4.4 Momentum and heat transfer analogies

Consider a fluid flows in a circular pipe in a laminar low (fig.6.6). The wall of the pipe is maintained at $T_w$temperature, which is higher than the flowing fluid temperature. The fluid being in relatively lower temperature than the wall temperature will get heated as it flows through the pipe. Moreover, the radial transport of the momentum in the pipe occurs as per the Newton's law of viscosity. For a circular pipe momentum transport and heat transport may be written in a similar way as shown in the eq. 4.28,

Momentum flux = momentum diffusivity × gradient of concentration of momentum

$$\tau = -\nu \frac{d}{dr}(\rho u_z) \qquad 4.28(a)$$

It may be noted that the fluid velocity (u_z) is a function of radius of the pipe.

Heat flux= thermal diffusivity × gradient of concentration of heat energy

$$q = \alpha \frac{d}{dy}(\rho c_p T) \qquad 4.28(b)$$

where $\alpha = \text{Thermal diffusion} = \frac{k}{\rho c_p}$

Now, the question comes, why are we discussion about the similarities? The answer is straight forward that it is comparatively easy to experimentally/theoretically evaluate the momentum transport under various conditions. However, the heat transport is not so easy to find out. Therefore, we will learn different analogies to find the heat transport relations.

Equation 4.28 is for the laminar flow but if the flow is turbulent, eddies are generated. Eddy is a lump/chunk of fluid elements that move together. Thus it may be assumed that the eddies are the molecules of the fluid and are responsible for the transport of momentum and heat energy in the turbulent flow. Therefore, in turbulent situation the momentum and heat transport is not only by the molecular diffusion but also by the eddy diffusivities.

Thus, turbulent transport of momentum and turbulent transport of heat may be represented by eq. 4.29a and 4.29b, respectively.

$$\tau = -(\nu + \epsilon_m)\frac{d}{dr}(\rho u_z) \qquad (4.29a)$$

$$q = (\alpha + \epsilon_h)\frac{d}{dr}(\rho c_p T) \qquad (4.29b)$$

The terms ϵ_m, and ϵ_h represent the eddy diffusivities for momentum and heat, respectively. At the wall of the pipe, the momentum equation (eq. 4.29a) becomes,

$$\tau_w = -(\nu + \epsilon_m)\frac{d}{dr}(\rho u_z)\Big|_{r=R} \qquad (4.30)$$

$$\tau_w = \frac{1}{2} f \rho \overline{u_z} \qquad (4.31)$$

Where *f* is the fanning friction factor (ratio of shear force to inertial force) and $\overline{u_z}$ is the average fluid velocity.

Equation eq.4.30 can be rearranged as,

$$-\frac{d\hat{u}_z}{dr}\Big|_{r=R} = \frac{\tau_w}{(\nu + \epsilon_m)\rho\bar{u}_z}$$

$$\text{where,} \qquad \hat{u}_z = \text{dimensionless velocity } \frac{u_z(r)}{\bar{u}_z}$$

Using eq.4.31 in the above equation,

$$-\frac{d\hat{u}_z}{dr}\Big|_{r=R} = \frac{f\bar{u}_z}{2(\nu+\epsilon_m)} \qquad (4.32)$$

The eq.4.32 is the dimensionless velocity gradient at the wall using momentum transport. We may get the similar relation using heat transport as shown below. Wall heat flux can be written as,

$$(\alpha + \epsilon_H)\frac{d}{dr}(\rho c_p T)\Big|_{r=R} = q'_w = h\,(T_w - T_{av})$$

Where T_{av} is the wall temperature and the T_{av} is the average temperature of the fluid. Thus, the dimensionless temperature gradient at the wall using heat transfer will be,

$$\frac{d\hat{T}}{dr}\Big|_{r=R} = \frac{h}{\rho c_p(\alpha+\varepsilon_H)} \qquad (4.33)$$

Where the heat transfer coefficient is represented by h and dimensionless temperature is represented by $\hat{T} = \frac{T}{T_w - T_{av}}$.

Based on the above discussion many researchers have given their analogies. These analogies are represented in the subsequent section.

4.4.1 Reynolds analogy

Reynolds has taken the following assumptions to find the analogy between heat and momentum transport.

1. Gradients of the dimensionless parameters at the wall are equal.
2. The diffusivity terms are equal. That is

Thus if we use the above assumptions along with the eq.4.32 and 4.33,

$$(\nu + \epsilon_M) = (\alpha + \epsilon_H)$$

Thus if we use the above assumptions along with the eq.4.32 and 4.33,

$$\frac{h}{\rho c_p \bar{v}_z} = \frac{f}{2}$$

On simplifying

$$\frac{\frac{hd}{k}}{\left(\frac{\mu C_P}{k}\right)\left(\frac{d\rho \bar{v}_z}{\mu}\right)} = \frac{f}{2}$$

$$\frac{Nu_d}{Pr\ Re_d} = \frac{f}{2}$$

$$St_d = \frac{f}{2} \qquad (4.34)$$

Equation 4.34 is known as Reynolds's analogy.
The above relation may also be written in terms of the Darcy's friction factor (fD) instead of fanning friction factor (f_D = 4f)
Where Stanton number (St) is defined as,

$$St_d = \frac{h}{\rho C_p \bar{v}_z} = \frac{h\,\Delta T}{\rho C_p \bar{v}_z\,\Delta T} = \frac{heat\ transfer\ by\ convection}{heat\ transfer\ by\ bulk\ flow}$$

The advantage of the analogy lies in that the *h* may not be available for certain geometries/situations however, for which *f* value may be available as it is easier to perform momentum transport experiments and then to calculate the *f*. Thus by using the eq.4.34 the *h* may be found out without involving into the exhaustive and difficult heat transfer experiments.

4.4.2 The Chilton-Colburn analogy

The Reynolds analogy does not always give satisfactory results. Thus, Chilton and Colburn experimentally modified the Reynolds' analogy. The empirically modified Reynolds' analogy is known as Chilton-Colburn analogy and is given by eq.4.35,

$$St = \frac{\frac{f}{2}}{Pr^{\frac{2}{3}}} \qquad (4.35a)$$

or

$$(St)(Pr^{\frac{2}{3}}) = \frac{f}{2} \qquad (4.35b)$$

It can be noted that for unit Prandtl number the Chilton-Colburn analogy becomes Reynolds analogy.

4.4.3 The Pradntl analogy

In the turbulent core the transport is mainly by eddies and near the wall, that is laminar sub-layer, the transport is by molecular diffusion. Therefore, Prandtl modified the above two analogies using universal velocity profile while driving the analogy (eq. 4.36).

$$St = \frac{\frac{f}{2}}{1+5\sqrt{\frac{f}{2}}(Pr-1)} \qquad (4.36)$$

4.4.4 The Van Karman analogy

Though Prandtl considered the laminar and turbulent laminar sublayers but did not consider the buffer zone. Thus, Van Karman included the buffer zone into the Prandtl analogy to further improve the analogy.

$$St = \frac{\frac{f}{2}}{1+5\sqrt{\frac{f}{2}}\left[(Pr-1)+\ln\left\{1+\frac{5}{6}(Pr-1)\right\}\right]} \tag{4.37}$$