical Oscillations)*
A particle of mass m is attached by a spring to a fixed point as given in the following diagram.

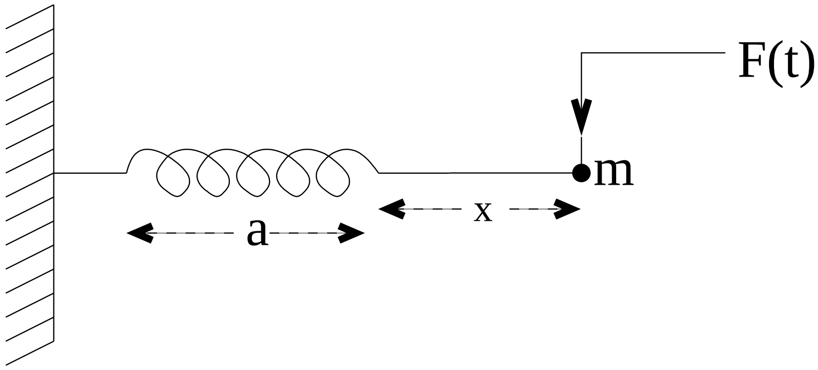

Figure 1.3.1: Mechanical oscillations

We assume that spring obeys Hook's law (tension is proportional to its extension) and resistance (damping) is proportional to the particle speed. The external force applied to the particle is $F(t)$. By equilibrium of forces we have

$$mF = m\frac{d^2x}{dt^2} + T + mk\frac{dx}{dt}$$

By Hook's Law, we get $T = \dfrac{\lambda x}{a}$ *and hence*

$$F = \frac{d^2x}{dt^2} + \frac{\lambda}{ma}x + k\frac{dx}{dt} \tag{1.3.5}$$

Equivalently, we get the following second order equation modelling the spring problem

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = F(t), \quad \omega^2 = \frac{\lambda}{ma} \tag{1.3.6}$$

Case 1: No resistance and no external force (Harmonic Oscillator)

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \quad \omega^2 = \frac{\lambda}{ma} \tag{1.3.7}$$

Eq. (1.3.7) is equivalent to the following system of first order differential equations

$$\frac{d\overline{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\dfrac{\lambda}{ma} & 0 \end{bmatrix} \overline{x} \tag{1.3.8}$$

It is easy to see the solution of the above system is given by (refer Section 4.4)

$$x(t) = x_0 \cos\omega(t-t_0) + \frac{x_0}{\omega}\sin\omega(t-t_0). \tag{1.3.9}$$

$$\dot{x}(t) = -x_0\omega\sin\omega(t-t_0) + x_0\cos\omega(t-t_0) \tag{1.3.10}$$

Equivalently, we have

$$x(t) = \alpha \cos(\omega t + \beta)$$

This is a simple harmonic motion with period $\dfrac{2\pi}{\omega}$ and amplitude α. ω is called its frequency.

We can easily draw plots of phase-space (normalized $\omega = 1$) and also solution curve of the ODE - Eq. (1.3.8).

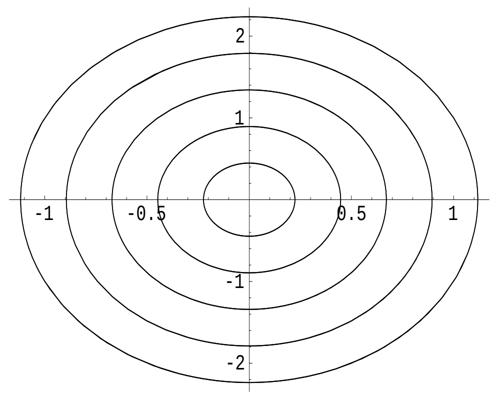

Figure 1.3.2: Phase space of the linear system

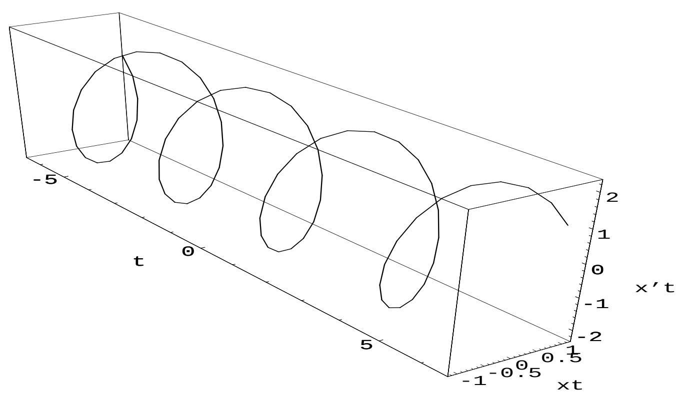

Figure 1.3.3: Solution curve of the linear system

Case 2: Solution with resistance

$$\frac{d\overline{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -k \end{bmatrix} \overline{x}(t) \tag{1.3.11}$$

(i) $k^2 - 4\omega^2 < 0$

The solution curve $x(t)$ is a damped oscillation given by (refer Section 4.4)

$$\begin{aligned} x(t) &= \exp(\frac{-kt}{2})(A\cos bt + B\sin bt) \\ b &= \frac{\sqrt{4\omega^2 - k^2}}{2} \end{aligned}$$

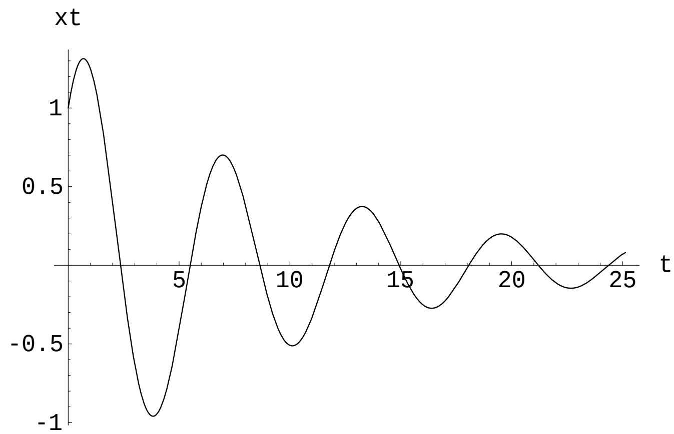

Figure 1.3.4: Solution curve for Case2(i)

The oscillation curves for the other cases are as under.
(ii) $(k^2 - 4\omega^2) = 0$

$$x(t) = A\exp(\frac{-kt}{2}) + Bt\exp(\frac{-kt}{2})$$

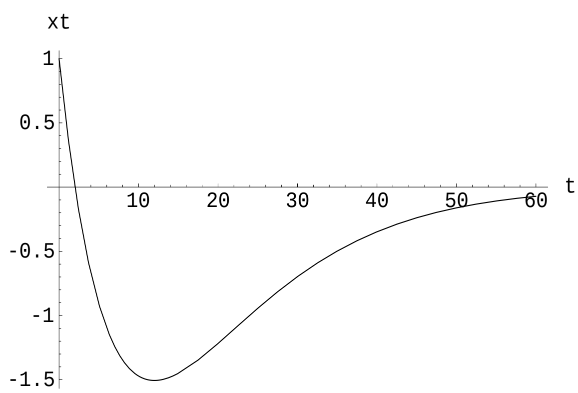

Figure 1.3.5: Solution surve for case2(ii)

(iii) $k^2 - 4\omega^2 > 0$

$$\begin{aligned} x(t) &= \exp(\frac{-kt}{2})\left[A_1 e^{ct} + A_2 e^{-ct}\right] \\ c &= \sqrt{k^2 - 4\omega^2} \end{aligned}$$

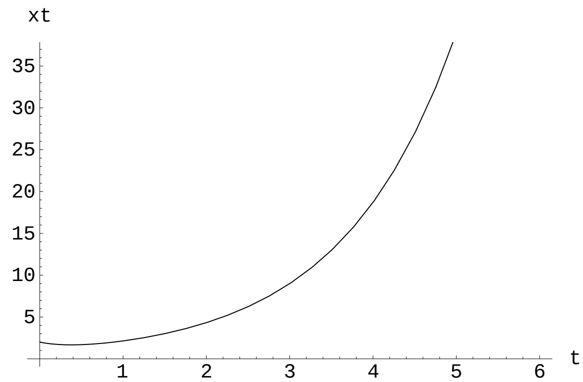

Figure 1.3.6: Solution curve for case2(iii)

Case 3: Effect of resistance with external force

$$\frac{d\overline{x}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -k \end{bmatrix} \overline{x} + \overline{F}(t), \overline{F}(t) = (0, F(t))$$

The solution $x(t)$ is given by

$$x = x_p + x_c$$

where x_c is the solution of the homogeneous equation with $F(t) = 0$. We have seen that $x_c \to 0$ as $t \to \infty$. A particular solution (response function) $x_p(t)$ of the above system is given by

$$x_p(t) = \frac{F_0}{D}\left[\cos\left(\beta t - \phi\right)\right]$$

corresponding to the external force (input function) $F(t) = F_0 \cos \beta t$.
The constant D is given by

$$D = \left[(\omega^2 - \beta^2)^2 + k^2\beta^2\right]^{1/2}$$

and

$$\sin\phi = \frac{k\beta}{D}, \cos\phi = \frac{\omega^2 - \beta^2}{D}$$

Thus the forced oscillations have the same time period as the applied force but with a phase change ϕ and modified amplitude

$$\frac{F_0}{D} = \frac{F_0}{[(\omega^2 - \beta^2)^2 + k^2\beta^2]^{1/2}}$$

Amplitude modification depends not only on the natural frequency and forcing frequency but also on the damping coefficient k.
As $k \to 0$ we have

$$x_p \to \frac{F_0}{(\omega^2 - \beta^2)} \to \infty \text{ as } \omega \to \beta.$$

For $k = 0$, the response is given by

$$x_p = \frac{F_0}{2\omega} t \sin wt$$

which implies that the system resonates with the same frequency but with rapidly increasing magnitude, as we see in the following graph.

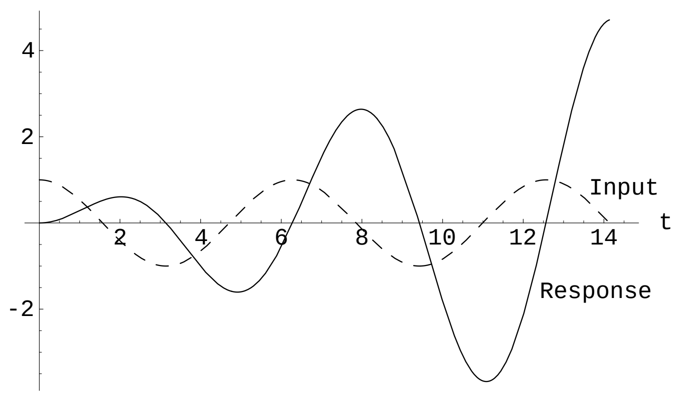

Figure 1.3.7: Input-Response curve

Example 1.3.2 *(Satellite Problem)*
A satellite can be thought of a mass orbiting around the earth under inverse square law field.

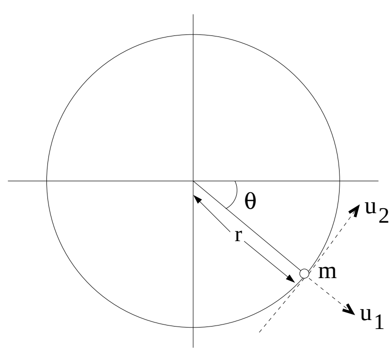

Figure 1.3.8: Satellite in orbit around the Earth

We assume that $m = 1$ and also the satellite has thrusting capacity with radial thrust u_1 and tangential thrust u_2.
Equating forces in normal and tangential direction on the orbiting satellite, we get

$$\begin{aligned}\left[\frac{d^2r}{dt^2} - r(t)(\frac{d\theta}{dt^2})^2\right] &= -\frac{k}{r^2(t)} + u_1(t) \\ \left[r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt}\right] &= u_2(t)\end{aligned}$$

This gives a pair of second order differential equations

$$\frac{d^2r}{dt^2} = r(t)(\frac{d\theta}{dt})^2 - \frac{k}{r^2(t)} + u_1(t) \tag{1.3.12a}$$

$$\frac{d^2\theta}{dt^2} = -\frac{2}{r(t)}\frac{d\theta}{dt}\frac{dr}{dt} + \frac{u_2(t)}{r(t)} \tag{1.3.12b}$$

] If $u_1 = 0 = u_2$, then one can show that Eq. (1.3.12) has a solution given by $r(t) = \sigma,\ \theta(t) = \omega t$ ($\sigma,\ \omega$ are constant and $\sigma^3\omega^2 = k$). Make the following change of variables:

$$x_1 = r - \sigma,\ x_2 = \dot{r},\ x_3 = \sigma(\theta - \omega t),\ x_4 = \sigma(\dot{\theta} - \omega)$$

This gives

$$r = x_1 + \sigma,\ \dot{r} = x_2,\ \theta = \frac{x_3}{\sigma} + \omega t,\ \dot{\theta} = \frac{x_4}{\sigma} + \omega$$

So Eq. (1.3.12) reduces to

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= (x_1 + \sigma)(\frac{x_4}{\sigma} + \omega)^2 - \frac{k}{(x_1 + \sigma)^2} + u_1 \\ \frac{dx_3}{dt} &= x_4 \\ \frac{dx_4}{dt} &= -2\sigma(\frac{x_4}{\sigma} + \omega)\frac{x_2}{(x_1 + \sigma)} + \frac{u_2\sigma}{(x_1 + \sigma)}\end{aligned} \tag{1.3.13}$$

Eq. (1.3.13) is a system of nonlinear ordinary diferential equations involving the forcing functions (controls) u_1 and u_2 and can be written in the compact form as

$$\frac{d\bar{x}}{dt} = f(\bar{x},\ \bar{u}),\ \overline{x}(t) \in \Re^4,\ \overline{u}(t) \in \Re^2 \tag{1.3.14}$$

Here f is a vector function with components f_1, f_2, f_3, f_4 given by

$$\begin{aligned} f_1(x_1,x_2,x_3,x_4;u_1,u_2) &= x_2 \\ f_1(x_1,x_2,x_3,x_4;u_1,u_2) &= (x_1+\sigma)(\frac{x_4}{\sigma}+\omega)^2 - \frac{k}{(x_1+\sigma)^2} + u_1 \\ f_1(x_1,x_2,x_3,x_4;u_1,u_2) &= x_4 \\ f_1(x_1,x_2,x_3,x_4;u_1,u_2) &= -2\sigma(\frac{x_4}{\sigma}+\omega)\frac{x_2}{(x_1+\sigma)} + \frac{u_2\sigma}{(x_1+\sigma)} \end{aligned}$$

We shall be interested in the solution, $\overline{x}(t) \in \Re^4$ of linearized equation corresponding to Eq. (1.3.14) in terms of the control vector $\overline{u}(t) \in \Re^2$.

1.4 Difference Equations

In some models (as we shall see subsequently), the state vector $\overline{x}(t)$ may not depend on t continuously. Rather, $\overline{x}(t)$ takes values at discrete set of points $\{t_1, t_2, \ldots, t_k, \cdots\}$
In such a situation we use difference quotients instead of differential quotients and that leads to difference equations. Suppose there exists a sequence of vector fields $f_i(\overline{x}) : \Omega \to \Re^n, \Omega \subseteq \Re^n$. Then the first order difference equation has the form

$$\overline{x}_{i+1} = f_i(\overline{x}_i), \bar{x}_i \in \Omega, i = 1, 2 \ldots \qquad (1.4.1(a))$$

If in addition

$$\overline{x}_0 = \overline{z}_0 \text{ (a given vector in } \Re^n) \qquad (1.4.1(b))$$

then Eq. (1.4.1(a)) - Eq. (1.4.1(b)) is called the IVP corresponding to a difference equation.
As for ODE, we can define orbit or trajectory and solution curve as discrete subsets of $\Re^n$ and $\Re^{n+1}$, respectively in an analogous way. A stationary point or equilibrium point of the difference equation Eq. (1.4.1) is a constant solution $\overline{x}$ such that

$$\overline{x} = f_i(\overline{x}), i \in \aleph$$

Eq. (1.4.1) is called linear if it has the form

$$\overline{x}_{i+1}(t) = A_i \overline{x}_i + \overline{b}_i, i \in \aleph$$

where $A_i \in \Re^{n \times n}$ and $\overline{b}_i \in \Re^n$.

For the scalar case $n = 1$, we have the linear difference equation

$$x_{i+1} = a_i x_i + b_i, i \in \aleph$$

Its solution is given by

$$x_i = (\Pi_{j=0}^{i-1}(a_j))x_0 + \sum_{j=0}^{i-1}(\Pi_{l=j+1}^{i} a_l) b_j$$

A k^{th} order difference equation in 1-dimension is given by

$$x_{i+1} = g_i(x_i, x_{i+1}, \ldots, x_{i+1-k}),\ i = k-1, k, \ldots$$

A linear k^{th} order difference equation is given by

$$x_{i+1} = \sum_{j=1}^{k} a_{ij} x_{i-j+1} + b_i.$$

This can be written as the first order system

$$\overline{x}_{i+1} = A_i \overline{x}_i + \overline{b}_i, \quad i \geq 0 \tag{1.4.2}$$

where

$$\bar{x}_i = \begin{bmatrix} x_i \\ \vdots \\ x_{i+k-1} \end{bmatrix}, \quad \bar{b}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{i+k} \end{bmatrix}$$

$$A_i = \begin{bmatrix} 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_{i+k-1,k} & a_{i+k-1,k-1} & \cdots & a_{i+k-1,1} \end{bmatrix}$$

For the homogeneous case $\overline{b}_i = 0$, with $A_i = A$ for all i, we get

$$\overline{x}_{i+1} = A \overline{x}_i \tag{1.4.3}$$

The solution of this system is of the form

$$\overline{x}_i = r^i \overline{c}$$

where scalar r and vector $\overline{c}$ are to be determined.
Plugging this representation in Eq. (1.4.3) we get

$$(A - rI)\overline{c} = 0 \tag{1.4.4}$$

This is an eigenvalue problem. If A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with linearly independent eigenvectors $\overline{c}_1, \ldots \overline{c}_n$, then the general solution of Eq. (1.4.3) is given by

$$\overline{x}_m = \sum_{i=1}^{n} d_i \lambda_i^m \overline{c}_i, \ \ m = 1, 2 \ldots \tag{1.4.5}$$

If the initial value is $\overline{x}_0$ then $\overline{d} = [d_1, \ldots, d_n]$ is given by $\overline{d} = C^{-1}(\overline{x}_0)$. Here $C = [\bar{c_1}, \bar{c_2}, \cdots, \bar{c_n}]$. So the solution to the IVP corresponding to Eq. (1.4.3) is completely given by Eq. (1.4.5).

Example 1.4.1 *(Discrete One Species Population Dynamics Model)*
We wish to measure population changes in a species with one year age distribution. Let $N_i(t)$ denote the number of i^{th} year old species and let b_i, d_i the corresponding birth and death rates, respectively $(0 \leq i \leq M)$. Then we have

$$\begin{aligned} N_0(t+\triangle t) &= b_0 N_0(t) + b_1 N_1(t) \\ &\quad + \ldots + b_M N_M(t), \ (\triangle t = 1) \qquad (1.4.6(a)) \\ N_{i+1}(t+\triangle t) &= (1-d_i)N_i(t), \ 0 \leq i \leq M-1 \qquad (1.4.6(b)) \end{aligned}$$

Let $\overline{N}(t) = (N_0(t), N_1(t), \ldots, N_M(t))$. Then Eq. (1.4.6) becomes

$$\overline{N}(t+\triangle t) = A\overline{N}(t) \qquad (1.4.7)$$

where

$$A = \begin{bmatrix} b_0 & b_1 & \cdots & b_M \\ 1-d_0 & 0 & \cdots & 0 \\ 0 & 1-d_1 & & \cdots \\ \vdots & & \vdots & \cdots \\ 0 & 0 & \cdots & 1-d_{M-1} \end{bmatrix}$$

If we denote $\overline{N}_m = \overline{N}(m\triangle t)$, then Eq. (1.4.7) can be viewed as a difference equation of the form

$$\overline{N}_{m+1} = A\overline{N}_m \qquad (1.4.8)$$

This is of the type given by Eq. (1.4.3) and hence the solution of the above equation is of the form

$$\overline{N}_m = \sum_{i=1}^{M} a_i \lambda_i^m \overline{\phi}_i$$

where $\{\lambda_i\}_{i=1}^M$ and $\{\overline{\phi}_i\}_{i=1}^M$ are eigenvalues (distinct) and linearly independent eigenvectors of A, respectively. $\overline{a} = [a_1, \ldots, a_m]$ is a constant vector. If the initial population $\overline{N}_0$ is known, the constant vector $\overline{a}$ is given by $\overline{a} = C\overline{N}_0$ where the nonsingular matrix C is given by

$$C = [\overline{\phi}_1, \overline{\phi}_2, \ldots, \overline{\phi}_M]$$

It is to be noticed that the population of each age group grows and decays depending upon the sign of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$.

1.5 Discretization

In this section we shall describe some well-known one-step numerical methods, which are used to solve ODE. However, we shall not be concerned with the stability and convergence aspects of these methods.
We also introduce to reader the concept of neural solution of ODE. This is also based on discretization but uses neural network approach (refer Haykin [6]).

It is interesting to observe that sometimes neural solution is a better way of approximating the solution of ODE.
Let us begin with scalar ordinary differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1.5.1}$$

corresponding to the scalar field $f(t, x) : I \times \Re \to \Re$.
Let $t_0, t_1, \ldots, t_N$ be a set of time points (called grid-points) wherein we would like to approximate the solution values $x(t_i), 0 \leq i \leq N$.
Integrating the above equation on $[t_i, t_i + 1]$ we get

$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(s, x(s))ds$$

If f is approximated by its values at t_i, we get the Euler's forward method

$$x(t_{i+1}) = x(t_i) + hf(t_i, x(t_i)), \ h = x_{i+1} - x_i \tag{1.5.2}$$

which is the so-called explicit, one-step method.
On the other hand, if f is approximated by its values at t_{i+1} we get Euler's backward method

$$x(t_{i+1}) = x(t_i) + hf(t_{i+1}, x(t_{i+1})) \tag{1.5.3}$$

which has to be solved implicitly for $x_{t_{i+1}}$.
If the integral $\int_{t_i}^{t_{i+1}} f(s, x(s))ds$ is approximated by the trapezoidal rule, we get

$$x(t_{i+1}) = x(t_i) + \frac{1}{2}h\left[f(t_i, x(t_i)) + f(t_{i+1}, x(t_{i+1}))\right] \tag{1.5.4}$$

which is again implicit.
Combining Eq. (1.5.2) and Eq. (1.5.4) to eliminate $x(t_{i+1})$, we get the Heun's method (refer Mattheij and Molenaar[9])

$$x(t_{i+1}) = x(t_i) + \frac{1}{2}h\left[f(t_i, x(t_i)) + f(t_{i+1}, x(t_i) + hf(t_i, x(t_i)))\right] \tag{1.5.5}$$

An important class of one-step numerical methods is Runge-Kutta method. Consider the integral equation

$$x(T) = x(t) + \int_t^T f(s, x(s))ds \tag{1.5.6}$$

Approximating the integral $\int_{t_i}^{t_{i+1}} f(s, x(s))ds$ by a general quadrature formula $\sum_{j=1}^{m} \beta_j f(t_{ij}, x(t_{ij}))$, we get

$$x(t_{i+1}) = x(t_i) + h\sum_{j=1}^{m} \beta_j f(t_{ij}, x(t_{ij})) \tag{1.5.7}$$

where $h = t_{i+1} - t_i$ and $t_{ij} = t_i + \rho_j h (0 \le \rho_j \le 1)$ are nodes on the interval $[t_i, t_{i+1}]$.
As $x(t_{ij})$ are unknowns, to find them we apply another quadrature formula for the integral $\int_{t_i}^{t_{ij}} f(s, x(s)) ds$ to get

$$x(t_{ij}) = x(t_i) + h \sum_{l=1}^{m} r_{jl} f(t_{il}, x(t_{il})) \tag{1.5.8}$$

Combining Eq. (1.5.7) and Eq. (1.5.8), we get the Runge-Kutta formula

$$\begin{aligned} x(t_{i+1}) &= x(t_i) + h \sum_{j=1}^{m} \beta_j k_j \qquad (1.5.9) \\ k_j &= f(t_i + \rho_j h, x(t_i) + h \sum_{l=1}^{m} r_{jl} k_l) \end{aligned}$$

Eq. (1.5.9) is explicit if $r_{jl} = 0, l \ge j$. If $\rho_1 = 0, \rho_2 = 1, r_{11} = 0 = r_{12}, r_{21} = 1, r_{22} = 0$ and $\beta_1 = 1/2 = \beta_2$, we get the Heun's formula given by Eq. (1.5.5). Also, it is easy to get the following classical Runge-Kutta method from Eq. (1.5.9)

$$x(t_{i+1}) = x(t_i) + h \left[\frac{1}{6} k_1 + \frac{1}{3} k_2 + \frac{1}{3} k_3 + \frac{1}{6} k_4 \right] \tag{1.5.10}$$

$$\begin{aligned} k_1 &= f(t_i, x(t_i)) \\ k_2 &= f(t_i + \frac{1}{2} h, x(t_i) + \frac{1}{2} h k_1) \\ k_3 &= f(t_i + \frac{1}{2} h, x(t_i) + \frac{1}{2} h k_2) \\ k_4 &= f(t_i + h, x(t_i) + h k_3) \end{aligned}$$

In case we have $f(t, \overline{x})$ as a vector field, we obtain a similar Runge - Kutta method, with scalars $x(t_i)$, $x(t_i + 1)$ being replaced by vectors $\overline{x}(t_i)$, $\overline{x}(t_{i+1})$, and scalars k_i being replaced by vectors $\overline{k}_i (1 \le i \le 4)$.

We now briefly discuss the concept of neural solution of the IVP associated with Eq. (1.5.1). A neural solution (refer Logaris et al [8]) $x_N(t)$ can be written as

$$x_N(t) = x_0 + t N(t, \overline{W}) \tag{1.5.11}$$

where $N(t, \overline{W})$ is the output of a feed forward neural network (refer Haykin [6]) with one input unit t and network weight vector $\overline{W}$. For a given input t, output of the network is $N = \sum_{i=1}^{k} v_i \sigma(z_i)$ where $z_i = w_i t - u_i$, w_i denotes the weight from the input unit t to the hidden unit i, v_i denotes the weight from

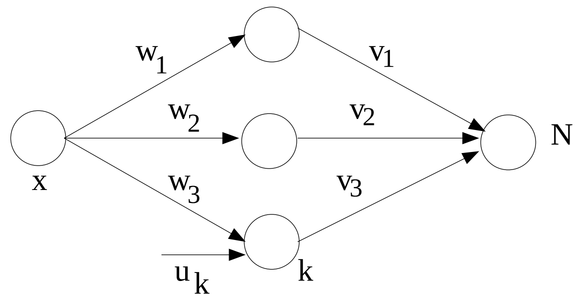

Figure 1.5.1: Neural network with one input

the hidden unit i to the output and u_i denotes the bias of the unit i, $\sigma(z)$ is the sigmoid transfer function

$$\sigma(z) = \frac{1}{1+e^{-z}}$$

and k is the number of hidden neurons.

We first determine the network parameters w_i, v_i and u_i in such a manner that $x_N(t)$ satisfies Eq. (1.5.1) in some sense. For this we discretize the interval $[t_0, t_f]$ as $t_0 < t_i < \ldots < t_m = t_f$.
As $x_N(t)$ is assumed to be a trial solution, it may not be exactly equal to $f(t_j, x_N(t_j))$ at point t_j, $0 \le j \le m$. Hence we train the parameter vector $\overline{W}$ in such a way that it minimizes the error function $\phi(\overline{W})$ given as

$$\phi(\overline{W}) = \sum_{j=0}^{m} \left(\frac{dx_N}{dt}|_{t=t_j} - f(t_j, x_N(t_j)) \right)^2$$

One can use any unconstrained optimization algorithms (refer Joshi and Kannan [7]) to minimize ϕ w.r.t. the neural network parameters to obtain their optimal values u_i^*, v_i^*, w_i^* to get the neutral solution $x_N(t)$ given by Eq. (1.5.11)

This methodology will be clear from the following example, wherein we compute numerical solutions by Euler's and Runge-Kutta methods and also neural solution. Further, we compare these solutions with the exact solution.

Example 1.5.1 *Solve the IVP*

$$\begin{aligned} \frac{dx}{dt} &= x - x^2, t \in [0,1] \qquad (1.5.12) \\ x(0) &= \frac{1}{2} \end{aligned}$$

The exact solution of this ODE (Bernoulli's differential equation) is given by

$$x(t) = \frac{1}{1+e^{-t}} \qquad (1.5.13)$$

(Refer Section 6 of this chapter).

For neural solution we take three data points $t_j = 0, 0.5, 1$ and three hidden neurons with a single hidden layer network. The neural solution is of the form

$$x_N(t) = \frac{1}{2} + t \sum_{i=1}^{3} \frac{v_i}{\left[1 + e^{(-w_i t + u_i)}\right]} \tag{1.5.14}$$

The weight vector $\overline{W} = (w_1, w_2, w_3, v_1, v_2, v_3, u_1, u_2, u_3)$ is computed in such a way that we minimize $\phi(\overline{W})$ given by

$$\begin{aligned}
\phi(\overline{W}) &= \sum_{j=1}^{3} \left[\frac{dx_N(t)}{dt}|_{t=t_j} - f(t_j, x_N(t_j)) \right]^2 \\
&= \sum_{j=1}^{3} \left[\left(\frac{1}{2} + \frac{v_1}{1+e^{u_1 - w_1 t_j}} + \frac{v_2}{1+e^{u_2 - w_2 t_j}} + \frac{v_3}{1+e^{u_3 - w_3 t_j}} \right) \right. \\
&\quad - \left(\frac{v_1}{1+e^{u_1 - w_1 t_j}} + \frac{v_2}{1+e^{u_2 - w_2 t_j}} + \frac{v_3}{1+e^{u_3 - w_3 t_j}} \right) t_j \\
&\quad + \left(\frac{e^{u_1 - w_1 t_j} v_1 w_1}{(1+e^{u_1 - w_1 t_j})^2} + \frac{e^{u_2 - w_2 t_j} v_2 w_2}{(1+e^{u_2 - w_2 t_j})^2} + \frac{e^{u_3 - w_3 t_j} v_3 w_3}{(1+e^{u_3 - w_3 t_j})^2} \right) t_j \\
&\quad \left. + \left(\frac{1}{2} + \left(\frac{v_1}{1+e^{u_1 - w_1 t_j}} + \frac{v_2}{1+e^{u_2 - w_2 t_j}} + \frac{v_3}{1+e^{u_3 - v_3 t_j}} \right) t_j \right)^2 \right]^2
\end{aligned}$$

We have used the steepest descent algorithm (refer to Algorithm 2.3.1) to compute the optimal weight vector $\overline{W}$. The neural solution is given by Eq. (1.5.14). We note that the neural solution is a continuous one. We can compare the values of this solution at a discrete set of points with the ones obtained by Euler and Runge-Kutta method (refer Conte and deBoor [3]). The following tables give the distinction between the three types of approximate solutions.

Table 1.5.1: Euler solution and actual solution

Euler's method	Actual solution	Absolute difference
0.525,	0.52498	0.00002
0.54994	0.54983	0.00010
0.57469	0.57444	0.00025
0.59913	0.59869	0.00044
0.62315	0.62246	0.00069
0.64663	0.64566	0.00097
0.66948	0.66819	0.00129
0.69161	0.68997	0.00163
0.71294	0.71095	0.00199
0.73340	0.73106	0.00234

Table 1.5.2: Runge-Kutta solution and actual solution

Runge Kutta method	Actual solution	Absolute difference
0.52498	0.52498	1.30339^{-9}
0.54983	0.54983	2.64162^{-9}
0.57444	0.57444	4.05919^{-9}
0.59869	0.59869	5.59648^{-9}
0.62246	0.62246	7.28708^{-9}
0.64566	0.64566	9.15537^{-9}
0.66819	0.66819	1.12147^{-8}
0.68997	0.68997	1.34665^{-8}
0.71095	0.71095	1.58997^{-8}
0.73106	0.73106	1.84916^{-8}

Table 1.5.3: Neural solution and actual solution

Neural Network sol	Actual sol	Absolute Difference
0.52511	0.52498	0.00013
0.54973	0.54983	0.00011
0.57383	0.57444	0.00062
0.59741	0.59869	0.00127
0.62048	0.62246	0.00198
0.64302	0.64566	0.00264
0.66503	0.66819	0.00316
0.68650	0.68997	0.00348
0.70743	0.71095	0.00352
0.72783	0.73106	0.003231

1.6 Techniques for Solving First Order Equations

The general form of a first order linear differential equation is

$$a(t)\frac{dx}{dt} + b(t)x + c(t) = 0 \qquad (1.6.1)$$

where $a(t)$, $b(t)$ and $c(t)$ are continuous functions in a given interval with $a(t) \neq 0$. Dividing by $a(t)$, we get the equivalent equation in normal form

$$\frac{dx}{dt} + P(t)x = Q(t) \qquad (1.6.2)$$

This equation is solved by multiplying Eq. (1.6.2) by $e^{\int P(t)dt}$ and integrating, to give us the solution

$$x(t) = e^{-\int P(t)dt}\left[\int e^{\int P(t)dt}Q(t)dt + c\right] \tag{1.6.3}$$

Exact Equations

The differential equation

$$M(t,x)dt + N(t,x)dx = 0 \tag{1.6.4}$$

or

$$M(t,x) + N(t,x)\frac{dx}{dt} = 0 = M(t,x)\frac{dt}{dx} + N(t,x)$$

is called exact if there is a differentiable function $u(t,x)$ such that

$$\frac{\partial u}{\partial t} = M, \quad \frac{\partial u}{\partial x} = N \tag{1.6.5}$$

If Eq. (1.6.5) holds, we have

$$\begin{aligned} M(t,x)dt + N(t,x)dx &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx \\ &= d(u(t,x)) \end{aligned}$$

Hence, intergrating the above wquation we get the solution

$$u(t,x) = C$$

of the exact equation given by Eq. (1.6.4).

The following theorem gives a test for the exactness of a differential equation.

Theorem 1.6.1 *If M and N are continuously differentiable function of $(t,x) \in D \subseteq \Re^2$ (with no hole in D), then the differential equation*

$$M(t,x)dt + N(t,x)dx = 0 \ \textit{ is exact iff } \ \frac{\partial M}{\partial x} = \frac{\partial N}{\partial t} \ \textit{ in } \ D \tag{1.6.6}$$

Proof : If the differential equation is exact, we have $\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}\right]$ and $\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x}\right]$. Since M and N are continuously differentiable in D, it follows that $\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x}$ and hence $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$ in D.

Conversly, let Eq. (1.6.6) holds.

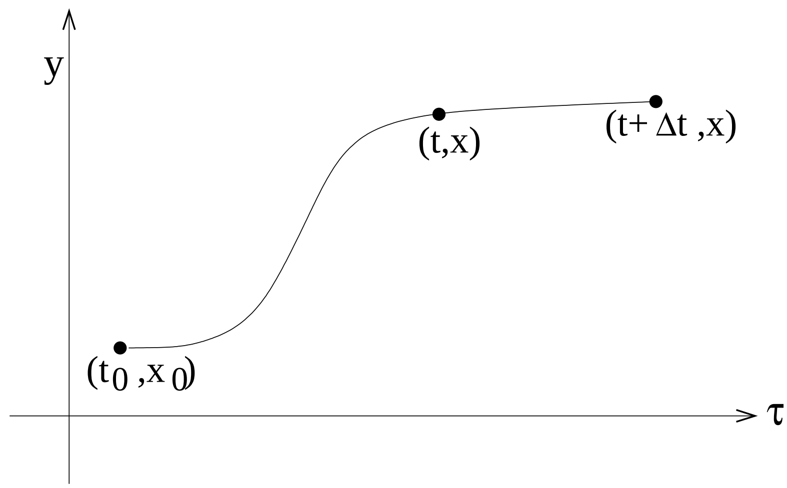

Figure 1.6.1: A curve in $\tau - y$ plane

We explicitily define the function $u(t,x)$ as the solution of the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= M(t,x), \quad \frac{\partial u}{\partial x} = N(t,x) \\ u(t_0, x_0) &= u_0 \end{aligned}$$

Then, $u(t,x)$ is given by

$$u(t,x) = u_0 + \int_{(t_0,x_0)}^{(t,x)} [M(\tau,y)d\tau + N(\tau,y)dy]$$

where the integral on the RHS is a line integral along a path joining (t_0, x_0) and (t,x). It can be shown that this line integral is independent of path if Eq. (1.6.6) holds.
This gives

$$\begin{aligned} u(t+\triangle t, x) - u(t,x) &= \int_{(t,x)}^{(t+\triangle t,x)} [M(\tau,y)d\tau + N(\tau,y)dy] \\ &= \int_t^{t+\triangle t} M(\tau,y)d\tau \ \ (\text{as } \ y = x \text{ and } dy = 0) \\ &= M(\tau_1, x)\triangle t, \quad t < \tau_1 < t + \triangle t \end{aligned}$$

(by meanvalue theorem for integrals).
Hence

$$\frac{\partial u}{\partial t} = \lim_{\triangle t \to 0} \left[\frac{u(t+\triangle t, x) - u(t,x)}{\triangle t} \right] = \lim_{\triangle t \to 0} [M(\tau_1, x)] = M(t,x)$$

Similarly, we have

$$\frac{\partial u}{\partial x} = N(t, x)$$

▮

It may happen that the equation $M(t,x)dt + N(t,x)dx = 0$ is not exact but after multiplying both sides of this equation by a function $\mu(t,x)$, it becomes exact. Such a function is called an integrating factor. For example, $xdy - ydx = (x^2+y^2)dx$ is not exact. But dividing by (x^2+y^2), we get

$$\frac{xdy - ydx}{x^2+y^2} = dx$$

which is equivalent to

$$\frac{\frac{x}{x^2}dy - \frac{y}{x^2}dx}{1 + \frac{y^2}{x^2}} = dx$$

This is now an exact differential equation with solution

$$tan^{-1}(\frac{y}{x}) = x + c$$

Change of Variables

There are a few common types of equations, wherein substitutions suggest themselves, as we see below

[A]

$$\frac{dx}{dt} = f(at + bx), \quad a, b \in \Re \tag{1.6.7}$$

We introduce the change of variable $X = at + bx$, which gives

$$x = \frac{1}{b}(X - at), \quad \frac{dx}{dt} = \frac{1}{b}(\frac{dX}{dt} - a)$$

and hence Eq. (1.6.7) becomes

$$\frac{1}{b}(\frac{dX}{dt} - a) = f(X)$$
$$\text{or} \quad \frac{dX}{dt} = a + bf(X)$$

which can be easily solved.

[B]

$$\frac{dx}{dt} = f(\frac{x}{t}) \tag{1.6.8}$$

in which RHS depends only on the ratio $\frac{x}{t}$.

Introduce the change of variable $u = \frac{x}{t}$. Thus

$$x = ut, \qquad \frac{dx}{dt} = u + t\frac{du}{dt}$$

Then Eq. (1.6.8) becomes

$$u + t\frac{du}{dt} = f(u)$$

or

$$\frac{du}{dt} = \frac{f(u) - u}{t}$$

which can be easily solved.

[C] Consider the equation

$$\frac{dx}{dt} = f\left(\frac{at + bx + p}{ct + dx + q}\right), \quad a, b, c, d, p, q, \in \Re \tag{1.6.9}$$

in which RHS depends only on the ratio of the linear expression.

We substitute $T = t - h,\ X = x - k$.

Then $\frac{dx}{dt} = \frac{dX}{dt}$ where we choose h and k such that

$$ah + bk + p = 0 = ch + dk + q \tag{1.6.10}$$

Then $\frac{dX}{dT} = f\left(\frac{at + bX}{ct + dX}\right)$ which is of type **[B]**.

Eq. (1.6.10) can always be solved for h, k except when $ad - bc = 0$. In that case we have $cx + dy = m(ax + by)$ and hence Eq. (1.6.9) becomes

$$\frac{dx}{dt} = f\left[\frac{at + bx + p}{m(at + bx) + q}\right]$$

which is of the type **[A]**.

[D] The equation

$$\frac{dx}{dt} + P(t)x = Q(t)x^m \tag{1.6.11}$$

in which the exponent is not necessarily an integer, is called Bernoulli's equation. Assume that $m \neq 1$. Then introduce the change of variable

$X = x^{1-m}$. Then $x = X^{(\frac{1}{1-m})}$, $\frac{dx}{dt} = \frac{1}{1-m} X^{(\frac{m}{1-m})} \frac{dX}{dt}$. Hence Eq. (1.6.11) becomes

$$\frac{1}{1-m} X^{(\frac{m}{1-m})} \frac{dX}{dt} + P(t) X^{(\frac{1}{1-m})} = Q(t) X^{(\frac{m}{1-m})}$$

Equivalently

$$\frac{dX}{dt} + (1-m)P(t)X = (1-m)Q(t)$$

which is a linear differential equation in X.

Example 1.6.1 *Solve*

$$(t + x^2)dx + (x - t^2)dt = 0$$
$$M(t,x) = x - t^2, \quad N(t,x) = t + x^2$$
$$\frac{\partial M}{\partial x} = 1 = \frac{\partial N}{\partial t}$$

and hence this equation is exact.
So we write the solution $u(t,x)$ as

$$u(t,x) = u_0 + \int_{(t_0,x_0)}^{(t,x)} \left[(\tau - y^2)d\tau + (\tau + y^2)dy\right]$$

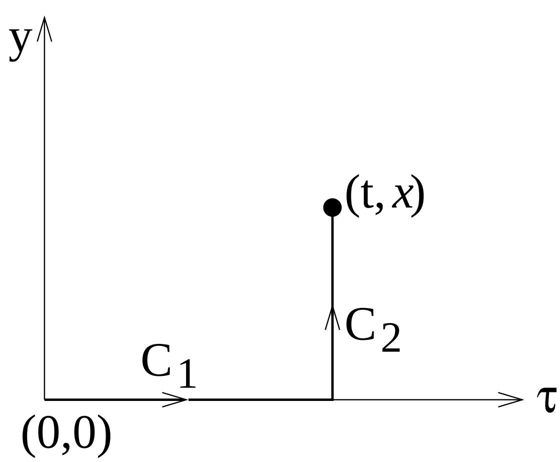

Figure 1.6.2: Path from $(0,0)$ to (t,x)

In the integrand on RHS, we go along the path C_1, C_2. Hence the integral become

$$\int_0^t (-\tau^2)d\tau + \int_0^x (t + y^2)dy = -\frac{t^3}{3} + tx + \frac{x^3}{3}$$

Hence, the solution is

$$\frac{-t^2}{3} + tx + \frac{x^3}{3} = C$$

Example 1.6.2 *Solve* $\dfrac{dx}{dt} = \dfrac{t+x-1}{t+4x+2}$

Solve $h+k-1 = 0 = h+4k+2$. *This gives* $h = 2, k = -1$. *Hence* $t = T+2, x = X-1$ *and* $\dfrac{dX}{dT} = \dfrac{T+X}{T+4X}$.

We now make substitution $X = TU$, *to get*

$$T\frac{dU}{dT} = \frac{1-4U^2}{1+4U}$$

which is equivalent to

$$\frac{1+4U}{1-4U^2}dU = \frac{dT}{T}$$

This yields

$$\begin{aligned} (1+2U)(1-2U)^3T^4 &= C \\ \text{or} \qquad (T+2X)(T-2X)^3 &= C \\ \text{or} \qquad (t+2x)(t-2x-4)^3 &= C \end{aligned}$$

Example 1.6.3 *Solve* $\dfrac{dx}{dt} + tx = t^3x^4$

This is a Bernoulli's equation. We make the change of variable $X = x^{-3}$ *and get an equivalent linear differential equation*

$$\frac{dx}{dt} - 3tX = -3t^3$$

The general solution of this equation is given by

$$\begin{aligned} X(t) &= e^{3\int tdt}\left[\int (3t^3)e^{-\int 3tdt} + C\right] \\ &= e^{\frac{3}{2}t^2}\left[\int (-3t^3)e^{-\frac{3}{2}t^2}dt + c\right] \end{aligned}$$

To compute $I = \int (-3t^3)e^{-\frac{3}{2}t^2}dt + c$*, put* $t^2 = u,\ 2tdt = du$*. This gives*

$$\begin{aligned} I &= \int e^{-\frac{3}{2}u}\left[-\frac{3}{2}udu\right] \\ &= -\frac{3}{2}\left[-\frac{2}{3}e^{-\frac{3}{2}u}u + \frac{2}{3}\int e^{-\frac{3}{2}u}du\right] \\ &= -\frac{3}{2}\left[-\frac{2}{3}e^{-\frac{3}{2}u}u - \frac{2}{3}\frac{2}{3}e^{-\frac{3}{2}u}\right] \\ &= e^{-\frac{3}{2}u}u + \frac{2}{3}e^{-\frac{3}{2}u} \\ &= e^{-\frac{3}{2}t^2}t^2 + \frac{2}{3}e^{-\frac{3}{2}t^2} \end{aligned}$$

and hence

$$\begin{aligned} X(t) &= e^{-\frac{3t^2}{2}}\left[e^{-\frac{3}{2}t^2}t^2 + \frac{2}{3}e^{-\frac{3}{2}t^2+c}\right] \\ &= \left[t^2 + \frac{2}{3} + ce^{\frac{3}{2}t^2}\right] \end{aligned}$$

Equivalently

$$x = \frac{1}{X^3} = \frac{1}{\left[t^2 + \frac{2}{3} + ce^{\frac{3}{2}t^2}\right]^3}$$

Example 1.6.4 *Solve the IVP*

$$\frac{dx}{dt} = x - x^2, \quad x(0) = \frac{1}{2}$$

We make the change of variable $X = \dfrac{1}{x}$ *and get an equivalent linear equation*

$$\frac{dX}{dt} + X = 1, \quad X(0) = 2$$

The unique solution $X(t)$ *is given by*

$$\begin{aligned} X(t) &= e^{-t}\left[\int e^t dt + t\right] \\ &= e^{-t} + 1 \end{aligned}$$

This gives

$$x(t) = \frac{1}{1 + e^{-t}}$$

For more on techniques for solving first order differential equation refer Golomb and Shanks[4].

For more on real life problems giving rise to mathematical models generated by ordinary differential equations, refer to Braun et al[1], Burghe and Borrie[2] and Haberman[5].