Outline

Operational Fundamentals of Linear Algebra
- Range and Null Space: Rank and Nullity
- Basis
- Change of Basis
- Elementary Transformations

Range and Null Space: Rank and Nullity

Range and Null Space: Rank and Nullity
Basis
Change of Basis
Elementary Transformations

Consider $\mathbf{A} \in R^{m \times n}$ as a mapping

$$\mathbf{A} : R^n \rightarrow R^m, \qquad \mathbf{A}\mathbf{x} = \mathbf{y}, \qquad \mathbf{x} \in R^n, \qquad \mathbf{y} \in R^m.$$

Observations

1. Every $\mathbf{x} \in R^n$ has an image $\mathbf{y} \in R^m$, but every $\mathbf{y} \in R^m$ need not have a pre-image in R^n.

   Range (or range space) as subset/subspace of co-domain: containing images of all $\mathbf{x} \in R^n$.

2. Image of $\mathbf{x} \in R^n$ in R^m is unique, but pre-image of $\mathbf{y} \in R^m$ need not be.
   It may be non-existent, unique or infinitely many.

   Null space as subset/subspace of domain: containing pre-images of only $\mathbf{0} \in R^m$.

Range and Null Space: Rank and Nullity

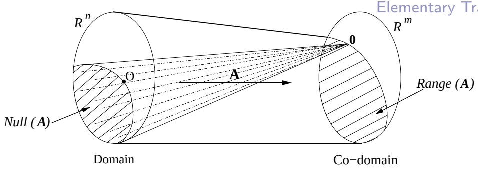

Figure: Range and null space: schematic representation

Question: What is the dimension of a vector space?

Linear dependence and independence: Vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r$ in a vector space are called linearly independent if

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r = \mathbf{0} \quad \Rightarrow \quad k_1 = k_2 = \cdots = k_r = 0.$$

$$\begin{aligned}
Range(\mathbf{A}) &= \{\mathbf{y} : \mathbf{y} = \mathbf{A}\mathbf{x}, \ \ \mathbf{x} \in R^n\} \\
Null(\mathbf{A}) &= \{\mathbf{x} : \mathbf{x} \in R^n, \ \ \mathbf{A}\mathbf{x} = \mathbf{0}\} \\
Rank(\mathbf{A}) &= \dim Range(\mathbf{A}) \\
Nullity(\mathbf{A}) &= \dim Null(\mathbf{A})
\end{aligned}$$

Basis

Take a set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_r$ in a vector space.
Question: Given a vector $\mathbf{v}$ in the vector space, can we describe it as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{V}\mathbf{k},$$

where $\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_r]$ and $\mathbf{k} = [k_1 \quad k_2 \quad \cdots \quad k_r]^T$?
Answer: Not necessarily.

Span, denoted as $< \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r >$: the subspace described/generated by a set of vectors.

Basis:

> *A basis of a vector space is composed of an ordered* minimal *set of vectors spanning the entire space.*

The basis for an n-dimensional space will have exactly n members, all linearly independent.

Basis

Orthogonal basis: $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ with

$$\mathbf{v}_j^T \mathbf{v}_k = 0 \quad \forall\, j \neq k.$$

Orthonormal basis:

$$\mathbf{v}_j^T \mathbf{v}_k = \delta_{jk} = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases}$$

Members of an **orthonormal** basis form an **orthogonal** matrix.
Properties of an orthogonal matrix:

$$\begin{aligned} \mathbf{V}^{-1} &= \mathbf{V}^T \text{ or } \mathbf{V}\mathbf{V}^T = \mathbf{I}, \quad \text{and} \\ \det \mathbf{V} &= +1 \text{ or } -1, \end{aligned}$$

Natural basis:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Change of Basis

Suppose $\mathbf{x}$ represents a vector (point) in R^n in some basis.
Question: If we change over to a new basis $\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$, how does the representation of a vector change?

$$\begin{aligned} \mathbf{x} &= \bar{x}_1\mathbf{c}_1 + \bar{x}_2\mathbf{c}_2 + \cdots + \bar{x}_n\mathbf{c}_n \\ &= [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_n] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}. \end{aligned}$$

With $\mathbf{C} = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_n]$,

new to old coordinates: $\mathbf{C}\bar{\mathbf{x}} = \mathbf{x}$ *and*
old to new coordinates: $\bar{\mathbf{x}} = \mathbf{C}^{-1}\mathbf{x}$.

Note: Matrix $\mathbf{C}$ is invertible. *How?*
Special case with $\mathbf{C}$ orthogonal:

orthogonal coordinate transformation.

Change of Basis

Question: And, how does basis change affect the representation of a linear transformation?

Consider the mapping $\mathbf{A} : R^n \rightarrow R^m, \quad \mathbf{Ax} = \mathbf{y}.$

Change the basis of the domain through $\mathbf{P} \in R^{n \times n}$ and that of the co-domain through $\mathbf{Q} \in R^{m \times m}$.

New and old vector representations are related as

$$\mathbf{P}\bar{\mathbf{x}} = \mathbf{x} \quad \text{and} \quad \mathbf{Q}\bar{\mathbf{y}} = \mathbf{y}.$$

Then, $\mathbf{Ax} = \mathbf{y} \Rightarrow \bar{\mathbf{A}}\bar{\mathbf{x}} = \bar{\mathbf{y}}$, with

$$\boxed{\bar{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{AP}}$$

Special case: $m = n$ and $\mathbf{P} = \mathbf{Q}$ gives a **similarity transformation**

$$\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$$

Elementary Transformations

Observation: Certain reorganizations of equations in a system have no effect on the solution(s).

Elementary Row Transformations:

1. interchange of two rows,
2. scaling of a row, and
3. addition of a scalar multiple of a row to another.

Elementary Column Transformations: Similar operations with columns, equivalent to a corresponding *shuffling* of the *variables* (unknowns).

Elementary Transformations

Equivalence of matrices: An elementary transformation defines an equivalence relation between two matrices.

Reduction to normal form:

$$\mathbf{A}_N = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Rank invariance: Elementary transformations do not alter the rank of a matrix.

Elementary transformation as matrix multiplication:

> *an elementary row transformation on a matrix is equivalent to a* pre-multiplication *with an* elementary matrix, *obtained through the same row transformation on the identity matrix (of appropriate size).*

Similarly, an elementary column transformation is equivalent to *post-multiplication* with the corresponding elementary matrix.

Points to note

- Concepts of range and null space of a linear transformation.
- Effects of change of basis on representations of vectors and linear transformations.
- Elementary transformations as tools to modify (simplify) systems of (simultaneous) linear equations.

Necessary Exercises: **2,4,5,6**