2. Heat conduction concepts, thermal resistance, and the overall heat transfer coefficient

It is the fire that warms the cold, the cold that moderates the heat. . . the general coin that purchases all things. . .

Don Quixote, **M. de Cervantes, 1615**

2.1 The heat diffusion equation

Objective

We must now develop some ideas that will be needed for the design of heat exchangers. The most important of these is the notion of an overall heat transfer coefficient. This is a measure of the general resistance of a heat exchanger to the flow of heat, and usually it must be built up from analyses of component resistances. Although we shall count radiation among these resistances, this overall heat transfer coefficient is most often dominated by convection and conduction.

We need to know values of $\overline{h}$ to handle convection. Calculating $\overline{h}$ becomes sufficiently complex that we defer it to Chapters 6 and 7. For the moment, we shall take the appropriate value of $\overline{h}$ as known information and concentrate upon its use in the overall heat transfer coefficient.

The heat conduction component also becomes more complex than the planar analyses we did in Chapter 1. But its calculation is within our present scope. Therefore we devote this Chapter to deriving the full heat conduction, or heat diffusion, equation, solving it in some fairly straightforward cases, and using our results in the overall coefficient. We undertake that task next.

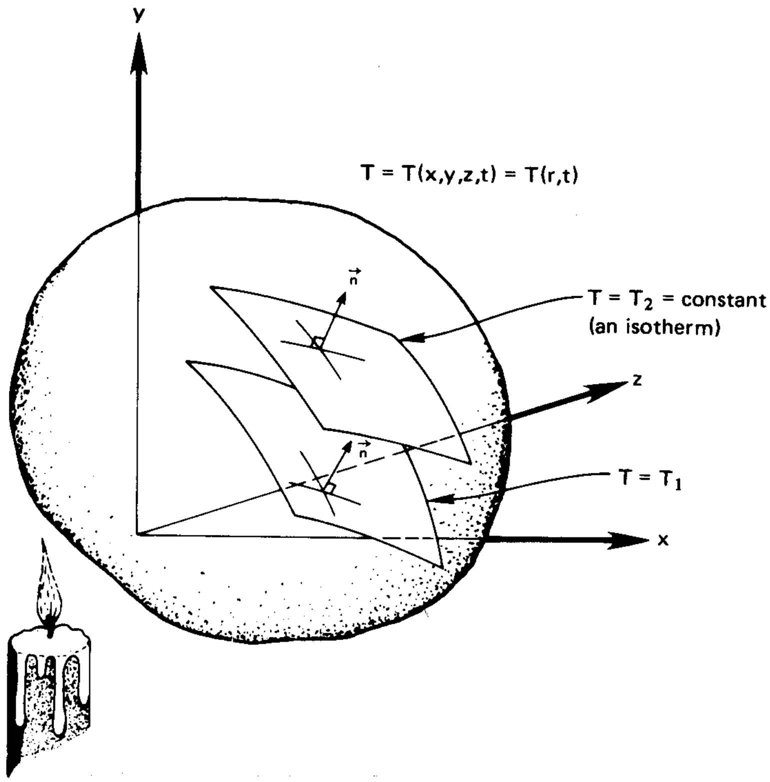

Figure 2.1 A three-dimensional, transient temperature field.

Consider the general temperature distribution in a three-dimensional body as depicted in Fig. 2.1. For some reason, say heating from one side, the temperature of the body varies with time and space. This field $T = T(x, y, z, t)$ or $T(\vec{r}, t)$, defines instantaneous isothermal surfaces, T_1, T_2, and so on.

We next consider a very important vector associated with the scalar, T. The vector that has both the magnitude and direction of the maximum increase of temperature at each point is called the *temperature gradient*, ∇T:

$$\nabla T \equiv \vec{i}\,\frac{\partial T}{\partial x} + \vec{j}\,\frac{\partial T}{\partial y} + \vec{k}\,\frac{\partial T}{\partial z} \tag{2.1}$$

Fourier's law

"Experience"—that is, physical observation—suggests two things about the heat flow that results from temperature nonuniformities in a body. These are:

$$\frac{\vec{q}}{|\vec{q}|} = -\frac{\nabla T}{|\nabla T|} \qquad \begin{cases}\text{This says that } \vec{q} \text{ and } \nabla T \text{ are exactly opposite one} \\ \text{another in direction}\end{cases}$$

and

$$|\vec{q}| \propto |\nabla T| \qquad \begin{cases}\text{This says that the magnitude of the heat flux is di-} \\ \text{rectly proportional to the temperature gradient}\end{cases}$$

Notice that the heat flux is now written as a quantity that has a specified direction as well as a specified magnitude. Fourier's law summarizes this physical experience succinctly as

$$\boxed{\vec{q} = -k\nabla T} \tag{2.2}$$

which resolves itself into three components:

$$q_x = -k\frac{\partial T}{\partial x} \qquad q_y = -k\frac{\partial T}{\partial y} \qquad q_z = -k\frac{\partial T}{\partial z}$$

The coefficient k—the thermal conductivity—also depends on position and temperature in the most general case:

$$k = k[\vec{r}, T(\vec{r}, t)] \tag{2.3}$$

Fortunately, most materials (though not all of them) are very nearly homogeneous. Thus we can usually write $k = k(T)$. The assumption that we really want to make is that k is constant. Whether or not that is legitimate must be determined in each case. As is apparent from Fig. 2.2 and Fig. 2.3, k almost always varies with temperature. It always rises with T in gases at low pressures, but it may rise or fall in metals or liquids. The problem is that of assessing whether or not k is approximately constant in the range of interest. We could safely take k to be a constant for iron between 0° and 40°C (see Fig. 2.2), but we would incur error between −100° and 800°C.

It is easy to prove (Problem 2.1) that if k varies linearly with T, and if heat transfer is plane and steady, then $q = k\Delta T/L$, with k evaluated at the average temperature in the plane. If heat transfer is not planar

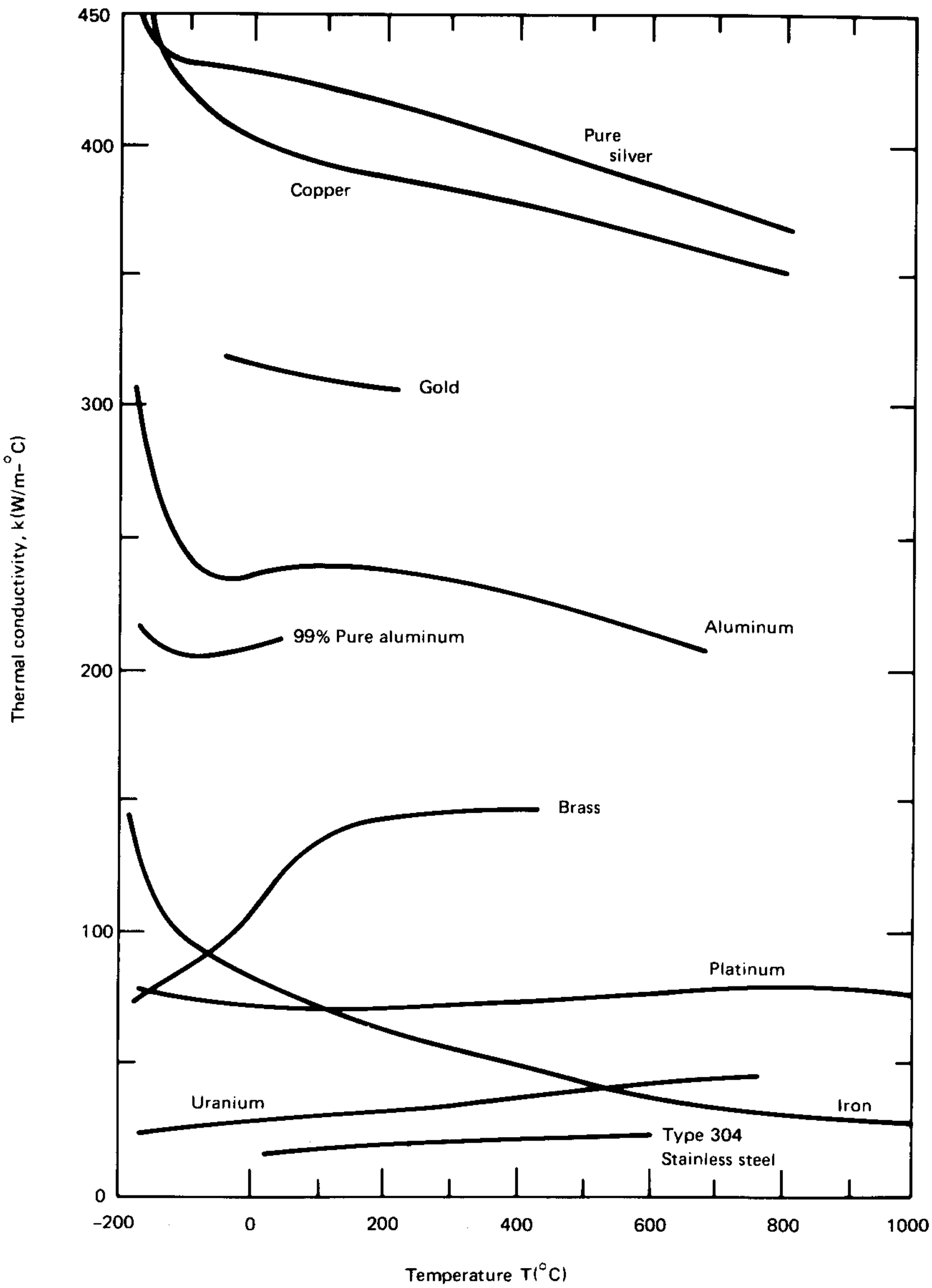

Figure 2.2 Variation of thermal conductivity of metallic solids with temperature

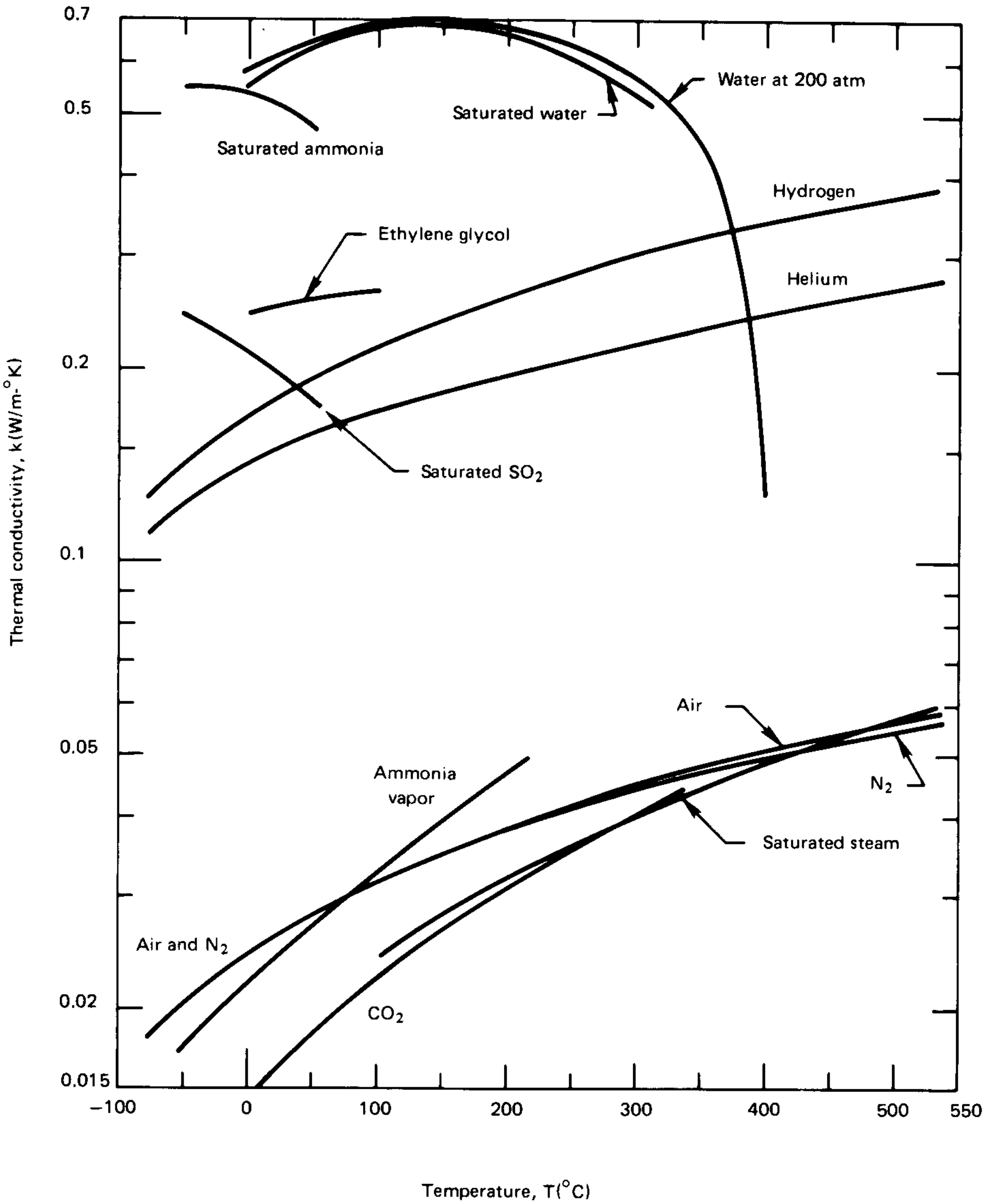

Figure 2.3 The temperature dependence of the thermal conductivity of liquids and gases that are either saturated or at 1 atm pressure.

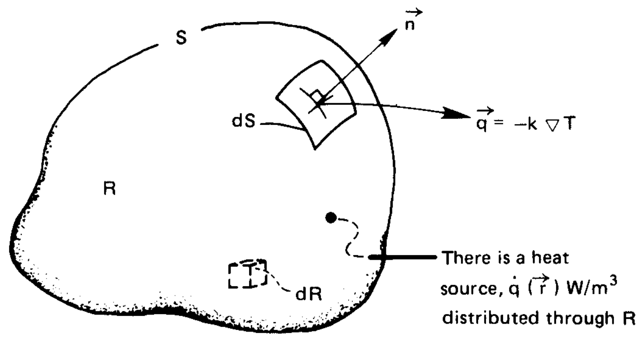

Figure 2.4 Control volume in a heat-flow field.

or if k is not simply $A + BT$, it can be much more difficult to specify a single accurate effective value of k. If ΔT is not large, one can still make a reasonably accurate approximation using a constant average value of k.

Now that we have Fourier's law in three dimensions, we see that heat conduction is more complex than it appeared to be in Chapter 1. We must now write the heat conduction equation in three dimensions. We begin, as we did in Chapter 1, with the First Law statement, eqn. (1.3):

$$Q = \frac{dU}{dt} \tag{1.3}$$

This time we apply eqn. (1.3) to a three-dimensional control volume, as shown in Fig. 2.4.[1] The control volume is a finite region of a conducting body, which we set aside for analysis. The surface is denoted as S and the volume and the region as R; both are at rest. An element of the surface, dS, is identified and two vectors are shown on dS: one is the unit normal vector, $\vec{n}$ (with $|\vec{n}| = 1$), and the other is the heat flux vector, $\vec{q} = -k\nabla T$, at that point on the surface.

We also allow the possibility that a volumetric heat release equal to $\dot{q}(\vec{r})$ W/m^3 is distributed through the region. This might be the result of chemical or nuclear reaction, of electrical resistance heating, of external radiation into the region or of still other causes.

With reference to Fig. 2.4, we can write the heat conducted *out* of dS, in watts, as

$$(-k\nabla T) \cdot (\vec{n}\,dS) \tag{2.4}$$

The heat generated (or consumed) within the region R must be added to the total heat flow *into* S to get the overall rate of heat addition to R:

$$Q = -\int_S (-k\nabla T) \cdot (\vec{n}\,dS) + \int_R \dot{q}\,dR \tag{2.5}$$

[1] Figure 2.4 is the three-dimensional version of the control volume shown in Fig. 1.8.

The rate of energy increase of the region R is

$$\frac{dU}{dt} = \int_R \left(\rho c \frac{\partial T}{\partial t}\right) dR \tag{2.6}$$

where the derivative of T is in partial form because T is a function of both $\vec{r}$ and t.

Finally, we combine Q, as given by eqn. (2.5), and dU/dt, as given by eqn. (2.6), into eqn. (1.3). After rearranging the terms, we obtain

$$\int_S k\nabla T \cdot \vec{n}\, dS = \int_R \left[\rho c \frac{\partial T}{\partial t} - \dot{q}\right] dR \tag{2.7}$$

To get the left-hand side into a convenient form, we introduce Gauss's theorem, which converts a surface integral into a volume integral. Gauss's theorem says that if $\vec{A}$ is any continuous function of position, then

$$\int_S \vec{A} \cdot \vec{n}\, dS = \int_R \nabla \cdot \vec{A}\, dR \tag{2.8}$$

Therefore, if we identify $\vec{A}$ with $(k\nabla T)$, eqn. (2.7) reduces to

$$\int_R \left(\nabla \cdot k\nabla T - \rho c \frac{\partial T}{\partial t} + \dot{q}\right) dR = 0 \tag{2.9}$$

Next, since the region R is arbitrary, the integrand must vanish identically.[2] We therefore get the *heat diffusion equation* in three dimensions:

$$\boxed{\nabla \cdot k\nabla T + \dot{q} = \rho c \frac{\partial T}{\partial t}} \tag{2.10}$$

The limitations on this equation are:

- Incompressible medium. (This was implied when no expansion work term was included.)

- No convection. (The medium cannot undergo any relative motion. However, it *can* be a liquid or gas as long as it sits still.)

[2]Consider $\int f(x)\, dx = 0$. If $f(x)$ were, say, $\sin x$, then this could only be true over intervals of $x = 2\pi$ or multiples of it. For eqn. (2.9) to be true for *any* range of integration one might choose, the terms in parentheses must be zero everywhere.

If the variation of k with T is small, k can be factored out of eqn. (2.10) to get

$$\boxed{\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.11}$$

This is a more complete version of the heat conduction equation [recall eqn. (1.14)] and α is the thermal diffusivity which was discussed after eqn. (1.14). The term $\nabla^2 T \equiv \nabla \cdot \nabla T$ is called the *Laplacian*. It arises thus in a Cartesian coordinate system:

$$\nabla \cdot k\nabla T \simeq k\nabla \cdot \nabla T = k\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial x}\right) \cdot \left(\vec{i}\frac{\partial T}{\partial x} + \vec{j}\frac{\partial T}{\partial y} + \vec{k}\frac{\partial T}{\partial z}\right)$$

or

$$\boxed{\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}} \tag{2.12}$$

The Laplacian can also be expressed in cylindrical or spherical coordinates. The results are:

- Cylindrical:

$$\nabla^2 T \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.13}$$

- Spherical:

$$\nabla^2 T \equiv \frac{1}{r}\frac{\partial^2 (rT)}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{2.14a}$$

or

$$\equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{2.14b}$$

where the coordinates are as described in Fig. 2.5.

2.2 Solutions of the heat diffusion equation

We are now in position to calculate the temperature distribution and/or heat flux in bodies with the help of the heat diffusion equation. In every

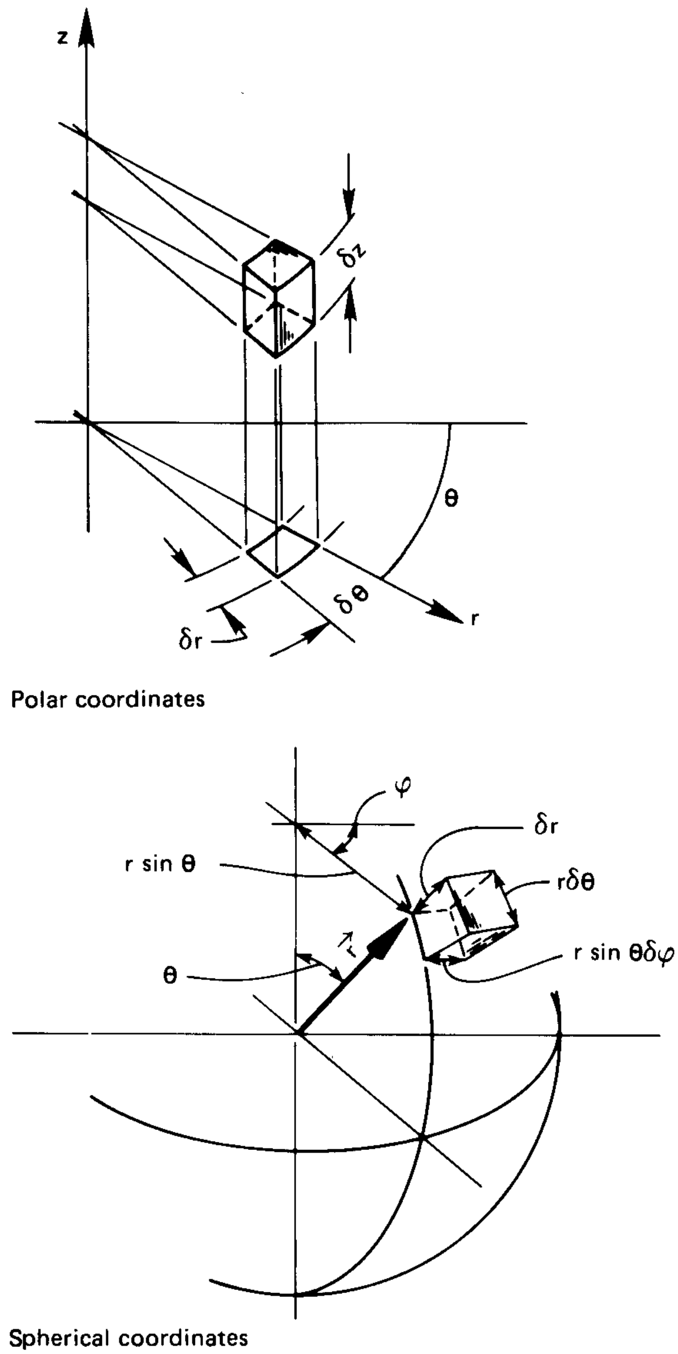

Figure 2.5 Cylindrical and spherical coordinate systems.

case, we first calculate $T(\vec{r}, t)$. Then, if we want the heat flux as well, we differentiate T to get q from Fourier's law.

The heat diffusion equation is a partial differential equation (p.d.e.) and the task of solving it may seem difficult, but we can actually do a lot with fairly elementary mathematical tools. For one thing, in one-dimensional steady-state situations the heat diffusion equation becomes an ordinary differential equation (o.d.e.); for another, the equation is linear and therefore not too formidable, in any case. Our procedure can be laid out, step by step, with the help of the following example.

Example 2.1 Basic Method

A large, thin concrete slab of thickness L is "setting." Setting is an exothermic process that releases $\dot{q}$ W/m^3. The outside surfaces are kept at the ambient temperature, so $T_w = T_\infty$. What is the maximum internal temperature?

Solution.

Step 1. *Pick the coordinate scheme that best fits the problem and identify the independent variables that determine T.* In the example, T will probably vary only along the thin dimension, which we will call the x-direction. (We should want to know that the edges are insulated and that L was much smaller than the width or height. If they are, this assumption should be quite good.) Since the interior temperature will reach its maximum value when the process becomes steady, we write $T = T(x \text{ only})$.

Step 2. *Write the appropriate d.e., starting with one of the forms of eqn. (2.11).*

$$\frac{\partial^2 T}{\partial x^2} + \underbrace{\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}}_{\substack{=0,\text{ since}\\ T \neq T(y \text{ or } z)}} + \frac{\dot{q}}{k} = \underbrace{\frac{1}{\alpha}\frac{\partial T}{\partial t}}_{\substack{=0,\text{ since}\\ \text{steady}}}$$

Therefore, since $T = T(x \text{ only})$, the equation reduces to the ordinary d.e.

$$\frac{d^2 T}{dx^2} = -\frac{\dot{q}}{k}$$

Step 3. *Obtain the general solution of the d.e.* (This is usually the easiest step.) We simply integrate the d.e. twice and get

$$T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

Step 4. *Write the "side conditions" on the d.e.—the initial and boundary conditions.* This is the trickiest part and the one that most seriously tests our physical or "practical" understanding any heat conduction problem.

Normally, we have to make two specifications of temperature on each position coordinate and one on the time coordinate to get rid of the constants of integration in the general solution. (These matters are discussed at greater length in Chapter 4.)

In this case we know two boundary conditions:

$$T(x = 0) = T_w \quad \text{and} \quad T(x = L) = T_w$$

Very Important Warning: Never, never introduce inaccessible information in a boundary or initial condition. Always stop and ask yourself, "Would I have access to a numerical value of the temperature (or other data) that I specify at a given position or time?" If the answer is no, then your result will be useless.

Step 5. *Substitute the general solution in the boundary and initial conditions and solve for the constants.* This process gets very complicated in the transient and multidimensional cases. Numerical methods are often needed to solve the problem. However, the steady one-dimensional problems are usually easy. In the example, by evaluating at $x = 0$ and $x = L$, we get:

$$\begin{aligned} T_w &= -0 + 0 + C_2 && \text{so} \quad C_2 = T_w \\ T_w &= -\frac{\dot{q}L^2}{2k} + C_1 L + \underbrace{C_2}_{=T_w} && \text{so} \quad C_1 = \frac{\dot{q}L}{2k} \end{aligned}$$

Step 6. *Put the calculated constants back in the general solution to get the particular solution to the problem.* In the example problem we obtain:

$$T = -\frac{\dot{q}}{2k}x^2 + \frac{\dot{q}}{2k}Lx + T_w$$

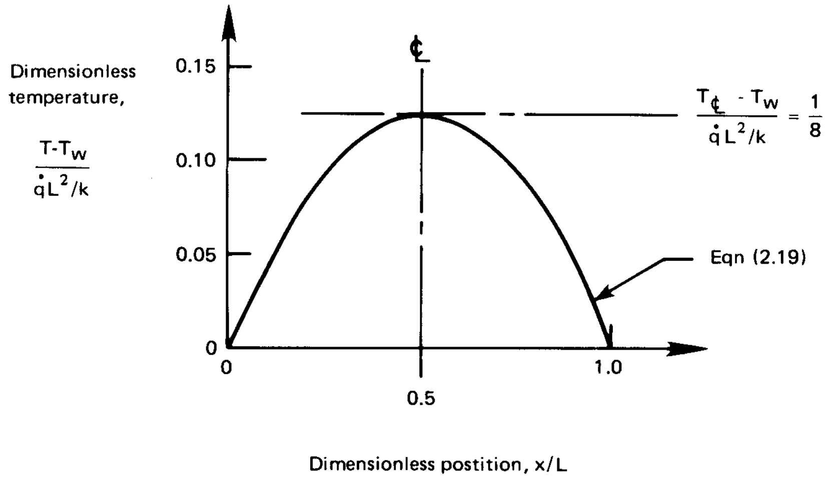

Figure 2.6 Temperature distribution in the setting concrete slab Example 2.1.

When we put this in neat dimensionless form, we can plot the result in Fig. 2.6 without having to know specific values of its parameters:

$$\frac{T - T_w}{\dot{q}L^2/k} = \frac{1}{2}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] \tag{2.15}$$

Step 7. *Play with the solution—look it over—see what it has to tell you. Make any checks you can think of to be sure it is correct.* In this case, the resulting temperature distribution is parabolic and, as we would expect, symmetrical. It satisfies the boundary conditions at the wall and maximizes in the center. By nondimensionalizing the result, we can represent all situations with a simple curve. That is highly desirable when the calculations are not simple, as they are here. (Even here T actually depends on *five* different things, and its solution is a single curve on a two-coordinate graph.)

Finally, we check to see if the heat flux at the wall is correct:

$$q_{\text{wall}} = -k\left.\frac{\partial T}{\partial x}\right|_{x=0} = k\left[\frac{\dot{q}}{k}x - \frac{\dot{q}L}{2k}\right]_{x=0} = -\frac{\dot{q}L}{2}$$

Thus, half of the total energy generated in the slab comes out of the front side, as we would expect. The solution appears to be correct.

Step 8. *If the temperature field is now correctly established, we can, if we wish, calculate the heat flux at any point in the body by substituting $T(\vec{r}, t)$ back into Fourier's law.* We did this already, in Step 7, to check our solution. ■

We offer additional examples in this section and the following one. In the process, we develop some important results for future use.

Example 2.2 The Simple Slab

A slab shown in Fig. 2.7 is at a steady state with dissimilar temperatures on either side and no internal heat generation. We want the temperature distribution and the heat flux through it.

Solution. These can be found quickly by following the steps set down in Example 2.1:

Step 1. $T = T(x)$ for steady x-direction heat flow

Step 2. $\dfrac{d^2T}{dx^2} = 0$, the steady 1-D heat equation with no $\dot{q}$

Step 3. $T = C_1 x + C_2$ is the general solution of that equation

Step 4. $T(x = 0) = T_1$ and $T(x = L) = T_2$ are the b.c.s

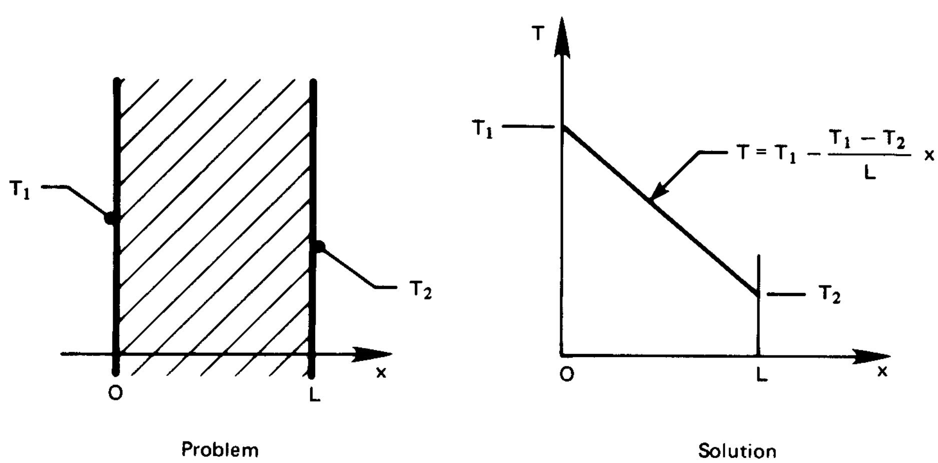

Figure 2.7 Heat conduction in a slab (Example 2.2).

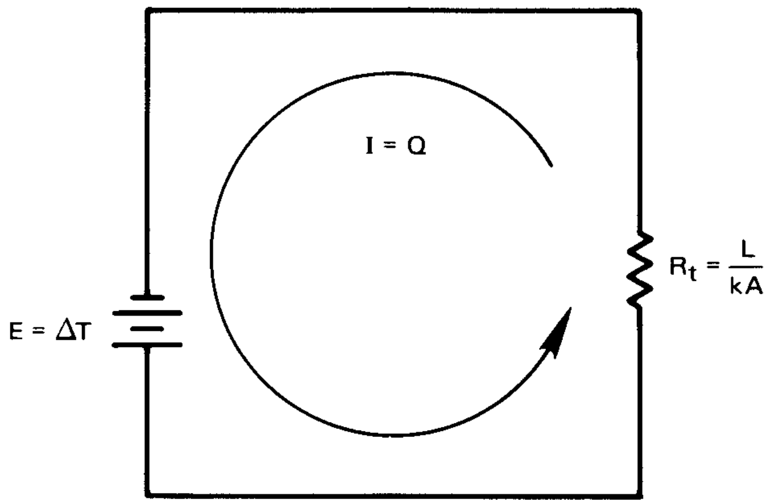

Figure 2.8 Ohm's law analogy to conduction through a slab.

Step 5. $T_1 = 0 + C_2$, so $C_2 = T_1$; and $T_2 = C_1 L + C_2$, so $C_1 = \dfrac{T_2 - T_1}{L}$

Step 6. $T = T_1 + \dfrac{T_2 - T_1}{L}x$; or $\dfrac{T - T_1}{T_2 - T_1} = \dfrac{x}{L}$

Step 7. We note that the solution satisfies the boundary conditions and that the temperature profile is linear.

Step 8. $q = -k\dfrac{dT}{dx} = -k\dfrac{d}{dx}\left(T_1 - \dfrac{T_1 - T_2}{L}x\right)$

so that $\boxed{q = k\dfrac{\Delta T}{L}}$ ■

This result, which is the simplest heat conduction solution, calls to mind Ohm's law. Thus, if we rearrange it:

$$Q = \frac{\Delta T}{L/kA} \qquad \text{is like} \qquad I = \frac{E}{R}$$

where L/kA assumes the role of a *thermal resistance*, to which we give the symbol R_t. R_t has the dimensions of (K/W). Figure 2.8 shows how we can represent heat flow through the slab with a diagram that is perfectly analogous to an electric circuit.

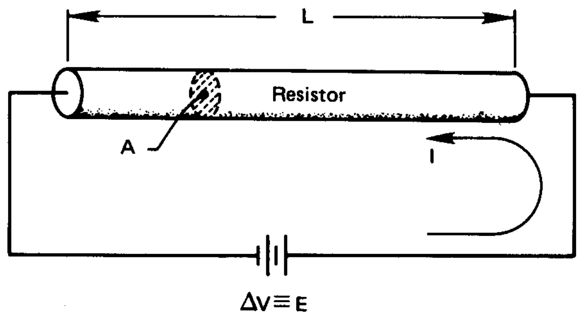

Figure 2.9 The one-dimensional flow of current.

2.3 Thermal resistance and the electrical analogy

Fourier's, Fick's, and Ohm's laws

Fourier's law has several extremely important analogies in other kinds of physical behavior, of which the electrical analogy is only one. These analogous processes provide us with a good deal of guidance in the solution of heat transfer problems. And, conversely, heat conduction analyses can often be adapted to describe those processes.

Let us first consider *Ohm's law* in three dimensions:

$$\text{flux of electrical charge} = \frac{\vec{I}}{A} \equiv \vec{J} = -\gamma\nabla V \tag{2.16}$$

$\vec{I}$ amperes is the vectorial electrical current, A is an area normal to the current vector, $\vec{J}$ is the flux of current or *current density*, γ is the electrical conductivity in cm/ohm·cm^2, and V is the voltage.

To apply eqn. (2.16) to a one-dimensional current flow, as pictured in Fig. 2.9, we write eqn. (2.16) as

$$J = -\gamma\frac{dV}{dx} = \gamma\frac{\Delta V}{L}, \tag{2.17}$$

but ΔV is the applied voltage, E, and the resistance of the wire is $R \equiv L/\gamma A$. Then, since $I = J\,A$, eqn. (2.17) becomes

$$I = \frac{E}{R} \tag{2.18}$$

which is the familiar, but restrictive, one-dimensional statement of Ohm's law.

Fick's law is another analogous relation. It states that during mass diffusion, the flux, $\vec{j}_1$, of a dilute component, 1, into a second fluid, 2, is proportional to the gradient of its mass concentration, m_1. Thus

$$\vec{j}_1 = -\rho \mathcal{D}_{12} \nabla m_1 \tag{2.19}$$

where the constant $\mathcal{D}_{12}$ is the binary diffusion coefficient.

Example 2.3

Air fills a thin tube 1 m in length. There is a small water leak at one end where the water vapor concentration builds to a mass fraction of 0.01. A desiccator maintains the concentration at zero on the other side. What is the steady flux of water from one side to the other if $\mathcal{D}_{12}$ is 2.84×10^{-5} m^2/s and $\rho = 1.18$ kg/m^3?

Solution.

$$\begin{aligned}\left|\vec{j}_{\text{water vapor}}\right| &= 1.18 \frac{\text{kg}}{\text{m}^3}\left(2.84 \times 10^{-5} \frac{\text{m}^2}{\text{s}}\right)\left(\frac{0.01 \text{ kg H}_2\text{O/kg mixture}}{1 \text{ m}}\right) \\ &= 3.35 \times 10^{-7} \frac{\text{kg}}{\text{m}^2 \cdot \text{s}}\end{aligned}$$

■

Contact resistance

The usefulness of the electrical resistance analogy is particularly apparent at the interface of two conducting media. No two solid surfaces ever form perfect thermal contact when they are pressed together. Since some roughness is always present, a typical plane of contact will always include tiny air gaps as shown in Fig. 2.10 (which is drawn with a highly exaggerated vertical scale). Heat transfer follows two paths through such an interface. Conduction through points of solid-to-solid contact is very effective, but conduction through the gas-filled interstices, which have low thermal conductivity, can be very poor. Thermal radiation across the gaps is also inefficient.

We treat the contact surface by placing an interfacial conductance, h_c, in series with the conducting materials on either side. The coefficient h_c is similar to a heat transfer coefficient and has the same units, W/m^{2}K. If ΔT is the temperature difference across an interface of area A, then $Q = A h_c \Delta T$. It follows that $Q = \Delta T / R_t$ for a contact resistance $R_t = 1/(h_c A)$ in K/W.

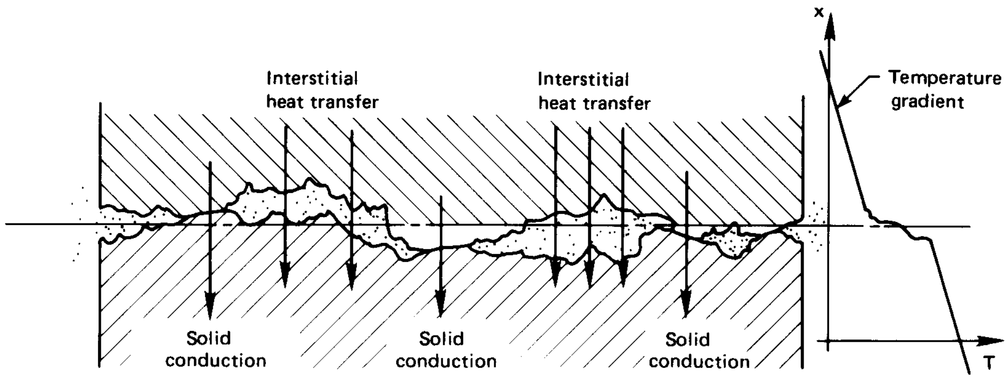

Figure 2.10 Heat transfer through the contact plane between two solid surfaces.

The interfacial conductance, h_c, depends on the following factors:

- The surface finish and cleanliness of the contacting solids.
- The materials that are in contact.
- The pressure with which the surfaces are forced together. This may vary over the surface, for example, in the vicinity of a bolt.
- The substance (or lack of it) in the interstitial spaces. Conductive shims or fillers can raise the interfacial conductance.
- The temperature at the contact plane.

The influence of contact pressure is usually a modest one up to around 10 atm in most metals. Beyond that, increasing plastic deformation of the local contact points causes h_c to increase more dramatically at high pressure. Table 2.1 gives typical values of contact resistances which bear out most of the preceding points. These values have been adapted from [2.1, Chpt. 3] and [2.2]. Theories of contact resistance are discussed in [2.3] and [2.4].

Example 2.4

Heat flows through two stainless steel slabs ($k = 18$ W/m·K) that are pressed together. The slab area is $A = 1$ m^2. How thick must the slabs be for contact resistance to be negligible?

Table 2.1 Some typical interfacial conductances for normal surface finishes and moderate contact pressures (about 1 to 10 atm). Air gaps not evacuated unless so indicated.

Situation	h_c (W/m^{2}K)
Iron/aluminum (70 atm pressure)	45,000
Copper/copper	10,000 – 25,000
Aluminum/aluminum	2,200 – 12,000
Graphite/metals	3,000 – 6,000
Ceramic/metals	1,500 – 8,500
Stainless steel/stainless steel	2,000 – 3,700
Ceramic/ceramic	500 – 3,000
Stainless steel/stainless steel (evacuated interstices)	200 – 1,100
Aluminum/aluminum (low pressure and evacuated interstices)	100 – 400

Solution. With reference to Fig. 2.11, the total or *equivalent* resistance is found by adding these resistances, which are in series:

$$R_{t_{\text{equiv}}} = \frac{L}{kA} + \frac{1}{h_c A} + \frac{L}{kA} = \frac{1}{A}\left(\frac{L}{18} + \frac{1}{h_c} + \frac{L}{18}\right)$$

Since h_c is about 3,000 W/m^{2}K,

$$\frac{2L}{18} \text{ must be } \gg \frac{1}{3000} = 0.00033$$

Thus, L must be large compared to $18(0.00033)/2 = 0.003$ m if contact resistance is to be ignored. If $L = 3$ cm, the error is about 10%. ■

Resistances for cylinders and for convection

As we continue developing our method of solving one-dimensional heat conduction problems, we find that other avenues of heat flow may also be expressed as thermal resistances, and introduced into the solutions that we obtain. We also find that, once the heat conduction equation has been solved, the results themselves may be used as new thermal resistances.

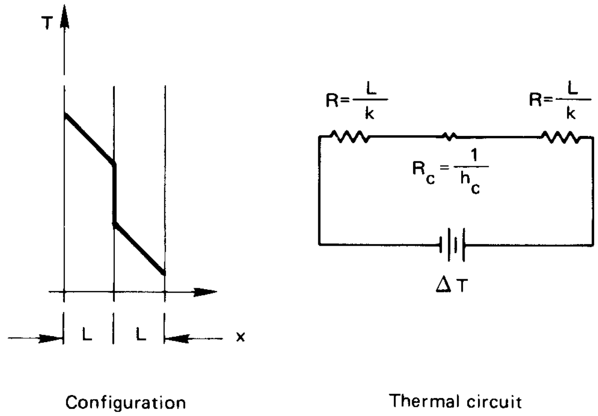

Figure 2.11 Conduction through two unit-area slabs with a contact resistance.

Example 2.5 Radial Heat Conduction in a Tube

Find the temperature distribution and the heat flux for the long hollow cylinder shown in Fig. 2.12.

Solution.

Step 1. $T = T(r)$

Step 2.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}}_{=0,\text{ since } T \neq T(\phi, z)} + \underbrace{\frac{\dot{q}}{k}}_{=0} = \underbrace{\frac{1}{\alpha}\frac{\partial T}{\partial t}}_{=0,\text{ since steady}}$$

Step 3. Integrate once: $r\dfrac{\partial T}{\partial r} = C_1$; integrate again: $T = C_1 \ln r + C_2$

Step 4. $T(r = r_i) = T_i$ and $T(r = r_o) = T_o$

Step 5.

$$\begin{aligned} T_i &= C_1 \ln r_i + C_2 \\ T_o &= C_1 \ln r_o + C_2 \end{aligned} \quad \Longrightarrow \quad \begin{cases} C_1 = \dfrac{T_i - T_o}{\ln(r_i/r_o)} = -\dfrac{\Delta T}{\ln(r_o/r_i)} \\ C_2 = T_i + \dfrac{\Delta T}{\ln(r_o/r_i)} \ln r_i \end{cases}$$

Step 6. $T = T_i - \dfrac{\Delta T}{\ln(r_o/r_i)}(\ln r - \ln r_i)$ or

$$\boxed{\frac{T - T_i}{T_o - T_i} = \frac{\ln(r/r_i)}{\ln(r_o/r_i)}} \tag{2.20}$$

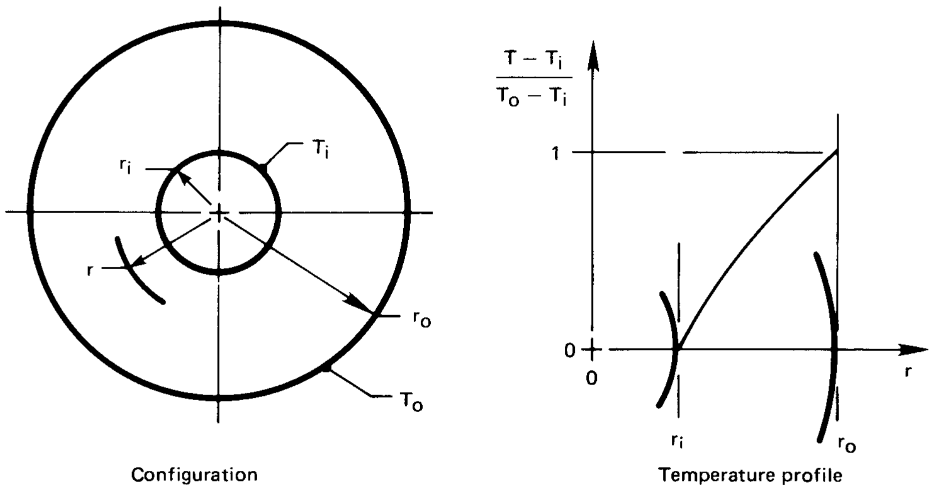

Figure 2.12 Heat transfer through a cylinder with a fixed wall temperature (Example 2.5).

Step 7. The solution is plotted in Fig. 2.12. We see that the temperature profile is logarithmic and that it satisfies both boundary conditions. Furthermore, it is instructive to see what happens when the wall of the cylinder is very thin, or when r_i/r_o is close to 1. In this case:

$$\ln(r/r_i) \simeq \frac{r}{r_i} - 1 = \frac{r - r_i}{r_i}$$

and

$$\ln(r_o/r_i) \simeq \frac{r_o - r_i}{r_i}$$

Thus eqn. (2.20) becomes

$$\frac{T - T_i}{T_o - T_i} = \frac{r - r_i}{r_o - r_i}$$

which is a simple linear profile. This is the same solution that we would get in a plane wall.

Step 8. At any station, r, with $\Delta T = T_i - T_o$:

$$q_{\text{radial}} = -k\frac{\partial T}{\partial r} = +\frac{k\Delta T}{\ln(r_o/r_i)}\frac{1}{r}$$

So the heat *flux* falls off inversely with radius. That is reasonable, since the same heat flow must pass through an increasingly large surface as the radius increases. Let us see if this is the case for a cylinder of length l:

$$Q\ (\mathrm{W}) = (2\pi r l)\, q = \frac{2\pi k l \Delta T}{\ln(r_o/r_i)} \neq f(r) \tag{2.21}$$

Finally, we again recognize Ohm's law in this result and write the thermal resistance for a cylinder:

$$R_{t_{\text{cyl}}} = \frac{\ln(r_o/r_i)}{2\pi l k} \left(\frac{\mathrm{K}}{\mathrm{W}}\right) \tag{2.22}$$

This can be compared with the resistance of a plane wall:

$$R_{t_{\text{wall}}} = \frac{L}{kA} \left(\frac{\mathrm{K}}{\mathrm{W}}\right)$$

Both resistances are inversely proportional to k, but each reflects a different geometry. ■

In the preceding examples, the boundary conditions were all the same—a temperature specified at an outer edge. Next let us suppose that the temperature is specified in the environment away from a body, with a heat transfer coefficient between the environment and the body.

Example 2.6 A Convective Boundary Condition

A convective heat transfer coefficient around the outside of the cylinder in Example 2.5 provides thermal resistance between the cylinder and an environment at $T = T_\infty$, as shown in Fig. 2.13. Find the temperature distribution and heat flux in this case.

Solution.

Step 1 through 3. These are the same as in Example 2.5.

Step 4. The first boundary condition is $T(r = r_i) = T_i$. The second boundary condition must be expressed as an energy balance at the outer wall (recall Section 1.3).

$$q_{\text{convection}} = q_{\substack{\text{conduction}\\ \text{at the wall}}}$$

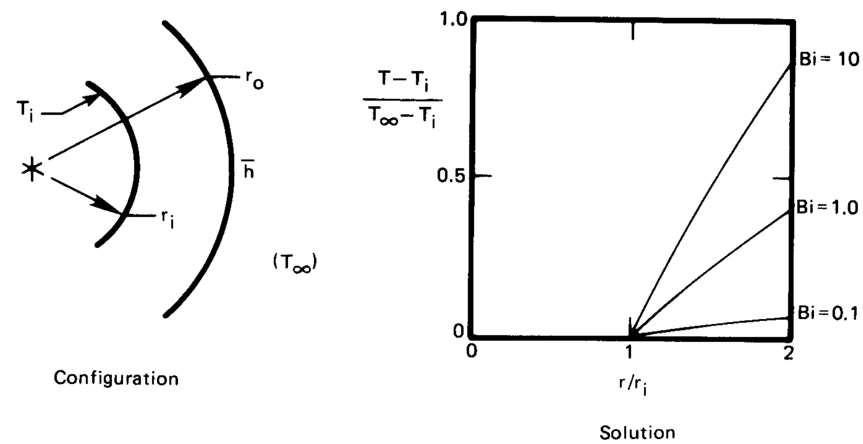

Figure 2.13 Heat transfer through a cylinder with a convective boundary condition (Example 2.6).

or

$$\overline{h}(T - T_\infty)_{r=r_o} = -k\left.\frac{\partial T}{\partial r}\right|_{r=r_o}$$

Step 5. From the first boundary condition we obtain $T_i = C_1 \ln r_i + C_2$. It is easy to make mistakes when we substitute the general solution into the second boundary condition, so we will do it in detail:

$$\overline{h}\Big[(C_1 \ln r + C_2) - T_\infty\Big]_{r=r_o} = -k\left[\frac{\partial}{\partial r}(C_1 \ln r + C_2)\right]_{r=r_o} \qquad (2.23)$$

A common error is to substitute $T = T_o$ on the lefthand side instead of substituting the entire general solution. That will do no good, because T_o is not an accessible piece of information. Equation (2.23) reduces to:

$$\overline{h}(T_\infty - C_1 \ln r_o - C_2) = \frac{kC_1}{r_o}$$

When we combine this with the result of the first boundary con-

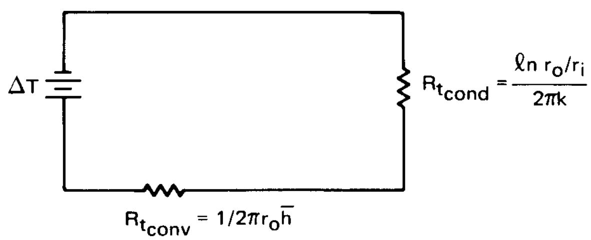

Figure 2.14 Thermal circuit with two resistances.

dition to eliminate C_2:

$$C_1 = -\frac{T_i - T_\infty}{k/(\overline{h}r_o) + \ln(r_o/r_i)} = \frac{T_\infty - T_i}{1/\text{Bi} + \ln(r_o/r_i)}$$

Then

$$C_2 = T_i - \frac{T_\infty - T_i}{1/\text{Bi} + \ln(r_o/r_i)} \ln r_i$$

Step 6.

$$T = \frac{T_\infty - T_i}{1/\text{Bi} + \ln(r_o/r_i)} \ln(r/r_i) + T_i$$

This can be rearranged in fully dimensionless form:

$$\frac{T - T_i}{T_\infty - T_i} = \frac{\ln(r/r_i)}{1/\text{Bi} + \ln(r_o/r_i)} \tag{2.24}$$

Step 7. Let us fix a value of r_o/r_i—say, 2—and plot eqn. (2.24) for several values of the Biot number. The results are included in Fig. 2.13. Some very important things show up in this plot. When Bi $\gg$ 1, the solution reduces to the solution given in Example 2.5. It is as though the convective resistance to heat flow were not there. That is exactly what we anticipated in Section 1.3 for large Bi. When Bi $\ll$ 1, the opposite is true: $(T - T_i)/(T_\infty - T_i)$ remains on the order of Bi, and internal conduction can be neglected. How big is big and how small is small? We do not really have to specify exactly. But in this case Bi < 0.1 signals constancy of temperature inside the cylinder with about $\pm 3\%$. Bi > 20 means that we can neglect convection with about 5% error.

Step 8. $q_{\text{radial}} = -k\dfrac{\partial T}{\partial r} = k\dfrac{T_i - T_\infty}{1/\text{Bi} + \ln(r_o/r_i)}\dfrac{1}{r}$

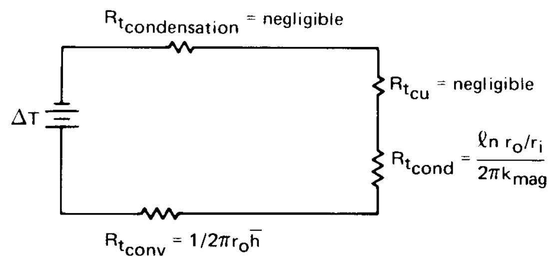

Figure 2.15 Thermal circuit for an insulated tube.

This can be written in terms of Q (W) $= q_{\text{radial}}\,(2\pi r l)$ for a cylinder of length l:

$$Q = \frac{T_i - T_\infty}{\dfrac{1}{\overline{h}\,2\pi r_o l} + \dfrac{\ln(r_o/r_i)}{2\pi k l}} = \frac{T_i - T_\infty}{R_{t_{\text{conv}}} + R_{t_{\text{cond}}}} \tag{2.25}$$

Equation (2.25) is once again analogous to Ohm's law. But this time the denominator is the sum of two thermal resistances, as would be the case in a series circuit. We accordingly present the analogous electrical circuit in Fig. 2.14.

The presence of convection on the outside surface of the cylinder causes a new thermal resistance of the form

$$R_{t_{\text{conv}}} = \frac{1}{\overline{h}A} \tag{2.26}$$

where A is the surface area over which convection occurs. ■

Example 2.7 Critical Radius of Insulation

An interesting consequence of the preceding result can be brought out with a specific example. Suppose that we insulate a 0.5 cm O.D. copper steam line with 85% magnesia to prevent the steam from condensing too rapidly. The steam is under pressure and stays at 150°C. The copper is thin and highly conductive—obviously a tiny resistance in series with the convective and insulation resistances, as we see in Fig. 2.15. The condensation of steam inside the tube also offers very little resistance.[3] But on the outside, a heat transfer coefficient of $\overline{h}$

[3] Condensation heat transfer is discussed in Chapter 8. It turns out that $\overline{h}$ is generally enormous during condensation so that $R_{t_{\text{condensation}}}$ is tiny.

= 20 W/m^{2}K offers fairly high resistance. It turns out that insulation can actually *improve* heat transfer in this case.

The two significant resistances, for a cylinder of unit length (l = 1 m), are

$$R_{t_{\text{cond}}} = \frac{\ln(r_o/r_i)}{2\pi k l} = \frac{\ln(r_o/r_i)}{2\pi(0.074)} \quad \text{K/W}$$
$$R_{t_{\text{conv}}} = \frac{1}{2\pi r_o \overline{h}} = \frac{1}{2\pi(20) r_o} \quad \text{K/W}$$

Figure 2.16 is a plot of these resistances and their sum. A very interesting thing occurs here. $R_{t_{\text{conv}}}$ falls off rapidly when r_o is increased, because the outside area is increasing. Accordingly, the total resistance passes through a minimum in this case. Will it always do so? To find out, we differentiate eqn. (2.25), again setting l = 1 m:

$$\frac{dQ}{dr_o} = \frac{(T_i - T_\infty)}{\left(\dfrac{1}{2\pi r_o \overline{h}} + \dfrac{\ln(r_o/r_i)}{2\pi k}\right)^2}\left(-\frac{1}{2\pi r_o^2 \overline{h}} + \frac{1}{2\pi k r_o}\right) = 0$$

When we solve this for the value of $r_o = r_{\text{crit}}$ at which Q is maximum and the total resistance is minimum, we obtain

$$\text{Bi} = 1 = \frac{\overline{h} r_{\text{crit}}}{k} \tag{2.27}$$

In the present example, adding insulation will *increase* heat loss instead of reducing it, until $r_{\text{crit}} = k/\overline{h} = 0.0037$ m or $r_{\text{crit}}/r_i = 1.48$. Indeed, insulation will not even start to do any good until $r_o/r_i = 2.32$ or $r_o = 0.0058$ m. We call r_{crit} the *critical radius* of insulation. ■

There is an interesting catch here. For most cylinders, $r_{\text{crit}} < r_i$ and the critical radius idiosyncrasy is of no concern. If our steam line had a 1 cm outside diameter, the critical radius difficulty would not have arisen. When cooling smaller diameter cylinders, such as electrical wiring, the critical radius must be considered, but one need not worry about it in the design of most large process equipment.

Resistance for thermal radiation

We saw in Chapter 1 that the net radiation exchanged by two objects is given by eqn. (1.34):

$$Q_{\text{net}} = A_1 \mathcal{F}_{1\text{-}2}\, \sigma \left(T_1^4 - T_2^4\right) \tag{1.34}$$

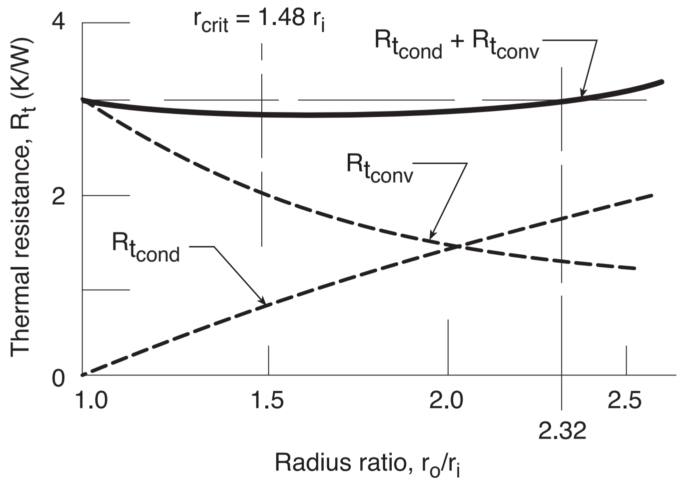

Figure 2.16 The critical radius of insulation (Example 2.7), written for a cylinder of unit length ($l = 1$ m).

When T_1 and T_2 are close, we can approximate this equation using a *radiation heat transfer coefficient*, h_{rad}. Specifically, suppose that the temperature difference, $\Delta T = T_1 - T_2$, is small compared to the mean temperature, $T_m = (T_1 + T_2)/2$. Then we can make the following expansion and approximation:

$$
\begin{aligned}
Q_{\text{net}} &= A_1 \mathcal{F}_{1\text{-}2}\, \sigma \left(T_1^4 - T_2^4\right) \\
&= A_1 \mathcal{F}_{1\text{-}2}\, \sigma (T_1^2 + T_2^2)(T_1^2 - T_2^2) \\
&= A_1 \mathcal{F}_{1\text{-}2}\, \sigma \underbrace{(T_1^2 + T_2^2)}_{=\,2T_m^2 + (\Delta T)^2/2} \underbrace{(T_1 + T_2)}_{=2T_m} \underbrace{(T_1 - T_2)}_{=\Delta T} \\
&\cong A_1 \underbrace{\left(4\sigma T_m^3\, \mathcal{F}_{1\text{-}2}\right)}_{\equiv h_{\text{rad}}} \Delta T
\end{aligned}
\tag{2.28}
$$

where the last step assumes that $(\Delta T)^2/2 \ll 2T_m^2$ or $(\Delta T/T_m)^2/4 \ll 1$. Thus, we have identified the radiation heat transfer coefficient

$$
\left.
\begin{aligned}
Q_{\text{net}} &= A_1 h_{\text{rad}} \Delta T \\
h_{\text{rad}} &= 4\sigma T_m^3 \mathcal{F}_{1\text{-}2}
\end{aligned}
\right\} \quad \text{for} \quad \left(\Delta T/T_m\right)^2/4 \ll 1
\tag{2.29}
$$

This leads us immediately to the introduction of a radiation thermal resistance, analogous to that for convection:

$$R_{t_{\text{rad}}} = \frac{1}{A_1 h_{\text{rad}}} \tag{2.30}$$

For the special case of a small object (1) in a much larger environment (2), the transfer factor is given by eqn. (1.35) as $\mathcal{F}_{1\text{-}2} = \varepsilon_1$, so that

$$h_{\text{rad}} = 4\sigma T_m^3 \varepsilon_1 \tag{2.31}$$

If the small object is black, its emittance is $\varepsilon_1 = 1$ and h_{rad} is maximized. For a black object radiating near room temperature, say $T_m = 300$ K,

$$h_{\text{rad}} = 4(5.67 \times 10^{-8})(300)^3 \cong 6 \text{ W/m}^2\text{K}$$

This value is of approximately the same size as $\overline{h}$ for natural convection into a gas at such temperatures. Thus, the heat transfer by thermal radiation and natural convection into gases are similar. Both effects must be taken into account. In forced convection in gases, on the other hand, $\overline{h}$ might well be larger than h_{rad} by an order of magnitude or more, so that thermal radiation can be neglected.

Example 2.8

An electrical resistor dissipating 0.1 W has been mounted well away from other components in an electronical cabinet. It is cylindrical with a 3.6 mm O.D. and a length of 10 mm. If the air in the cabinet is at 35°C and at rest, and the resistor has $\overline{h} = 13$ W/m²K for natural convection and $\varepsilon = 0.9$, what is the resistor's temperature? Assume that the electrical leads are configured so that little heat is conducted into them.

Solution. The resistor may be treated as a small object in a large isothermal environment. To compute h_{rad}, let us estimate the resistor's temperature as 50°C. Then

$$T_m = (35 + 50)/2 \cong 43°\text{C} = 316 \text{ K}$$

so

$$h_{\text{rad}} = 4\sigma T_m^3 \varepsilon = 4(5.67 \times 10^{-8})(316)^3(0.9) = 6.44 \text{ W/m}^2\text{K}$$

Heat is lost by natural convection and thermal radiation acting in parallel. To find the equivalent thermal resistance, we combine the two parallel resistances as follows:

$$\frac{1}{R_{t_{\text{equiv}}}} = \frac{1}{R_{t_{\text{rad}}}} + \frac{1}{R_{t_{\text{conv}}}} = Ah_{\text{rad}} + A\overline{h} = A(h_{\text{rad}} + \overline{h})$$

Thus,

$$R_{t_{\text{equiv}}} = \frac{1}{A(h_{\text{rad}} + \overline{h})}$$

A calculation shows $A = 133\text{ mm}^2 = 1.33 \times 10^{-4}\text{ m}^2$ for the resistor surface. Thus, the equivalent thermal resistance is

$$R_{t_{\text{equiv}}} = \frac{1}{(1.33 \times 10^{-4})(13 + 6.44)} = 386.8\text{ K/W}$$

Since

$$Q = \frac{T_{\text{resistor}} - T_{\text{air}}}{R_{t_{\text{equiv}}}}$$

We find

$$T_{\text{resistor}} = T_{\text{air}} + Q \cdot R_{t_{\text{equiv}}} = 35 + (0.1)(386.8) = 73.68\ ^\circ\text{C}$$

We guessed a resistor temperature of 50°C in finding h_{rad}. Recomputing with this higher temperature, we have $T_m = 327$ K and $h_{\text{rad}} = 7.17\text{ W/m}^2\text{K}$. If we repeat the rest of the calculation, we get a new value $T_{\text{resistor}} = 72.3$°C. Further iteration is not needed.

Since the use of h_{rad} is an approximation, we should check its applicability:

$$\frac{1}{4}\left(\frac{\Delta T}{T_m}\right)^2 = \frac{1}{4}\left(\frac{72.3 - 35.0}{327}\right)^2 = 0.00325 \ll 1$$

In this case, the approximation is a very good one. ■

Example 2.9

Suppose that power to the resistor in Example 2.8 is turned off. How long does it take to cool? The resistor has $k \cong 10$ W/m·K, $\rho \cong 2000\text{ kg/m}^3$, and $c_p \cong 700$ J/kg·K.

Solution. The lumped capacity model, eqn. (1.22), may be applicable. To find out, we check the resistor's Biot number, noting that

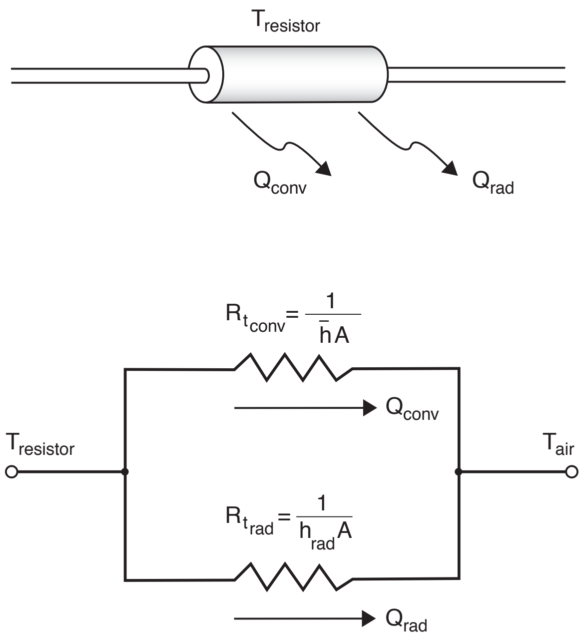

Figure 2.17 An electrical resistor cooled by convection and radiation.

the parallel convection and radiation processes have an *effective* heat transfer coefficient $h_{\text{eff}} = \overline{h} + h_{\text{rad}} = 20.17\ \text{W/m}^2\text{K}$. Then,

$$\text{Bi} = \frac{h_{\text{eff}} r_o}{k} = \frac{(20.17)(0.0036/2)}{10} = 0.0036 \ll 1$$

so eqn. (1.22) can be used to describe the cooling process. The time constant is

$$\boldsymbol{T} = \frac{\rho c_p V}{h_{\text{eff}} A} = \frac{(2000)(700)\pi(0.010)(0.0036)^2/4}{(20.17)(1.33 \times 10^{-4})} = 53.1\ \text{s}$$

From eqn. (1.22) with $T_0 = 72.3°\text{C}$

$$T_{\text{resistor}} = 35.0 + (72.3 - 35.0)e^{-t/53.1}\ °\text{C}$$

Ninety-five percent of the total temperature drop has occured when $t = 3\boldsymbol{T} = 159$ s. ■

2.4 Overall heat transfer coefficient, U

Definition

We often want to transfer heat through composite resistances, such as the series of resistances shown in Fig. 2.18. It is very convenient to have a number, U, that works like this[4]:

$$\boxed{Q = UA\,\Delta T} \tag{2.32}$$

This number, called the *overall heat transfer coefficient*, is defined largely by the system, and in many cases it proves to be insensitive to the operating conditions of the system.

In Example 2.6, for instance, two resistances are in series. We can use the value Q given by eqn. (2.25) to get

$$U = \frac{Q\ (\mathrm{W})}{[2\pi r_o l\,(\mathrm{m}^2)]\ \Delta T\ (\mathrm{K})} = \frac{1}{\dfrac{1}{\overline{h}} + \dfrac{r_o \ln(r_o/r_i)}{k}} \quad (\mathrm{W/m^2K}) \tag{2.33}$$

We have based U on the outside area, $A_o = 2\pi r_o l$, in this case. We might instead have based it on inside area, $A_i = 2\pi r_i l$, and obtained

$$U = \frac{1}{\dfrac{r_i}{\overline{h} r_o} + \dfrac{r_i \ln(r_o/r_i)}{k}} \tag{2.34}$$

It is therefore important to remember which area an overall heat transfer coefficient is based on. It is particularly important that A and U be consistent when we write $Q = UA\,\Delta T$.

In general, for any composite resistance, the overall heat transfer coefficient may be obtained from the equivalent resistance. The equivalent resistance is calculated taking account of series and parallel resistors, as in Examples 2.4 and 2.8. Then, because $Q = \Delta T/R_{t_\mathrm{equiv}} = UA\,\Delta T$, it follows that $UA = 1/R_{t_\mathrm{equiv}}$.

Example 2.10

Estimate the overall heat transfer coefficient for the tea kettle shown in Fig. 2.19. Note that the flame convects heat to the thin aluminum. The heat is then conducted through the aluminum and finally convected by boiling into the water.

[4]This U must not be confused with internal energy. The two terms should always be distinct in context.

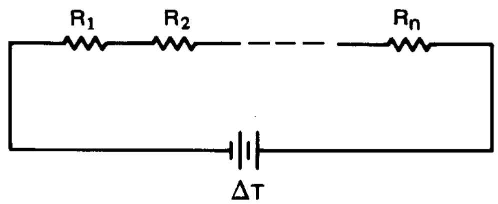

Figure 2.18 A thermal circuit with many resistances in series. The equivalent resistance is $R_{t_{\text{equiv}}} = \sum_i R_i$.

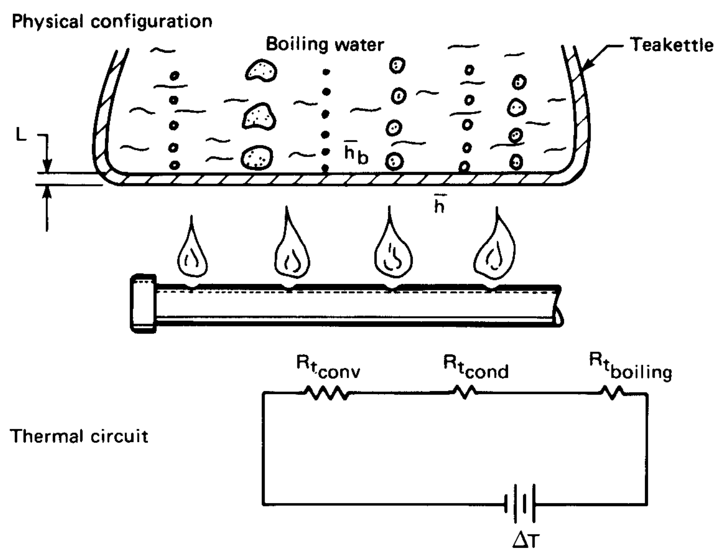

Figure 2.19 Heat transfer through the bottom of a tea kettle.

Solution. We need not worry about deciding which area to base A on, in this case, because the area normal to the heat flux vector does not change. We simply write the heat flow

$$Q = \frac{\Delta T}{\sum R_t} = \frac{T_{\text{flame}} - T_{\text{boiling water}}}{\dfrac{1}{\overline{h}A} + \dfrac{L}{k_{\text{Al}}A} + \dfrac{1}{\overline{h}_b A}}$$

and apply the definition of U

$$U = \frac{Q}{A\Delta T} = \frac{1}{\dfrac{1}{\overline{h}} + \dfrac{L}{k_{\text{Al}}} + \dfrac{1}{\overline{h}_b}}$$

Let us see what typical numbers would look like in this example: $\overline{h}$ might be around 200 W/m^{2}K; L/k_{Al} might be 0.001 m/(160 W/m·K) or 1/160,000 W/m^{2}K; and $\overline{h}_b$ is quite large—perhaps about 5000 W/m^{2}K.

Thus:

$$U \simeq \frac{1}{\dfrac{1}{200} + \dfrac{1}{160{,}000} + \dfrac{1}{5000}} = 192.1 \text{ W/m}^2\text{K} \qquad \blacksquare$$

It is clear that the first resistance is dominant, as is shown in Fig. 2.19. Notice that in such cases

$$UA \longrightarrow 1/R_{t_{\text{dominant}}} \tag{2.35}$$

where A is any area (inside or outside) in the thermal circuit.

Experiment 2.1

Boil water in a paper cup over an open flame and explain why you can do so. [Recall eqn. (2.35) and see Problem 2.12.]

Example 2.11

A wall consists of alternating layers of pine and sawdust, as shown in Fig. 2.20). The sheathes on the outside have negligible resistance and $\overline{h}$ is known on the sides. Compute Q and U for the wall.

Solution. So long as the wood and the sawdust do not differ dramatically from one another in thermal conductivity, we can approximate the wall as a parallel resistance circuit, as shown in the figure.[5] The equivalent thermal resistance of the circuit is

$$R_{t_{\text{equiv}}} = R_{t_{\text{conv}}} + \frac{1}{\left(\dfrac{1}{R_{t_{\text{pine}}}} + \dfrac{1}{R_{t_{\text{sawdust}}}}\right)} + R_{t_{\text{conv}}}$$

Thus

$$Q = \frac{\Delta T}{R_{t_{\text{equiv}}}} = \frac{T_{\infty_1} - T_{\infty_r}}{\dfrac{1}{\overline{h}A} + \dfrac{1}{\left(\dfrac{k_p A_p}{L} + \dfrac{k_s A_s}{L}\right)} + \dfrac{1}{\overline{h}A}}$$

[5] For this approximation to be exact, the resistances must be equal. If they differ radically, the problem must be treated as two-dimensional.

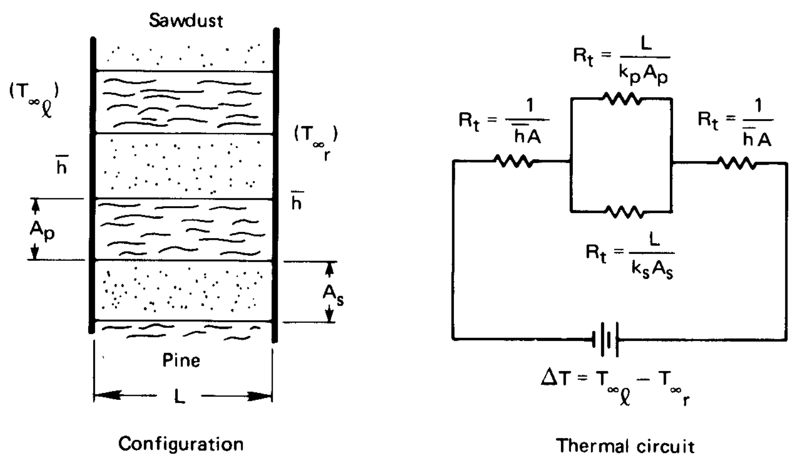

Figure 2.20 Heat transfer through a composite wall.

and

$$U = \frac{Q}{A\Delta T} = \frac{1}{\dfrac{2}{\bar{h}} + \dfrac{1}{\left(\dfrac{k_p}{L}\dfrac{A_p}{A} + \dfrac{k_s}{L}\dfrac{A_s}{A}\right)}} \qquad \blacksquare$$

The approach illustrated in this example is very widely used in calculating U values for the walls and roofs houses and buildings. The thermal resistances of each structural element — insulation, studs, siding, doors, windows, etc. — are combined to calculate U or $R_{t_{\text{equiv}}}$, which is then used together with weather data to estimate heating and cooling loads [2.5].

Typical values of U

In a fairly general use of the word, a heat exchanger is anything that lies between two fluid masses at different temperatures. In this sense a heat exchanger might be designed either to impede or to enhance heat exchange. Consider some typical values of U shown in Table 2.2, which were assembled from a variety of technical sources. If the exchanger is intended to improve heat exchange, U will generally be much greater than 40 W/m^{2}K. If it is intended to impede heat flow, it will be less than 10 W/m^{2}K—anywhere down to almost perfect insulation. You should have some numerical concept of relative values of U, so we recommend

Table 2.2 Typical ranges or magnitudes of U

Heat Exchange Configuration	U (W/m^{2}K)
Walls and roofs dwellings with a 24 km/h outdoor wind:	
• Insulated roofs	0.3–2
• Finished masonry walls	0.5–6
• Frame walls	0.3–5
• Uninsulated roofs	1.2–4
Single-pane windows	$\sim 6^{\dagger}$
Air to heavy tars and oils	As low as 45
Air to low-viscosity liquids	As high as 600
Air to various gases	60–550
Steam or water to oil	60–340
Liquids in coils immersed in liquids	110–2,000
Feedwater heaters	110–8,500
Air condensers	350–780
Steam-jacketed, agitated vessels	500–1,900
Shell-and-tube ammonia condensers	800–1,400
Steam condensers with 25°C water	1,500–5,000
Condensing steam to high-pressure boiling water	1,500–10,000

† Main heat loss is by infiltration.

that you scrutinize the numbers in Table 2.2. Some things worth bearing in mind are:

- The fluids with low thermal conductivities, such as tars, oils, or any of the gases, usually yield low values of $\overline{h}$. When such fluid flows on one side of an exchanger, U will generally be pulled down.

- Condensing and boiling are very effective heat transfer processes. They greatly improve U but they cannot override one very small value of $\overline{h}$ on the other side of the exchange. (Recall Example 2.10.)

- For a high U, *all* resistances in the exchanger must be low.

- The highly conducting liquids, such as water and liquid metals, give high values of $\overline{h}$ and U.

Fouling resistance

Figure 2.21 shows one of the simplest forms of a heat exchanger—a pipe. The inside is new and clean on the left, but on the right it has built up a layer of scale. In conventional freshwater preheaters, for example, this scale is typically $MgSO_4$ (magnesium sulfate) or $CaSO_4$ (calcium sulfate) which precipitates onto the pipe wall after a time. To account for the resistance offered by these buildups, we must include an additional, highly empirical resistance when we calculate U. Thus, for the pipe shown in Fig. 2.21,

$$U\Big|_{\substack{\text{older pipe}\\ \text{based on } A_i}} = \frac{1}{\dfrac{1}{h_i} + \dfrac{r_i \ln(r_o/r_p)}{k_{\text{insul}}} + \dfrac{r_i \ln(r_p/r_i)}{k_{\text{pipe}}} + \dfrac{r_i}{r_o h_o} + R_f}$$

where R_f is a fouling resistance for a unit area of pipe (in m^2K/W). And clearly

$$\boxed{R_f \equiv \frac{1}{U_{\text{old}}} - \frac{1}{U_{\text{new}}}} \tag{2.36}$$

Some typical values of R_f are given in Table 2.3. These values have been adapted from [2.6] and [2.7]. Notice that fouling has the effect of adding a resistance in series on the order of 10^{-4} m^2K/W. It is rather like another heat transfer coefficient, $\overline{h}_f$, on the order of 10,000 W/m^2K in series with the other resistances in the exchanger.

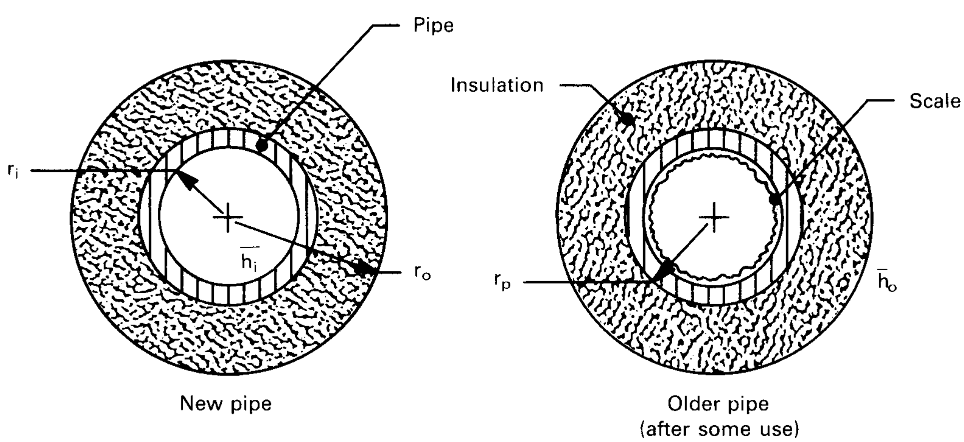

Figure 2.21 The fouling of a pipe.

Table 2.3 Some typical fouling resistances for a unit area.

Fluid and Situation	*Fouling Resistance* R_f (m^2K/W)
Distilled water	0.0001
Seawater	0.0001 – 0.0004
Treated boiler feedwater	0.0001 – 0.0002
Clean river or lake water	0.0002 – 0.0006
About the worst waters used in heat exchangers	< 0.0020
No. 6 fuel oil	0.0001
Transformer or lubricating oil	0.0002
Most industrial liquids	0.0002
Most refinery liquids	0.0002 – 0.0009
Steam, non-oil-bearing	0.0001
Steam, oil-bearing (e.g., turbine exhaust)	0.0003
Most stable gases	0.0002 – 0.0004
Flue gases	0.0010 – 0.0020
Refrigerant vapors (oil-bearing)	0.0040

The tabulated values of R_f are given to only one significant figure because they are very approximate. Clearly, exact values would have to be referred to specific heat exchanger configurations, to particular fluids, to fluid velocities, to operating temperatures, and to age [2.8, 2.9]. The resistance generally drops with increased velocity and increases with temperature and age. The values given in the table are based on reasonable maintenance and the use of conventional shell-and-tube heat exchangers. With misuse, a given heat exchanger can yield much higher values of R_f.

Notice too, that if $U \lesssim 1{,}000$ W/m^{2}K, fouling will be unimportant because it will introduce a negligibly small resistance in series. Thus, in a water-to-water heat exchanger, for which U is on the order of 2000 W/m^{2}K, fouling might be important; but in a finned-tube heat exchanger with hot gas in the tubes and cold gas passing across the fins on them, U might be around 200 W/m^{2}K, and fouling will be usually be insignificant.