Chapter 6

Series Solution

This chapter lays emphasis on the classical theory of series solution for ODE by using power series expansion around ordinary and singular points.
We mainly focus on Legendre, Hermite and Bessel differential equations. We also deduce various interesting properties of Legendre and Hermite polynomials as well as Bessel functions.

6.1 Preliminaries

A power series in powers of $(t - t_0)$ is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k (t - t_0)^k = c_0 + c_1(t - t_0) + c_2(t - t_0)^2 + \cdots \tag{6.1.1}$$

where t is a variable and $c_0, c_1, c_2, \cdots$ are constants. Recall that a series $\sum_{k=0}^{\infty} c_k (t - t_0)^k$ is convergent at a point t if the limit of the partial sums $s_n(t) = \sum_{k=0}^{n} c_k (t - t_0)^k$ exists. This limit $f(t)$ is denoted as the sum of the series at the point t. A series which does not converge is said to be a divergent series.

Example 6.1.1 *Consider the geometric power series* $\sum_{k=0}^{\infty} t^k = 1 + t + t^2 + \cdots$
The partial sums $s_n(t) = 1 + t + \cdots + t^n$ *satisfy the relation*

$$ts_n(t) = t + t^2 + \cdots + t^n + t^{n+1}$$

and hence $s_n(t) = \dfrac{1 - t^{n+1}}{1 - t}$.

This gives $f(t) = \lim s_n(t) = \dfrac{1}{1 - t}$ *for* $|t| < 1$.

Definition 6.1.1 *A power series* $\sum_{k=0}^{\infty} c_k(t-t_0)^k$ *is said to converge absolutely at a point t if the series* $\sum_{k=0}^{\infty} |c_k(t-t_0)^k|$ *converges.*

One can show that if the series converges absolutely, then it also converges. However, the converse is not true.

We have the following tests for checking the convergence or divergence of a series of real numbers.

(i) **Comparision test**

(a) Let a series $\sum_{k=0}^{\infty} a_k$ of real numbers be given and let there exists a convergent series $\sum_{k=0}^{\infty} b_k$ of nonnegative real numbers such that

$$|a_k| \leq b_k, \qquad k \geq 1$$

Then the orginal series $\sum_{k=0}^{\infty} a_k$ converges.

(b) Let a series $\sum_{k=0}^{\infty} a_k$ of real numbers be given and let there exists a divergent series $\sum_{k=0}^{\infty} d_k$ of nonnegative real numbers such that

$$|a_k| \geq d_k, \qquad k \geq 1$$

Then the orginal series $\sum_{k=0}^{\infty} a_k$ diverges.

(ii) **Ratio test**

Let the series $\sum_{k=0}^{\infty} a_k($ with $a_n \neq 0)$ be such that $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L.$

(a) The series $\sum_{k=0}^{\infty} a_k$ converges absolutely if $L < 1$.

(b) The series $\sum_{k=0}^{\infty} a_k$ diverges if $L > 1$.

(c) No conclusion if $L = 1$.

(iii) **Root test**

If the series $\sum_{k=0}^{\infty} a_k$ is such that $\lim_{k \to \infty} \left((|a_k|)^{\frac{1}{k}} \right) = L$.

(a) The series $\sum_{k=0}^{\infty} a_k$ converges absolutely of $L < 1$.

(b) The series $\sum_{k=0}^{\infty} a_k$ diverges if $L > 1$.

(c) No conclusion if $L = 1$.

Theorem 6.1.1 *If the power series given by Eq. (6.1.1) converges at a point $t = t_1$, then it converges absolutely for every t for which $|t - t_0| < |t_1 - t_0|$.*

Proof : Since the series given by Eq. (6.1.1) converges for $t = t_1$, it follows that the partial sums $s_n(t_1)$ converge and hence $s_n(t_1)$ is Cauchy. This implies that

$$s_{n+1}(t_1) - s_n(t_1) \to 0 \text{ as } n \to \infty$$

This, in turn, implies that $c_k(t_1 - t_0)^k \longrightarrow 0$ as $k \to \infty$. Hence the elements $c_k(t_1 - t_0)^k$ of the series given by Eq. (6.1.1) are bounded at $t = t_1$. That is,

$$|c_k(t_1 - t_0)^k| \leq M \text{ for all } k \geq 1 \tag{6.1.2}$$

Eq. (6.1.2) implies that

$$\begin{aligned} |c_k(t - t_0)^k| &= \left| c_k(t_1 - t_0)^k \left(\frac{t - t_0}{t_1 - t_0} \right)^k \right| \\ &\leq M \left| \frac{t - t_0}{t_1 - t_0} \right|^k \end{aligned} \tag{6.1.3}$$

If $|t - t_0| < |t_1 - t_0|$, the series $\sum_{k=0}^{\infty} \left| \frac{t - t_0}{t_1 - t_0} \right|^k$ converges and hence by comparision test the series $\sum_{k=0}^{\infty} c_k(t - t_0)^k$ converges absolutely. ∎

Definition 6.1.2 *If the series $\sum_{k=0}^{\infty} c_k(t - t_0)^k$ converges absolutely for $|t - t_0| < r$ and diverges for $|t - t_0| > r$, then r is called the radius of convergence.*

The power series $\sum_{k=1}^{\infty} \frac{t^k}{k}$ converges for $|t| < 1$ and diverges for $|t| > 1$. At $t = 1$, it diverges and $t = -1$, it converges. Thus, the radius of convergence of this series is 1.

The radius of convergence of the power series given by Eq. (6.1.1) may be determined from the coefficients of the series as follows.

Theorem 6.1.2 *(Radius of convergence)*

(i) Suppose that the sequence $\{|c_k|^{\frac{1}{k}}\}$ *converges . Let* L *denotes its limit. Then, if* $L \neq 0$, *the radius of convergence* r *of the power series is* $\frac{1}{L}$.

(ii) If $L = 0$, *then* $r = \infty$ *and hence the series Eq. (6.1.1) converges for all* t.

(iii) If $\{|c_k|^{\frac{1}{k}}\}$ *does not converge, but it is bounded, then* $r = \frac{1}{l}$ *where* $l = sup\{|c_k|^{\frac{1}{k}}\}$. *If this sequence is not bounded, then* $r = 0$ *and hence the series is convergent only for* $t = t_0$.

Let $\sum_{k=0}^{\infty} c_k(t-t_0)^k$ be a power series with non zero radius of convergence r. Then the sum of the series is a function $f(t)$ of t and we write

$$f(t) = \sum_{k=0}^{\infty} c_k(t-t_0)^k \tag{6.1.4}$$

One can easily show the following.

(i) The function $f(t)$ in Eq. (6.1.4) is continuous at $t = t_0$.

(ii) The same function $f(t)$ can not be represented by two different power series with the same centre. That is, if

$$f(t) = \sum_{k=0}^{\infty} c_k(t-t_0)^k = \sum_{k=0}^{\infty} d_k(t-t_0)^k$$

in a disk: $|t - t_0| < r$, then $c_k = d_k$ for all $k \geq 0$.

We can carry out the standard operations on power series with ease - addition and subtraction, multiplication, term by term differentiation and integration.

(i) **Addition and Subtraction**

Two power series $\sum_{k=0}^{\infty} c_k(t-t_0)^k$ and $\sum_{k=0}^{\infty} d_k(t-t_0)^k$ can be added and subtracted in the common radius of convergence.

$$\begin{aligned} \text{If } f(t) &= \sum_{k=0}^{\infty} c_k(t-t_0)^k \text{ in } |t-t_0| < r_1 \quad \text{and} \\ g(t) &= \sum_{k=0}^{\infty} d_k(t-t_0)^k \text{ in } |t-t_0| < r_2 \end{aligned}$$

Then $f(t) \pm g(t) = \sum_{k=0}^{\infty} (c_k \pm d_k)(t-t_0)^k$ in $|t - t_0| < r = min(r_1, r_2)$.

(ii) **Multiplication**

$$\begin{aligned}\text{If } f(t) &= \sum_{k=0}^{\infty} c_k(t-t_0)^k \text{ in } |t-t_0| < r_1 \quad \text{and} \\ g(t) &= \sum_{k=0}^{\infty} d_k(t-t_0)^k \text{ in } |t-t_0| < r_2\end{aligned}$$

Then $h(t) = f(t)g(t)$ defined within the radius of convergence of each series and

$$h(t) = \sum_{k=0}^{\infty} c_k(t-t_0)^k$$

where $c_k = \sum_{m=0}^{k} c_m d_{k-m}$. That is

$$h(t) = c_0 d_0 + (c_0 d_1 + c_1 d_0)\,t + (c_0 d_2 + c_1 d_1 + c_2 d_0)\,t^2 + \cdots$$

The series converges absolutely within the radius of convergence of each series.

(iii) **Differentiation**

$$\text{Let } f(t) = \sum_{k=0}^{\infty} c_k(t-t_0)^k \text{ in } |t-t_0| < r$$

Then

$$\frac{df}{dt} = \dot{f} = \sum_{k=1}^{\infty} c_k k(t-t_0)^{k-1} \text{ in } |t-t_0| < r$$

(iv) **Integration**

$$\text{Let } f(t) = \sum_{k=0}^{\infty} c_k(t-t_0)^k \text{ in } |t-t_0| < r$$

Then

$$\int f(t)dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(t-t_0)^{k+1} \text{ in } |t-t_0| < r$$

Definition 6.1.3 *We shall say that a function $f(t)$ is analytic at $t = t_0$ if it can be expanded as a sum of a power series of the form $\sum_{k=0}^{\infty} c_k(t-t_0)^k$ with a radius of convergence r. It is clear that if $f(t)$ is analytic at t_0, then $c_k = \frac{f^{(k)}(t_0)}{k!}$, $k = 0, 1, 2, \cdots$ ($f^k(t_0)$ denotes the k^{th} derivative of f at t_0).*

Example 6.1.2 *We have the binomial expansion for $(1-t)^{-k}$ for a fixed positive number k to give us*

$$(1-t)^{-k} = 1 + kt + \frac{k(k+1)}{2!}t^2 + \frac{k(k+1)...(k+r-1)}{r!}t^r + \cdots \quad for \ |t| < 1$$

We denote by

$$u_r = \frac{k(k+1)...(k+r-1)}{r!}$$

Then

$$(1-t)^{-k} = \sum_{r=0}^{\infty} u_r t^r, \quad |t| < 1$$

As a power series can be differentiated term by term in its interval of convergence, we have

$$k(1-t)^{-k-1} = \sum_{r=1}^{\infty} r u_r t^{r-1}, \quad |t| < 1$$

This gives

$$k(1-t)^{-k}\frac{1}{(1-t)} = \sum_{r=1}^{\infty} r u_r t^{r-1}, \quad |t| < 1$$

Again, using the power series expansion for $\frac{1}{1-t} = 1 + t + t^2 + \cdots$, we get

$$k\left(\sum_{r=0}^{\infty} t^r\right)\left(\sum_{r=0}^{\infty} u_r t^r\right) = \sum_{r=1}^{\infty} r u_r t^{r-1}$$

Using the product formula for LHS, we get

$$k\sum_{r=0}^{\infty} t^r \sum_{l=0}^{r} u_l = \sum_{r=0}^{\infty} (r+1)u_{r+1}t^r$$

Uniqueness of power series representation gives

$$(r+1)u_{r+1} = k\sum_{l=0}^{r} u_l \tag{6.1.5a}$$

where

$$u_r = \frac{k(k+1).....(k+r-1)}{r!}, \quad k \text{ is a fixed integer} \tag{6.1.5b}$$

We now make an attempt to define the concept of uniform convergence. To define this concept, we consider the following series whose terms are functions $\{f_k(t)\}_{k=0}^{\infty}$

$$\sum_{k=0}^{\infty} f_k(t) = f_0(t) + f_1(t) + f_2(t) + \cdots \tag{6.1.6}$$

Note that for $f_k(t) = c_k(t-t_0)^k$, we get the power series.

Definition 6.1.4 *We shall say that the series given by Eq. (6.1.6) with sum $f(t)$ in an interval $I \subset \Re$ is uniformly convergent if for every $\epsilon > 0$, we can find $N = N(t)$, not depending on t, such that*

$$|f(t) - s_n(t)| < \epsilon \text{ for all } n \geq N(\epsilon)$$

where $s_n(t) = f_0(t) + f_1(t) + \cdots + f_n(t)$.

Theorem 6.1.3 *A power series $\sum_{k=0}^{\infty} c_k(t-t_0)^k$ with nonzero radius of convergennt is uniformly convergent in every closed interval $|t - t_0| \leq p$ of radius $p < r$.*

Proof : For $|t - t_0| \leq p$ and any positive integers n and l we have

$$\left|c_{n+1}(t-t_0)^{n+1} + \cdots + c_{n+l}(t-t_0)^{n+l}\right| \leq |c_{n+1}|\, p^{n+1} + \cdots + |c_{n+l}|\, p^{n+l} \qquad (6.1.7)$$

The series $\sum_{k=0}^{\infty} c_k(t-t_0)^k$ converges absolutely if $|t - t_0| \leq p < r$ (by Theorem 6.1.1) and hence by Cauchy convergence, given $\epsilon > 0$, $\exists\, N(\epsilon)$ such that

$$|c_{n+1}|\, p^{n+1} + \cdots + |c_{n+p}|\, p^{n+l} < \epsilon \quad \text{for } n \geq N(\epsilon), \quad l = 1, 2, ..$$

Eq. (6.1.7) gives

$$\begin{aligned} &\left|c_{n+1}(t-t_0)^{n+1} + \cdots + c_{n+p}(t-t_0)^{n+l}\right| \\ &\leq |c_{n+1}|\, p^{n+1} + \cdots + |c_{n+p}|\, p^{n+l} < \epsilon \quad \text{for } n \geq N(\epsilon), \quad |t-t_0| \leq p < r \end{aligned}$$

This implies that, given $\epsilon > 0, \exists\, N(\epsilon)$ such that $|f(t) - s_n(t)| < \epsilon$ for $n \geq N(\epsilon)$ and all t where $s_n(t) = \sum_{k=0}^{n} c_k(t-t_0)^k$. This proves the uniform convergence of the power series inside the interval $|t - t_0| \leq p < r$. ∎

Example 6.1.3 *The geometric series $1 + t + t^2 + \cdots$ is uniformly convergent in the interval $|t| \leq p < 1$. It is not uniformly convergent in the whole interval $|t| < 1$.*

6.2 Linear System with Analytic Coefficients

We now revisit the non-autonomous system in $\Re^n$

$$\begin{aligned} \frac{d\bar{x}}{dt} &= A(t)\bar{x} + g(t) \qquad (6.2.1) \\ \bar{x}(0) &= \bar{x}_0 \end{aligned}$$

where the matrix $A(t)$ is analytic at $t = 0$ and hence has the power series representation

$$A(t) = \sum_{k=0}^{\infty} A_k t^k \tag{6.2.2}$$

in its interval of convergence $|t| < r$. Here each A_k is $n \times n$ matrix.

So, the homogeneous system corresponding to Eq. (6.2.1) is given by

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \sum_{k=0}^{\infty} A_k t^k \bar{x} \\ \bar{x}(0) &= \bar{x}_0 \end{aligned} \tag{6.2.3}$$

We shall look for analytic solution of Eq. (6.2.3), which is of the form $\bar{x}(t) = \sum_{k=0}^{\infty} \bar{c}_k t^k$. The vector coefficients $\bar{c}_k$ are to be determined. The point $t = 0$ is called ordinary point of the above system.

The following theorem gives the analytic solution of the system given by Eq. (6.2.3)

Theorem 6.2.1 *The homogeneous system given by Eq. (6.2.3) has analytic solution* $\bar{x}(t) = \sum_{k=0}^{\infty} \bar{c}_k t^k$ *in the interval of convergence* $|t| < r$. *This solution* $\bar{x}(t)$ *is uniquely determined by the initial vector* $\bar{x}_0$.

Proof : Let $\bar{x}(t) = \sum_{k=0}^{\infty} \bar{c}_k t^k$, where the vector coefficient $\bar{c}_k$ are yet to determined. Plugging this representation in Eq. (6.2.3) we get

$$\begin{aligned} \sum_{k=1}^{\infty} k\bar{c}_k t^{k-1} &= \left(\sum_{k=0}^{\infty} A_k t^k\right)\left(\sum_{k=0}^{\infty} \bar{c}_k t^k\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} A_{k-l}\bar{c}_l\right) t^k \end{aligned}$$

Equivalently, we have

$$\sum_{k=0}^{\infty}(k+1)\bar{c}_{k+1} t^k = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} A_{k-l}\bar{c}_l\right) t^k \tag{6.2.4}$$

Uniqeness fo power series in the interval $|t| < r$, gives

$$(k+1)\bar{c}_{k+1} = \sum_{l=0}^{k} A_{k-l}\bar{c}_l \tag{6.2.5}$$

and hence

$$(k+1)\,\|\bar{c}_{k+1}\| \le \sum_{l=0}^{k} \|A_{k-l}\| \;\; \|\bar{c}_l\| \tag{6.2.6}$$

By Theorem 6.1.3, the power series $\sum_{k=0}^{\infty} A_k t^k$ converges absolutely and uniformly in the interval $|t| \le p < r$ and hence the terms $A_k p^k$ must be uniformly bounded. That is, $\exists\, M$ such that

$$\|A_k\| p^k \le M, \quad k \ge 0 \tag{6.2.7}$$

Using the above inequality in Eq. (6.2.6), we get

$$(k+1)\|\bar{c}_{k+1}\| \le \sum_{l=0}^{k} M \frac{\|\bar{c}_l\|}{p^{k-l}}$$

Put $d_l = p^l \|\bar{c}_l\|$, then the above inequality becomes

$$(k+1) d_{k+1} \le Mp \sum_{l=0}^{k} d_l \tag{6.2.8}$$

Using Eq. (6.2.8) inductively, we get

$$\begin{aligned}
d_1 &\le Mpd_0 \\
d_2 &\le \frac{Mp}{2}(d_0 + d_1) \\
&\le \frac{1}{2}\left(Mp + M^2p^2\right) d_0 \\
&\vdots \\
d_k &\le \frac{Mp(Mp+1)(Mp+2) + \cdots + (Mp+k-1)}{k!} d_0
\end{aligned} \tag{6.2.9}$$

To claim that Eq. (6.2.8) holds for all k, we need to prove this inequality by induction. So, let this inequality be true for all $r < k$. By Eq. (6.2.8), we have

$$\begin{aligned}
(r+1) d_{r+1} &\le Mp\|\bar{c}_0\| \sum_{l=0}^{r} d_l \\
&\le Mp\|\bar{c}_0\| \sum_{l=0}^{r} \frac{Mp(Mp+1)(Mp+2) + \cdots + (Mp+l-1)}{l!}
\end{aligned}$$

Using the notation of Example 6.1.2, set

$$u_l = \frac{Mp(Mp+1)(Mp+2) + \cdots + (Mp+l-1)}{l!}$$

Hence using Eq. (6.1.5), we get

$$\begin{aligned}(r+1)d_{r+1} &\le Mp\|\bar{c}_0\|\sum_{l=0}^{r} u_l \\ &= \|\bar{c}_0\|(r+1)u_{r+1}\end{aligned}$$

This gives

$$(r+1)d_{r+1} \le (r+1)\frac{Mp(Mp+1)(Mp+2)+\cdots+(Mp+r)}{(r+1)!}\|\bar{c}_0\|$$

That is

$$d_{r+1} \le \frac{Mp(Mp+1)(Mp+2)+\cdots+(Mp+r)}{(r+1)!}\|\bar{c}_0\|$$

This proves the induction.
Hence, it follows that

$$\|\bar{c}_k\| = \frac{d_k}{p^k} \le \frac{Mp(Mp+1)(Mp+2)+\cdots+(Mp+k-1)}{k!p^k}\|\bar{c}_0\|$$

This gives

$$\begin{aligned}\|\bar{x}\| &= \left\|\sum_{k=0}^{\infty}\bar{c}_k t^k\right\| \\ &\le \|\bar{c}_0\| \times \left(\sum_{k=0}^{\infty}\frac{Mp(Mp+1)(Mp+2)+\cdots+(Mp+k-1)}{k!}\left(\frac{|t|}{p}\right)^k\right)\end{aligned}$$

That is

$$\|\bar{x}\| \le \frac{\|\bar{c}_0\|}{\left(1-\frac{|t|}{p}\right)^{Mp}} \tag{6.2.10}$$

provided $|t| \le p < r$.
This proves the existence of analytic solution of the system given by Eq. (6.2.3), This solution is uniquely determined by the initial value $\bar{x}_0$. For if $\bar{x}(t)$ and $\bar{y}(t)$ are two different solutions of the initial value problem given by Eq. (6.2.3), then $\bar{z}(t) = \bar{x}(t) - \bar{y}(t)$ is the solution of the initial value problem

$$\begin{aligned}\frac{d\bar{z}}{dt} &= A(t)\bar{z}(t) \\ \bar{z}(0) &= \bar{0}\end{aligned}$$

Since $\bar{c}_0 = 0$, it follows that $\bar{z} = \bar{0}$, thereby implying that $\bar{x} = \bar{y}$. ▮

Example 6.2.1 *(Legendre Differential Equation)*

$$(1-t^2)\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + n(n+1)x = 0 \tag{6.2.11}$$

By substituting $x_1 = x,\ x_2 = \dfrac{dx}{dt}$, *this differential equation is equivalent to the following linear homogeneous system*

$$\frac{d\bar{x}}{dt} = A(t)\bar{x}(t)$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ \dfrac{-n(n+1)}{1-t^2} & \dfrac{2t}{1-t^2} \end{pmatrix}$$

As $A(t)$ is analytic in the interval $|t| < 1$, it follows by Theorem 6.2.1 that Eq. (6.2.11) has a unique analytic solution of the form $\sum\limits_{k=0}^{\infty} c_k t^k$. We shall now determine the coefficients c_k. We substitute

$$x(t) = \sum_{k=0}^{\infty} c_k t^k, \quad \frac{dx}{dt} = \sum_{k=1}^{\infty} k c_k t^{k-1}, \quad \frac{d^2x}{dt^2} = \sum_{k=2}^{\infty} k(k-1)c_k t^{k-2}$$

in Eq. (6.2.11) to get

$$(1-t^2)\sum_{k=2}^{\infty} k(k-1)c_k t^{k-2} - 2t\sum_{k=1}^{\infty} k c_k t^{k-1} + n(n+1)\sum_{k=1}^{\infty} c_k t^k = 0$$

Equivalently, we have

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)c_{k+2} + c_k\left[n(n+1) - (k-1)k - 2k\right]\right] t^k = 0$$

The uniqueness of series representaion in $|t| < 1$ gives

$$\begin{aligned} c_{k+2} &= -\frac{[n(n+1) - (k-1)k - 2k]}{(k+2)(k+1)} c_k \\ &= -\frac{(n-k)(n+k+1)}{(k+2)(k+1)} c_k, \qquad k = 0, 1, 2, \cdots \end{aligned} \tag{6.2.12}$$

This gives us

$$\begin{aligned} c_2 &= -\frac{n(n+1)}{2!}c_0, \quad c_4 = -\frac{(n+3)(n-2)}{4.3}c_2 = \frac{(n+3)(n+1)n(n-2)}{4!}c_0 \\ c_3 &= -\frac{(n+2)(n-1)}{3!}c_1, \quad c_5 = -\frac{(n+4)(n-3)}{5.4}c_3 \\ &\qquad = \frac{(n+4)(n+2)(n-3)(n-1)}{5!}c_1 \end{aligned}$$

and hence the coefficients c_k's are inductively defined.
So, we have

$$\begin{aligned}
x(t) &= c_0 x_1(t) + c_1 x_2(t) \\
x_1(t) &= 1 - \frac{n(n+1)}{2!}t^2 + \frac{(n-2)n(n+1)(n+3)}{4!}t^4 - \cdots \\
x_2(t) &= t - \frac{(n-1)(n+2)}{3!}t^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}t^5 - \cdots
\end{aligned}$$

Since RHS of Eq. (6.2.12) is zero for $k = n$, we have $0 = c_{n+2} = c_{n+4} = \cdots$ and hence if n is even, $x_1(t)$ reduces to a polynomial of degree n and if n is odd, the same is true for $x_2(t)$. These polynomials multiplied by some constants, are called Legendre polynomials.

In Eq. (6.2.12), writing c_k in terms of c_{k+2}, we get

$$c_k = -\frac{(k+2)(k+1)}{(n-k)(n+k+1)}c_{k+2} \quad (k \leq n-2)$$

This gives

$$c_{n-2} = -\frac{n(n-1)}{2(2n-1)}c_n$$

We fix c_n and then compute the lower order coefficients. It is standard to choose $c_n = 1$ when $n = 0$ and $\frac{2n!}{2^n(n!)^2} = \frac{1.3.5...(2n-1)}{n!}$ for $n \neq 0$. This defines

$$\begin{aligned}
c_{n-2} &= -\frac{n(n-1)}{2(2n-1)}\frac{(2n)!}{2^n(n!)^2} \\
&= -\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n(n-1)!n(n-1)(n-2)!} \\
&= -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}
\end{aligned}$$

In general

$$c_{n-2m} = \frac{(-1)^m(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

So we get a solution of Legendre differential equation given by Eq. (6.2.11) as

$$P_n(t) = \sum_{m=0}^{M} \frac{(-1)^m(2n-2m)!}{2^n m!(n-m)!(n-2m)!}t^{n-2m}$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$ whichever is an integer. Some of the Legendre polynomials are given as below

$$\begin{aligned} P_0(t) &= 1,\ P_1(t) = t,\ P_2(t) = \frac{1}{2}(3t^2-1),\ P_3(t) = \frac{1}{2}(5t^3-3t) \\ P_4(t) &= \frac{1}{8}(35t^4-30t^2+3),\ P_5(t) = \frac{1}{8}(63t^5-70t^3+15t) \end{aligned}$$

We shall discuss the properties of Legendre differential equation in section 4.

Example 6.2.2 *(Hermite Differential Equation)*
In the study of the linear harmonic oscillator in quantum mechanics, one encounters the following deifferential equation

$$\frac{d^2u}{dt^2} + \left[\lambda - t^2\right] u(t) = 0 \tag{6.2.13}$$

over the interval $(-\infty,\ \infty)$*. As we look for the solution* $u \in L_2(-\infty,\ \infty)$*, we put* $u(t) = e^{\left(\frac{-t^2}{2}\right)} x(t)$*. So Eq. (6.2.13) becomes*

$$\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + (\lambda - 1)\, x = 0$$

We take $\lambda = 2n+1$ *and get the equation*

$$\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2nx = 0 \tag{6.2.14}$$

Eq. (6.3.13) (or alternatively Eq. (6.2.14)) is called Hermite differential equation.

As the coefficient of the differential equation given by Eq. (6.2.14) are analytic in $(-\infty, \infty)$, we get a series solution of the form

$$x(t) = \sum_{k=0}^{\infty} c_k t^k \quad \text{for} \quad -\infty < t < \infty \tag{6.2.15}$$

Plugging the representation for $x(t)$, $\dfrac{dx}{dt}$ and $\dfrac{dx^2}{dt^2}$ in Eq. (6.2.14) we see that c_k satisfy the recurrence relation

$$\begin{aligned} c_{k+2} &= \frac{2k+1-(2n+1)}{(k+2)(k+1)} c_k \\ &= \frac{2(k-n)}{(k+2)(k+1)} c_k \end{aligned} \tag{6.2.16}$$

It is clear that $c_{n+2} = 0$ and hence the solution given by Eq. (6.2.15) is a polynomial. As in the previous example, we normalize c_n and put it equal to 2^n and then compute the coefficients in descending order. This gives us the solution, which is denoted by $H_n(t)$, the Hermite polynomial. It is given by

$$H_n(t) = \sum_{m=0}^{M} \frac{(-1)^m n!}{m!(n-2m)!} (2t)^{n-2m}$$

We shall discuss the properties of the Hermite polynomials in section 4.

6.3 Linear System with Regular Singularities

A system of the form

$$\frac{d\bar{x}}{dt} = \frac{1}{t-t_0}A(t)\bar{x} \tag{6.3.1}$$

where $A(t)$ is analytic at t_0, is said to have a regular singularity at t_0.

As before, we can assume that $t_0 = 0$ and $A(t)$ has series representation of the form

$$A(t) = \sum_{k=0}^{\infty} A_k t^k \tag{6.3.2}$$

in the interval of the convergence $|t| < r$.
Eq. (6.3.1) then reduces to the equation of the form

$$\frac{d\bar{x}}{dt} = \frac{1}{t}\left(\sum_{k=0}^{\infty} A_k t^k\right)\bar{x} \tag{6.3.3}$$

We prove the following theorem, giving the existence of a series solution of the equation given by Eq. (6.3.3)

Theorem 6.3.1 *The differential equation given by Eq. (6.3.3) has a series solution of the form* $\bar{x}(t) = t^{\mu}\sum_{k=0}^{\infty}\bar{c}_k t^k$ *in the interval* $|t| < r$*, provided* μ *is an eigenvalue of* A_0 *and no other eigenvalue of the form* $\mu + n$ *exists for* A_0*, where* n *is a positive integer.*

Proof : We introduce a new dependent variable $\bar{y}(t) = t^{\mu}\bar{x}(t)$. This gives

$$\begin{aligned}\frac{d\bar{y}}{dt} &= \frac{1}{t}\left[A(t) - \mu I\right]\bar{y}(t) \\ &= \frac{1}{t}\left[\sum_{k=0}^{\infty} A_k t^k - \mu I\right]\bar{y}(t)\end{aligned}$$

Assuming that $\bar{y}(t) = \sum_{k=0}^{\infty}\bar{c}_k t^k$, we get

$\sum_{k=1}^{\infty} \bar{c}_k k t^k = \sum_{k=0}^{\infty} \left[\left(\sum_{l=0}^{k} A_{k-l}\bar{c}_l \right) - \mu \bar{c}_k \right] t^k$. This gives us the recurrence relations

$$\begin{aligned}
(A_0 - \mu I)\bar{c}_0 &= 0 \\
(A_0 - (\mu + I))\,\bar{c}_1 &= -A_1\bar{c}_0 \\
&\vdots \\
(A_0 - (\mu + n)I)\bar{c}_n &= -\sum_{l=0}^{n-1} A_{n-l}\bar{c}_l \\
&\vdots
\end{aligned} \tag{6.3.4}$$

Since $|(A_0 - (\mu+n)I)\bar{c}_n| \neq 0$ for all $n \geq 1$, the above relations iteratively define $\bar{c}_0, \bar{c}_1, \cdots, \bar{c}_n, \cdots$ We shall now show that the series $x(t) = t^{\mu}\sum_{k=0}^{\infty} \bar{c}_k t^k$ converges.

As in Section 6.2, for some positive numbers M, and $p < r$ we have

$$\|A_k\| p^k \leq M, \quad k = 0, 1, 2, \cdots \tag{6.3.5}$$

We first observe the following

(i) $|A_0 - (\mu + k)I| \neq 0$ for all positive integer k

(ii) $\lim\limits_{k\to\infty} \left| A_0 - \dfrac{(\mu+k)}{k} I \right|$
$= \lim\limits_{k\to\infty} \left| \dfrac{(A_0 - \mu I)}{k} - \dfrac{k}{k} I \right|$
$= n$

In view of property (ii), it follows that there exists a positive quantity $\epsilon > 0$ such that

$$\frac{\|A_0 - (\mu + k)I\|}{k} > \epsilon, \quad k = 1, 2, 3, \cdots$$

So, the recurrence relation given by Eq. (6.3.4) provides

$$(k+1)\bar{c}_{k+1} = [A_0 - (\mu + k + 1)I]^{-1}\,(k+1)\sum_{l=0}^{k} A_{k+1-l}\bar{c}_l$$

Using Eq. (6.3.5) and (ii) we get

$$(k+1)\,\|\bar{c}_{k+1}\| \leq \frac{M(k+1)}{\epsilon}\sum_{l=0}^{k}\frac{\|\bar{c}_l\|}{p^{k+1-l}}$$

Equivalently,

$$(k+1)\,\|\bar{c}_{k+1}\|\,p^{k+1} \leq \frac{M}{\epsilon}(k+1)\sum_{l=0}^{k}\|\bar{c}_l\| p^l$$

Proceeding as in Theorem 6.2.1, we have the relation

$$\|\bar{c}_k\| \leq \frac{\left(\frac{M}{\epsilon}\right)\left(\frac{M}{\epsilon}+1\right)\cdots\left(\frac{M}{\epsilon}+k-1\right)}{k!p^k}\|\bar{c}_0\|$$

for all $k \geq 1$.

Applying this estimate to the series expansion for $\bar{x}(t)$, we get

$$\begin{aligned}
\|\bar{x}(t)\| &\leq \left\| t^\mu \sum_{k=0}^{\infty} \bar{c}_k t^k \right\| \\
&\leq |t|^\mu \|\bar{c}_0\| \sum_{k=0}^{\infty} \frac{\left(\frac{M}{\epsilon}\right)\left(\frac{M}{\epsilon}+1\right)\cdots\left(\frac{M}{\epsilon}+k-1\right)}{k!}\left(\frac{|t|}{p}\right)^k \\
&= \frac{|t|^\mu \|\bar{c}_0\|}{\left(1-\dfrac{|t|}{p}\right)^{\frac{M}{\epsilon}}}
\end{aligned}$$

in the interval of uniform convergence $|t| \leq p < r$.
This proves the theorem. ∎

Example 6.3.1 *(Bessel's Equation)*

$$\begin{aligned}
t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + (t^2-\mu^2)x(t) &= 0 \qquad (6.3.6)\\
\Longleftrightarrow \frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} + (1-\frac{\mu^2}{t^2})x(t) &= 0
\end{aligned}$$

We use the substitution $x_1 = x(t), \;\; x_2 = t\dot{x}(t)$ *to get*

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{1}{t}x_2 \\
\frac{dx_2}{dt} &= \frac{dx_1}{dt} + t\frac{d^2x_1}{dt^2} \\
&= \frac{dx_1}{dt} - \frac{dx_1}{dt} - (t-\frac{\mu^2}{t})x_1(t) \\
&= -\frac{1}{t}(t^2-\mu^2)x_1(t)
\end{aligned}$$

This reduces Eq. (6.3.6) to

$$\begin{aligned}
\frac{d\bar{x}}{dt} &= \frac{1}{t}\begin{pmatrix} 0 & 1 \\ \mu^2-t^2 & 0 \end{pmatrix}\bar{x} \\
&= \frac{1}{t}\left[A_0 + A_2t^2\right]\bar{x}
\end{aligned}$$

where $A_0 = \begin{pmatrix} 0 & 1 \\ \mu^2 & 0 \end{pmatrix}$, $\quad A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. *It is clear that* A *is analytic in* $(-\infty, \infty)$ *and* A_0 *has eigenvalues* $\pm\mu$.

Case 1: μ *is not an integer*

Then Eq. (6.3.6) has solution of the form $x(t) = t^\mu \sum_{k=0}^{\infty} \bar{c}_k t^k$ *in* $(-\infty, \infty)$.

For the eigenvalue $\mu > 0$*, plugging the representation*

$$\begin{aligned} x(t) &= \sum_{m=0}^{\infty} c_m x^{m+\mu}, \quad \frac{dx}{dt} = \sum_{m=0}^{\infty} c_m (m+\mu) t^{m+\mu-1} \\ \frac{d^2x}{dt^2} &= \sum_{m=0}^{\infty} c_m (m+\mu)(m+\mu-1) t^{m+\mu-2} \end{aligned}$$

in Eq. (6.3.6) we get the recurrence relation

$$\begin{aligned} c_{2m} &= -\frac{1}{2^2 m(\mu+m)} c_{2m-2}, \ m = 1, 2 \cdots \\ c_{2m+1} &= 0, \ \ m = 0, 1, 2, \cdots \end{aligned}$$

This gives

$$\begin{aligned} c_2 &= -\frac{c_0}{2^2(\mu+1)} \\ c_4 &= -\frac{c_2}{2^2 2(\mu+2)} = \frac{c_0}{2^4 2!(\mu+1)(\mu+2)} \\ &\vdots \\ c_{2m} &= \frac{(-1)^m c_0}{2^{2m} m!(\mu+1)(\mu+2)\cdots(\mu+m)}, \\ &\qquad\qquad m = 1, 2, \cdots \\ c_{2m+1} &= 0, \ \ m = 0, 1, 2, 3, \cdots \end{aligned} \tag{6.3.7}$$

Now define the gamma function $\Gamma(\mu)$ *as*

$$\Gamma(\mu) = \int_0^\infty e^{-t} t^{\mu-1} dt, \ \ \mu > 0$$

which is a convergent integral. $\Gamma(\mu)$ *satisfies the property that* $\Gamma(\mu+1) = \mu\Gamma(\mu)$ *and* $\Gamma(n+1) = n!$ *If we put* $c_0 = \frac{1}{2^\mu}\Gamma(\mu+1)$*, then* c_{2m} *is given by*

$$\begin{aligned} c_{2m} &= \frac{(-1)^m}{2^{2m+\mu} m!(\mu+1)(\mu+2)\cdots(\mu+m)\Gamma(\mu+1)} \\ &= \frac{(-1)^m}{2^{2m+\mu} m!\Gamma(\mu+m+1)} \end{aligned}$$

This gives us the first solution $J_\mu(t)$, called Bessl's function of order μ of the differential equation given by Eq. (6.3.6), as

$$J_\mu(t) = t^\mu \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{2^{2m+\mu} m! \Gamma(\mu + m + 1)}$$

Extending gamma function for $\mu < 0$, we get the representation for the second solution $J_{-\mu}(t)$ as

$$J_{-\mu}(t) = t^{-\mu} \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{2^{2m-\mu} m! \Gamma(-\mu + m + 1)}$$

$J_\mu(t)$ and $J_\mu(t)$ are linearly independent.

Case 2: *$\mu = n$ integer*
We have

$$J_n(t) = t^n \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{2^{2m+n} m! (n + m)}$$

One can easily derive that

$$J_{-n}(t) = (-1)^n J_n(t)$$

This gives

$$\begin{aligned} J_0(t) &= \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{2^{2m} (m!)^2} \\ &= 1 - \frac{t^2}{2^2 (1!)^2} + \frac{t^4}{2^4 (2!)^2} - \frac{t^6}{2^6 (3!)^2} + \cdots \\ J_1(t) &= \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+1}}{2^{2m+1} (k!)(k+1)!} \\ &= t - \frac{t^3}{2^3 (1!)(2!)} + \frac{t^5}{2^5 (2!)(3!)} - \frac{t^7}{2^7 (3!)(4!)} + \cdots \end{aligned}$$

J_0 looks similar to cosine function and J_1 looks similar to sine function. The zeros of these functions are not completely regularly spaced and also oscillations are damped as we see in the following graphics.

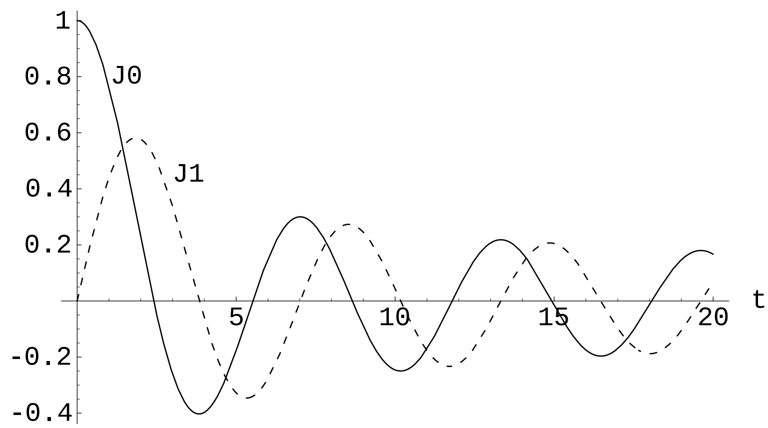

Figure 6.3.1: Sketch of J_0 and J_1

Example 6.3.2 *(Bessel's equation of order zero)*
Consider the equation

$$x\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$$

This is equivalent to the first order system of the form

$$\frac{d\bar{x}}{dt} = -\frac{1}{t}\begin{pmatrix} 0 & 1 \\ -t^2 & 0 \end{pmatrix}\bar{x} \tag{6.3.8}$$

where $x_1 = x,\ x_2 = t\dfrac{dx}{dt}$.

As we have seen before, a solution of this system is given by

$$x_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{4^k (k!)^2} \tag{6.3.9}$$

$$x_2(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{4^k (k!)^2} \tag{6.3.10}$$

To get another solution $\bar{y}(t)$ *linearly independent of* $\bar{x}(t)$ *and satisfying Eq. (6.3.8), we set*

$$\bar{y}(t) = \psi(t)\bar{z}(t) \ \ where \ \ \psi(t) = \begin{pmatrix} 1 & x_1 \\ 0 & x_2 \end{pmatrix}.$$

This gives

$$\frac{d\bar{y}}{dt} = \frac{d\phi}{dt}\bar{z}(t) + \psi(t)\frac{d\bar{z}}{dt} = A(t)\psi(t)\bar{x}(t)$$

That is

$$\begin{pmatrix} 0 & \frac{1}{t}x_2 \\ & \\ 0 & -tx_1 \end{pmatrix} \bar{z} + \begin{pmatrix} 1 & x_1 \\ & \\ 0 & x_2 \end{pmatrix} \frac{dz}{dt} = \begin{pmatrix} 0 & \frac{1}{t}x_2 \\ & \\ -t & -tx_1 \end{pmatrix} \bar{z}(t)$$

Solving for $\frac{dz}{dt}$ *we get*

$$\begin{pmatrix} 1 & x_1 \\ & \\ 0 & x_2 \end{pmatrix} \frac{dz}{dt} = \begin{pmatrix} 0 & 0 \\ & \\ -t & 0 \end{pmatrix} \bar{z}(t)$$

and hence

$$\frac{dz}{dt} = \begin{pmatrix} t\frac{x_1}{x_2} & 0 \\ & \\ -\frac{t}{x_2} & 0 \end{pmatrix} \bar{z}(t)$$

So, we need to solve

$$\frac{dz_1}{dt} = t\frac{x_1}{x_2}z_1 = -\frac{\dot{x_2}}{x_2}z_1 \tag{6.3.11a}$$

$$\frac{dz_2}{dt} = -\frac{t}{x_2}z_1 \tag{6.3.11b}$$

Integrating Eq. (6.3.11(a)) - Eq. (6.3.11(b)) we get

$$z_1 = \frac{c_1}{x_2}, \quad z_2 = -c_1 \int \frac{t}{x_2^2(t)} dt$$

Hence $\bar{y}(t) = \psi(t)\bar{z}(t)$ *is given by*

$$\bar{y}(t) = \begin{pmatrix} \frac{c_1}{x_2} - c_1 x_1 \int \frac{tdt}{x_2^2(t)} \\ \\ -c_1 x_2 \int \frac{t}{x_2^2(t)} dt \end{pmatrix}$$

That is

$$\begin{aligned} y_1(t) &= \frac{c_1}{x_2} - c_1 x_1 \int \left[\frac{-\dot{x_2}/x_1}{x_2^2}\right] dt \\ &= -c_1 x_1 \int \frac{\dot{x_2}}{x_2 x_1^2} dt \\ &= -c_1 x_1 \int \frac{1}{t x_1^2} dt \end{aligned}$$

$$y_2(t) = -c_1 x_2 \int \frac{t}{x_2^2} dt$$

Using the represntations of Eq. (6.3.9) - Eq. (6.3.10), we get

$$\begin{aligned} x_1(t) &= 1 - \frac{t^2}{4} + \cdots \\ x_2(t) &= -\frac{t^2}{2} + \frac{t^4}{16} + \cdots \end{aligned}$$

This gives

$$\begin{aligned} x_1 \int \frac{dt}{t x_1^2} &= x_1 \int \frac{1 + \frac{t^2}{2} + \cdots}{t} dt \\ &= x_1 lnt + x_1 (\frac{t^2}{4} + \cdots) \end{aligned}$$

and

$$\begin{aligned} x_2 \int \frac{t}{x_2^2} dt &= 4x_2 \int \frac{t\left(1 + \frac{t^2}{4} + \cdots\right)}{t^4} dt \\ &= \frac{-2x_2}{t^2} + x_2 lnt + \cdots \end{aligned}$$

Hence, it follows that the second solution $\bar{y}(t)$, linearly independent of $x(t)$, is given by

$$y(t) = x(t) lnt + x(t)\left(1 + \frac{t^2}{4} + \cdots\right)$$

To be more precise, one can show that

$$y(t) = J_0(t) lnt + \sum_{m=1}^{\infty} \frac{(-1)^m h_m t^{2m}}{2^{2m}(m!)^2}$$

where $h_m = \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right)$.

The function $Y_0(t)$ defined as

$$Y_0(t) = a y(t) + b J_0(t), a = \frac{2}{\pi},\ b = r - ln2$$

where r is the Euler's constant $\lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{m} - lnn\right)$ is called the Bessel's function of the second kind (order zero). Hence, it is given by

$$Y_0(t) = \frac{2}{\pi}\left[J_0(t)\left(ln\frac{t}{2} + r\right) + \sum_{m=1}^{\infty} \frac{(-1)^m h_m t^{2m}}{2^{2m}(m!)^2}\right]$$

6.4 Properties of Legendre, Hermite and Bessel Functions

Legendre polynomial $P_n(t)$ is a solution of the differential equation

$$(1-t^2)\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + n(n+1)x(t) = 0 \tag{6.4.1}$$

In terms of the notation of Chapter 6, this is eigenvalue problem of the form

$$Lx = \lambda x \tag{6.4.2}$$

where L is the second order differential operator

$$Lx = (1-t^2)\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} = \frac{d}{dt}\left((1-t^2)\frac{dx}{dt}\right) \tag{6.4.3}$$

We can think of L as an operator defined on $L_2[-1,1]$ with $D(L)$ as the set of all functions having second order derivatives.

Then Green's formula (to be done in Section 7.1) gives

$$\int_{-1}^{1} (yLx - xL^*y)\,dt = J(x,y)\Big|_{-1}^{1}$$

where $J(x,y)|_{-1}^{1} = a_0\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)\Big|_{-1}^{1}$ with $a_0(t) = (1-t^2)$.

As $a_0(1) = a_0(-1) = 0$, it follows that $J(x,y)|_{-1}^{1} = 0$ and hence

$$\int_{-1}^{1} (yLx - xL^*y)\,dt = 0 \quad \forall x,\ y \in D(L) = D(L^*)$$

That is $L^* = L$ and $D(L^*) = D(L)$.

We appeal to Theorem 7.5.1, of Chapter 7 to claim that the eigenfunctions of L form a complete orthonormal set. We have already proved in Example 6.2.1 that Eq. (6.4.1) has sequence of eigenvalues $\{n(n+1)\}$ with $\{P_n(t)\}$ corresponding eigenfunctions and hence they are orthogonal. However, for the sake of completeness we prove this result directly.

Theorem 6.4.1 *The set of Legendre polymonials $\{P_n(t)\}$ satisfying Eq. (6.4.1) are orthogonal set of polynomials in $L_2[-1,1]$.*

Proof : We have

$$\frac{d}{dt}\left((1-t^2)\frac{dP_n}{dt}\right) = -n(n+1)P_n(t) \tag{6.4.4}$$

$$\frac{d}{dt}\left((1-t^2)\frac{dP_m}{dt}\right) = -m(m+1)P_m(t) \tag{6.4.5}$$

Multiplying Eq. (6.4.4) by $P_m(t)$ and Eq. (6.4.5) by $P_n(t)$ and substracting the equations and integrating, we get

$$\begin{aligned}
&\int_{-1}^{1} [m(m+1) - n(n+1)]\, P_m(t)P_n(t)dt \\
= &\int_{-1}^{1} \left[\frac{d}{dt}\left((1-t^2)\frac{dP_n}{dt}\right)P_m(t) - \frac{d}{dt}\left((1-t^2)\frac{dP_m}{dt}P_n(t)\right)\right] dt \\
= &\left[(1-t^2)\left[\dot{P}_n(t)P_m(t) - \dot{P}_m(t)P_n(t)\right]\right]_{-1}^{1} \\
&-\int_{-1}^{1}\left[(1-t^2)\left[\dot{P}_n(t)\dot{P}_m(t) - \dot{P}_m(t)\dot{P}_n(t)\right]\right] \\
= &\ 0
\end{aligned}$$

Hence $(P_m, P_n) = \int_{-1}^{1} P_m(t)P_n(t)dt = 0$. That is $P_m \perp P_n$. ∎

Theorem 6.4.2 *The Legendre polynomials $P_n(t)$ satisfy the Rodrigues formula*

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left[(t^2-1)^n\right], \quad n = 0, 1, 2, \cdots \tag{6.4.6}$$

Proof : We have

$$P_n(t) = \frac{1}{2^n n!}\sum_{k=0}^{M}\frac{(-1)^k n!}{k!(n-k)!}\frac{(2n-2k)!}{(n-2k)!}t^{n-2k}$$

Since $\frac{d^n}{dt^n}\left[t^{2n-2k}\right] = \frac{(2n-2k)!}{(n-2k)!}t^{n-2k}$, it follows that

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left[\sum_{k=0}^{M}\frac{(-1)^k n!}{k!(n-k)!}t^{2n-2k}\right]$$

It is now clear that the sum $\left[\sum_{k=0}^{M}\frac{(-1)^k n!}{k!(n-k)!}t^{2n-2k}\right]$ is the binomial expansion of $(t^2-1)^n$ and hence

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$$

∎

Definition 6.4.1 *Let $\{f_n(t)\}$ be a sequence of functions in an interval I. A function $F(t,u)$ is said to be a generating function of this sequence if*

$$F(t,u) = \sum_{n=0}^{\infty} f_n(t)u^n \tag{6.4.7}$$

We have the following theorem giving the generating function for the sequence $\{P_n(t)\}$ *of Legendre polynomials.*

Theorem 6.4.3 *For Legendre polynomials* $\{P_n(t)\}$*, we have*

$$(1-2tu+u^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(t)u^n \tag{6.4.8}$$

Proof : Let $|t| \le r$ (arbitary positive number) and $|u| < (1+r^2)^{\frac{1}{2}} - r$. Then

$$\begin{aligned}
|2tu - u^2| &\le 2|t||u| + |u^2| \\
&\le 2r(1+r^2)^{\frac{1}{2}} - 2r^2 + (1+r^2) + r^2 - 2r(1+r^2)^{\frac{1}{2}} \\
&= 1
\end{aligned}$$

So expanding $(1-2tu+u^2)^{-\frac{1}{2}}$ in a binomial series, we get

$$\begin{aligned}
[1-u(2t-u)]^{-\frac{1}{2}} &= 1 + \frac{1}{2}u(2t-u) + \frac{1}{2}\frac{3}{4}u^2(2t-u)^2 + \cdots \\
&\quad + \frac{1.3......(2n-1)}{1.2.....(2n)}u^n(2t-u)^n + \cdots
\end{aligned}$$

The coefficient of u^n in this expression is

$$\begin{aligned}
& \frac{1.3......(2n-1)}{2.4....(2n)}(2t)^n - \frac{1.3......(2n-3)}{2.4....(2n-2)}(2t)^{n-2} + \cdots \\
= \; & \frac{1.3......(2n-1)}{n!}\left[t^n - \frac{n(n-1)}{(2n-1)^2}t^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4}t^{n-4} + \cdots\right] \\
= \; & P_n(t)
\end{aligned}$$

∎

Theorem 6.4.4 *The Legendre polynomials satisfy the following recurrence relation*

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t), \quad n = 1, 2, ... \tag{6.4.9}$$

Proof : Differentiating the relation given by Eq. (6.4.8) with respect to u, we get

$$(t-u)(1-2tu+u^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} nP_n(t)u^{n-1}$$

This gives

$$(t-u)(1-2tu+u^2)^{-\frac{1}{2}} = (1-2tu+u^2)\sum_{n=0}^{\infty} nP_n(t)u^{n-1}$$

That is

$$(t-u)\sum_{n=1}^{\infty} P_n(t)u^n = (1-2tu+u^2)\sum_{n=0}^{\infty} nP_n(t)u^{n-1}$$

Equivalently,

$$\begin{aligned}
& \sum_{n=0}^{\infty} tP_n(t)u^n - \sum_{n=0}^{\infty} P_n(t)u^{n+1} \\
= \; & \sum_{n=1}^{\infty} nP_n(t)u^{n-1} - 2\sum_{n=1}^{\infty} ntP_n(t)u^n \\
& + \sum_{n=0}^{\infty} nP_n(t)u^{n+1}
\end{aligned}$$

Rewriting, we get

$$\begin{aligned}
& \sum_{n=1}^{\infty} tP_n(t)u^n - \sum_{n=1}^{\infty} P_{n-1}(t)u^n \\
= \; & \sum_{n=1}^{\infty}(n+1)P_{n+1}(t)u^n - 2\sum_{n=1}^{\infty} ntP_n(t)u^n \\
& + \sum_{n=1}^{\infty}(n-1)P_{n-1}(t)u^n
\end{aligned}$$

Comparing the coefficient of u^n, we get

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t)$$

∎

Corollary 6.4.1

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(t)dt = \frac{2}{2n+1} \tag{6.4.10}$$

Proof : Recurrence relation gives

$$\begin{aligned}
(2n+1)tP_n(t) &= (n+1)P_{n+1}(t) + nP_{n-1}(t) \\
(2n-1)tP_{n-1}(t) &= nP_n(t) + (n-1)P_{n-2}(t)
\end{aligned}$$

These two equations give

$$\begin{aligned}
0 &= \int_{-1}^{1} \left[tP_n(t)P_{n-1}(t) - tP_{n-1}(t)P_n(t)\right] dt \\
&= \frac{(n+1)}{(2n+1)}\int_{-1}^{1} P_{n+1}(t)P_{n-1}(t) + \frac{n}{(2n+1)}\int_{-1}^{1} P_{n-1}^2(t)dt \\
&\quad -\frac{(n-1)}{(2n-1)}\int_{-1}^{1} P_{n-2}(t)P_n(t) - \frac{n}{(2n-1)}\int_{-1}^{1} P_n^2(t)dt
\end{aligned}$$

This implies that

$$\int_{-1}^{1} P_n^2(t)dt = \frac{(2n-1)}{(2n+1)} \int_{-1}^{1} P_{n-1}^2(t)dt$$

and hence

$$\begin{aligned}\int_{-1}^{1} P_n^2(t)dt &= \frac{(2n-1)}{(2n+1)} \frac{(2n-3)}{(2n-1)} \cdots \frac{2}{1} \int_{-1}^{1} P_0^2(t)dt \\ &= \frac{2}{(2n+1)}\end{aligned}$$

∎

We now examine the properties of the Hermite polynomials. Recall that Hermite polynomial $H_n(t)$ satisfy the differential equation

$$\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2nx = 0$$

and is given by

$$H_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^n n!}{k!(n-2k)!}(2t)^{n-2k}$$

We have the following theorem concerning the orthogonality of $\{H_n(t)\}$ in $L_2(-\infty,\ \infty)$ with respect to the weight function e^{-t^2}.

Theorem 6.4.5 *The Hermite Polynomials $H_n(t)$ are orthogonal set of polynomials in the space $L_2(-\infty,\ \infty)$ with respect to the weight function e^{-t^2}.*

Theorem 6.4.6 *For Hermite polynomials $H_n(t)$, we have the following formulae.*

(i) Rodgriues formula

$$H_n(t) = (-1)e^{t^2}\frac{d^n}{dt^n}e^{-t^2}$$

(ii) Generating function

$$e^{2tu-u^2} = \sum_{n=0}^{\infty} H_n(t)\frac{u^n}{n!}$$

(iii) Recurrence relation

$$H_{n+1}(t) = 2tH_n(t) - 2nH_{n-1}(t), \quad H_0 = 1, H_1 = 2t$$

We now enunciate the properties of the Bessel's function $J_n(t)$.

Theorem 6.4.7 *For each fixed nonnegative integer n, the sequence of Bessel functions $J_n(k_m t)$, where k_m are the zeros of $J_n(k)$, form an orthogonal set in $L_2[0,1]$ with respect to the weight function t. That is*

$$\int_0^1 tJ_n(k_l t)J_n(k_m t)dt = 0, \quad l \neq m \tag{6.4.11}$$

Proof : The Bessel function $J_n(t)$ satisfies the differential equation (refer Eq. (6.3.6))

$$t^2\ddot{J}_n(t) + t\dot{J}_n(t) + (t^2 - n^2)J_n(t) = 0$$

Set $t = ks$, then the above equation reduces to the differential equation

$$\frac{d}{ds}\left[s\dot{J}_n(ks)\right] + \left(-\frac{n^2}{s} + k^2 s\right)J_n(ks) = 0 \tag{6.4.12}$$

Let k_m be the zeros of the Bessel function $J_n(k)$, then we have

$$\frac{d}{ds}\left[s\dot{J}_n(k_l s)\right] + \left(-\frac{n^2}{s} + k_l^2 s\right)J_n(k_l s) \quad = \quad 0 \tag{6.4.13}$$

and

$$\frac{d}{ds}\left[s\dot{J}_n(k_m s)\right] + \left(-\frac{n^2}{s} + k_m^2 s\right)J_n(k_m s) = 0 \tag{6.4.14}$$

Eq. (6.4.13) - Eq. (6.4.14) give

$$\begin{aligned}
& (k_l^2 - k_m^2)\int_0^1 sJ_n(k_l s)J_n(k_m s)ds \\
= & \int_0^1 \left[\frac{d}{ds}\left[s\dot{J}_n(k_l s)\right]J_n(k_m s) - \frac{d}{ds}\left[s\dot{J}_n(k_m s)\right]J_n(k_l s)\right]ds \\
= & \ 0, \quad k_l \neq k_m
\end{aligned}$$

using the fact that $J_n(k_l) = 0 = J_n(k_m)$. ∎

Theorem 6.4.8 *The generating function for the sequence $\{J_n(t)\}$ of Bessel functions is given by*

$$exp\left(\frac{1}{2}t\left(u - \frac{1}{u}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(t)u^n \tag{6.4.15}$$

and hence $J_{-n}(t) = (-1)^n J_n(t)$.

Proof : Expanding $exp\left(\frac{1}{2}t\left(u-\frac{1}{u}\right)\right)$ in powers of u, we get

$$\begin{aligned} exp\left(\frac{1}{2}t\left(u-\frac{1}{u}\right)\right) &= \left(exp\frac{1}{2}tu\right)\left(exp\frac{1}{2}\left(\frac{-t}{u}\right)\right) \\ &= \left(\sum_{k=0}^{\infty}\frac{(tu)^k}{2^k k!}\right)\left(\sum_{l=0}^{\infty}\frac{(-t)^l}{2^l l!\ u^l}\right) \\ &= \sum_{-\infty}^{\infty} c_n(t)u^n \end{aligned}$$

The coefficient $c_n(t)$ of u^n in the above expression is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{t}{2}\right)^{2k+n}}{k!(k+n)!} = J_n(t)$ and hence we get the generating function relation given by Eq. (6.4.15) for $J_n(t)$.

If we replace u by $-\dfrac{1}{v}$ in Eq. (6.4.15), we get

$$\begin{aligned} exp\left(\frac{1}{2}\left(v-\frac{1}{v}\right)t\right) &= \sum_{n=-\infty}^{\infty} J_n(t)(-1)^n v^{-n} \\ &= \sum_{n=-\infty}^{\infty} J_{-n}(t)v^n \end{aligned}$$

Hence, it follows that $\qquad J_{-n}(t) = (-1)^n J_n(t)$

∎

Theorem 6.4.9 *The Bessel functions $J_\mu(t)$ satisfy the following recurrence relations*

$$J_{\mu-1}(t) + J_{\mu+1}(t) = \frac{2\mu}{t}J_\mu(t) \qquad (6.4.16a)$$

$$J_{\mu-1}(t) - J_{\mu+1}(t) = 2J_\mu(t) \qquad (6.4.16b)$$

Proof : We have

$$J_\mu(t) = t^\mu\left(\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{2^{2k+\mu}(k!)\Gamma(\mu+k+1)}\right)$$

Multiplying $J_\mu(t)$ by t^μ and pulling $t^{2\mu}$ under the summation, we have

$$t^\mu J_\mu(t) = \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+2\mu}}{2^{2k+\mu}(k!)\Gamma(\mu+k+1)}$$

Differentiating the above expression we have

$$\begin{aligned}
\frac{d}{dt}\left(t^{\mu}J_{\mu}(t)\right) &= \sum_{k=0}^{\infty}\frac{(-1)^k 2(k+\mu)t^{2k+2\mu-1}}{2^{2k+\mu}(k!)\Gamma(\mu+k+1)} \\
&= t^{\mu}t^{\mu-1}\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{2^{2k+\mu-1}(k!)\Gamma(\mu+k)} \\
&= t^{\mu}J_{\mu-1}(t)
\end{aligned}$$

Thus, we have

$$\frac{d}{dt}\left(t^{\mu}J_{\mu}(t)\right) = t^{\mu}J_{\mu-1}(t)$$

and

$$\frac{d}{dt}\left(t^{-\mu}J_{\mu}(t)\right) = -t^{-\mu}J_{\mu+1}(t)$$

Expanding the LHS of the above expressions, we have

$$\begin{aligned}
\mu t^{\mu-1}J_{\mu} + t^{\mu}\dot{J}_{\mu} &= t^{\mu}J_{\mu-1} \\
-\mu t^{\mu-1}J_{\mu} + t^{\mu}\dot{J}_{\mu} &= -t^{\mu}J_{\mu+1}
\end{aligned}$$

Adding and substracting the above relations, we get

$$\begin{aligned}
J_{\mu-1} + J_{\mu+1} &= \frac{2\mu}{t}J_{\mu}(t) \\
J_{\mu-1} - J_{\mu+1} &= 2\dot{J}_{\mu}(t)
\end{aligned}$$

▮

Corollary 6.4.2 *From the definition of $J_{\mu}(t)$, we have*

$$J_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\sin t, \quad J_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\cos t$$

and hence the above recurrence relations give

$$J_{\frac{3}{2}}(t) = \sqrt{\frac{2}{\pi t}}\left(\frac{\sin t}{t} - \cos t\right)$$

$$J_{-\frac{3}{2}}(t) = \sqrt{\frac{2}{\pi t}}\left(-\frac{\cos t}{t} - \sin t\right)$$

We have the following graphics for the above functions.

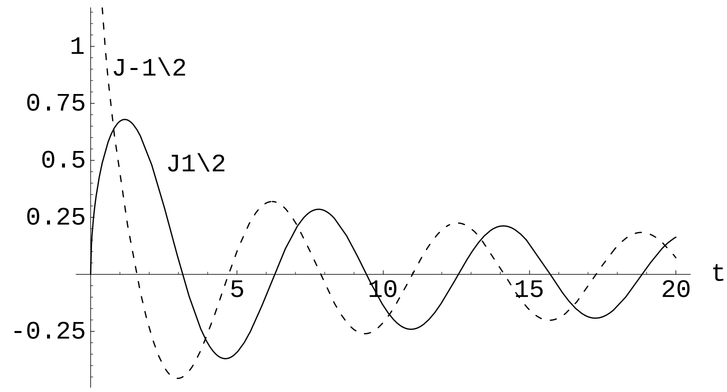

Figure 6.4.2: Sketch of $J_{-1/2}(t)$ and $J_{1/2}(t)$

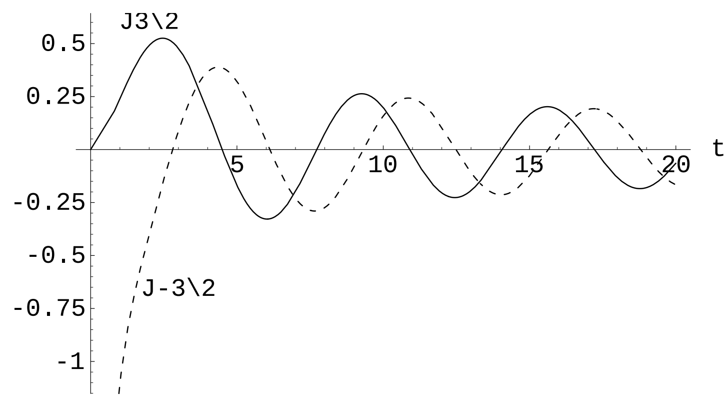

Figure 6.4.3: Sketch of $J_{3/2}(t)$ and $J_{-3/2}(t)$

For more details on varioius topics in this chapter, refer Agarwal and Gupta [1], Brown and Churchill [2], Hochstadt [3] and Kreyzig [4].

6.5 Exercises

1. Let the functions $f, g : \Re \to \Re$ be defined as follows

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}, \quad g(t) = \begin{cases} \dfrac{1-\cos t}{t^2}, & t \neq 0 \\ \dfrac{1}{2}, & t = 0 \end{cases}$$

   Show that f and g are analytic at $t = 0$.

2. Find the series solution of the following IVPs

   (i) $\ddot{x} + t\dot{x} - 2x = 0, \ \ x(0) = 1, \ \ \dot{x}(0) = 0$

   (ii) $t(2-t)\ddot{x} - 6(t-1)\dot{x} - 4x = 0, \ \ x(1) = 1, \ \ \dot{x}(1) = 0$

   (iii) $\ddot{x} + e^t\dot{x} + (1+t^2)x = 0, \ \ x(0) = 1, \ \ \dot{x}(0) = 0$

   (iv) $\ddot{x} - (\sin t)x = 0, \ \ x(\pi) = 1, \ \ \dot{x}(\pi) = 0$

3. Show that the general solution of the Chebyshev differential equation

$$(1-t^2)\ddot{x} - t\dot{x} + a^2x = 0$$

   is given by

$$\begin{aligned} x(t) &= c_0\left[1 + \sum_{n=1}^{\infty} \frac{(-a)^2(2^2-a^2)......((2n-2)^2-a^2)}{(2n)!}t^{2n}\right] \\ &+ c_1\left[t + \sum_{n=1}^{\infty} \frac{(1-a^2)(3^2-a^2)......((2n-1)^2-a^2)}{(2n-1)!}t^{2n+1}\right] \end{aligned}$$

4. Show that the two linearly independent solutions of the following differential equation

$$t^2\ddot{x} + t(t-\frac{1}{2})\dot{x} + \frac{1}{2}x = 0$$

   are given by

$$\begin{aligned} x_1(t) &= |t|\sum_{n=0}^{\infty}(-1)^n\frac{(2t)^n}{(2n+1)(2n-1)......3.1} \\ x_2(t) &= |t|^{\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!} \end{aligned}$$

5. Show that the two linearly independent solutions of the differential equation

$$t(1-t)\ddot{x} + (1-t)\dot{x} - x = 0$$

are given by

$$
\begin{aligned}
x_1(t) &= 1+\sum_{n=1}^{\infty}\frac{1.2.5......((n-1)^2+1)}{(n!)^2}t^n \\
x_2(t) &= x_1(t)ln|t| + 2\left(\sum_{n=1}^{\infty}\frac{1.2.5.....((n-1)^2+1)}{(n!)^2}\right. \\
& \quad \left. \times \left(\sum_{k=1}^{n}\frac{k-2}{k((k-1)^2+1)}\right)t^n\right.
\end{aligned}
$$

6. Using Rolle's theorem show that between two consucative zeros of $J_n(t)$, there exists precisely one zero of $J_{n+1}(t)$.

7. Prove the following identities for Bessel functions $J_\mu(t),\ \mu \in \Re$.

   (i) $\int t^\mu J_{\mu-1}(t)dt = t^\mu J_\mu(t) + c$

   (ii) $\int t^{-\mu} J_{\mu+1}(t)dt = -t^{-\mu} J_\mu(t) + c$

   (iii) $\int J_{\mu+1}(t)dt = \int J_{\mu-1}(t)dt - 2J_\mu(t)$

8. Show that

$$
\|J_n(k_{lm}t)\|^2 = \int_0^1 tJ_n^2(k_{lm}t)dt = \frac{1}{2}J_{n+1}^2(k_{lm})
$$

9. Using Rodrigues formula, show that

$$
P_n(0) = \begin{cases} 0, & n \text{ is odd} \\ (-1)^{n/2}\dfrac{1.3....n-1}{2.4.....n}, & n \text{ is even} \end{cases}
$$

10. Let $p(t)$ be a polynomial of degree n. Show that $p(t)$ is orthogonal to all Legendre polynomials of degree strictly less than n.

References

[1] Ravi P. Agarwal and Ramesh C. Gupta, *Essentials of Ordinary differential Equations, McGraw Hill Book Company, 1993*

[2] James Ward Brown and Rulev Churchill, *Fourier Series and Boundary Value Problems, McGraw Hill, Inc., 1993*

[3] Harry Hochstadt, *Differential Equations, A Modern Approach, Dover Publications, Inc., 1964*

[4] Erwin Kreyszig, *Advanced Engineering Mathematics, John Wiley and Sons , Inc., 1999*