Solution Paper – II

Mathematical Methods in Engineering & Science

Exercise 1

Determine if the following system is consistent or not

$$\begin{cases} 3x_1 + 4x_2 + x_3 = 1 \\ 2x_1 + 3x_2 = 0 \\ 4x_1 + 3x_2 - x_3 = -2. \end{cases}$$

Solution.

Step 1: To eliminate the variable x_1 from the second and third equations we perform the operations $r_2 \leftarrow 3r_2 - 2r_1$ and $r_3 \leftarrow 3r_3 - 4r_1$ obtaining the system

$$\begin{cases} 3x_1 + 4x_2 + x_3 = 1 \\ x_2 - 2x_3 = -2 \\ -7x_2 - 7x_3 = -10. \end{cases}$$

Step 2: Now, to eliminate the variable x_3 from the third equation we apply the operation $r_3 \leftarrow r_3 + 7r_2$ to obtain

$$\begin{cases} 3x_1 + 4x_2 + x_3 = 1 \\ x_2 - 2x_3 = -2 \\ -21x_3 = -24. \end{cases}$$

Solving the system by the method of backward substitution we find the unique solution $x_1 = -\frac{3}{7}, x_2 = \frac{2}{7}, x_3 = \frac{8}{7}$. Hence the system is consistent ■

Exercise 2

Show that $(5 + 4s - 7t, s, t)$, *where* $s, t \in \mathbb{R}$, *is a solution to the equation*

$$x_1 - 4x_2 + 7x_3 = 5.$$

Solution

$x_1 = 5 + 4s - 7t, x_2 = s$, and $x_3 = t$ is a solution to the given equation because

$$x_1 - 4x_2 + 7x_3 = (5 + 4s - 7t) - 4s + 7t = 5. ■$$

A linear equation can have infinitely many solutions, exactly one solution or no solutions at all.

Exercise — 3 Find the complex Fourier series of the function

$$f(t) = \begin{cases} 0 & -\pi < t < 0, \\ 1 & 0 < t < \pi. \end{cases}$$

Solution — Since the period is 2π, so $p = \pi$, and the complex Fourier series is given by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{int}$$

with

$$c_0 = \frac{1}{2\pi} \int_0^{\pi} \mathrm{d}t = \frac{1}{2},$$

$$c_n = \frac{1}{2\pi} \int_0^{\pi} \mathrm{e}^{-int} \mathrm{d}t = \frac{1 - \mathrm{e}^{-in\pi}}{2\pi n\mathrm{i}} = \begin{cases} 0 & n = \text{even}, \\ \frac{1}{\pi ni} & n = \text{odd}. \end{cases}$$

Therefore the complex series is

$$f(t) = \frac{1}{2} + \frac{1}{\mathrm{i}\pi} \left(\cdots - \frac{1}{3}\mathrm{e}^{-\mathrm{i}3t} - \mathrm{e}^{-\mathrm{i}t} + \mathrm{e}^{\mathrm{i}t} + \frac{1}{3}\mathrm{e}^{\mathrm{i}3t} + \cdots \right).$$

It is clear that

$$c_{-n} = \frac{1}{\pi(-n)\mathrm{i}} = \frac{1}{\pi n(-\mathrm{i})} = c_n^*$$

as we expect, sine $f(t)$ is real. Furthermore, since

$$\mathrm{e}^{int} - \mathrm{e}^{-int} = 2\mathrm{i} \sin nt,$$

the Fourier series can be written as

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots \right).$$

This is also what we expected, since $f(t) - \frac{1}{2}$ is an odd function, and

$$a_n = c_n + c_{-n} = \frac{1}{\pi n\mathrm{i}} + \frac{1}{\pi(-n)\mathrm{i}} = 0,$$

$$b_n = \mathrm{i}(c_n - c_{-n}) = \mathrm{i} \left(\frac{1}{\pi n\mathrm{i}} - \frac{1}{\pi(-n)\mathrm{i}} \right) = \frac{2}{\pi n}.$$

Exercise 4
Use Gauss-Jordan elimination to transform the following matrix first into row-echelon form and then into reduced row-echelon form

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

Solution.
By following the steps in the Gauss-Jordan algorithm we find

Step 1: $r_3 \leftarrow \frac{1}{3}r_3$

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{pmatrix}$$

Step 2: $r_1 \leftrightarrow r_3$

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

Step 3: $r_2 \leftarrow r_2 - 3r_1$

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

Step 4: $r_2 \leftarrow \frac{1}{2}r_2$

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

Step 5: $r_3 \leftarrow r_3 - 3r_2$

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Step 6: $r_1 \leftarrow r_1 + 3r_2$

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 5r_3$ and $r_2 \leftarrow r_2 - r_3$

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} \blacksquare$$

It can be shown that no matter how the elementary row operations are varied, one will always arrive at the same reduced row-echelon form; that is the reduced row echelon form is unique (See Theorem 68). On the contrary row-echelon form is **not** unique. However, the number of leading 1's of two different row-echelon forms is the same (this will be proved in Chapter 4). That is, two row-echelon matrices have the same number of nonzero rows. This number is called the **rank** of A and is denoted by $rank(A)$. In Chapter 6, we will prove that if A is an $m \times n$ matrix then $rank(A) \leq n$ and $rank(A) \leq m$.

Exercise - 5 Find the Fourier series for $f(t)$ which is defined as

$$f(t) = t \text{ for } -L < t \leq L, \quad \text{and} \quad f(t+2L) = f(t).$$

Solution

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right),$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} t\,\mathrm{d}t = 0,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} t\cos\frac{n\pi t}{L}\mathrm{d}t = \frac{1}{L}\left[\frac{L}{n\pi}t\sin\frac{n\pi t}{L} + \left(\frac{L}{n\pi}\right)^2\cos\frac{n\pi t}{L}\right]_{-L}^{L} = 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} t\sin\frac{n\pi t}{L}\mathrm{d}t$$

$$= \frac{1}{L}\left[-\frac{L}{n\pi}t\cos\frac{n\pi t}{L} + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi t}{L}\right]_{-L}^{L} = -\frac{2L}{n\pi}\cos n\pi.$$

Thus

$$f(t) = \frac{2L}{\pi}\sum_{n=1}^{\infty} -\frac{1}{n}\cos n\pi \sin\frac{n\pi t}{L} = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{L}$$

$$= \frac{2L}{\pi}\left(\sin\frac{\pi t}{L} - \frac{1}{2}\sin\frac{2\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} - \cdots\right). \quad (1.19)$$

The convergence of this series is shown in Fig. 1.3, where S_N is the partial sum defined as

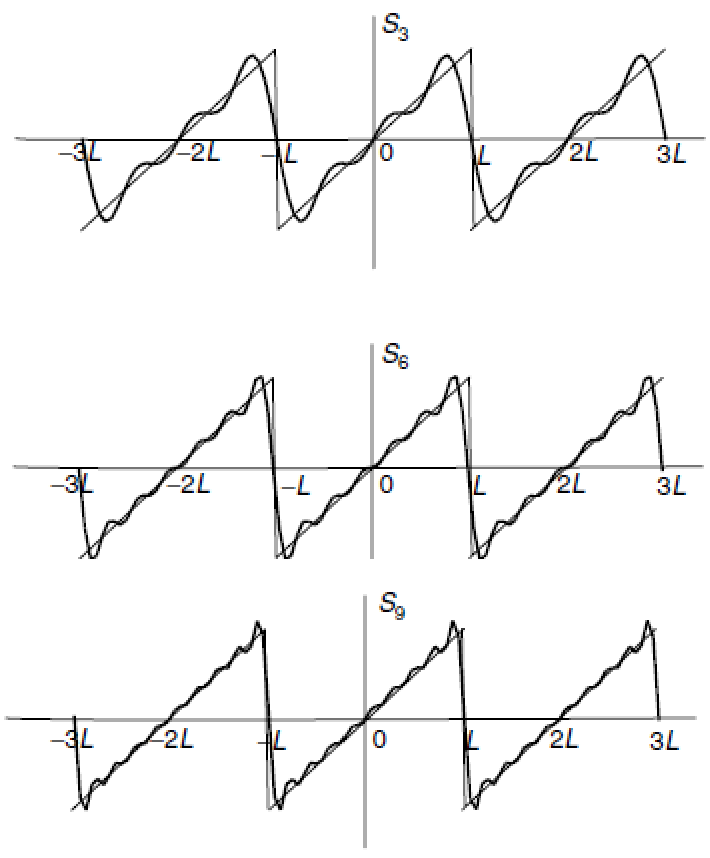

Fig. 1.3. The convergence of the Fourier series for the periodic function whose definition in one period is $f(t) = t,\ -L < t < L$. The first N terms approximations are shown as S_N

$$S_N = \frac{2L}{\pi}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{L}.$$

Note the increasing accuracy with which the terms approximate the function. With three terms, S_3 already looks like the function. Except for the Gibbs' phenomenon, a very good approximation is obtained with S_9.

Exercise - 6 Find the Fourier series of the function whose definition in one period is

$$f(t) = t^3, \quad -L < t < L.$$

Solution — Integrating the Fourier series for t^2 in the required range term-by-term

$$\int t^2 \mathrm{d}t = \int \left[\frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}t\right]\mathrm{d}t,$$

we obtain

$$\frac{1}{3}t^3 = \frac{L^2}{3}t + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{L}{n\pi}\sin\frac{n\pi}{L}t + C.$$

We can find the integration constant C by looking at the values of both sides of this equation at $t = 0$. Clearly $C = 0$. Furthermore, since in the range of $-L < t < L$,

$$t = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{L}t,$$

therefore the Fourier series of t^3 in the required range is

$$t^3 = \frac{2L^3}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{L}t + \frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi}{L}t.$$

Exercise - 7 Find the domain of $f(x, y) = x\ln(y^2 - x)$.

Solution. The expression $x\ln(y^2 - x)$ is defined only when $y^2 - x > 0$. That is $y^2 > x$. The curve $y^2 = x$ separates the plane into two regions, one satisfying the inequality $y^2 > x$, the other satisfying $y^2 < x$. To find out which region is determined by the inequality $y^2 > x$. Pick any point in one of the regions and test whether it satisfies the inequality. If it does, then by 'connectivity', that whole region is the one satisfying $y^2 > x$, otherwise, it must be the other region. For example, pick the point $(3, 2)$. Since $2^2 > 3$, the region satisfying $y^2 > x$ is the one containing $(3, 2)$. Thus, domain of f is $\{(x, y) \in \mathbb{R}^2 \mid y^2 > x\}$.

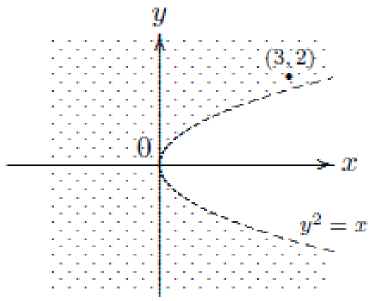

Figure 39 Domain of $x\ln(y^2 - x)$

Exercise - 8 Find the distance between the skew lines:

$$L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t$$

$$L_2 : x = 2s, y = 3 + s, z = -3 + 4s$$

Solution. As L_1 and L_2 are skew, they are contained in two parallel planes respectively. A normal to these two parallel planes is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \langle 13, -6, -5 \rangle.$$

Let $s = 0$ in L_2. We get the point $(0, 3, -3)$ on L_2. Therefore, an equation of the plane containing L_2 is $\langle x - 0, y - 3, z - (-3) \rangle \cdot \langle 13, -6, -5 \rangle = 0$. That is $13x - 6y - 5z + 3 = 0$. Let $= 0$ in L_1. We get the point $(1, -2, 4)$ on L_1. Thus, the distance between L_1 and L_2 is give by

$$\frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}}.$$

Exercise - 9 Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ t - 2, & 2 \le t. \end{cases}$$

Solution.
We do this by definition:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\, dt = \int_0^2 e^{-st} dt + \int_2^\infty (t-2)e^{-st} dt \\ &= \frac{1}{-s}e^{-st}\Big|_{t=0}^{2} + (t-2)\frac{1}{-s}e^{-st}\Big|_{t=2}^{\infty} - \int_2^A \frac{1}{-s}e^{-st} dt \\ &= \frac{1}{-s}(e^{-2s}-1) + (0-0) + \frac{1}{s}\,\frac{1}{-s}e^{-st}\Big|_{t=2}^{\infty} = \frac{1}{-s}(e^{-2s}-1) + \frac{1}{s^2}e^{-2s} \end{aligned}$$

Exercise - 10 Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and find all eigenvectors.

Solution. The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = 0$, or

$$(\lambda - 1)(\lambda - 3) = 0.$$

Hence $\lambda = 1$ or 3. The eigenvector equation $(A - \lambda I_n)X = 0$ reduces to

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{aligned} (2-\lambda)x + y &= 0 \\ x + (2-\lambda)y &= 0. \end{aligned}$$

Taking $\lambda = 1$ gives

$$\begin{aligned} x + y &= 0 \\ x + y &= 0, \end{aligned}$$

which has solution $x = -y$, y arbitrary. Consequently the eigenvectors corresponding to $\lambda = 1$ are the vectors $\begin{bmatrix} -y \\ y \end{bmatrix}$, with $y \neq 0$.

Taking $\lambda = 3$ gives

$$\begin{aligned} -x + y &= 0 \\ x - y &= 0, \end{aligned}$$

which has solution $x = y$, y arbitrary. Consequently the eigenvectors corresponding to $\lambda = 3$ are the vectors $\begin{bmatrix} y \\ y \end{bmatrix}$, with $y \neq 0$.